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On Peixoto's Conjecture for Flows on Non-Orientable 2-Manifolds

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Resumo

Seja M uma variedade bidimensional compacta não-orientável de gênero 4. Então existe uma família de campos vetoriais de Kupka-Smale quase-minimais $\{X_\mu \in \mathfrak{X}^\infty(M)\}$, dependendo suavemente de $\mu \in [0,\epsilon)$, tal que, para alguma caixa de fluxo $V \subset M$ de X_0, e para todo $\mu, \nu \in [0,\epsilon)$, (i) $X_\mu|_V$ é uma caixa de fluxo; (ii) Se $\mu \neq \nu$, X_μ é uma perturbação C^∞–twist de X_ν localizada em V; e (iii) X_μ e X_ν são topologicamente equivalentes.

Abstract

Let M be a non-orientable compact 2-manifold of genus 4. Then there exists a family of quasi-minimal, Kupka-Smale vector fields $\{X_\mu \in \mathfrak{X}^\infty(M)\}$, depending smoothly on $\mu \in [0,\epsilon)$, such that, for some flow box $V \subset M$ of X_0, and for all $\mu, \nu \in [0,\epsilon)$, (i) $X_\mu|_V$ is a flow box; (ii) If $\mu \neq \nu$, X_μ is a C^∞–twist perturbation of X_ν localized in V; and (iii) X_μ and X_ν are topologically equivalent.

1. INTRODUCTION

The importance of the C^r-Closing Lemma Problem lies in the fact that a positive answer to it would lead to very deep positive conclusions in Dynamical Systems, in topics related to the Generic, Stability and Bifurcation Theories. As a consequence of its role, there are several useful C^r-Closing Lemmas. As a sample of some of the important results, we wish to mention Peixoto's C^r-Connecting Lemma [24], Pugh's C^1-Closing Lemma [26], Mañe's C^1-Ergodic Closing Lemma [19], Gutierrez C^r-counterexample [9], Herman's C^r-Closing Lemma [17, 18] , Hayashi's C^1-Connecting Lemma [14, 15, 16]. Besides the articles that will be quoted in this paper, in the same way as above, we wish to mention [12, 13, 20], [22]–[34]. As a sample of some very recent results that use C^1-Closing Lemma results, we wish to mention [3, 5].

Let $\mathfrak{X}^r(M)$, $0 \leq r \leq \infty$, denote the space of C^r–vector fields (with the C^r–topology) on a compact, connected, boundaryless, C^∞, 2-manifold M. Our (compact) flow boxes $V \subset M$ of X will be either the standard ones or those such that $X|_V$ is topogically equivalent to the constant vector field $(1,0)$ on the cylinder $[0,1] \times I\!R/Z\!\!Z$. Notice that in the second case, for all $t \in (0,1)$, $\{t\} \times I\!R/Z\!\!Z$ is a tranversal circle to the vector field $(1,0)$.
To state our main result we shall need the following

Definition 1.1.1 (twist perturbation of a vector field). *Let M be a 2-manifold, $X \in \mathfrak{X}^r(M)$ and $V \subset M$ be a compact flow box of X. Given $Y \in \mathfrak{X}^r(M)$ with support in V, we say that $X + Y$ is a C^r-twist perturbation of X, localized in V, if $X|_{V^o}$ is transversal to $Y|_{V^o}$, where V^o denotes the interior of V.*

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Points of $\mathbb{R}/\mathbb{Z}$ will be denoted as if they were points of $\mathbb{R}$; in this way, we shall use expressions of the form $x + a$ and $x - a$ when refering to points of $\mathbb{R}/\mathbb{Z}$. Also, if $x < y$ are real numbers such that $y - x < 1$, the subinterval (x, y) of $\mathbb{R}$ determines a unique subinterval of $\mathbb{R}/\mathbb{Z}$.

