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**A General Approximation Scheme for Attractors of
Abstract Parabolic Problems with Hyperbolic
Equilibrium Points**

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Resumo

Neste artigo consideramos problemas parabólicos semilineares da forma $u' = Au + f(u)$, onde A gera um semigrupo analítico compacto com decaimento exponencial em um espaço de Banach E e f é globalmente Lipschitz contínua e limitada de E^α em E ($E^\alpha = D((-A)^\alpha)$ com a norma do gráfico). Estas hipóteses asseguram que o problema tem um atrator global. Consideramos a aproximação deste problema por um esquema bastante geral que inclui o método dos elementos finitos, o método das diferenças finitas e o método das projeções. Ao aplicarmos este esquema de aproximação ao problema acima obtemos a semicontinuidade superior para a dinâmica assintótica (atratores) deste problema. Também provamos que, se todas as soluções de equilíbrio para problema são hiperbólicas então, existe um número ímpar delas. Adicionalmente, se impomos que toda solução global converge quando $t \rightarrow \pm\infty$, (como no caso de sistemas gradientes) então, provamos a semicontinuidade inferior de atratores para estas aproximações. A principal hipótese sobre a aproximação é a convergência compacta de resolventes que pode ser aplicada a muitos outros problemas não relacionados a discretização.

A GENERAL APPROXIMATION SCHEME FOR ATTRACTORS OF ABSTRACT PARABOLIC PROBLEMS WITH HYPERBOLIC EQUILIBRIUM POINTS

ALEXANDRE N. CARVALHO* AND SERGEY PISKAREV†

ABSTRACT. In this paper we consider semilinear problems of the form $u' = Au + f(u)$ where A generates an exponentially decaying compact analytic semigroup in a Banach space E and f is globally Lipschitz and bounded map from E^α into E ($E^\alpha = D((-A)^\alpha)$ with the graph norm). These assumptions ensure that the problem has a global attractor. Under a very general approximation scheme we prove that the dynamics of such problem behaves upper semicontinuously.

We also prove that, if all equilibrium solutions of this problem are hyperbolic, then there is an odd number of such equilibrium solutions. Additionally, if we impose that every global solution converges as $t \rightarrow \pm\infty$, (e.g. gradient systems), then we prove that under this approximation scheme the attractors also behave lower semicontinuously.

This general approximation scheme includes finite element method, projection and finite difference methods. The main assumption on the approximation is the compact convergence of resolvents which may be applied to many other problems not related to discretization.

1. INTRODUCTION

In this paper we give rather general conditions under which the asymptotic dynamics (global attractors) of a parabolic problem behaves continuously with respect to perturbations of the equation. This is done from a functional analytic point of view on general approximation scheme. By continuity of attractors we understand upper and lower semicontinuity of attractors. Next we introduce some terminology to be able to present the results that we will prove in this paper.

Let E be a Banach space, Λ be a topological spaces and $\mathcal{A}_\lambda \subset E$, $\lambda \in \Lambda$. Denote by $\text{dist}(\cdot, \cdot) : E \times E \rightarrow \mathbb{R}^+$ the metric induced by the norm in E .

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Definition 1.1. By upper and lower semicontinuity of the family of sets $\{\mathcal{A}_\lambda\}_{\lambda \in \Lambda}$ at $\lambda = \lambda_0$ we understand the following

- (1) We say that $\{\mathcal{A}_\lambda\}$ is upper semicontinuous at λ_0 if $\sup_{u_\lambda \in \mathcal{A}_\lambda} \text{dist}(u_\lambda, \mathcal{A}) \xrightarrow{\lambda \rightarrow \lambda_0} 0$.
- (2) We say that $\{\mathcal{A}_\lambda\}$ is lower semicontinuous at λ_0 if $\sup_{u \in \mathcal{A}} \text{dist}(u, \mathcal{A}_\lambda) \xrightarrow{\lambda \rightarrow \lambda_0} 0$.

In order to prove upper and lower semicontinuity we employ the following result

Lemma 1.1. Let $\{\mathcal{A}_\lambda\}_{\lambda \in \Lambda}$ be as in Definition 1.1.

- (1) If any sequence $\{u_{\lambda_n}\}$ with $u_{\lambda_n} \in \mathcal{A}_{\lambda_n}$, $\lambda_n \rightarrow \lambda_0$, has a convergent subsequence with limit belonging to $\mathcal{A}$, then $\{\mathcal{A}_\lambda\}_{\lambda \in \Lambda}$ is upper semicontinuous at λ_0 .
- (2) If $\mathcal{A}$ is compact and for any $u \in \mathcal{A}$ there is a sequence $\{u_{\lambda_n}\}$ with $u_{\lambda_n} \in \mathcal{A}_{\lambda_n}$, $\lambda_n \rightarrow \lambda_0$, which converges to u , then $\{\mathcal{A}_\lambda\}_{\lambda \in \Lambda}$ is lower semicontinuous at λ_0 .

The proof of this result is straight forward from the definition.

Next we introduce the notion of a global attractor for continuous semigroups. For that we follow [30] (see also [9, 55, 67]). Let E be a Banach space, by a *continuous semigroup* we understand a one parameter family $\{T(t) : t \geq 0\}$ of (nonlinear) operators such that

- (i) $T(0) = I$,
- (ii) $T(t+s) = T(t)T(s)$, $t, s \geq 0$,
- (iii) $\mathbb{R}^+ \times E \ni (t, x) \mapsto T(t)x \in E$ is continuous.

For each $x \in E$, the positive orbit $\gamma^+(x)$ through x is defined as $\gamma^+(x) = \{T(t)x : t \geq 0\}$. A backward solution through x is a continuous function $\phi : (-\infty, 0] \rightarrow E$ such that $\phi(0) = x$ and, for any $s \leq 0$, $T(t)\phi(s) = \phi(t+s)$ for $0 \leq t \leq -s$. A global solution through u^0 is a function $\phi : \mathbb{R} \rightarrow E$ such that $\phi(0) = x$, $s \in \mathbb{R}$ and $t \geq 0$ we have that $T(t)\phi(s) = \phi(t+s)$.

Backward or global solutions may not exist and its existence may depend on the choice of x . Also, when a backward solution exists, it may not be unique. Let the negative orbit through x be defined as

$$\gamma^-(x) = \cup_{t \geq 0} H(t, x),$$

where $H(t, x) = \{y \in E : \text{there is a backward solution through } x \text{ defined by } \phi : (-\infty, 0] \rightarrow E \text{ with } \phi(0) = x \text{ and } \phi(-t) = y\}$.

A complete orbit $\gamma(x)$ through x is defined as $\gamma(x) = \gamma^-(x) \cup \gamma^+(x)$. A set $B \subset E$ is said to attract a set C under $T(t)$ if $\text{dist}(T(t)C, B) \rightarrow 0$ as $t \rightarrow \infty$. A set $S \subset E$ is said to be invariant under $T(\cdot)$ if, for any $x \in S$, there is a complete orbit $\gamma(x)$ through x such that $\gamma(x) \subset S$ or equivalently if $T(t)S = S$ for any $t \geq 0$.

Definition 1.2. A set $\mathcal{A} \subseteq E$ is said to be a global attractor for semigroup $\{T(t) : t \geq 0\}$ if it is compact, invariant and attracts any bounded subsets of E .

Let $\{T(t, \lambda) : t \geq 0\}$ be a family of semigroups. Assume that for each $\lambda \in \Lambda$ the semigroup $\{T(t, \lambda) : t \geq 0\}$ has a global attractors $\mathcal{A}_\lambda$.

To obtain the upper semicontinuity of attractors is a relatively simple matter being enough to prove that the union of the attractors is relatively compact in E and that the family of semigroups behaves continuously with respect to λ .

Theorem 1.1. *Assume that the set $\cup\{\mathcal{A}_\lambda : \lambda \in \Lambda\}$ is relatively compact in E and that for $u_\lambda \xrightarrow{\lambda \rightarrow \lambda_0} u_{\lambda_0}$ in E one has $\|T(t, \lambda)u_\lambda - T(t, \lambda_0)u_{\lambda_0}\| \rightarrow 0$ as $\lambda \rightarrow \lambda_0$ for any $t \in \mathbb{R}^+$. Then, $\{\mathcal{A}_\lambda\}_{\lambda \in \Lambda}$ is upper semicontinuous at $\lambda = \lambda_0$.*

Proof: If $\{u_{\lambda_n}\}$ is a sequence with $u_{\lambda_n} \in \mathcal{A}_{\lambda_n}$, $\lambda_n \rightarrow \lambda_0$, we may assume, from the compactness of $\cup\{\mathcal{A}_\lambda : \lambda \in \Lambda\}$, that u_{λ_n} is convergent to some $u_{\lambda_0} \in E$. To show that $u_{\lambda_0} \in \mathcal{A}_{\lambda_0}$ it is enough to show that there is a complete bounded solution through u_{λ_0} . Let $\mathbb{R} \ni t \mapsto \phi_{\lambda_n}(t) \in E$ be a complete solution through u_{λ_n} . Since the union of the attractors is bounded we have that $\sup_{n \geq 1} \sup_{t \in \mathbb{R}} \|\phi_{\lambda_n}(t)\|_E < \infty$. From the convergence properties of the semigroups we have that there is a complete bounded orbit through u_{λ_0} and the result is proved. $\square$

The upper semicontinuity has been the subject of study by a number of authors, among them we cite [1, 3, 5, 6, 8], [11] – [16], [29] – [36], [38, 40, 41], [44] – [47], [62] – [65].

On the other hand, the lower semicontinuity of attractors is a much more complicated matter. We will not attempt to describe this procedure in this degree of generality and will leave for a more specific case, still in this introduction. The pioneer work done in [43] established the procedure to obtain the lower semicontinuity of attractors for gradient semigroups. In the sequel we cite the work [42] by the same authors. In the later the set of equilibria does not depend upon the parameter which considerably simplifies the problem. Other works used the reduction to finite dimension to obtain lower semicontinuity or even better continuity results as in [11, 13, 14, 15, 46, 64]. More recently the work [5] brought new light into the subject, in this case the set of equilibria also depends upon the parameter and reduction to finite dimension is not available. The results in [5] are oriented in such a manner that the continuity of attractors is drawn from the continuity of the spectrum of the involved linear operators. Our approach resembles the steps described in [5] in a rather general abstract formulation.

The aim of the present paper is to consider approximation/perturbation of attractors. Actually, the problem of approximation/perturbation of attractors is naturally motivated by the following arguments. The semigroup $\{T(t) : t \geq 0\}$ usually comes from a differential equation that is a mathematical model for a phenomenon (physical, biological, economical, etc). One central question in the modelling is the stability under perturbations. In fact, in every step of modelling there is always a degree of uncertainty in the determination of the

parameters of the mathematical model. Roughly speaking everything in the mathematical model is determined approximately. Therefore, it is crucial to determine whether the solutions depend continuously on all the parameters of the model. We tackle this question from the point of view of the asymptotic dynamics; that is, we will be satisfied with the proof that the attractors behave continuously on all the parameters of the mathematical model.

One of the important aspects of developing approximation/perturbation philosophy is the subject of numerical analysis. In the setting of present paper we consider just discretization in space and consider just convergence aspects, leaving the order of convergence and discretization in time for forthcoming papers.

In fact we will concentrate on the continuity of attractor on a very general discretization scheme, which include finite element, projection and finite difference methods and may as well apply to many perturbations that are not concerned directly with discretization.

In [11] the authors study the continuity of attractors of a one dimensional parabolic problem in space with respect to discretization. Of course the proof of the results in [11] cannot be extended to higher dimensions. The spectral gap and the special convergence of eigenvalues and eigenfunctions cannot be extended to higher dimensional domains and enter in a crucial way in the proof of the results in [11].

In [31, 45, 51, 52] the authors also consider the continuity of dynamics with respect to discretization.

To state the main results of this paper we specialize to the class of problems considered here. Let E be a complex Banach space and $A : D(A) \subseteq E \rightarrow E$ be a closed linear operator with compact resolvent, such that

$$(1.1) \quad \|(\lambda I - A)^{-1}\| \leq \frac{M}{1 + |\lambda|} \text{ for any } \operatorname{Re} \lambda \geq 0.$$

In such situation $\theta(A) = \inf\{\operatorname{Re} \lambda : \lambda \in \sigma(A)\} < 0$. Let $(-A)^\alpha$, $\alpha \in \mathbb{R}^+$, denote the fractional power operators [54] associated to A and denote by E^α the corresponding fractional power spaces; that is, $E^\alpha := D((-A)^\alpha)$ endowed with the graph norm $\|x\|_{E^\alpha} = \|(-A)^\alpha x\|_E$.

If A is as above and $0 \leq \alpha < 1$ is fixed, consider in the space E^α the following semilinear autonomous parabolic problem

$$(1.2) \quad \begin{aligned} u' &= Au + f(u), \quad t > 0, \\ u(0) &= u^0 \in E^\alpha, \end{aligned}$$

where $f(\cdot) : E^\alpha \subseteq E \rightarrow E$ is a globally Lipschitz, bounded and continuously Fréchet differentiable function.

Remark 1.1. *The fact that $-A$ is sectorial with compact resolvent enters in a crucial way to prove the continuity of the dynamics near equilibria through the fact that the linearization*

of the right hand side of (1.2) has only a finite number of eigenvalues with positive real part. This is the main difficulty to generalize such results to the case when A generates C_0 -semigroups.

Under the above assumptions the mild solution of (1.2) defined for all $t \geq 0$ (see [48], [80]). Let $u(\cdot) = T(\cdot)u^0 : \mathbb{R}^+ \rightarrow E^\alpha$ be the only mild solution of (1.2). With this notation we have defined a family of nonlinear operators in E^α $\{T(t) : t \geq 0\}$ which is a continuous semigroup. It is well known that nonlinear semigroup $T(\cdot)$ is given by the Variation of Constants Formula

$$(1.3) \quad T(t)u^0 = e^{(t-t_0)A}T(t_0)u^0 + \int_{t_0}^t e^{(t-s)A}f(T(s)u^0)ds, \quad t \geq t_0.$$

Under all above assumptions it is not difficult to prove (see [30, 48]) that

Theorem 1.2. *The problem (1.2) has a global attractor $\mathcal{A}$ in E^α .*

In the models that give rise to (1.2), the nonlinearity $f(\cdot)$ usually does not satisfy the condition above but only conditions that ensure local well posedness and existence of a global attractor. After obtaining some “nice” bounds on the attractor we can change the nonlinearity, without affecting the attractor, so that it satisfy the above conditions. This is seen in the following example.

Example 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Consider the second order strongly elliptic operator*

$$(1.4) \quad Lu = \sum_{i,j=1}^n a_{ij}(x)u_{x_i x_j} + \sum_{j=1}^n b_j(x)u + (c(x) + \nu)u,$$

where the coefficients a_{ij}, b_j, c are smooth. Consider the associated parabolic problem

$$(1.5) \quad \begin{aligned} u_t(t, x) &= -Lu(t, x) + f(u(t, x)), \quad t > 0, \quad x \in \Omega, \\ u(t, x) &= 0, \quad t > 0, \quad x \in \partial\Omega, \\ u(0, x) &= u^0(x) \in H_0^1(\Omega). \end{aligned}$$

Let $E = L^2(\Omega)$ and define the operator $A : D(A) \subset E \rightarrow E$ by $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $Au = -Lu$ for all $u \in D(A)$. It is well known that A generates an analytic and compact C_0 -semigroup $\{\exp(tA) : t \geq 0\}$. Assume that ν is chosen such that the spectrum of A is located to the left of the imaginary axis. Then, we can define the fractional powers $(-A)^\alpha$ of $-A$ as before. It is well known that $E^1 = D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $E^{\frac{1}{2}} = H_0^1(\Omega)$.