DEFINITION 1.1.2 (smooth/affine/isometric *iet*). *Let* Γ *be either the the unit interval* $[0,1)$ *or the unit circle* $\mathbb{R}/\mathbb{Z}$. *Given a finite subset* $S = \{a_0, a_1, \cdots, a_n\}$ *of* Γ, *with* $0 = a_0 < a_1 < \cdots < a_n = 1$, *a smooth interval exchange transformation, shortly smooth iet , will be an injective transformation* $T : \Gamma \backslash S \to \Gamma$ *such that* $T|_{(a_{i-1},a_i)}$ *(i.e.* T *restricted to the interval* (a_{i-1}, a_i)*) is a smooth–diffeomorphism,* $1 \le i \le n$, *and the range of* T *is all* Γ *but n points. If, moreover,* $T|_{(a_{i-1},a_i)}$ *is an affine (isometric) transformation for all* $1 \le i \le n$, *we call* T *an affine (isometric) iet . As usual, the term iet will refer to an isometric iet .*

In this paper, we prove the following

THEOREM 1.1.3 (smooth *iet* version). *There exist a minimal isometric iet* $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ *and a family of diffeomorphisms* $\{G_\mu : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, 0 \le \mu < \epsilon\}$, *with* G_0 *being the identity map, depending smoothly on* $\mu \in [0, \epsilon)$, *such that, for all* $(\mu_0, p_0) \in [0, \epsilon) \times \mathbb{R}/\mathbb{Z}$,

(1) $\frac{d}{d\mu}|_{\mu=\mu_0} G_\mu(p_0) > 0$ *(in particular for all* $\mu \in (0, \epsilon)$, G_μ *has no fixed points);*
(2) $G_{\mu_0} \circ T$ *is a smooth iet* C^∞*-conjugate to* T.

Given a smooth *iet* T, defined on $\mathbb{R}/\mathbb{Z}$, there exist a compact 2–manifold M, containning $\mathbb{R}/\mathbb{Z}$, and a vector field $X \in \mathfrak{X}^\infty(M)$ transversal to $\mathbb{R}/\mathbb{Z}$ such that the first return Poincaré map $\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ induced by X is precisely T (see [8]). Using this, the result above can be extended to vector fields as follows:

THEOREM 1.1.4 (vector field version). *Let* M *be a non-orientable compact 2-manifold of genus 4. Then there exists a family of quasi-minimal, Kupka-Smale vector fields* $\{X_\mu \in \mathfrak{X}^\infty(M)\}$, *depending smoothly on* $\mu \in [0, \epsilon)$, *such that, for some flow box* $V \subset M$ *of* X_0, *which can be taken to be homeomorphic to either a rectangle or a cylinder, and for all* $\mu, \nu \in [0, \epsilon)$,

(1) $X_\mu|_V$ *is a flow box;*
(2) *If* $\mu \ne \nu$, X_μ *is a* C^∞*-twist perturbation of* X_ν *localized in* V;
(3) X_μ *and* X_ν *are topologically equivalent.*

Now we relate our results to Peixoto's Conjecture. Let Σ^r be the subset of $\mathfrak{X}^r(M)$ formed by the Morse-Smale C^r–vector fields. M. Peixoto states in [24] the following conjecture.

(PC) Let M be a non-orientable 2-manifold. $X \in \mathfrak{X}^r(M)$ is structurally stable if and only if $X \in \Sigma^r$. Moreover, Σ^r is open and dense in $\mathfrak{X}^r(M)$.

Peixoto [24] proved this Conjecture for M orientable. As a consequence of Peixoto's work and Pugh's C^1–Closing Lemma [26, 30], it follows that Σ^1 is dense in $\mathfrak{X}^1(M)$. There are some partial results concerning **(PC)** in class C^r, $r \geq 1$: The conjecture is true both for the projective plane $\mathbf{P}^2$ and for the Klein bottle $\mathbf{K}^2$ (see in [21] the proof that flows on $\mathbf{K}^2$ do not have non-trivial recurrence). Gutierrez [6] showed that **(PC)** is true for the torus with one cross-cap.

By [24], **(PC)** is true if, and only if, it is possible to give an affirmative answer for the following C^r–Connecting Lemma question:

(CL) Let $X \in \mathfrak{X}^r(M)$ have finitely many singularities, all hyperbolic (at least one singularity). Suppose that X has a non-trivial recurrent trajectory. Does there exist an arbitrarily small C^r–perturbation of X such that the resulting vector field has one more saddle connection than X?

We recall that, when M is orientable, **(CL)** has a positive answer by arbitrarily small C^r–twist perturbations [24]. Also, when M is non-orientable, there are many cases in which **(CL)** has a positive answer, again by arbitrarily small C^r–twist perturbations [11].