Concerning the nonlinear term $f(\cdot)$, it was shown in [4], [7] that under some growth conditions, the problem (1.5) is locally well posed in $E^{\frac{1}{2}}$. These growth restrictions are expressed as follows:

(G)_E: Let $f(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ be a $C^2(\mathbb{R}, \mathbb{R})$ function and assume one of the following

i) $n = 1$,

ii) $n = 2$ and f satisfies, for every $\eta > 0$, that there exists $c_\eta > 0$ such that

$$(1.6) \quad |f(x, u) - f(x, v)| \leq c_\eta (e^{\eta|u|^{\frac{N}{N-1}}} + e^{\eta|v|^{\frac{N}{N-1}}}) |u - v|,$$

iii) $n > 2$ and f satisfies

$$(1.7) \quad |f(x, u) - f(x, v)| \leq c|u - v|(|u|^{\rho-1} + |v|^{\rho-1} + 1),$$

with exponent ρ such that

$$1 < \rho \leq \frac{n+2}{n-2}.$$

It is also known (see [4],[7]) that, under these assumptions, these solutions are classical. If in addition to the assumptions above we assume that

$$(1.8) \quad \limsup_{|u| \rightarrow \infty} \frac{f(u)}{u} < 0,$$

then all solutions of (1.5) are globally defined. In this case we have, for each $u^0 \in E^{\frac{1}{2}}$ a globally defined solution $t \mapsto u(t, u^0) \in E^{\frac{1}{2}}$, $t \geq 0$.

Under the above hypotheses, the nonlinear semigroup $\{T(t, \cdot) : t \geq 0\}$ associated to (1.5) has a global attractor $\mathcal{A}$ in $E^{\frac{1}{2}}$ and the attractor $\mathcal{A}$ satisfies (see [8])

$$(1.9) \quad \sup_{u \in \mathcal{A}} \|u\|_{L^\infty(\Omega)} < \infty.$$

This bound enable us to cut the nonlinearity $f(\cdot)$ in such a way that it becomes bounded with bounded derivatives up to second order, but the problem with such changed nonlinearity has the same attractor. Hereafter we may assume that $f(\cdot)$ is bounded with bounded derivatives up to second order.

The equilibrium solutions of (1.2) are those solutions which are independent of time $u(t) = u$, $t \geq 0$; that is, the solutions of equation in E^α

$$(1.10) \quad Au + f(u) = 0.$$

We denote by $\mathcal{E}$ the set of solutions to (1.10); that is, $\mathcal{E} = \{u \in E^\alpha : Au + f(u) = 0\}$.

Definition 1.3. We say that a solution u^* of (1.10), i.e. $u^* \in \mathcal{E}$, is hyperbolic if the spectrum $\sigma(A + f'(u^*))$ is disjoint from the imaginary axis, i.e. $\sigma(A + f'(u^*)) \cap i\mathbb{R} = \emptyset$.

About the solutions of (1.10) we can prove the following result

Proposition 1.1. (see Proposition 2.1) If all solutions of (1.10) are hyperbolic, then $\mathcal{E}$ has $2k+1$ elements for some non-negative integer k .

This feature can be seen in [17], so called Chafee–Infante example, for which one can check that the number of equilibrium points is always odd.

Definition 1.4. *The unstable manifold of the equilibrium set $\mathcal{E}$ is the set $W^u(\mathcal{E}) = \{\eta \in E^\alpha : u(t, \eta) \text{ is defined for all } t \leq 0 \text{ and } u(t, \eta) \rightarrow \mathcal{E} \text{ as } t \rightarrow -\infty\}$.*

The unstable manifold of an equilibrium solution $u^ \in \mathcal{E}$ is defined by $W^u(u^*) = \{\eta \in E^\alpha : u(t, \eta) \text{ is defined for all } t \leq 0 \text{ and } u(t, \eta) \rightarrow u^* \text{ as } t \rightarrow -\infty\}$.*

We now specialize to a class of equations for which we can describe the structure of attractor. We assume that any bounded global solution $\phi(\cdot) : \mathbb{R} \rightarrow E^\alpha$ of (1.2) has limits

$$(1.11) \quad \phi(t) \rightarrow \phi_{-\infty}^* \text{ as } t \rightarrow -\infty \text{ and } \phi(t) \rightarrow \phi_{\infty}^* \text{ as } t \rightarrow \infty.$$

In this case ϕ_{∞}^* and $\phi_{-\infty}^*$ are in $\mathcal{E}$ and the attractor has a representation $\mathcal{A} = \cup_{u^* \in \mathcal{E}} W^u(u^*)$. Such structure of attractor one gets for example in case of gradient system with isolated equilibria.

Definition 1.5. *We say that $\{T(t) : t \geq 0\}$ is gradient if there is a continuous function $V : E^\alpha \rightarrow \mathbb{R}$ such that function $\mathbb{R}^+ \ni t \mapsto V(T(t)u^0) \in \mathbb{R}$ is non-increasing and $\mathcal{E} = \{u^0 \in E^\alpha : V(T(t)u^0) = V(u^0), \forall t \in \mathbb{R}\}$.*

Theorem 1.3. *(see Theorem 2.3) If A , α and f are as before and if $\{T(t) : t \geq 0\}$ is gradient then the attractor $\mathcal{A}$ for (1.2) can be characterized as the unstable manifold of the equilibrium set $\mathcal{E}$; that is, $\mathcal{A} = W^u(\mathcal{E})$. If in addition, each $u^* \in \mathcal{E}$ is isolated then $\mathcal{E}$ is a finite set and $\mathcal{A} = \cup_{u^* \in \mathcal{E}} W^u(u^*)$, i.e. attractor consists of the union of the unstable manifolds of equilibrium solutions.*

Example 1.2. *If the operator L in Example 1.1 is in the divergence form, then the semigroup associated to (1.5) is gradient.*

In the mean time our general assumption (1.11) admits the case $\phi_{-\infty}^* = \phi_{+\infty}^*$, which is not allowed in case of gradient systems.

Our results on lower semicontinuity of attractors require that the attractor $\mathcal{A}$ of (1.2) is characterized in the form

$$(1.12) \quad \mathcal{A} = \bigcup_{u^* \in \mathcal{E}} W^u(u^*)$$

with $\mathcal{E}$ being a finite set consisting only of hyperbolic equilibria of (1.2). Other results on upper semicontinuity of attractor or on continuity of the set of equilibria do not require such special characterization.

Definition 1.6. *We say that an invariant set Γ is asymptotically stable if there is a neighborhood V of Γ such that, for any $v \in V$ $\text{dist}(T(t)v, \Gamma) \rightarrow 0$ as $t \rightarrow \infty$.*

In fact, in the proof of lower semicontinuity of attractors, we may assume that the attractor is the union of unstable manifolds of hyperbolic equilibria $W^u(u_i^*)$, $1 \leq i \leq n^*$, together with asymptotically stable invariant set Γ ; that is,

$$\mathcal{A} = \bigcup_{1 \leq i \leq n^*} W^u(u_i^*) \cup \Gamma$$

which can be approximated (upper and lower semicontinuously). As an example, under the same assumptions on A and $f(\cdot)$ we can approximate uniformly stable periodic orbits of (1.2), see [10, 27].

Example 1.3. *In Chafee–Infante example [17] the attractor is the union of unstable manifolds of equilibrium points.*

Next we introduce the general discretization scheme and state the main results of the paper. The general approximation scheme [22] – [24], [66] – [75], can be described in the following way. Let E_n and E be Banach spaces, and let $\{p_n\}$ be a sequence of linear bounded operators $p_n : E \rightarrow E_n$, $p_n \in B(E, E_n)$, $n \in \mathbb{N} = \{1, 2, \dots\}$, with the following property:

$$(1.13) \quad \|p_n x\|_{E_n} \rightarrow \|x\|_E \text{ as } n \rightarrow \infty \text{ for any } x \in E.$$

We will also use the operators $p_n^\alpha = (-A_n)^{-\alpha} p_n (-A)^\alpha \in B(E^\alpha, E_n^\alpha)$ with the same property (1.13), but for the spaces E^α, E_n^α . Operators A_n and A are supposed to be related with conditions (1.1), (A) and (B_1) .

Definition 1.7. *A sequence of elements $\{x_n\}$, $x_n \in E_n$, $n \in \mathbb{N}$, is said to be $\mathcal{P}$ -convergent to $x \in E$ iff $\|x_n - p_n x\|_{E_n} \rightarrow 0$ as $n \rightarrow \infty$; we write this as $x_n \xrightarrow{\mathcal{P}} x$.*

Definition 1.8. *A sequence of elements $\{x_n\}$, $x_n \in E_n$, $n \in \mathbb{N}$, is said to be $\mathcal{P}$ -compact if for any $\mathbb{N}' \subseteq \mathbb{N}$ there exist $\mathbb{N}'' \subseteq \mathbb{N}'$ and $x \in E$ such that $x_n \xrightarrow{\mathcal{P}} x$, as $n \rightarrow \infty$ in $\mathbb{N}''$.*

Definition 1.9. *A sequence of closed linear operators $\{A_n\}$, $A_n \in \mathcal{C}(E_n)$, $n \in \mathbb{N}$, is said to be compatible with a closed linear operator $A \in \mathcal{C}(E)$ iff for each $x \in D(A)$ there is a sequence $\{x_n\}$, $x_n \in D(A_n) \subseteq E_n$, $n \in \mathbb{N}$, such that $x_n \xrightarrow{\mathcal{P}} x$ and $A_n x_n \xrightarrow{\mathcal{P}} Ax$. We write this as (A_n, A) are compatible.*

Consider in Banach spaces E_n also the family of parabolic problems

$$(1.14) \quad \begin{aligned} u'_n &= A_n u_n + f_n(u_n), \quad t > 0, \\ u_n(0) &= u_n^0 \in E_n^\alpha, \end{aligned}$$

where $u_n^0 \xrightarrow{\mathcal{P}} u^0$, operators (A_n, A) are compatible, $f_n : E_n^\alpha \rightarrow E_n$ is bounded, and Lipschitz continuous. Consider the associated elliptic problems in E_n^α

$$(1.15) \quad A_n u_n + f_n(u_n) = 0.$$

We denote by $\mathcal{E}_n$ the set of solutions to (1.14); that is, $\mathcal{E}_n = \{u_n \in E_n^\alpha : A_n u_n + f_n(u_n) = 0\}$.

Definition 1.10. A sequence of bounded linear operators $B_n \in B(E_n), n \in \mathbb{N}$, is said to be $\mathcal{PP}$ -convergent to the bounded linear operator $B \in B(E)$ if for every $x \in E$ and for every sequence $\{x_n\}, x_n \in E_n, n \in \mathbb{N}$, such that $x_n \xrightarrow{\mathcal{P}} x$ one has $B_n x_n \xrightarrow{\mathcal{P}} Bx$. We write this as $B_n \xrightarrow{\mathcal{PP}} B$.

We have the following version of Trotter–Kato Theorem on the general approximation scheme.

Theorem 1.4. [60] Let operators A and A_n generate analytic C_0 -semigroups. The following conditions (A) and (B_1) are equivalent to condition (C_1) .

(A) *Compatibility.* There exists $\lambda \in [\cap_n \rho(A_n)] \cap \rho(A)$ such that the resolvents converge:

$$(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1};$$

(B_1) *Stability.* There are some constants $M_2 \geq 1$ and ω_2 such that

$$\|(\lambda I_n - A_n)^{-1}\| \leq \frac{M_2}{|\lambda - \omega_2|}, \quad \operatorname{Re} \lambda > \omega_2, n \in \mathbb{N};$$

(C_1) *Convergence.* For any finite $\mu > 0$ and some $0 < \theta < \frac{\pi}{2}$, we have

$$\max_{\eta \in \Sigma(\theta, \mu)} \|\exp(\eta A_n) u_n^0 - p_n \exp(\eta A) u^0\|_{E_n} \rightarrow 0$$

as $n \rightarrow \infty$, whenever $u_n^0 \rightarrow u^0$. Here, $\Sigma(\theta, \mu) = \{z \in \Sigma(\theta) : |z| \leq \mu\}$ and $\Sigma(\theta) = \{z \in \mathbb{C} : |\arg z| \leq \theta\}$.

The above Theorem ABC gives a necessary and sufficient condition for convergence of mild solutions of (1.14) to mild solution of (1.2) in case $f_n \equiv 0, f \equiv 0$. In this trivial case under our above assumptions on A the attractor $\mathcal{A} = \{0\}$, since $\|\exp(tA)\| \leq Me^{-bt} \rightarrow 0$ as $t \rightarrow \infty$ and equation $Au + f(u) = 0$ has only one solution $u^* = 0$. One may conjecture that from conditions (A) and (B_1) of Theorem 1.4, it follows also that $\|\exp(tA_n)\| \leq \tilde{M}e^{-\tilde{b}t}$ as $t \rightarrow \infty$ for some $\tilde{b} > 0$ and $\tilde{M} \geq 1$. In general it is not true as we can see from counterexample.

Example 1.4. Discrete convergence of resolvents (without condition $\Delta_{cc} \neq \emptyset$) is not enough to get uniform estimates (and also the statement of Theorem 6.5), which is clear from the following example noted to us by Professor Cesar Palencia (private communication). Let us consider a Banach space $E = l_2$, the Hilbert space formed by all the square summable sequences, i.e $x = (x_1, x_2, \dots)$, $\sum_{k=1}^\infty |x_k|^2 < \infty$. Then, the operator

$$A = \operatorname{diag}(-\alpha_1, -\alpha_2, -\alpha_3, \dots)$$

with $1 = \alpha_1 \leq \alpha_2 \leq \dots$ generates on E an analytic C_0 -semigroup with $\|e^{tA}\| = e^{-t}$, $t \geq 0$, i.e. it is exponentially decreasing semigroup. The spectrum $\sigma(A) = \{-\alpha_1, -\alpha_2, -\alpha_3, \dots\}$.

One could consider approximation of A by the operators

$$A_n = \text{diag}(-\alpha_1, -\alpha_2, \dots, -\alpha_n, M_n, 0, 0, 0, \dots),$$

where $M_n = -\omega I_n + L_n$, with $0 < \omega < 1$, is $n \times n$ -matrix and L_n has got all of its entries equal to zero, except the first upper diagonal which is formed by ones. The spectrum of M_n reduces to $-\omega$. Moreover, since

$$\|\exp(tM_n)\| \leq e^{-\omega t} e^{\|L_n\|t} \leq e^{(1-\omega)t},$$

it is clear that, on finite intervals of $t \in [0, T]$, we have the strong convergence of the semi-groups generated by

$$A_n = \text{diag}(-\alpha_1, -\alpha_2, \dots, -\alpha_n, M_n, 0, 0 \dots)$$

to $\exp(tA) = \text{diag}(e^{-\alpha_1 t}, e^{-\alpha_2 t}, e^{-\alpha_3 t}, \dots)$. In the mean time, the $(1, n)$ element of $\exp(tM_n)$ is nothing but $e^{-\omega t} \frac{t^{n-1}}{(n-1)!}$ so that, by Stirling formula

$$\|e^{(n-1)M_n}\| \geq e^{-\omega(n-1)} \frac{(n-1)^{n-1}}{(n-1)!} \sim \frac{e^{(n-1)(1-\omega)}}{(2\pi n)^{1/2}} \text{ as } n \rightarrow \infty,$$

and since $0 < \omega < 1$ we have then

$$\sup\{\|e^{(n-1)M_n}\| : n \geq 1\} = +\infty.$$

It is necessary to stress that in this example $\text{Re } \sigma(A_n) \leq -\omega$, one has uniform convergence $\exp(tA_n) \rightarrow \exp(tA)$ in $t \in [0, T]$ for any finite $T > 0$, and $\|\exp(tA)\| \leq e^{-t}$, $t \geq 0$, but in the mean time $\|\exp(t_n A_n)\| \rightarrow \infty$ as $t_n = n - 1 \rightarrow \infty$.

So, this counterexample shows that it can happen $\|\exp(t_n A_n)\| \rightarrow \infty$ as $t_n \rightarrow \infty$. Since splitting of the solutions of the problem (1.2) in the neighborhood of any equilibrium point involves (see (5.4)) estimates like $\|\exp(t\bar{A}_n)\| \leq \bar{M}e^{-\bar{\omega}t}$, $t \geq 0$, with some $\bar{M} \geq 1$, $\bar{\omega} > 0$, for operator $\bar{A}_n$, which is the restriction of A_n on the corresponding subspaces, one needs to put on (A_n, A) additional assumption besides the discrete convergence of resolvents (A) and stability (B_1).

Example 1.5. It is well known [50] that the operator A which generates analytic C_0 -semigroup can be associated one-to-one with a sesquilinear form $\sigma : E^{\frac{1}{2}} \times E^{\frac{1}{2}} \rightarrow \mathbb{C}$ such that

$$|\sigma(u, v)| \leq c_1 \|u\|_{E^{\frac{1}{2}}} \|v\|_{E^{\frac{1}{2}}}, \quad u, v \in E^{\frac{1}{2}},$$

$$\text{Re } \sigma(u, u) \geq c_2 \|u\|_{E^{\frac{1}{2}}}^2, \quad u \in E^{\frac{1}{2}},$$

$$\sigma(u, v) = \langle -Au, v \rangle, \quad u \in D(A), \quad v \in E^{\frac{1}{2}}.$$

If Ω is a convex polygon consider a subdivision of it which consists of a regular triangulation where each triangle has largest diameter equal to h . In this case denote by S_h the set of all functions in $E^{\frac{1}{2}}$ which are linear in each triangular element.

So we are given a family S_h of finite dimensional subspaces of $H_0^1(\Omega)$ with the standard [68] approximation property

$$\inf_{\chi \in S_h} (\|v - \chi\|_E + h\|v - \chi\|_{E^{1/2}}) \leq Ch^2\|v\|_{E^1} \text{ for } v \in H^2(\Omega) \cap H_0^1(\Omega).$$

We also introduce analogy of our connecting mappings $\{p_n^\alpha\}$ as $\tilde{u}_h = \tilde{P}_h u$, i.e. the projection of $u \in E^{\frac{1}{2}}$ on $E_h^{\frac{1}{2}}$ with respect to the $H^1(\Omega)$ inner product and we have

$$\|u - \tilde{u}^h\|_{E^{\frac{1}{2}}} \leq Ch\|u\|_{E^1} \quad \text{and} \quad \|u - \tilde{u}^h\|_E \leq Ch^2\|u\|_{E^1}, \quad u \in E^1$$

(see [18, 19]). Incidentally we stand $P_h u$ for the projection of $u \in E$ on $S_h = E_h^{\frac{1}{2}}$ with respect to the $L^2(\Omega)$ inner product, these operators play the role of connecting mappings $\{p_n\}$. In this framework, our finite element approximation $A_h : S_h \rightarrow S_h$ of A is defined by

$$\langle -A_h \phi_h, \psi_h \rangle = \sigma(\phi_h, \psi_h), \quad \phi_h, \psi_h \in E_h^{\frac{1}{2}}.$$

In other words, A_h is the operator associated with the sesquilinear form $\sigma_h(\cdot, \cdot)$ which is the restriction of $\sigma(\cdot, \cdot)$ to $E_h^{\frac{1}{2}} \times E_h^{\frac{1}{2}}$. With such setting one can prove that there exist a constant C and an acute angle θ such that for any $x \in E$ and $z \in \{z \in \mathbb{C} : \theta \leq |\arg z| \leq \pi\}$ we have for instance

$$\|(zI - A)^{-1}x - (zI_h - A_h)^{-1}P_h x\|_E \leq Ch^2\|x\|_E.$$

This estimate shows that in such example one has $\mathcal{P}$ -convergence with uniform convergence of the resolvents. Since our resolvent $(\lambda I - A)^{-1}$ is compact for some λ , then because of inequality (where $\mu(\cdot)$ is measure of noncompactness)

$$\mu((zI_h - A_h)^{-1}x_h) \leq \mu((zI - A)^{-1}x_h) + \|(zI - A)^{-1}x_h - (zI_h - A_h)^{-1}x_h\|_E \leq Ch^2\|x_h\|_E$$

resolvents converge compactly as $h \rightarrow 0$. So we will use compact convergence property as one of the basic assumption on general approximation scheme to get the estimates like (5.3).

Let $u_h(t) : [0, \infty) \rightarrow E_h^{\frac{1}{2}}$ be the solution of

$$(1.16) \quad \langle u'_h(t), \phi_h \rangle + \sigma(u_h(t), \phi_h) = 0, \quad \phi_h \in E_h^{\frac{1}{2}},$$

and $u_h(0) = u_h^0$. Then we can write instead of (1.16)

$$(1.17) \quad u'_h(t) = A_h u_h(t), \quad u_h(0) = u_h^0 \in E_h^{\frac{1}{2}},$$

and we also have $u_h(t) = \exp(tA_h)u_h^0$. Moreover, for $x \in E^{\frac{1}{2}}$ we have (see [18]) for example that

$$\|\exp(tA)x - \exp(tA_h)\tilde{P}_h x\|_{E^{\frac{1}{2}}} \leq Ch t^{-\frac{1}{2}} \|x\|_{E^{\frac{1}{2}}}.$$

With this we conclude that if we apply for our modelling problem (1.5), with $n = 2$, finite element method, then all kind of assumptions we put in the present paper are naturally/automatically verified and one can get the results on approximation of attractor.

For the case of finite difference methods approximation of eigenvalues and regular convergence of operators (where the notion of discrete compactness play crucial role) has been considered in [76]. For more references concerning discretization of problems like (1.2), see for example, [10, 27, 56, 57, 59, 61].

Next we state the main results in this paper. These results hold under the following assumption

- [A1] Let $f(\cdot)$ and $f_n(\cdot)$ be globally bounded and Lipschitz continuous and such that $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{P}\mathcal{P}} (\lambda I - A)^{-1}$ compactly (see Definition 3.7) for some λ , conditions (1.1) and (B_1) hold and $f_n(x_n) \xrightarrow{\mathcal{P}} f(x)$, whenever $x_n \xrightarrow{\mathcal{P}^\alpha} x$.

Theorem 1.5. (see Theorem 4.6) Assume that [A1] holds, then $T_n(t)u_n^0 \xrightarrow{\mathcal{P}^\alpha} T(t)u^0$ uniformly in $t \in [0, T]$, $T < \infty$, whenever $u_n^0 \xrightarrow{\mathcal{P}^\alpha} u^0$.

Remark 1.2. The condition (B_1) and compact convergence of resolvents are independent in the following sense. Let we have the same operator A as in Example 1.4, with $\alpha_n = n^2$, $n \in \mathbb{N}$, but we approximate it with A_n in the form

$$A_n = \text{diag}(-\alpha_1, -\alpha_2, -\alpha_3, \dots, -\alpha_{n-1}, \alpha_n, -\alpha_{n+1}, \dots),$$

one can see that resolvents are compact and $A_n^{-1} \rightarrow A^{-1}$ compactly. But, since $\alpha_n \rightarrow +\infty$, the estimate (B_1) does not hold. So this means that compact convergence of resolvents is not enough to get the condition (B_1) .

Definition 1.11. Let $\mathcal{A} \subset E^\alpha$ and $\mathcal{A}_n \subset E_n^\alpha$, $n \in \mathbb{N}$. Denote by $\text{dist}(\cdot, \cdot)$ the metric induced by the norm in E_n^α , $n \in \mathbb{N}$, i.e. $\text{dist}(x_n, y_n) = \|x_n - y_n\|_{E_n^\alpha}$.

- (1) We say that $\{\mathcal{A}_n\}$ is $\mathcal{P}^\alpha$ -upper semicontinuous at infinity if any neighborhood of $\mathcal{A}$, i.e. $\mathcal{O}(\mathcal{A})$, contains all but finitely many $\mathcal{A}_n$'s; that is, $\sup_{u_n \in \mathcal{A}_n} \text{dist}(u_n, p_n^\alpha \mathcal{A}) \xrightarrow{n \rightarrow \infty} 0$.
- (2) We say that $\{\mathcal{A}_n\}$ is $\mathcal{P}^\alpha$ -lower semicontinuous at infinity if, for any $\epsilon > 0$, $\mathcal{O}_\epsilon(\mathcal{A}_n) \supset p_n^\alpha(\mathcal{A})$ for all but finitely many n 's; that is, $\sup_{u \in \mathcal{A}} \text{dist}(p_n^\alpha u, \mathcal{A}_n) \xrightarrow{n \rightarrow \infty} 0$.

Theorem 1.6. (see Theorem 5.1) Let $\mathcal{A}$, $\mathcal{A}_n$ be the attractors for (1.2) and (1.14), respectively and assume that [A1] holds. If $\{x_n^*\}$, $x_n^* \in \mathcal{A}_n$, $\mathcal{P}^\alpha$ -converges to some point x^* , then $x^* \in \mathcal{A}$ and the $\{\mathcal{A}_n\}$ is $\mathcal{P}^\alpha$ -upper semicontinuous at infinity.

Theorem 1.7. Assume also that problem (1.10) has exactly m solutions $u^1, \dots, u^m$ and that all of them are hyperbolic and that [A1] holds. Assume also that for any point u^i and any $\epsilon > 0$ there exists $\delta > 0$ such that $\|f'_n(u_n) - f'_n(p_n^\alpha u^i)\|_{E_n} \leq \epsilon$, whenever $\|u_n - p_n^\alpha u^i\|_{E_n^\alpha} \leq \delta$.

Then, there exists n_0 such that for all $n \geq n_0$ the problem (1.15) has exactly m solutions $u_n^1, \dots, u_n^m$. Moreover, we have

$$\|u_n^k - p_n^\alpha u^k\|_{E_n^\alpha} \xrightarrow{n \rightarrow \infty} 0, \quad k = 1, \dots, m,$$

and there exist $\rho > 0$ such that local unstable manifolds $W_\rho^u(u_n^k)$, $k = 1, \dots, m$, behave continuously in E_n^α as $n \rightarrow \infty$.

Theorem 1.8. (see Theorem 5.2) Let $\mathcal{A}$, $\mathcal{A}_n$ be the attractors for (1.2) and (1.14), respectively. Assume $\mathcal{A}$ can be written in the form (1.12) and that $\mathcal{E}$ consists only of hyperbolic equilibrium. Assume also that for any point u^i and any $\epsilon > 0$ there exists $\delta > 0$ such that $\|f'_n(u_n) - f'_n(p_n^\alpha u^i)\|_{E_n} \leq \epsilon$, whenever $\|u_n - p_n^\alpha u^i\|_{E_n^\alpha} \leq \delta$. If $x^* \in \mathcal{A}$, then there is a sequence of points $\{x_n\}$, with $x_n^* \in \mathcal{A}_n$, such that $x_n^* \xrightarrow{\mathcal{P}^\alpha} x^*$ and the $\{\mathcal{A}_n\}$ is $\mathcal{P}^\alpha$ -lower semicontinuous at infinity.

Proof: If $x^* \in \mathcal{A}$, then it is on the unstable manifold of some equilibrium point. Following back in time by some trajectory through x^* one can come, in time $-T < 0$, to a point y in the neighborhood of some equilibrium point, which as we know by Theorem 1.7 is approximated by points $y_n \in \mathcal{A}_n$. Then, $T_n(t)y_n \in \mathcal{A}_n$ approximates x^* and we obtain the result. $\square$

In what follows we describe the algorithm that we followed to show continuity (upper and lower semicontinuity) of attractors. As presented here, the problem (1.14) is a perturbation of the problem (1.2).

- The perturbation of the unbounded operator A may be very singular and the continuity of its approximation by A_n is given in terms of continuity of resolvent operators.
- Continuity of resolvents (plus some $\mathcal{P}$ -compactness and stability) ensures the continuity (upper and lower) of the spectrum and of generalized eigenspaces.
- The continuity of resolvent operators (plus some stability) implies the continuity of linear semigroups. This is accomplished through a version of Trotter-Kato Theorem (Theorem 1.4) referred as Theorem ABC.
- Using the variation of constants formula (plus some compactness) we can obtain the continuity of nonlinear semigroups given in Theorem 4.6.
- The continuity of nonlinear semigroups (plus some uniform boundedness and $\mathcal{P}$ -compactness) imply the upper semicontinuity of attractors.
- Assuming that all points in $\mathcal{E}$ (solutions of (2.4)) are hyperbolic equilibrium we obtain continuity of the set of equilibria.

- The continuity of the set of equilibria implies the continuity of the linear unstable manifold of the linearization around equilibria. Using the continuity of the linear unstable manifolds we obtain the continuity of the nonlinear local (near the equilibrium) unstable manifolds.
- Assuming that the attractor for the problem (1.2) can be written in the form (1.12) and using the continuity of the nonlinear semigroups and the continuity of the local unstable manifolds (plus uniform boundedness and $\mathcal{P}$ -compactness of the attractors) we obtain lower semicontinuity of attractors.

This paper is organized as follows. In Section 2 we establish the existence of attractors for (1.2), prove that if all equilibrium points for (1.2) are hyperbolic then there is an odd number of them, prove the existence of unstable manifolds in the way needed for the approximation and characterize the attractors that can be approximated lower semicontinuously. In Section 3 we establish the results on convergence of the spectrum (upper and lower semicontinuously). Section 4 is devoted to approximation of semiflows. Finally in Section 5 we prove the upper semicontinuity of attractors, the continuity of hyperbolic equilibria and unstable manifolds and the lower semicontinuity of attractors. We conclude the paper with an Appendix where we have placed some basic results on convergence of resolvents and uniform estimates we needed throughout the paper.

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2. ATTRACTORS FOR ABSTRACT PARABOLIC PROBLEMS

We assume that the operator A in (1.1) generates analytic C_0 -semigroup with estimate $\|e^{tA}\| \leq Me^{-t\omega}$, $t \geq 0$, where $\omega > 0$, resolvent $(\lambda I - A)^{-1}$ is compact for some $\lambda \in \rho(A)$, function $f(\cdot) : E^\alpha \rightarrow E$ is a globally Lipschitz, bounded, i.e. there is a constant $K > 0$, such that $\|f(x)\|_E \leq K$ for all $x \in E^\alpha$, and continuously Fréchet differentiable function.

2.1. Preliminaries on invariant and compact sets.

Let us show that the conditions imposed on A and $f(\cdot)$ in (1.2) are enough to guarantee that semigroup $\{T(t) : t \geq 0\}$ has a global attractor. Recall that for any set $B \subset E^\alpha$, the sets $\gamma^+(B) = \cup_{u^0 \in B} \gamma^+(u^0)$, $\gamma^-(B) = \cup_{u^0 \in B} \gamma^-(u^0)$, $\gamma(B) = \cup_{u^0 \in B} \gamma(u^0)$, are respectively, the positive orbit, the negative orbit, the complete orbit through B when they exist.