Hence, it is natural to wonder whether, in the non-orientable case, we may make use of an arbitrary C^r–twist perturbation in order to give an affirmative answer for **(CL)**. Theorem 1.1.4 gives a negative answer to this question.

Finally, the Theorems 1.1.3 and 1.1.4 above are relevant to the C^r-closing Lemma problem because the positive answer to the C^r–Closing Lemma, given in [7], for a large class of flows on orientable two-manifolds, was obtained by means of twist perturbations (see also [1, 2, 4]).

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2. A FLOW ON THE TORUS WITH TWO CROSS-CAPS

Let $\varphi : I\!R \times M \to M$ be a flow on M and $\gamma = \gamma(t)$ a trajectory of φ. We denote by $\omega(\gamma)(\alpha(\gamma))$ the ω-limit set (α-limit set) of γ. We say that γ is ω–recurrent (α–recurrent) if $\gamma \subset \omega(\gamma)(\gamma \subset \alpha(\gamma))$. A recurrent trajectory is a trajectory which is ω–recurrent or α–recurrent. A fixed point and a closed orbit are called trivial recurrent trajectories.

Given a C^1 flow $\varphi : I\!R \times M \to M$ with a recurrent trajectory γ and a point $p \in \gamma$, Peixoto proved that there exists a smooth circle Γ, transversal to φ, passing through the point p (see [6, 24]). We shall analyse flows in terms of their action on transverse circles. Let us recall now the example of flow given by Gutierrez in [10].

Gutierrez constructed in [10] an *iet* $T : I\!R/Z\!\!Z \to I\!R/Z\!\!Z$ and a suspension of T which is a quasi-minimal C^∞–flow $\varphi : I\!R \times M \to M$, on the the torus with two cross-caps M, in such a way that

(1) φ has two hyperbolic saddle points as its only singularities;

(2) $I\!R/Z\!\!Z$ is a subset of M, φ is transversal to $I\!R/Z\!\!Z$, and the *iet* $T : I\!R/Z\!\!Z \to I\!R/Z\!\!Z$ is precisely the Poincaré return map induced by φ on $I\!R/Z\!\!Z$; moreover, for some real positive numbers a, b, c, d, e,

• $\mathrm{dom}(T) = I\!R/Z\!\!Z \setminus \{a, a+b, a+b+c, 1\}$, where $\mathrm{dom}(T)$ is the the domain of definition of T;

• the numbers $a, a+e-c, a-c-d+2e, \cdots$, are ordered (modulo 1) according to Figures 1 and 2. In particular,
$0 < a < b < e < a+b < 1-e < a+b+c < a+b+c+d = 1$, $d < e$,
$a+b+c+e = 1+e-d$;

• T operates upon the intervals below, by reversing orientation, in the following way:

$$\begin{aligned} (a, a+b) &\mapsto (a+e, a+b+e), \\ (a+b+c, 1) &\mapsto (e-d, e); \end{aligned}$$

• T operates upon the intervals below, by preserving orientation, in the following way:

$$\begin{aligned} (0, a) &\mapsto (e, a+e); \\ (a+b, a+b+c) &\mapsto (a+b+e, e-d). \end{aligned}$$

In next section, given a small number $\delta > 0$, we will introduce an affine *iet* T_δ which is topologically conjugate to T and 2δ-close to T, in the uniform C^0-topology.

3. TOPOLOGICAL CONJUGACY

DEFINITION 3.3.1 (T_δ). *Let $\delta > 0$ be small and let T_δ be the affine iet satisfying*

•*T_δ operates upon the intervals below, linearly, diffeomorphically, and reversing orientation, in the following way:*

$$\begin{aligned} (a, a+e-c+\delta] &\mapsto [a+b+c, a+b+e+\delta), \\ [a+e-c+\delta, a-c-d+2e+2\delta] &\mapsto [1-e-\delta, a+b+c], \\ [a-c-d+2e+2\delta, 2a-c+e] &\mapsto [b+c+\delta, 1-e-\delta], \\ [2a-c+e, a+b) &\mapsto (a+e+\delta, b+c+\delta], \\ (a+b+c, a+b+e+\delta] &\mapsto [c, e+\delta), \\ [a+b+e+\delta, a+2e+2\delta] &\mapsto [c+b-e, c], \\ [a+2e+2\delta, b+c+e+2\delta] &\mapsto [a, c+b-e], \\ [b+c+e+2\delta, 1) &\mapsto (e-d+\delta, a]; \end{aligned}$$

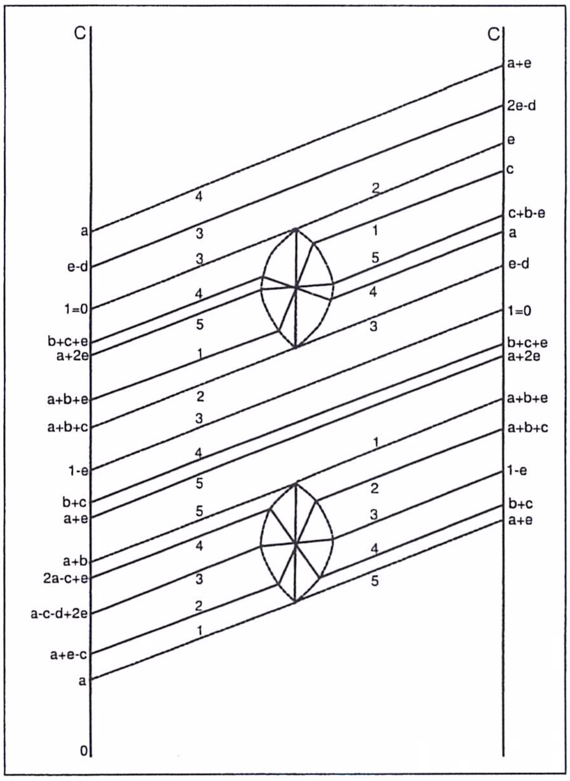

FIG. 1. The first return map T induced by φ on $\mathrm{R}/\mathbb{Z}$

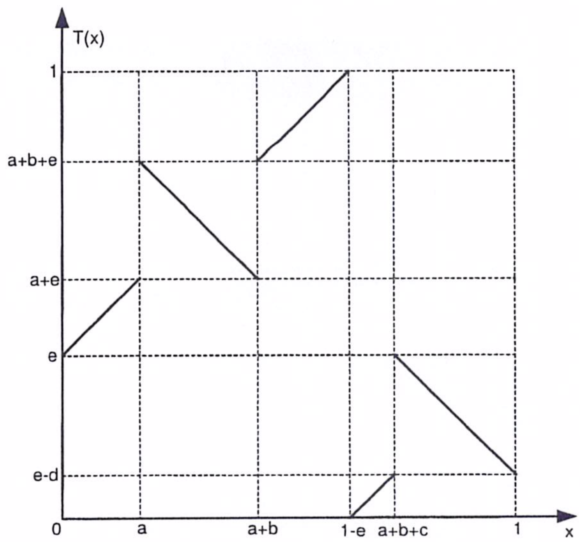

FIG. 2. The isometric *iet* T

•T_δ *operates upon the intervals below, linearly, diffeomorphically and preserving orientation, in the following way:*

$$
\begin{aligned}
(a+b, a+e+\delta] &\mapsto (a+b+e+\delta, a+2e+2\delta], \\
[a+e+\delta, b+c+\delta] &\mapsto [a+2e+2\delta, b+c+e+2\delta], \\
[b+c+\delta, 1-e-\delta) &\mapsto [b+c+e+2\delta, 1), \\
(1-e-\delta, a+b+c) &\mapsto (0, e-d+\delta), \\
(0, e-d+\delta] &\mapsto (e+\delta, 2e-d+2\delta], \\
[e-d+\delta, a) &\mapsto [2e-d+2\delta, a+e+\delta).
\end{aligned}
$$