For any set $B \subset E^\alpha$, define the ω - limit set $\omega(B)$ of B and the α - limit set $\alpha(B)$ of B as

$$\omega(B) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} T(t)B}, \quad \alpha(B) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} H(t, B)}.$$

We start with some basic lemmas:

Lemma 2.1. *If $S \subset E^\alpha$ is invariant under $\{T(t) : t \geq 0\}$ and $v \in S$, then there is a complete orbit through v .*

Lemma 2.2. *A point v belongs to $\omega(B)$ if and only if there is a sequence $t_n \rightarrow +\infty$ and a sequence $v_n \in B$ such that $T(t_n)v_n \rightarrow v$.*

Lemma 2.3. *For any bounded set $B \subset E^\alpha$ the set $\gamma^+(B)$ is bounded and $T(t)\gamma^+(B)$ is precompact for any $t > 0$.*

Proof: It is well known that $T(t) : E^\alpha \rightarrow E^\alpha$ is compact. Using the Variation of Constants Formula and standard estimates, it follows from the assumptions on A and $f(\cdot)$ that $\gamma^+(B)$ is bounded in E^α whenever B is bounded in E^α . Therefore, since $\gamma^+(B)$ is bounded, we have that $\{T(s)B : s \geq t\} = T(t)\gamma^+(B)$ is compact and the result follows. $\square$

Lemma 2.4. *For any $u^0 \in E^\alpha$ the set $\omega(u^0)$ is non-empty, connected, compact, invariant and attracts u^0 .*

See [30] for a proof.

Lemma 2.5. *Suppose that $u^0 \in E^\alpha$ is such that there is a negative orbit $\phi : (-\infty, 0] \rightarrow E^\alpha$ through u^0 and such that $\overline{\phi((-\infty, 0])}$ is compact. Define*

$$\alpha_\phi(u^0) = \{v \in E^\alpha : \exists t_n \rightarrow \infty \text{ such that } \phi(-t_n) \rightarrow v\}.$$

Then, $\alpha_\phi(u^0)$ is non-empty, connected, compact and invariant.

Proof: It is easy to see that $\alpha_\phi(u^0) = \bigcap_{t \geq 0} \overline{\phi((-\infty, t])}$ and from this it is immediate that $\alpha_\phi(u^0)$ is non-empty, compact and connected.

It remains to prove that $\alpha_\phi(u^0)$ is invariant. In fact, if $v \in \alpha_\phi(u^0)$, then there is a sequence $t_n \rightarrow +\infty$ such that $\phi(-t_n) \rightarrow v$. From the continuity of $T(t) : E^\alpha \rightarrow E^\alpha$ we obtain that $T(t)\phi(-t_n) = \phi(t - t_n) \rightarrow T(t)v$ and therefore $T(t)v \in \alpha_\phi(u^0)$. On the other hand, if $w \in \alpha_\phi(u^0)$, there is a sequence $t_n \rightarrow \infty$ (w.l.o.g. we can assume $t_n \geq t$, $n \in \mathbb{N}$) such that $\phi(-t_n) \rightarrow w$. Since $\{\phi(-t_n - t) : n \in \mathbb{N}\}$ is relatively compact, taking subsequences if necessary, there is a $z \in E^\alpha$ such that $\phi(-t_n - t) \rightarrow z$ and $z \in \alpha_\phi(u^0)$. It follows from the uniqueness of the limit that $T(t)z = w$. $\square$

Remark 2.1. *It follows immediately from the invariance that, if $u_0 \in E^\alpha$ and $\omega(u_0) = \{u^*\}$, then u^* is an equilibrium. Similar statements hold for $\alpha(u_0)$ and $\alpha_\phi(u_0)$.*

Lemma 2.6. *The orbit of any bounded set $B \subset E^\alpha$ is bounded and for each bounded set $B \subset E^\alpha$ there is an instant τ_B and a constant N (independent of B) such that*

$$(2.1) \quad \sup_{t \geq \tau_B} \sup_{w \in T(t)B} \|w\|_{E^\alpha} \leq N.$$

Furthermore,

$$(2.2) \quad \sup_{\substack{B \subset E^\alpha \\ B \text{ bounded}}} \sup_{v \in \omega(B)} \|v\|_{E^\alpha} \leq N.$$

Proof: The first statement follows trivially from the Variation of Constants Formula and (2.2) follows from (2.1) and from the definition of $\omega(B)$. It remains to prove (2.1). Given a bounded set $B \subset E^\alpha$ and $v \in B$ we have from (1.3) that

$$(2.3) \quad T(t)v = \exp(tA)v + \int_0^t \exp((t-s)A)f(T(s)v)ds, \quad t \geq 0.$$

From the hypothesis on $f(\cdot)$ and on A we have that there is a constant K such that $\|f(w)\|_E \leq K$ for all $w \in E^\alpha$ and

$$\|T(t)v\|_{E^\alpha} \leq Me^{-\beta t}\|v\|_{E^\alpha} + MK \int_0^t (t-s)^{-\alpha} e^{-\beta(t-s)} ds.$$

From the fact that v lies in a bounded subset of E^α we have that there exists τ_B such that

$$\|T(t)v\|_{E^\alpha} \leq 1 + MK \int_0^\infty \xi^{-\alpha} e^{-\beta\xi} d\xi =: N, \quad t \geq \tau_B.$$

Since N is independent of $B \subset E^\alpha$ the result follows. $\square$

Lemma 2.7. *If $B \subset E^\alpha$ is a bounded set, then $\omega(B)$ is non-empty, compact, invariant, and attracts B under $\{T(t) : t \geq 0\}$. Furthermore, if B is connected, then $\omega(B)$ is connected.*

See [30] for a proof.

Theorem 2.1. *If $B_N = \{u \in E^\alpha : \|u\|_{E^\alpha} \leq N\}$, then $\omega(B_N)$ is the global attractor for $\{T(t) : t \geq 0\}$.*

Proof. It is clear that $\omega(B_N)$ is compact, invariant, connected and attracts B_N . First note that for any bounded set B there is a τ_B such that $T(t)B \subset B_N$ for all $t \geq \tau_B$. Since B_N is attracted by $\omega(B_N)$ and since B_N attracts B the result follows. $\square$

2.2. Equilibrium points.

Now we are going to describe the structure of the attractor for (1.2). We will start with the elements from the attractor which are the most obvious, the equilibrium solutions. The

equilibrium solutions of (1.2) are those solutions which are independent of time $u(t) = u$, $t \geq 0$; that is, the solutions of equation

$$(2.4) \quad Au + f(u) = 0.$$

We say that a solution u^* of (2.4), i.e. equilibrium point u^* , is hyperbolic if the spectrum $\sigma(A + f'(u^*))$ is disjoint from the imaginary axis, i.e. $\sigma(A + f'(u^*)) \cap i\mathbb{R} = \emptyset$.

Let Ω be an open convex set in a Banach space F , and let $\mathcal{B} : \overline{\Omega} \rightarrow F$ be a compact operator having no fixed points on the boundary $\partial\Omega$ of Ω . Then for the vector field $\mathcal{V}(x) = x - \mathcal{B}x$, the rotation (degree) $\gamma(I - \mathcal{B}; \partial\Omega)$ being a integer-valued characteristics of this field is well defined. Let z^* be an isolated fixed point of the operator $\mathcal{B}$ in the ball $B(z^*, r_0)$ of radius r_0 centered at a point z^* . Then $\gamma(I - \mathcal{B}; \partial B(z^*, r_0)) = \gamma(I - \mathcal{B}; \partial B(z^*, r))$ for $0 < r < r_0$. This common value of the rotation is called the index of the fixed point z^* and is denoted by $\text{ind}(z^*; I - \mathcal{B})$.

Proposition 2.1. *Assume that all equilibrium points of (1.2) are hyperbolic. Then, there is only a finite number of them and this number is odd.*

Proof. First we observe that since $f(\cdot) : E^\alpha \rightarrow E$ is bounded we have that all solutions of (2.4) satisfies

$$(2.5) \quad Iu + A^{-1}f(u) = 0$$

and therefore $\|u\|_{E^1} \leq \|f(u)\|_E \leq K$. So if we consider the ball of radius more than $\|A^{-1+\alpha}\|K$, then the operator $A^{-1}f(\cdot)$ maps the ball $B(0, \|A^{-1+\alpha}\|K) \subset E^\alpha$ into itself. By Schauder fixed point Theorem (see Theorem 21.5 [53]) $\gamma(I + A^{-1}f(\cdot), \partial B(0, \|A^{-1+\alpha}\|K)) = 1$, i.e. exists at least one fixed point

$$Iu^* + A^{-1}f(u^*) = 0 \text{ with } u^* \in B(0, \|A^{-1}\|K).$$

Since the operator $A^{-1}f(\cdot) : E^\alpha \rightarrow E^\alpha$ is compact we have that the set $\mathcal{E} = \{u : Au + f(u) = 0\}$ is compact in E^α . Moreover, any fixed point u^* is isolated and $|\text{ind}(u^*, I + A^{-1}f(\cdot))| = 1$, since u^* is hyperbolic and $\mathcal{N}(I + A^{-1}f'(u^*)) = \{0\}$, (see [53], Theorem 21.6). If the number of the fixed points is infinite, i.e. we have a sequence $\{u_i^*\}_{i=1}^\infty$, then the sequence $-A^{-1}f(u_i^*) = u_i^* \rightarrow u_\infty^*$ converges on some subsequence $i \in N' \subset N$, which is a contradiction with the fact that each fixed point u_∞^* is isolated. So the number of the equilibrium points is finite. Now by Theorem 20.6 [53]

$$1 = \gamma(I + A^{-1}f(\cdot), \partial B(0, \|A^{-1}\|K)) = \sum_{i=1}^d \text{ind}(u_i^*, I + A^{-1}f(\cdot))$$

and therefore the number $d = 2k + 1$ for some integer $k \geq 0$. $\square$

2.3. Local unstable manifolds.

Next we show that there are many other solutions which are in the attractor besides the stationary solutions. It is easy to see that every mild solution of (1.2) remains bounded of any $u^0 \in E^\alpha$; that is, $\sup_{t \geq 0} \|u(t, u^0)\|_{E^\alpha} < \infty$. If we show that the mild solution $u(\cdot, u^0)$ is defined in $\mathbb{R}$, takes value in E^α and is bounded, then it must be in the attractor $\mathcal{A}$. So it is enough to show that a solution is defined and bounded in $\mathbb{R}^-$ to obtain a solution in the attractor. We define the unstable set of $u^* \in \mathcal{E}$ as $W^u(u^*) = \{\eta \in E^\alpha : u(t, \eta) \text{ is defined for all } t \leq 0 \text{ and } u(t, \eta) \rightarrow u^* \text{ as } t \rightarrow -\infty\}$.

Before we give more details about how to construct $W^u(u^*)$ at least in a neighborhood of u^* let us look for a moment at a linear problem. If we consider the change of variables $v(\cdot) = u(\cdot) - u^*$ in the problem (1.2), add and subtract $f'(u^*)v(\cdot)$ to its right hand side we obtain

$$(2.6) \quad \begin{aligned} v'(t) &= (A + f'(u^*))v(t) + f(v(t) + u^*) - f(u^*) - f'(u^*)v(t), \\ v(0) &= u^0 - u^* = v^0. \end{aligned}$$

In this equation, for v^0 very small, the part $f(v(t) + u^*) - f(u^*) - f'(u^*)v(t)$ is also very small. It is natural then to consider what happens when we neglect the nonlinearity; that is, what happens to the Cauchy problem

$$(2.7) \quad \begin{aligned} v'(t) &= (A + f'(u^*))v(t), \\ v(0) &= v^0. \end{aligned}$$

Just to say a word about the operator $A + f'(u^*)$, the operator $f'(u^*) \in L(E^\alpha, E)$ and therefore the operator $A + f'(u^*)$ is defined in $D(A)$ and it is an analytic C_0 -semigroup generator with compact resolvent. It is well-known that such generator has the spectrum consisting just of isolated eigenvalues with generalized eigenspaces of finite dimension (so called discrete spectrum). We also know that under above assumptions (see [48]) there are constants $M, b > 0$ such that

$$(2.8) \quad \|\exp(tA)\|_{B(E^{\alpha_1}, E^{\alpha_2})} \leq M t^{\alpha_1 - \alpha_2} e^{-bt}, \quad \alpha_2 > \alpha_1 \geq 0, t \geq 0.$$

The part σ^+ of the spectrum of operator $A + f'(u^*)$ to the right of the imaginary axis consists of a finite number of eigenvalues with finite multiplicity (see Figure 4). Let $U(\sigma^+) \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\}$ be an open connected neighborhood of σ^+ which has a closed rectifiable curve, $\partial U(\sigma^+)$, as boundary. We decompose E^θ using the projection

$$(2.9) \quad Q(\sigma^+) := Q(\sigma^+, A + f'(u^*)) := \frac{1}{2\pi i} \int_{\partial U(\sigma^+)} \left(\zeta I - (A + f'(u^*)) \right)^{-1} d\zeta$$

defined by σ^+ , we have that there is a $\omega > 0$ such that

$$(2.10) \quad \begin{aligned} & \left\| \exp \left(t(A + f'(u^*)) \right) Q(\sigma^+) \right\|_{B(E)} \leq M e^{\beta t}, \quad t \leq 0, \\ & \left\| \exp \left(t(A + f'(u^*)) \right) (I - Q(\sigma^+)) \right\|_{B(E, E^\alpha)} \leq M t^{-\alpha} e^{-\beta t}, \quad t \geq 0, \alpha \geq 0. \end{aligned}$$

If $Q(\sigma^+)$ is the spectral projection defined by the spectrum of $(A + f'(u^*))$ to the right of the imaginary axis, we have that for $v_0 \in Q(\sigma^+)E^\alpha$, then the solution $v(t, v_0)$ of (2.7) exists for all negative time and $v(t, v_0) \rightarrow 0$ as $t \rightarrow -\infty$ and $v(t) + u^* \rightarrow u^*$ as $t \rightarrow -\infty$. That serves as inspiration to what we want to do. When we perturb (2.7) with a very small nonlinearity we should observe solutions of (2.6) that exist for all negative time. Of course, the initial data for which such solutions exist will no longer be in $Q(\sigma^+)E^\alpha$, but in a nonlinear manifold near it (see Figure 1).