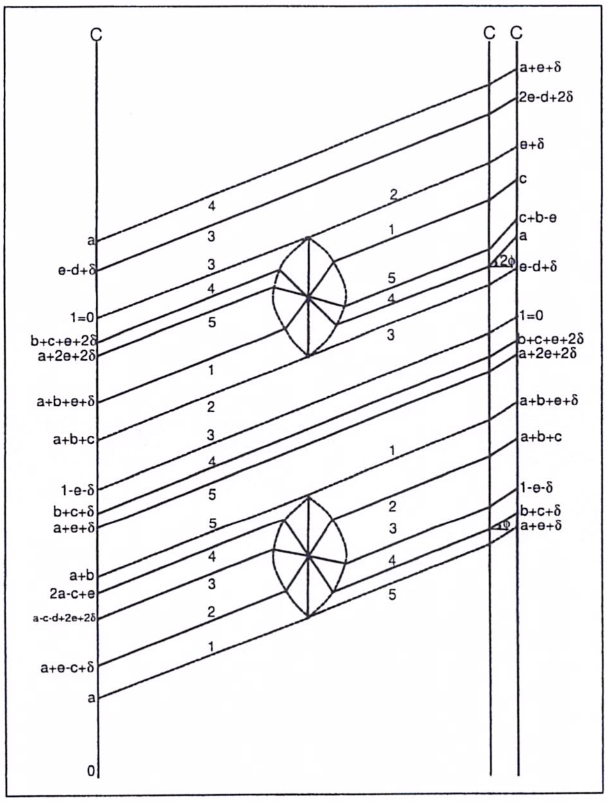

FIG. 3. The affine *iet* T_δ

PROPOSITION 3.3.2 (topological conjugacy). *Given $\delta > 0$, there exist a fixed-point-free homeomorphism*

$$H = H_\delta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$$

and a piecewise affine homeomorphism

$$h = h_\delta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$$

such that

(1) $T_\delta = H \circ T$

(2) $\mathrm{fix}(h) \supset \mathrm{dis}(T_\delta) = \mathrm{dis}(T) = \mathbb{R}/\mathbb{Z} \setminus \mathrm{dom}(T_\delta) = \{a, a+b, a+b+c, 1\}$, *where* $\mathrm{fix}(h)$ *(resp.* $\mathrm{dis}(T)$*) denotes the set of fixed points of h (resp. the discontinuity point set of T);*

(3) $(h^{-1} \circ T \circ h)(x) = T_\delta(x) = (H \circ T)(x), \forall x \in \mathrm{dom}(T_\delta)$; *in particular, T and T_δ are topologically conjugate;*

(4) $h = h_\delta \to I$, *in the uniform C^0-topology, as δ goes to 0, where* $I : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ *denotes the identity transformation; and*

(5) the derivative $(h_\delta)'$ of h_δ is a piecewise constant map which converges uniformly to the constant map 1 as $\delta \to 0$.

Proof. The existence of h follows from the definition of T_δ. Put

$$\begin{aligned}
h(a) &= a,\\
h(a+e-c+\delta) &= a+e-c,\\
h(a-c-d+2e+2\delta) &= a-c-d+2e,\\
h(2a-c+e) &= 2a-c+e,\\
h(a+b) &= a+b,\\
h(a+e+\delta) &= a+e,\\
h(b+c+\delta) &= b+c,\\
h(1-e-\delta) &= 1-e,\\
h(a+b+c) &= a+b+c,\\
h(a+b+e+\delta) &= a+b+e,\\
h(a+2e+2\delta) &= a+2e,\\
h(b+c+e+2\delta) &= b+c+e,\\
h(1) &= 1,\\
h(e-d+\delta) &= e-d.
\end{aligned}$$

We may extend h linearly to the other points so that h becomes a piecewise affine homeomorphism of the unit circle $\mathbb{R}/\mathbb{Z}$. It follows at once that h satisfies (2)–(5). ∎