Theorem 2.2. *Assume that u^* is a hyperbolic equilibrium for (1.2), which is an isolated solution of (2.4). Then, there exist a neighborhood U of u^* in E^α such that the local unstable manifold $W^u(u^*)$ is given by*

$$W_\delta^u(u^*) = \{w(\cdot) \in W^u(u^*) : \|w(\cdot) - u^*\|_{E^\alpha} < \delta\} = \{w(\cdot) : (v(\cdot), z(\cdot)) \in E^\alpha : u \in U\}.$$

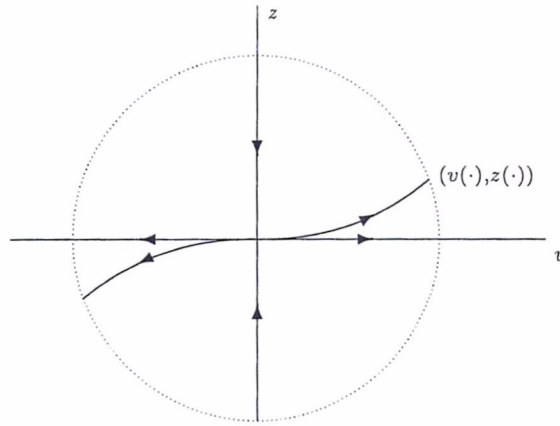


Figure 1

Proof: Rewriting (1.2) for $w(t) = u(t) - u^*$ to deal with the neighborhood of $u^* \in \mathcal{E}$ we arrive at

$$(2.11) \quad w'(t) = (A + f'(u^*))w(t) + f(w(t) + u^*) - f(u^*) - f'(u^*)w(t).$$

Now we decompose the equation (2.11) in the following way. If $w(\cdot)$ is a mild solution of (2.11) we write $v(t) = Q(\sigma^+)w(t)$ and $z(t) = (I - Q(\sigma^+))w(t)$. Hence, if B denotes

the restriction of $(A + f'(u^*))$ to $Q(\sigma^+)E^\alpha$ and by $\tilde{A}$ the restriction of $(A + f'(u^*))$ to $(I - Q(\sigma^+))E^\alpha$, we have that

$$v'(t) = Bv(t) + Q(\sigma^+)f(v(t) + z(t) + u^*) - Q(\sigma^+)f(u^*) - Q(\sigma^+)f'(u^*)(v(t) + z(t))$$

and

$$z'(t) = \tilde{A}z(t) + (I - Q(\sigma^+))f(v(t) + z(t) + u^*) - (I - Q(\sigma^+))f(u^*) - (I - Q(\sigma^+))f'(u^*)(v(t) + z(t)).$$

See also the Figure 4 for location of the spectrum parts. We write

$$H(v, z) = Q(\sigma^+)f(v(t) + z(t) + u^*) - Q(\sigma^+)f(u^*) - Q(\sigma^+)f'(u^*)(v(t) + z(t))$$

and

$$G(v, z) = (I - Q(\sigma^+))f(v(t) + z(t) + u^*) - (I - Q(\sigma^+))f(u^*) - (I - Q(\sigma^+))f'(u^*)(v(t) + z(t)).$$

Hence, we have that, at $(0, 0)$ the functions H and G are zero with zero derivatives at $(0, 0)$. From the continuous differentiability of H and G we obtain that given $\rho > 0$ and $\delta > 0$ such that if $\|v\|_{E^\alpha} + \|z\|_{E^\alpha} < \delta$, we have

$$(2.12) \quad \begin{cases} \|H(v, z)\|_E \leq \rho, \\ \|G(v, z)\|_E \leq \rho, \\ \|H(v, z) - H(\tilde{v}, \tilde{z})\|_E \leq \rho(\|v - \tilde{v}\|_{E^\alpha} + \|z - \tilde{z}\|_{E^\alpha}), \\ \|G(v, z) - G(\tilde{v}, \tilde{z})\|_E \leq \rho(\|v - \tilde{v}\|_{E^\alpha} + \|z - \tilde{z}\|_{E^\alpha}). \end{cases}$$

With the notation above, the equation (2.11) can be rewritten in the following form

$$(2.13) \quad \begin{aligned} v'(t) &= Bv(t) + H(v(t), z(t)), \\ z'(t) &= \tilde{A}z(t) + G(v(t), z(t)), \end{aligned}$$

where H, G satisfy (2.12) for all $v(\cdot) \in Q(\sigma^+)E^\alpha$, $z(\cdot) \in (I - Q(\sigma^+))E^\alpha$. Also, for some positive $M, \omega > 0$, because of analyticity of semigroup, i.e. the property $\omega(A) = \theta(A)$, and splitting the space into $E^\alpha = Q(\sigma^+)E^\alpha \oplus (I - Q(\sigma^+))E^\alpha$, one has

$$(2.14) \quad \begin{cases} \|\exp(t\tilde{A})z\|_{E^\alpha} \leq Me^{-\omega t}\|z\|_{E^\alpha}, & t \geq 0, \\ \|\exp(t\tilde{A})z\|_{E^\alpha} \leq Mt^{-\alpha}e^{-\omega t}\|z\|_E, & t \geq 0, \\ \|\exp(tB)v\|_{E^\alpha} \leq Me^{\omega t}\|v\|_E, & t \leq 0. \end{cases}$$

We have to noted that $\|\exp(t\tilde{A})z\|_{E^\alpha} = \|A^\alpha \exp(t\tilde{A})z\| = \|A^\alpha \tilde{A}^{-\alpha} \tilde{A}^\alpha \exp(t\tilde{A})z\|$ and since $A^\alpha \tilde{A}^{-\alpha}$ is bounded we get inequalities (2.14). Here we note also that $Q(\sigma^+)E^\alpha$ and $Q(\sigma^+)E$ are the same finite dimensional subspace of E and that in this sense the E^α -norm is equivalent to the E -norm in it; that is,

$$\|Q(\sigma^+)AQ(\sigma^+)\|_E = \left\| \frac{1}{2\pi} \int_{\partial U(\sigma^+)} A(\lambda I - A)^{-1} d\lambda \right\| \leq \text{constant},$$

where $U(\sigma^+)$ is an open connected neighborhood of σ^+ which has a closed rectifiable curve, $\partial U(\sigma^+)$, as boundary.

Now we will show that for a suitably small $\rho > 0$, there exists local unstable manifold for $u^* \in \mathcal{E}$

$$W^u = \{(v(\cdot), z(\cdot)) : z(\cdot) \in (I - Q(\sigma^+))E^\alpha, v(\cdot) \in Q(\sigma^+)E^\alpha\}.$$

The solution $(v(t), z(t))$ must go to zero as $t \rightarrow -\infty$ and in particular it must stay bounded. Since

$$z(t) = \exp((t - t_0)\tilde{A})z(t_0) + \int_{t_0}^t \exp((t - s)\tilde{A}) G(v(s), z(s)) ds,$$

making $t_0 \rightarrow -\infty$, we have that

$$z(t) = \int_{-\infty}^t \exp((t - s)\tilde{A}) G(v(s), z(s)) ds.$$

The mild solution of (2.13) is a vector-function $\zeta(t) = \begin{pmatrix} v(t) \\ z(t) \end{pmatrix}$, which satisfies equation

$$(2.15) \quad \zeta(t) = \Theta(\eta, \zeta(t)) \equiv \begin{pmatrix} \exp((t - \tau)\tilde{A})\eta + \int_{\tau}^t \exp((t - s)\tilde{A}) H(v(s), z(s)) ds \\ \int_{-\infty}^t \exp((t - s)\tilde{A}) G(v(s), z(s)) ds \end{pmatrix}.$$

Now, let us consider (2.15) in the space $\Upsilon = C_0((-\infty, \tau]; Q(\sigma^+)E^\alpha) \times C_0((-\infty, \tau]; (I - Q(\sigma^+))E^\alpha)$ with sup norms and condition $v(-\infty) = z(-\infty) = 0$. Let us show that $\Theta(\eta, \cdot) : \Upsilon \rightarrow \Upsilon$.

Take $\zeta(\cdot) \in \Upsilon$, then according to (2.12), for any $\epsilon > 0$ there exists a number $-\infty < t_\epsilon < 0$ such that $\|G(\zeta(s))\|_{(I - Q(\sigma^+))E^\alpha}, \|H(\zeta(s))\|_{Q(\sigma^+)E^\alpha} \leq \epsilon$ as $s \leq t_\epsilon$.

From (2.14) it follows that $e^{(\cdot - \tau)B}\eta \in C_0((-\infty, \tau]; Q(\sigma^+)E^\alpha)$ for any $\eta \in Q(\sigma^+)E^\alpha$ and that, for $t \leq t_\epsilon$,

$$\begin{aligned} \left\| A^\alpha \int_{\tau}^t e^{(t-s)B} H(\zeta(s)) ds \right\| &\leq C \left\| \int_{\tau}^t e^{(t-s)B} H(\zeta(s)) ds \right\| \\ &\leq C \left\| e^{(t-t_\epsilon)A} \int_{\tau}^{t_\epsilon} e^{(t_\epsilon-s)B} H(\zeta(s)) ds + \int_{t_\epsilon}^t e^{(t-s)B} H(\zeta(s)) ds \right\| \\ &\leq CM^2 e^{\omega(t-t_\epsilon)} \int_{\tau}^{t_\epsilon} e^{\omega(t_\epsilon-s)} K ds + M\epsilon \int_{t_\epsilon}^t e^{\omega(t-s)} ds \\ &\leq CM^2 e^{\omega(t-t_\epsilon)} \frac{K}{\omega} + \epsilon \frac{M}{\omega}. \end{aligned}$$

Hence, given $\epsilon > 0$,

$$0 \leq \limsup_{t \rightarrow -\infty} \left\| A^\alpha \int_{\tau}^t \exp((t - s)B) H(\zeta(s)) ds \right\| \leq \epsilon \frac{M}{\omega}.$$

This shows that the first coordinate of $\Theta(\eta, \zeta(t))$ is in $C_0((-\infty, \tau]; Q(\sigma^+)E^\alpha)$. It is easy to see that the second coordinate of $\Theta(\eta, \zeta(t))$ is in $C_0((-\infty, \tau]; (I - Q(\sigma^+))E^\alpha)$ too.

If we put $\eta = \eta_0 = 0$ and $\zeta(t) = \zeta_0(t) = \begin{pmatrix} v_0(t) \\ z_0(t) \end{pmatrix} = 0, t \in (-\infty, \tau]$, then $\zeta_0(t) = \Theta(0, \zeta_0(t))$ and operator $\Theta(\cdot, \cdot)$ is continuous in both arguments. Moreover, Fréchet derivative $\Theta'_\zeta(\eta_0, \zeta_0) : \Upsilon \rightarrow \Upsilon$ is given by

$$\Theta'_\zeta(\eta_0, \zeta_0)h = \begin{pmatrix} \int_{-\infty}^t e^{(t-s)\bar{A}} H'_v(\zeta_0(s)) h^1(s) ds & \int_{-\infty}^t e^{(t-s)\bar{A}} H'_z(\zeta_0(s)) h^2(s) ds \\ \int_{-\infty}^t e^{(t-s)\bar{A}} G'_v(\zeta_0(s)) h^1(s) ds & \int_{-\infty}^t e^{(t-s)\bar{A}} G'_z(\zeta_0(s)) h^2(s) ds \end{pmatrix} = 0$$

for any $h = \begin{pmatrix} h^1(t) \\ h^2(t) \end{pmatrix} \in \Upsilon$, since functions G, H have derivatives equal to 0 at zero. Therefore continuous operator $(I - \Theta'_\zeta(\eta_0, \zeta_0))^{-1}$ exists. By Theorem 54.2 from [53] it follows that there are $\delta, \rho > 0$ such that for $\|\eta - \eta_0\|_{Q(\sigma^+)E^\alpha} \leq \delta$ equation (2.15) has in the ball $\|\zeta - \zeta_0\|_\Upsilon \leq \rho$ unique solution $\zeta(\cdot)$, which depends on η continuously.

From this it follows that the unstable manifold is the graph of the function

$$S : \{\eta \in Q(\sigma^+)E^\alpha : \|\eta\|_{E^\alpha} \leq \delta\} \rightarrow (I - Q(\sigma^+))E^\alpha$$

defined by $S(\eta) = z(\tau)$ where $z(\tau)$ is the second coordinate of the unique solution to (2.15). It is clear from the above reasoning that S is a continuous function. $\square$

2.4. Characterization of the Attractors.

Next we assume that the problem (1.2) has the property (1.11). In case of systems such that all equilibrium points are isolated one can prove the following

Lemma 2.8. *For any $u^0 \in E^\alpha$, $\omega(u^0) = \{u^*\}$ for some $u^* \in \mathcal{E}$. As a consequence of that we conclude that $T(t)u^0 \rightarrow u^*$ as $t \rightarrow \infty$.*

Proof: First note that if $v \in \omega(u^0)$ then there is a sequence $t_n \rightarrow \infty$ such that $T(t_n)u^0 \rightarrow v$. Since $V(T(t)u^0)$ is decreasing and bounded below we have that there is an $\ell \in \mathbb{R}$ such that $V(T(t)u^0) \rightarrow \ell$ as $t \rightarrow \infty$. It is easy to see that $V(T(t)v) = \ell$ for all $t \in \mathbb{R}$. From this it follows that $v \in \mathcal{E}$. Since the set $\mathcal{E}$ consist only of a finite number of points and since $\omega(u^0)$ is connected we have that $\omega(u^0) = \{u^*\}$ for some $u^* \in \mathcal{E}$. $\square$

The next result implies that if there is a negative orbit through u^0 which is bounded, then u^0 is in the unstable manifold of some $u^* \in \mathcal{E}$.

Lemma 2.9. *Suppose that $u^0 \in E^\alpha$ is such that there is a negative orbit $\phi : (-\infty, 0] \rightarrow E^\alpha$ through u^0 and such that $\overline{\phi((-\infty, 0])}$ is compact. Define*

$$\alpha_\phi(u^0) = \{v \in E^\alpha : \exists t_n \rightarrow \infty \text{ such that } \phi(-t_n) \rightarrow v\}.$$

Then, there is a $u^ \in \mathcal{E}$ such that $\alpha_\phi(u^0) = \{u^*\}$. As a consequence of that we conclude that $\phi(t) \rightarrow u^*$ as $t \rightarrow -\infty$.*

Proof. First note that if $v \in \alpha_\phi(u^0)$, then there is a sequence $t_n \rightarrow \infty$ such that $\phi(-t_n) \rightarrow v$. Since $V(\phi(-t))$ is increasing and bounded above we have that there is an $r \in \mathbb{R}$ such that $V(\phi(-t)) \rightarrow r$ as $t \rightarrow \infty$. It is easy to see that $V(T(t)v) = r$ for all $t \in \mathbb{R}$. From this it follows that $v \in \mathcal{E}$. Since the set $\mathcal{E}$ consist only of a finite number of points and since $\alpha_\phi(u^0)$ is connected we have that $\alpha_\phi(u^0) = \{u^*\}$ for some $u^* \in \mathcal{E}$. Furthermore, for any sequence $t_n \rightarrow \infty$, $\{\phi(-t_n)\}$ has a subsequence which converges to u^* and therefore $\phi(t) \rightarrow u^*$ as $t \rightarrow -\infty$. $\square$

Lemma 2.10. *Let S be a relatively compact invariant subset of E^α under $\{T(t) : t \geq 0\}$. If $v(\cdot) \in S$ such that (1.11) is satisfied, then there are $v^+, v^- \in \mathcal{E}$ such that $v(\cdot) \in W^u(v^+)$ and $v(\cdot) \in W^s(v^-)$.*

Proof: Let $v(\cdot) \in S$. Since S is invariant, there is a complete orbit $\mathbb{R} \ni t \mapsto T(t)v(\tau) \in S$ through $v(\cdot)$. Since $\bar{S}$ is compact this complete orbit is compact. The result now follows from the previous lemmas. $\square$

From this lemma one gets the following result.

Theorem 2.3. *Assume that A has compact resolvent and let $\mathcal{A}$ be the attractor for the semi-flow $\{T(t) : t \geq 0\}$. If (1.11) is satisfied then, $\mathcal{A} = \bigcup_{u^* \in \mathcal{E}} W^u(u^*)$.*

3. APPROXIMATION OF SPECTRUM OF LINEAR OPERATORS

The most important role in approximations of equation $Bx = y$ and approximations of spectrum of an operator B is played by the notions of *stable* and *regular* convergence. These notions are used in different areas of numerical analysis (see [2, 20, 25, 26, 72, 75]).

3.1. Stable and Regular convergences.

Definition 3.1. *A sequence of operators $\{B_n\}, B_n \in B(E_n), n \in \mathbb{N}$, is said to be stably convergent to an operator $B \in B(E)$ iff $B_n \xrightarrow{PP} B$ and $\|B_n^{-1}\|_{B(E_n)} = O(1), n \rightarrow \infty$. We will write this as: $B_n \xrightarrow{PP} B$ stably.*

Definition 3.2. A sequence of operators $\{B_n\}, B_n \in B(E_n)$, is called *regularly convergent* to the operator $B \in B(E)$ iff $B_n \xrightarrow{\mathcal{PP}} B$ and the following implication holds:

$$\|x_n\|_{E_n} = O(1) \text{ \& } \{B_n x_n\} \text{ is } \mathcal{P}\text{-compact} \implies \{x_n\} \text{ is } \mathcal{P}\text{-compact}.$$

We write this as: $B_n \xrightarrow{\mathcal{PP}} B$ regularly.

Let $\Lambda \subseteq \mathbb{C}$ be some open connected set, and let $B \in B(E)$. For an isolated point $\lambda \in \sigma(B)$, the corresponding maximal invariant space (or generalized eigenspace) will be denoted by $\mathcal{W}(\lambda; B) = P(\lambda)E$, where $P(\lambda) = \frac{1}{2\pi i} \int_{|\zeta - \lambda| = \delta} (\zeta I - B)^{-1} d\zeta$ and δ is small enough so that there are no points of $\sigma(B)$ in the disc $\{\zeta : |\zeta - \lambda| \leq \delta\}$ different from λ . The isolated point $\lambda \in \sigma(B)$ is a *Riesz point* of B if $\lambda I - B$ is a Fredholm operator of index zero and $P(\lambda)$ is of finite rank. Denote by

$$\mathcal{W}(\lambda, \delta; B_n) = \bigcup_{\substack{\lambda_n \in \sigma(B_n), \\ |\lambda_n - \lambda| < \delta}} \mathcal{W}(\lambda_n, B_n).$$

It is clear that $\mathcal{W}(\lambda, \delta; B_n) = P_n(\lambda)E_n$, where

$$P_n(\lambda) = \frac{1}{2\pi i} \int_{|\zeta - \lambda| = \delta} (\zeta I - B_n)^{-1} d\zeta.$$

The following theorems state the complete picture of the approximation of the spectrum.

Theorem 3.1. [21, 73, 74] Assume that $L_n(\lambda) = \lambda I - B_n$ and $L(\lambda) = \lambda I - B$ are Fredholm operators of index zero for any $\lambda \in \Lambda$ and $L_n(\lambda) \rightarrow L(\lambda)$ stably for any $\lambda \in \rho(B) \cap \Lambda \neq \emptyset$. Then

- (i) for any $\lambda_0 \in \sigma(B) \cap \Lambda$, there exists a sequence $\{\lambda_n\}$, $\lambda_n \in \sigma(B_n)$, $n \in \mathbb{N}$, such that $\lambda_n \rightarrow \lambda_0$ as $n \rightarrow \infty$;
- (ii) if for some sequence $\{\lambda_n\}$, $\lambda_n \in \sigma(B_n)$, $n \in \mathbb{N}$, one has $\lambda_n \rightarrow \lambda_0 \in \Lambda$ as $n \rightarrow \infty$, then $\lambda_0 \in \sigma(B)$;
- (iii) for any $x \in \mathcal{W}(\lambda_0, B)$, there exists a sequence $\{x_n\}$, $x_n \in \mathcal{W}(\lambda_0, \delta; B_n)$, $n \in \mathbb{N}$, such that $x_n \rightarrow x$ as $n \rightarrow \infty$;
- (iv) there exists $n_0 \in \mathbb{N}$ such that $\dim \mathcal{W}(\lambda_0, \delta; B_n) \geq \dim \mathcal{W}(\lambda_0, B)$ for any $n \geq n_0$.

Remark 3.1. The inequality in (iv) can be strict for all $n \in \mathbb{N}$ as is shown in [72].

Theorem 3.2. [72] Assume that $L_n(\lambda)$ and $L(\lambda)$ are Fredholm operators of index zero for all $\lambda \in \Lambda$. Suppose that $L_n(\lambda) \rightarrow L(\lambda)$ regularly for any $\lambda \in \Lambda$ and $\rho(B) \cap \Lambda \neq \emptyset$. Then statements (i)–(iii) of Theorem 3.1 hold and also

- (iv) there exists $n_0 \in \mathbb{N}$ such that $\dim \mathcal{W}(\lambda_0, \delta; B_n) = \dim \mathcal{W}(\lambda_0, B)$ for all $n \geq n_0$;

(v) any sequence $\{x_n\}$, $x_n \in \mathcal{W}(\lambda_0, \delta; B_n)$, $n \in \mathbb{N}$, with $\|x_n\|_{E_n} = 1$ is $\mathcal{P}$ -compact and any limit point of this sequence belongs to $\mathcal{W}(\lambda_0, B)$.

The Theorem 3.2 gives us necessary and sufficient conditions for the equality

$$\dim \mathcal{W}(\lambda_0, \delta; B_n) = \dim \mathcal{W}(\lambda_0, B), \quad \forall n \geq n_0,$$

which is needed in the approximation of unstable manifolds.

3.2. Regions of convergence.

Theorems 3.1 and 3.2 have been generalized to the case of closed operators in [77] by using the following notions introduced by T. Kato [50].

Definition 3.3. The region of stability $\Delta_s = \Delta_s(\{A_n\})$, $A_n \in \mathcal{C}(B_n)$, is defined as the set of all $\lambda \in \mathbb{C}$ such that $\lambda \in \rho(A_n)$ for almost all n and such that the sequence $\{\|(\lambda I_n - A_n)^{-1}\|\}_{n \in \mathbb{N}}$ is bounded. The region of convergence $\Delta_c = \Delta_c(\{A_n\})$, $A_n \in \mathcal{C}(E_n)$, is defined as the set of all $\lambda \in \mathbb{C}$ such that $\lambda \in \Delta_s(\{A_n\})$ and such that the sequence of operators $\{(\lambda I_n - A_n)^{-1}\}_{n \in \mathbb{N}}$ is $\mathcal{PP}$ -convergent to some operator $S(\lambda) \in B(E)$.

Definition 3.4. A sequence of operators $\{K_n\}$, $K_n \in \mathcal{C}(E_n)$, is called regularly compatible with an operator $K \in \mathcal{C}(E)$ if (K_n, K) are compatible and, for any bounded sequence $\|x_n\|_{E_n} = O(1)$ such that $x_n \in D(K_n)$ and $\{K_n x_n\}$ is $\mathcal{P}$ -compact, it follows that $\{x_n\}$ is $\mathcal{P}$ -compact, and the $\mathcal{P}$ -convergence of $\{x_n\}$ to some x and that of $\{K_n x_n\}$ to some y as $n \rightarrow \infty$ in $\mathbb{N}' \subseteq \mathbb{N}$ imply that $x \in D(K)$ and $Kx = y$.

Definition 3.5. The region of regularity $\Delta_r = \Delta_r(\{A_n\}, A)$, is defined as the set of all $\lambda \in \mathbb{C}$ such that (K_n, K) , where $K_n = \lambda I - A_n$ and $K = \lambda I - A$ are regularly compatible.

The relationships between these regions are given by the following statement.

Proposition 3.1. [77] Suppose that $\Delta_c \neq \emptyset$ and $\mathcal{N}(S(\lambda)) = \{0\}$ at least for one point $\lambda \in \Delta_c$ so that $S(\lambda) = (\lambda I - A)^{-1}$. Then (A_n, A) are compatible and

$$\Delta_c = \Delta_s \cap \rho(A) = \Delta_s \cap \Delta_r = \Delta_r \cap \rho(A).$$

It is shown in [77] that the conditions (A_n, A) are compatible, $\lambda I_n - A_n$ and $\lambda I - A$ are Fredholm operators with index zero for any $\lambda \in \Lambda$ and $\rho(A) \cap \Lambda \neq \emptyset$ imply (i)–(iv) of Theorem 3.1 when $\rho(A) \cap \Lambda \subseteq \Delta_s$ and imply (i)–(iii) of Theorem 3.1 and (iv)–(v) of Theorem 3.2, when $\Lambda \subseteq \Delta_r$.

Definition 3.6. A Riesz point $\lambda_0 \in \sigma(A)$ is said to be strongly stable in Kato's sense if $\dim \mathcal{W}(\lambda_0, \delta; B_n) \leq \dim \mathcal{W}(\lambda_0, B)$ for all $n \geq n_0$.

Theorem 3.3. [77] The Riesz point $\lambda_0 \in \sigma(A)$ is strongly stable in Kato's sense iff $\lambda_0 \in \Lambda \cap \Delta_r \cap \sigma(A)$.

We consider now the important class of operators which have compact resolvents, this is say our case of problem (1.2). In this case, it is natural to consider approximate operators which “preserve” this property.

Definition 3.7. A sequence of operators $\{B_n\}$, $B_n : E_n \rightarrow E_n$, $n \in \mathbb{N}$, converges compactly to an operator $B : E \rightarrow E$ if $B_n \xrightarrow{\mathcal{PP}} B$ and the following compactness condition holds:

$$\|x_n\|_{E_n} = O(1) \implies \{B_n x_n\} \text{ is } \mathcal{P}\text{-compact.}$$

Definition 3.8. The region of compact convergence of resolvents, $\Delta_{cc} = \Delta_{cc}(A_n, A)$, where $A_n \in \mathcal{C}(E_n)$ and $A \in \mathcal{C}(E)$ is defined as the set of all $\lambda \in \Delta_c \cap \rho(A)$ such that $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$ compactly.

Theorem 3.4. Assume that $\Delta_{cc} \neq \emptyset$. Then for any $\zeta \in \Delta_s$ the following implication holds:

$$(3.1) \quad \|x_n\|_{E_n} = O(1) \ \& \ \|(\zeta I_n - A_n)x_n\|_{E_n} = O(1) \implies \{x_n\} \text{ is } \mathcal{P}\text{-compact.}$$

Conversely, if for some $\zeta \in \Delta_c \cap \rho(A)$ implication (3.1) holds, then $\Delta_{cc} \neq \emptyset$.

Proof. Let $(\mu I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\mu I - A)^{-1}$ compactly for some $\mu \in \Delta_{cc}$. Then for $\|x_n\|_{E_n} = O(1)$ and $\|(\zeta I_n - A_n)x_n\|_{E_n} = O(1)$, from the Hilbert identity

$$(3.2) \quad (\zeta I_n - A_n)^{-1} - (\mu I_n - A_n)^{-1} = (\mu - \zeta)(\zeta I_n - A_n)^{-1}(\mu I_n - A_n)^{-1},$$

we obtain $x_n = (\mu I_n - A_n)^{-1}(\zeta I_n - A_n)x_n - (\zeta - \mu)(\mu I_n - A_n)^{-1}x_n$, and it follows that $\{x_n\}$ is $\mathcal{P}$ -compact. Conversely, let implication (3.1) hold for some $\zeta_0 \in \Delta_c \cap \rho(A)$. We show that $\zeta_0 \in \Delta_{cc}$. Taking a bounded sequence $\{y_n\}$, $n \in \mathbb{N}$, we obtain $\|(\zeta_0 I_n - A_n)^{-1}y_n\|_{E_n} = O(1)$ for $n \in \mathbb{N}$. Let us apply implication (3.1) to the sequence $x_n = (\zeta_0 I_n - A_n)^{-1}y_n$. It is easy to see that $\{x_n\}$ is $\mathcal{P}$ -compact. Hence $\zeta_0 \in \Delta_{cc}$. $\square$

Corollary 3.1. Assume that $\Delta_{cc} \neq \emptyset$. Then $\Delta_{cc} = \Delta_c \cap \rho(A)$.

Proof. It is clear that $\Delta_{cc} \subseteq \Delta_c \cap \rho(A)$. To prove that $\Delta_{cc} \supseteq \Delta_c \cap \rho(A)$, let us consider the Hilbert identity (3.2). Now let $\mu \in \Delta_{cc}$. Then $\mu \in \Delta_{cc} \cap \Delta_c \cap \rho(A)$. Hence, for every $\zeta \in \Delta_c \cap \rho(A)$ and for any bounded sequence $\{x_n\}$, $n \in \mathbb{N}$, the sequence $\{(\zeta I_n - A_n)^{-1}x_n\}$ is $\mathcal{P}$ -compact. $\square$

Theorem 3.5. Assume that $\Delta_{cc} \neq \emptyset$. Then $\Delta_r = \mathbb{C}$.

Proof. Take any point $\lambda_1 \in \mathbb{C}$. We have to show that $(\lambda_1 I_n - A_n, \lambda_1 I - A)$ are regularly compatible. Assume that $\|x_n\|_{E_n} = O(1)$ and that $\{(\lambda_1 I_n - A_n)x_n\}$ is $\mathcal{P}$ -compact. To show that $\{x_n\}$ is $\mathcal{P}$ -compact, we take $\mu \in \Delta_{cc}$. Assume now that $\|x_n\| \leq \text{constant}$ and $(\lambda_1 I_n - A_n)x_n \xrightarrow{\mathcal{P}} y$, as $n \rightarrow \infty$ in $\mathbb{N}' \subseteq \mathbb{N}$. Applying operator $(\mu I_n - A_n)^{-1}$ one gets first that $\{x_n\}$ is $\mathcal{P}$ -compact and then assuming $x_n \xrightarrow{\mathcal{P}} x$ as $n \in \mathbb{N}'' \subset \mathbb{N}'$ one obtains

$$(\mu I_n - A_n)^{-1}(\lambda_1 I_n - A_n)x_n = (\lambda_1 - \mu)(\mu I_n - A_n)^{-1}x_n + x_n \xrightarrow{\mathcal{P}}$$

$$\xrightarrow{\mathcal{P}} (\mu I - A)^{-1} y = (\lambda_1 - \mu)(\mu I - A)^{-1} x + x$$

as $n \rightarrow \infty$ in $\mathbb{N}' \subseteq \mathbb{N}$. Then it follows that $x \in D(A)$ and $(\lambda_1 I - A)x = y$. $\square$

From the Theorem 3.5 it follows that in case $\Delta_{cc} \neq \emptyset$ all the assumptions of Theorem 3.2 are satisfied and a complete picture of approximation of spectrum is valid.

4. DISCRETIZATION OF SEMIGROUPS

As the matter of fact most of the results on approximation of attractors as we will see later follow from convergence of linear operators and families of operators.

The general approximation scheme is described in the Introduction. We just mention that from (1.13) and the Uniform Boundedness Principle (see [49]) one has

Lemma 4.1. *There is a constant $C \geq 1$ such that $\|p_n\|_{L(E, E_n)} \leq C$, $\forall n \in N$.*

Let us recall that each mild solution of problem (1.2) satisfies the equation

$$(4.1) \quad u(t) = (Ku)(t) \equiv \exp(tA)u^0 + \int_0^t \exp((t-s)A)f(u(s))ds, \quad t \in [0, T], T < \infty.$$

4.1. Compact convergence of some operator families.

According to Theorem 1.4 assume that operators A_n generate (see (1.1)) analytic C_0 -semigroups with estimate $\|e^{tA_n}\| \leq Me^{-t\tilde{\omega}}$, $t \geq 0$, where $\tilde{\omega} > 0$, resolvents $(\lambda I_n - A_n)^{-1}$ are compact for some $\lambda \in \rho(A_n)$, converge compactly $(\lambda I_n - A_n)^{-1} \rightarrow (\lambda I - A)^{-1}$ for some $\lambda \in \cap_{n=1}^{\infty} \rho(A_n) \cap \rho(A)$, functions $f_n(\cdot) : E_n^\alpha \rightarrow E_n$ are globally Lipschitz, bounded, i.e. there is a constant $\tilde{K} > 0$, such that $\|f_n(w_n)\|_{E_n} \leq \tilde{K}$ for all $w_n \in E_n^\alpha$, and they are continuously Fréchet differentiable functions. Assume also that $f_n(x_n) \xrightarrow{\mathcal{P}} f(x)$ and $f'_n(x_n) \xrightarrow{\mathcal{PP}} f'(x)$ whenever $x_n \xrightarrow{\mathcal{P}^\alpha} x$.