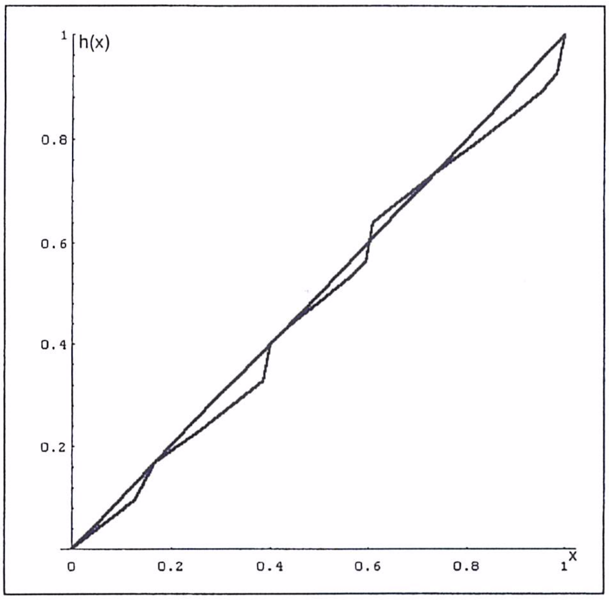

FIG. 4. The homeomorphism h in local coordinates

4. C^∞–CONJUGACY

In this section we show that the homeomorphism h conjugating T and T_δ and the fixed-point-free homeomorphism H can be substituted by a C^∞–diffeomorphism g and a fixed-point-free C^∞–diffeomorphism G, respectively, in such a way that the relation $g^{-1} \circ T \circ g = G \circ T$ remains true.

Recall $\mathrm{dis}(T) \subset \mathrm{fix}(h)$ and that

$$\begin{aligned} \mathrm{dom}(T) &= \mathbb{R}/\mathbb{Z} \backslash \{a, a+b, a+b+c, 1\} \\ \mathrm{dis}(T) &= \{a, a+b, a+b+c, 1\} \\ \mathrm{dom}(T^{-1}) &= \mathbb{R}/\mathbb{Z} \backslash \{e, a+e, a+b+e, e-d\} \\ \mathrm{dis}(T^{-1}) &= \{e, a+e, a+b+e, e-d\} \end{aligned}$$

Let $\mathcal{G}$ be the set of C^∞–diffeomorphisms $g : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ such that

(P1) $g|_U = I|_U$, where U is a small neighborhood of $\mathrm{fix}(h)$;

(P2) $g'(x) \geq \frac{1}{2}$, $\forall x \in \mathbb{R}/\mathbb{Z}$;

LEMMA 4.4.1. *If $g \in \mathcal{G}$ then*

$$\text{(P3)} \quad T(g(x)) - T(x) = T'(x)(g(x) - x), \quad \forall x \in \text{dom}(T)$$

(where $T'(x) \in \{-1, 1\}$).

Proof. An immediate consequence of (P1) is that for any $x \in \text{dom}(T)$, x and $g(x)$ lie in the same interval of the partition associated to the *iet* T; that is, for each $x \in$ dom(T), there exists $1 \leq i \leq n$ such that x, $g(x) \in (a_{i-1}, a_i)$. This implies the lemma. ∎

We prove now the main result of this section.

PROPOSITION 4.4.2. *Let $g \in \mathcal{G}$.*

(1) The map

$$G = g^{-1} \circ T \circ g \circ T^{-1}$$

is well defined in $\text{dom}(T^{-1})$ *and extends to a C^∞-diffeomorphism $G : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ such that $\forall x \in \text{dom}(T)$, $(g^{-1} \circ T \circ g)(x) = (G \circ T)(x)$. In particular, the isometric iet T and the smooth iet $G \circ T$ are C^∞-conjugate.*

(2) Fix $\delta > 0$ small and let $h = h_\delta$ be as in Proposition 3.3.2. If $g \in \mathcal{G}$ is C^0-close enough to h and G is as above, then $G : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a fixed-point-free C^∞-diffeomorphism.

Proof. Extend G to the whole $\mathbb{R}/\mathbb{Z}$ by defining $G(q) = g^{-1}(q), \forall q \in \text{dis}(T^{-1})$. Notice that G is bijective, $G|_{\text{dom}(T^{-1})}$ is smooth, and $\left(G|_{\text{dom}(T^{-1})}\right)^{-1}$ is smooth. If x is in a small neighborhood W_q of $q \in \text{dis}(T^{-1})$, $x \neq q$, then $T^{-1}(x)$ is in a neighborhood U_p of p for some $p \in \text{dis}(T) \subset \text{fix}(h)$ and by (P1) and by definition of G, we get

$$G(x) = g^{-1}(x),\ \forall x \in W_q$$

Therefore, G is smooth at any point $q \in \text{dis}(T^{-1})$ and as, by (P2), $G'(q) \neq 0$, we obtain that G^{-1} is a C^∞–diffeomorphism. By definition of G, $(g^{-1} \circ T \circ g)(x) = (G \circ T)(x)$, $\forall x \in$ dom(T). This proves (1). If g is C^0-close to h, then G will also be C^0-close to H; therefore, as H is a fixed-point-free homeomorphism, we will obtain that G is also fixed-point-free. ∎

5. MAIN RESULTS

In this section we prove Theorems 1.1.3 and 1.1.4. We start by proving Theorem 1.1.3.