Let $t \mapsto T_n(t)u_n^0$ be the solution of (1.14). We know that all solutions of (1.14) are globally defined, hence $t \mapsto T_n(t)(\cdot)$ is a continuous nonlinear semigroup for each $n \in N$. Consider the operator equations

$$(4.2) \quad u_n(t) = (K_n u_n)(t) \equiv \exp(tA_n)u_n^0 + \int_0^t \exp((t-s)A_n)f_n(u_n(s))ds, \quad t \in [0, T], T < \infty.$$

Consider the operators $K : F \rightarrow F$ and $K_n : F_n \rightarrow F_n$ defined by (4.1) and (4.2) where

$$F = C([0, T]; E^\alpha) \equiv \{u(t) : \|u\|_F = \max_{t \in [0, T]} \|u(t)\|_{E^\alpha} < \infty\}$$

and

$$F_n = C([0, T]; E_n^\alpha) \equiv \{u_n(t) : \|u_n\|_{F_n} = \max_{t \in [0, T]} \|u_n(t)\|_{E_n^\alpha} < \infty\}.$$

Theorem 4.1. *Assume that the operator A generates analytic and compact C_0 -semigroup. Assume that the function $f(\cdot)$ is sufficiently smooth. Then operator K is compact.*

Proof: The operator K defined by (4.1) is compact in F . Indeed, we obtain that the operator

$$\mathcal{F}_\epsilon(u^k)(t) = \exp(\epsilon A) \int_0^{t-\epsilon} A^\alpha \exp((t-s-\epsilon)A) f(u^k(s)) ds$$

maps any bounded set of functions $\{u^k(\cdot)\}$, $\|u^k(\cdot)\|_F \leq C$, into a compact set in E^α for any $0 < \epsilon < t$.

Since interval $[0, T]$ is finite and $\int_0^\epsilon \leq C\epsilon$ we see that $\|\mathcal{F}_\epsilon(u^k)(t) - \mathcal{F}(u^k)(t)\| \leq C\epsilon$ for any $t \in (0, T]$, where

$$\mathcal{F}(u^k)(t) = \int_0^t A^\alpha \exp((t-s)A) f(u^k(s)) ds$$

and $0 < \epsilon < t$. Then it follows that the operator $\mathcal{F}(\cdot)(t) : F \rightarrow E^\alpha$ is compact for the same $t > 0$. For $t = 0$, the operator $\mathcal{F}(\cdot)(0)$ is also compact. Moreover, the set of functions $\{F_k(\cdot)\}$, $F_k(t) = \mathcal{F}(u^k)(t)$, $t \in [0, T]$, is an equibounded and equicontinuous family, since for $0 < t_1 < t_2$, we obtain

$$\|F_k(t_2) - F_k(t_1)\|_{E^\alpha} \leq C \left(\int_0^{t_1} \|\exp((t_2-s)A) - \exp((t_1-s)A)\|_{E^\alpha} ds + |t_2 - t_1|^{1-\alpha} \right),$$

and $\exp(\cdot A)$ is uniformly continuous in $t > 0$. By the generalized Arzela–Ascoli theorem, the same way as in Theorem 4.2, it follows that operator K is compact. $\square$

Theorem 4.2. *Let the condition [A1] be satisfied. Then the operators $K_n \xrightarrow{\mathcal{PP}} K$ compactly.*

Proof: To show that $K_n \xrightarrow{\mathcal{PP}} K$ compactly, we assume that $\|u_n\|_{F_n} = O(1)$. We know from Proposition 6.1 that condition $\Delta_{cc} \neq \emptyset$ implies that $\exp(tA_n) \rightarrow \exp(tA)$ compactly. Now $\{K_n u_n\}$ is $\mathcal{P}$ -compact by the generalized Arzela–Ascoli theorem. To show this, we verify the vanishing of the noncompactness measure $\mu(\{(K_n u_n)(t)\}) = 0$ for all $t \in [0, T]$. Let us consider the relation

$$(K_n u_n)(t) = (-A)^\alpha \exp(tA_n) u_n^0 + \psi_n^\tau(t) + \varphi_n^\tau(t),$$

where

$$\psi_n^\tau(t) = \exp(\tau A_n) \int_0^{t-\tau} (-A)^\alpha \exp((t-s-\tau)A_n) f_n(u_n(s)) ds,$$

$$\varphi_n^\tau(t) = \int_{t-\tau}^t (-A)^\alpha \exp((t-s)A_n) f_n(u_n(s)) ds.$$

By virtue of the boundedness of $\|f_n(u_n(\cdot))\|_{F_n}$, we can choose the term $\|\varphi_n^\tau(\cdot)\|_{F_n}$ sufficiently small with τ small enough and $\mu(\{\psi_n^\tau\}) = 0$. The sequence $\{(-A)^\alpha e^{tA_n} u_n^0\}$ is also $\mathcal{P}$ -compact.

It is clear that $\{K_n u_n\}$ is uniformly bounded in n . To show equicontinuity one can write for $t_1 \leq t_2$

$$\begin{aligned} & \|A_n^\alpha \int_0^{t_2} \exp((t_2 - s)A_n) f_n(u_n(s)) ds - A_n^\alpha \int_0^{t_1} \exp((t_1 - s)A_n) f_n(u_n(s)) ds\|_{E_n} \\ & \leq \left\| \int_{t_1}^{t_2} A_n^\alpha e^{(t_2-s)A_n} f_n(u_n(s)) ds \right\|_{E_n} + \left\| \int_0^{t_1} A_n^\alpha (e^{(t_2-s)A_n} - e^{(t_1-s)A_n}) f_n(u_n(s)) ds \right\|_{E_n}. \end{aligned}$$

The first term tends to zero as $|t_2 - t_1| \rightarrow 0$, since $\int_{t_1}^{t_2} \frac{ds}{(t_2-s)^\alpha} = -(t_2-s)^{1-\alpha} \Big|_{t_1}^{t_2} = (t_2-t_1)^{1-\alpha}$. To estimate the second term for any $\epsilon > 0$ and some $0 < \delta \leq \epsilon^{\frac{1}{1-\alpha}}$ consider the cases $t_1 \leq \delta$ and $t_1 > \delta$. If $t_1 \leq \delta$ and $|t_2 - t_1| \leq \delta$, then as for the first term $\|\int_0^{t_2} \dots ds\| \leq c\delta^{1-\alpha} \leq C\epsilon$. If $t_1 > \delta$, then one can divide the second term into $\int_0^{t_1} \dots ds = \int_0^{t_1-\delta} \dots ds + \int_{t_1-\delta}^{t_1} \dots ds$. So, if $|t_2 - t_1| \leq \delta$, then $\int_{t_1-\delta}^{t_1} \dots ds \approx c\delta^{1-\alpha} \leq C\epsilon$ as before. Since $t_1 - s, t_2 - s \in [\delta, t_1]$ for $s \in [0, t_1 - \delta]$, then $\int_0^{t_1} A_n^\alpha (e^{(t_2-s)A_n} - e^{(t_1-s)A_n}) f_n(u_n(s)) ds$ tends to zero as $|t_2 - t_1| \rightarrow 0$ because of convergence $\|A_n^\alpha (e^{(t_2-s)A_n} - e^{(t_1-s)A_n})\|_{E_n} \rightarrow 0$ as $|t_2 - t_1| \rightarrow 0$, i.e. arguments $t_2 - s, t_1 - s$ are separated from zero for more than δ .

Let $M, \omega > 0$ and $\Omega^{M, \omega} = \{(v(\cdot), z(\cdot)) \in C_0((-\infty, 0]; Q(\sigma^+)E^\alpha) \times C_0((-\infty, 0]; (I - Q(\sigma^+))E^\alpha) : \|v(t)\|_{E^\alpha}, \|z(t)\|_{E^\alpha} \leq Me^{\omega t}, t \leq 0\}$. Using Gronwall's inequality one can show right from (2.15) that solution of (2.15) belongs to $\Omega^{M, \omega}$.

Theorem 4.3. *Assume that resolvent $(\lambda I - A)^{-1}$ is compact for some $\lambda \in \rho(A)$. Then the operator $\Theta(\eta, \cdot) : \Omega^{M, \omega} \subset \Upsilon \rightarrow \Upsilon$ is compact.*

Proof. Let us take the sequence of bounded functions $\{v^k(\cdot)\}, \{z^k(\cdot)\}, v^k, z^k \in \Omega^{M, \omega}, k \in \mathbb{N}$. Then the operators

$$A^\alpha \int_0^t e^{(t-s)(A+f'(u^*))} Q(\sigma^+) H(v^k(s), z^k(s)) ds,$$

and

$$A^\alpha \int_{-\infty}^t e^{(t-s)(A+f'(u^*))} (I - Q(\sigma^+)) G(v^k(s), z^k(s)) ds$$

map this sequences into compact sequence. Indeed, one can very easy to check uniform boundedness and compactness for any fixed $t < 0$. To check equicontinuity note that

$$\|A^\alpha \int_0^t e^{(t-s)(A+f'(u^*))} Q(\sigma^+) H(v^k(s), z^k(s)) ds\| \leq \epsilon, \text{ whenever } \|v^k(s)\|, \|z^k(s)\| \leq \delta \text{ for } t \leq t_\epsilon.$$

So on the interval $[t_\epsilon, 0]$ we have to apply Arzela-Ascoli Theorem and check equicontinuity like in Theorem 4.2.

The same way as we introduced operator $\Theta(\eta, u)$ in (2.15) we can introduce the operator $\Theta_n(\eta_n, u_n)$. Normally we assume that $\eta_n \xrightarrow{p^\alpha} \eta$ or even $\eta_n = p_n^\alpha \eta$. Rewriting (1.14) for $w_n(t) =$

$u_n(t) - u_n^*$ to deal with the neighborhood of u_n^* we arrive at

$$(4.3) \quad w'_n(t) = A_n w_n(t) + f'_n(u_n^*) w_n(t) + f_n(w_n(t) + u_n^*) - f_n(u_n^*(t)) - f'_n(u_n^*) w_n(t).$$

If $Q_n(\sigma_n^+)$ is as in Proposition 5.5 and $W_n(\sigma_n^+) = Q_n(\sigma_n^+)E_n^\alpha$, then $W_n(\sigma_n^+)$ is isomorphic to $W = Q(\sigma^+)E^\alpha$ through an isomorphism $\mathfrak{I}_n = Q_n(\sigma_n^+)p_n^\alpha$, which is bounded with bounded inverse $\mathfrak{I}_n^{-1}$ and the norms of $\mathfrak{I}_n$ and $\mathfrak{I}_n^{-1}$ are uniformly bounded in $n \geq n_0$. Recall that $Q_n(\sigma_n^+)$ are projectors such that $Q_n(\sigma_n^+) \xrightarrow{\mathcal{P}^\alpha} Q(\sigma^+)$ compactly for $Q(\sigma^+)$ from Section 2.3. We identify all spaces $W_n(\sigma_n^+)$ and $Q_n(\sigma_n^+)E_n$ through these isomorphisms to the fixed space W .

Now we decompose the equation (4.3) in the following way. If w_n is a solution to (4.3) we write

$$w_n(t) = v_n(t) + z_n(t), \text{ where } v_n(t) = Q_n(\sigma_n^+)w_n(t), \quad z_n(t) = (I_n - Q_n(\sigma_n^+))w_n(t).$$

Hence, if we denote by B_n the bounded linear operator given by the restriction of A_n to $Q_n(\sigma_n^+)E_n$, we have

$$v'_n = B_n v_n(t) + Q_n(\sigma_n^+) \left(f_n(v_n + z_n + u_n^*) - f_n(u_n^*) - f'_n(u_n^*)(v_n(t) + z_n(t)) \right)$$

and

$$z'_n = \tilde{A}_n z_n + f'_n(u_n^*) z_n + (I_n - Q_n(\sigma_n^+)) \left(f_n(v_n + z_n + u_n^*) - f_n(u_n^*) - f'_n(u_n^*)(v_n + z_n) \right).$$

Here we denoted by $\tilde{A}_n = (A_n + f'_n(u_n^*))|_{(I_n - Q_n(\sigma_n^+))E_n^\alpha}$. We write

$$H_n(v_n, z_n) = Q_n(\sigma_n^+) \left(f_n(v_n + z_n + u_n^*) - f_n(u_n^*) - f'_n(u_n^*)(v_n + z_n) \right)$$

and

$$G_n(v_n, z_n) = (I_n - Q_n(\sigma_n^+)) \left(f_n(v_n + z_n + u_n^*) - f_n(u_n^*) - f'_n(u_n^*)(v_n + z_n) \right).$$

Hence, we have that, $H_n(0, 0) = 0, G_n(0, 0) = 0$. From the continuous differentiability of H_n and G_n and estimates (5.4), which valid uniformly in n we obtain that given $\rho > 0$ there exist $n_0 > 0$ and $\delta > 0$ such that if $\|v_n\|_{W_n(\sigma_n^+)} + \|z_n\|_{E_n^\alpha} < \delta$ and $n \geq n_0$ we have

$$(4.4) \quad \begin{aligned} & \|H_n(v_n, z_n)\|_{W_n(\sigma_n^+)} \leq \rho, \\ & \|G_n(v_n, z_n)\|_{E_n} \leq \rho, \\ & \|H_n(v_n, z_n) - H_n(\tilde{v}_n, \tilde{z}_n)\|_{W_n(\sigma_n^+)} \leq \rho(\|v_n - \tilde{v}_n\|_{W_n(\sigma_n^+)} + \|z_n - \tilde{z}_n\|_{E_n^\alpha}), \\ & \|G_n(v_n, z_n) - G_n(\tilde{v}_n, \tilde{z}_n)\|_{E_n} \leq \rho(\|v_n - \tilde{v}_n\|_{W_n(\sigma_n^+)} + \|z_n - \tilde{z}_n\|_{E_n^\alpha}). \end{aligned}$$

The fact that we can choose ρ and δ uniformly for $n \geq n_0$ satisfying the inequalities above is the key point to obtain that the local unstable manifolds are defined in a small “neighborhood” of the equilibrium point u_n uniformly for $n \geq n_0$. Then, equation (4.3) can be

rewritten in the following form

$$(4.5) \quad \begin{aligned} v'_n(t) &= (A_n + f'_n(u_n^*))Q_n(\sigma_n^+)v_n(t) + H_n(v_n(t), z_n(t)), \\ z'_n(t) &= (A_n + f'_n(u_n^*))(I_n - Q_n(\sigma_n^+))z_n(t) + G_n(v_n(t), z_n(t)), \end{aligned}$$

where H_n, G_n satisfy (4.4) for all $v_n(\cdot) \in W_n(\sigma_n^+)$, $z_n(\cdot) \in (I_n - Q_n(\sigma_n^+))E_n^\alpha$.