Theorem 1.3 (smooth *iet* version) *There exist a minimal isometric iet $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ and a family of diffeomorphisms $\{G_\mu : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, 0 \leq \mu < \epsilon\}$, with G_0 being the identity map, depending smoothly on $\mu \in [0, \epsilon)$, such that, for all $(\mu_0, p_0) \in [0, \epsilon) \times \mathbb{R}/\mathbb{Z}$,*

(1) $\frac{d}{d\mu}\big|_{\mu=\mu_0} G_\mu(p_0) > 0$;
(2) $G_{\mu_0} \circ T$ *is a smooth iet* C^∞*-conjugate to* T.

Proof. By Proposition 4.4.2, there exist a diffeomorphism g with Properties (P1) to (P3) and a fixed-point-free diffeomorphism G such that $g^{-1} \circ T \circ g = G \circ T$ at every point of $\mathrm{dom}(T)$. Given $\mu \in [0,1]$, let $g_\mu : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be defined by

$$g_\mu(x) = \mu g(x) + (1-\mu)x \tag{1}$$

Notice that $\{g_\mu : \mu \in [0,1]\}$ provides a smooth isotopy between the identity map $g_0 = I$ and $g_1 = g$; also by (P2),

$$\frac{dg_\mu}{dx}(x) = \mu g'(x) + 1 - \mu \geq \frac{\mu}{2} + 1 - \mu = 1 - \frac{\mu}{2} \geq \frac{1}{2},\ \forall x \in \mathbb{R}/\mathbb{Z}$$

Thus, by construction, g_μ is a diffeomorphism of the unit circle satisfying (P1)-(P3), for each $\mu \in [0,1]$, that is, $g_\mu \in \mathcal{G}$. Given $\mu \in [0,1]$, by Proposition 4.4.2, there exists a diffeomorphism $G_\mu : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ satisfying

$$(G_\mu \circ T)(x) = \big((g_\mu)^{-1} \circ T \circ g_\mu\big)(x),\ \forall x \in \mathrm{dom}(T) \tag{2}$$

The proof of this Theorem follows at once from the following lemma

Lemma 5.5.1. *Given* $x \in \mathbb{R}/\mathbb{Z}$, *the map* $\mu \in [0,1] \mapsto G_\mu(x) \in \mathbb{R}/\mathbb{Z}$ *is differentiable. Moreover, there exist* $\epsilon > 0$ *and* $\sigma > 0$ *such that* $\forall (\mu_0, x) \in [0,\epsilon) \times \mathbb{R}/\mathbb{Z}$, $\frac{d}{d\mu}\Big|_{\mu=\mu_0} G_\mu(x) > \sigma$.

Proof. Let $u \in (0,1]$ be a real number. Then, from (2),

$$\big(T \circ g_u\big)(x) = \big(g_u \circ G_u \circ T\big)(x),\ \forall x \in \mathrm{dom}(T) \tag{3}$$

From (1), we reach

$$g_u(y) = y + u(g(y) - y),\ \forall y \in \mathbb{R}/\mathbb{Z} \tag{4}$$

From (4) and from property (P3) of g_u, we obtain

$$\frac{(T \circ g_u)(x) - (T \circ g_0)(x)}{u} = \frac{T'(x) \cdot \big(g_u(x) - x\big)}{u} = T'(x) \cdot \big(g(x) - x\big) \tag{5}$$

From equations (3)-(5), we get

$$\begin{aligned} T'(x) \cdot \big(g(x) - x\big) &\overset{(5)}{=} \{(T \circ g_u)(x) - (T \circ g_0)(x)\}/u = \\ &\overset{(3)}{=} \{(g_u \circ G_u \circ T)(x) - (g_0 \circ G_0 \circ T)(x)\}/u = \\ &\overset{(4)}{=} \frac{G_u(T(x)) - G_0(T(x))}{u} + g(G_u(T(x))) - G_u(T(x)). \end{aligned}$$

When $u \to 0$, we get

$$\frac{d}{d\mu}\Big|_{\mu=0} G_\mu(T(x)) = T'(x) \cdot \big(g(x) - x\big) - \big(g(T(x)) - T(x)\big) \tag{6}$$

And from Property (P3) of g, we reach

$$\frac{d}{d\mu}\Big|_{\mu=0} G_\mu(T(x)) = T(g(x)) - g(T(x)), \quad \forall x \in \mathrm{dom}(T) \tag{7}$$

As $G = g^{-1} \circ T \circ g \circ T^{-1}$ has no fixed points, we have that $T(g(x)) - g(T(x)) \neq 0$, $\forall x \in \mathrm{dom}(T)$. Therefore,

$$\frac{d}{d\mu}\Big|_{\mu=0} G_\mu(T(x)) \neq 0, \ \forall x \in \mathrm{dom}(T).$$

That is,

$$\frac{d}{d\mu}\Big|_{\mu=0} G_\mu(y) \neq 0, \quad \forall y \in \mathrm{dom}(T^{-1}). \tag{8}$$

If $q \in \mathrm{dis}(\mathrm{T}^{-1})$ then $G_\mu(q) = (g_\mu)^{-1}(q)$, $\forall \mu$. Besides, we have by Figure 4 that $q - g(q) > 0$ (the points $(q, g(q))$ with $q \in \mathrm{dis}(T^{-1})$ lie below the diagonal). Observe that

$$\begin{aligned} \big(g_\mu \circ (g_\mu)^{-1}\big)(x) &= x, \quad \forall x \in I\!R/\mathbb{Z} \\ \mu \cdot g((g_\mu)^{-1}(x)) + (1-\mu)(g_\mu)^{-1}(x) &= x, \quad \forall x \in I\!R/\mathbb{Z}, \end{aligned}$$

Therefore, by derivating the previous equation with respect to μ in $\mu = 0$, we get

$$\frac{d}{d\mu}\Big|_{\mu=0} G_\mu(q) = \frac{d}{d_\mu}\Big|_{\mu=0} (g_\mu)^{-1}(q) = q - g(q) > 0. \tag{9}$$

We remark that equations (7) and (9) are compatible, that is, for any $q \in \mathrm{dis}(T^{-1})$,

$$\lim_{\mathrm{y}\to\mathrm{q}} \frac{d}{d\mu}\Big|_{\mu=0} G_\mu(y) = q - g(q)$$

Hence, the map $(\mu, x) \in [0,1] \times I\!R/\mathbb{Z} \mapsto \frac{d}{d\mu}\Big|_{\mu=0} G_\mu(x)$ is continuous. This implies the lemma. ■

COROLLARY 5.5.2. *For all $\mu \in (0, \epsilon)$, G_μ has no fixed points.*

Proof. It follows immediatly from Theorem 1.1.3.

We now prove Theorem 1.1.4.

Theorem 1.4 (Vector field version) *Let M be a non-orientable compact 2-manifold of genus 4. Then there exists a family of quasi-minimal, Kupka-Smale flows $\{X_\mu \in \mathfrak{X}^\infty(M)\}$, depending smoothly on $\mu \in [0, \epsilon)$, such that, for some flow box $V \subset M$ of X_0, which can be taken to be homeomorphic to either a rectangle or a cylinder, and for all $\mu, \nu \in [0, \epsilon)$,*

(1) *$X_\mu|_V$ is a flow box;*
(2) *If $\mu \neq \nu$, X_μ is a C^∞-twist perturbation of X_ν localized in V;*
(3) *X_μ and X_ν are topologically equivalent.*

Proof. Let T be as in Theorem 1.1.3. From Theorem 1.1.3, we know that $\{G_\mu \circ T\}$ is a family of smooth *iet* 's conjugate to T. Each *iet* $\{G_\mu \circ T\}$ may be suspended to obtain a smooth vector field X_μ on a non-orientable compact manifold M of genus 4 (see [8]). We may assume that M contains $I\!R/Z\!\!Z$ and does not depend on μ. By definition of suspension, each X_μ is tranversal to $I\!R/Z\!\!Z$ and $G_\mu \circ T : I\!R/Z\!\!Z \to I\!R/Z\!\!Z$ is the forward Poincaré map induced by X_μ. This family $\{X_\mu\}$ can be constructed to satisfy the condition of the theorem in the case in which V is a cylinder. The other case is similar. ∎

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