Theorem 4.4. *The operators $\Theta_n(\eta_n, \cdot) : \Omega_n^{M, \omega} \subset \Upsilon_n \rightarrow \Upsilon_n$, $\Theta(\eta, \cdot) : \Omega^{M, \omega} \subset \Upsilon \rightarrow \Upsilon$, converge $\Theta_n(\eta_n, \cdot) \rightarrow \Theta(\eta, \cdot)$ compactly.*

Proof. We recall that

$$\|u_n(\cdot)\|_{\Upsilon_n} = \left\| \begin{pmatrix} u_n^1(\cdot) \\ u_n^2(\cdot) \end{pmatrix} \right\|_{\Upsilon_n} = \sup_{-\infty < t \leq 0} \|u_n^1(t)\|_{Q_n(\sigma_n^+)E_n^\alpha} + \sup_{-\infty < t \leq 0} \|u_n^2(t)\|_{(I_n - Q_n(\sigma_n^+))E_n^\alpha}$$

and $u_n^1(-\infty) = u_n^2(-\infty) = 0$. So the discrete convergence $\Upsilon_n \ni u_n(\cdot) \rightarrow u(\cdot) \in \Upsilon$ means that $\sup_{-\infty < t \leq 0} \|u_n^1(t) - p_n^\alpha u^1(t)\|_{Q_n(\sigma_n^+)E_n^\alpha} + \sup_{-\infty < t \leq 0} \|u_n^2(t) - p_n^\alpha u^2(t)\|_{(I_n - Q_n(\sigma_n^+))E_n^\alpha} \rightarrow 0$ as $n \rightarrow \infty$. Let us check that condition $\|u_n(\cdot)\|_{\Upsilon_n} = O(1)$ implies that sequence $\{\Theta_n(\eta_n, u_n(\cdot))\}$ is compact. We have to apply Arzela-Ascoli Theorem to the sequences

$$\{A_n^\alpha \int_0^t e^{(t-s)(A_n + f'_n(u_n^*))} Q_n(\sigma_n^+) H_n(v_n^k(s), z_n^k(s)) ds\},$$

$$\{A_n^\alpha \int_{-\infty}^t e^{(t-s)(A_n + f'_n(u_n^*))} (I_n - Q_n(\sigma_n^+)) G_n(v_n^k(s), z_n^k(s)) ds\}.$$

One has $\|A_n^\alpha \int_0^t e^{(t-s)A_n} H_n(v_n^k(s), z_n^k(s)) ds\| \leq \epsilon$ as $t \leq t_\epsilon$, because of (4.4). So we have to apply Arzela-Ascoli Theorem for sequences of functions

$$\{A_n^\alpha \int_0^t e^{(t-s)(A_n + f'_n(u_n^*))} Q_n(\sigma_n^+) H_n(v_n^k(s), z_n^k(s)) ds\}$$

and $\{A_n^\alpha \int_{-\infty}^t e^{(t-s)(A_n + f'_n(u_n^*))} (I_n - Q_n(\sigma_n^+)) G_n(v_n^k(s), z_n^k(s)) ds\}$ just on $[t_\epsilon, 0]$. This can be done as in Theorem 4.2.

4.2. Approximation of semiflows.

We will need the following result

Theorem 4.5. [28] *Let A be generator of analytic C_0 -semigroup, the resolvent $(\lambda I - A)^{-1}$ be compact for a certain $\lambda \in \rho(A)$, and let the operator K be given by formula (4.1). If $u^*(\cdot)$ is a unique mild solution of problem (1.2), then $\text{ind}(u^*(\cdot); I - K) = 1$.*

Theorem 4.6. *Let $\Delta_{cc} \neq \emptyset$. Assume also that conditions (A), (B_1) and condition $f_n(x_n) \xrightarrow{\mathcal{P}} f(x)$, whenever $x_n \xrightarrow{\mathcal{P}^\alpha} x$ are satisfied. Then $T_n(t)u_n^0 \xrightarrow{\mathcal{P}^\alpha} T(t)u^0$ uniformly in $t \in [0, T]$, $T < \infty$, whenever $u_n^0 \xrightarrow{\mathcal{P}^\alpha} u^0$.*

Proof. We know (see Theorem 4.5) that $\text{ind}(u^*, I - K) = 1$. By Theorem 4.2 operators $K_n \xrightarrow{\mathcal{PP}} K$ compactly, so according to Theorem 3 from [75] one has $\gamma(I_n - K_n, \partial\Omega_n) = 1$ for $n \geq n_0$. This means that there is a sequence $\{u_n^*\}, n \in N$, which is a solution of problems (4.2) and $\{u_n^*\}, u_n^* \in F_n$, is compact; that is one can write $u_n^*(t) \xrightarrow{\mathcal{P}^\alpha} u^*(t)$ uniformly in $t \in [0, T]$ as $n \rightarrow \infty$.

5. APPROXIMATION OF ATTRACTORS

Next we intend to obtain that the attractors behave continuously with respect to parameters under rather general conditions.

To prove upper or lower semicontinuity of attractors we use the following result

Lemma 5.1. *Let $\mathcal{A} \subset E^\alpha$ and $\mathcal{A}_n \subset E_n^\alpha$, $n \in \mathbb{N}$.*

- (1) *If any sequence $\{u_n\}$ with $u_n \in \mathcal{A}_n$ has a $\mathcal{P}^\alpha$ -convergent subsequence with limit belonging to $\mathcal{A}$, then $\{\mathcal{A}_n\}$ is $\mathcal{P}^\alpha$ -upper semicontinuous at infinity.*
- (2) *If $\mathcal{A}$ is compact and for any $u \in \mathcal{A}$ there is a sequence $\{u_n\}$ with $u_n \in \mathcal{A}_n$, which $\mathcal{P}^\alpha$ -converges to u , then $\{\mathcal{A}_n\}$ is $\mathcal{P}^\alpha$ -lower semicontinuous at infinity.*

The proof of this lemma is straight forward from the definitions of upper and lower semicontinuity.

5.1. Upper Semicontinuity of Attractors.

In this subsection we obtain that the family of attractors $\{\mathcal{A}_n\}$ is upper-semicontinuous. In order to do that we use the results on continuity of nonlinear semiflows (Subsection 4.2) and the following lemma.

We assume (see Lemma 6.4) that operators A_n generate analytic C_0 -semigroups with estimate $\|e^{tA_n}\| \leq M e^{-t\tilde{\omega}}$, $t \geq 0$, where $\tilde{\omega} > 0$, resolvents $(\lambda I_n - A_n)^{-1}$ are compact for some $\lambda \in \rho(A_n)$, converges compactly $(\lambda I_n - A_n)^{-1} \rightarrow (\lambda I - A)^{-1}$ for some $\lambda \in \cap_{n=1}^\infty \rho(A_n) \cap \rho(A)$, functions $f_n(\cdot) : E_n^\alpha \rightarrow E_n$ are globally Lipschitz, bounded, i.e. there is a constant $\tilde{K} > 0$, such that $\|f_n(w_n)\|_{E_n^\alpha} \leq \tilde{K}$ for all $w_n \in E_n^\alpha$, and they are continuously Fréchet differentiable functions.

Lemma 5.2. *Given $\alpha \leq \gamma < 1$, there is a constant N_γ independent of n such that*

$$\sup_{n \in \mathbb{N}} \sup_{u_n \in \mathcal{A}_n} \|u_n\|_{E_n^\gamma} \leq N_\gamma.$$

Furthermore, any sequence $\{u_n\}, u_n \in \mathcal{A}_n$, is $\mathcal{P}^\alpha$ -compact.

Proof. The first part of the lemma follows immediately from the invariance of the attractor and of the variation of constants formula. For the second part we only need to apply Lemma 6.5. $\square$

Remark 5.1. We know by definition that any attractor $\mathcal{A}_n$ is a compact set. The above lemma is stating that the set of the attractors $\mathcal{A}_n$ is collectively $\mathcal{P}^\alpha$ -compact.

Theorem 5.1. Let $\mathcal{A}$, $\mathcal{A}_n$ be the attractors for (1.2) and (1.14), respectively. If $\{x_n^*\}$, $x_n^* \in \mathcal{A}_n$, $\mathcal{P}^\alpha$ -converges to some point x^* , then $x^* \in \mathcal{A}$ and the $\{\mathcal{A}_n\}$ is $\mathcal{P}^\alpha$ -upper semicontinuous at infinity.

Proof. Assume that $x_n^* \in \mathcal{E}_n$. In such case by Proposition 5.1 the point $x^* \in \mathcal{A}$. Assume now that x_n^* does not belong to $\mathcal{E}_n$. In this case x_n^* can converge to some point in $\mathcal{E}$ and it is nothing to prove. Let us consider the case, when the limit point $x_n^* \rightarrow x^*$ is not in $\mathcal{E}$. In this situation consider the trajectories back in time starting from this points as $u_n(0) = x_n^*$ and $u(0) = x^*$. On interval $(-\infty, 0]$ the complete trajectories $u_n(t)$, $t \in (-\infty, 0]$, has to be uniformly bounded in n and t , since functions $f_n(\cdot)$ are uniformly bounded in n and all attractors $\mathcal{A}_n$ are uniformly bounded in n . From Lemma 5.2, any sequence $\{y_n\}$ with $y_n \in \mathcal{A}_n$, $n \in \mathbb{N}$, is discretely compact in E_n^α . Now taking bounded trajectories $u_n(\cdot)$ through y_n and considering the sequence $\{u_n(-1)\}$ one can find that $\{u_n(-1)\}$ is also discretely compact. Taking a subsequence $n \in \mathbb{N}' \subset \mathbb{N}$ we can find $u_n(-1) \xrightarrow{\mathcal{P}^\alpha} \tilde{u}(-1)$ as $n \rightarrow \infty$, $n \in \mathbb{N}'$. It follows that $T_n(t)u_n(-1) \rightarrow T(t)\tilde{u}(-1)$ for any $t \geq 0$ and in particular $T_n(1)u_n(-1) = x_n^* \rightarrow T(1)\tilde{u}(-1) = x^*$. Now taking the sequence $u_n(-2)$ one can find that trajectory $T(t)\tilde{u}(-2)$ is bounded for $t \geq 0$ and coincide with trajectory $T(t)\tilde{u}(-1)$ for $t \geq 0$ for common t . In this way one can construct trajectory $u(t)$, $t \in \mathbb{R}$, which is bounded and contains the point x^* . So $x^* \in \mathcal{A}$.

5.2. Approximation of the Set of Equilibria.

Let us consider in Banach spaces E_n^α the following family of nonlinear problems

$$(5.1) \quad A_n u_n + f_n(u_n) = 0.$$

Proposition 5.1. Assume that $\Delta_{cc} \neq \emptyset$ and that the problems (5.1) have solutions $\{u_n^*\}$, $n \in \mathbb{N}$. Then, taking subsequences if necessary, there is a solution u^* of (2.4) such that $\|u_n^* - p_n^\alpha u^*\|_{E_n^\alpha} \rightarrow 0$ as $n \rightarrow \infty$.

Proof. It is clear that

$$u_n^* + A_n^{-1} f_n(u_n^*) = 0.$$

Since $A_n^{-1} \xrightarrow{\mathcal{PP}} A^{-1}$ compactly by Lemma 6.5 and $f_n(\cdot)$ are bounded uniformly in $n \in \mathbb{N}$, we have that there is a subsequence, which we again denote by u_n^* and a point u^* such that $u_n^* \xrightarrow{\mathcal{P}^\alpha} u^*$. It follows from the continuity properties of $f_n(\cdot)$ and from convergence of resolvents $A_n^{-1} \xrightarrow{\mathcal{PP}} A^{-1}$ that

$$u^* + A^{-1} f^*(u^*) = 0,$$

which is equivalent to say that u^* is a solution of (2.4). $\square$

Proposition 5.2. *Assume that $A_n^{-1} \xrightarrow{\mathcal{PP}} A^{-1}$ compactly and $f_n(x_n) \xrightarrow{\mathcal{P}} f(x)$ whenever $x_n \xrightarrow{\mathcal{P}^\alpha} x$. Assume that u^* is hyperbolic solution of (2.4). Then there is n_0 such that for $n \geq n_0$ the equations (5.1) have at least one sequence of solutions $\{u_n^*\}$. Any sequence $\{u_n^*\}$, $n \in N$, is $\mathcal{P}^\alpha$ -compact and $\mathcal{P}^\alpha$ -limit point of $\{u_n^*\}$, $n \in N$, is the solution u^* of (2.4), i.e. $u_n^* \xrightarrow{\mathcal{P}^\alpha} u^*$ as $n \in \mathbb{N}' \subset \mathbb{N}$.*

Proof: As in Theorem 2.1 there is a ball $B(u^*, \delta)$ such that there are no other fixed points in it except u^* , and we get $|\text{ind}(u^*, I + A^{-1}f(\cdot))| = 1$. The compact convergence $A_n^{-1}f_n(\cdot) \xrightarrow{\mathcal{PP}} A^{-1}f(\cdot)$ follows from boundedness and continuity of $f_n(\cdot)$ and compact convergence $A_n^{-1} \xrightarrow{\mathcal{PP}} A^{-1}$. Now compact convergence $A_n^{-1}f_n(\cdot) \rightarrow A^{-1}f(\cdot)$ implies (see [75], Theorem 3) that $\text{ind}(u^*, I + A^{-1}f(\cdot)) = \gamma(I_n + A_n^{-1}f_n(\cdot), \partial B(p_n u^*, \delta))$, $n \geq n_0$, which means that there is at least one fixed point u_n^* exists in any ball $B(p_n u^*, \delta)$, $n \geq n_0$. This sequence $\{u_n^*\}$ is $\mathcal{P}^\alpha$ -compact and converge on subsequences to u^* .

In case we have information on differentiability of corresponding functions the result could be more sharp.

Proposition 5.3. *Assume that problem (2.4) has a solution u^* , which is hyperbolic point. Assume also that the Fréchet derivatives of functions $f_n(\cdot)$ at the balls of the points $p_n^\alpha u^*$ exist and $f'_n(u_n)$ are uniformly in n continuous in the balls $B(p_n^\alpha u^*, \delta)$ for some $\delta > 0$ and $\Delta_{cc} \neq \emptyset$.*

Then, there exists n_0 and $\delta > 0$ so that problems (5.1) have exactly one solution, u_n^ , in $\{w_n : \|w_n - p_n^\alpha u^*\|_{E_n^\alpha} \leq \delta\}$ for $n \geq n_0$. Furthermore, $\|u_n^* - p_n^\alpha u^*\|_{E_n^\alpha} \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. We want to prove that in a neighborhood of the solution u^* of (2.4) there is a unique solution u_n^* of (5.1). Furthermore, since $f'_n(\cdot)$ is continuous uniformly in n , there is a $\delta > 0$ such that $\|u_n - p_n^\alpha u^*\|_{E_n^\alpha} < \delta$ implies $M\|f'_n(u_n) - f'_n(p_n^\alpha u^*)\|_{B(E_n^\alpha, E_n)} < \frac{1}{2C}$.

From Lemma 6.6, $\sigma(A_n + f'_n(p_n^\alpha u^*))$ is disjoint from the imaginary axis and there is a constant $M > 0$ such that

$$\|(A_n + f'_n(p_n^\alpha u^*))^{-1}\|_{B(E_n, E_n^\alpha)} \leq M.$$

If u_n is a solution to (5.1), then

$$\begin{aligned} 0 &= A_n u_n + f'_n(p_n^\alpha u^*) u_n + f_n(u_n) - f'_n(p_n^\alpha u^*) u_n \\ &= (A_n + f'_n(p_n^\alpha u^*)) \left(u_n + (A_n + f'_n(p_n^\alpha u^*))^{-1} (f_n(u_n) - f'_n(p_n^\alpha u^*) u_n) \right). \end{aligned}$$

Since $(A_n + f'_n(p_n^\alpha u^*))$ is invertible, to show that u_n is a solution to (5.1) is equivalent to show that it is a fixed point of mapping

$$\Phi_n(u_n) = -(A_n + f'_n(p_n^\alpha u^*))^{-1} (f_n(u_n) - f'_n(p_n^\alpha u^*) u_n).$$

It is clear from the convergence of the resolvent and the continuity properties of $f_n(\cdot)$ and $f'_n(p_n^\alpha u^*)$ that

$$(5.2) \quad \Phi_n(p_n^\alpha u^*) \rightarrow -(A + f'(u^*))^{-1}(f(u^*) - f'(u^*)u^*) = u^*.$$

Next we show that, for suitably small $\delta > 0$ and $n \geq n_0$, Φ_n is a contraction in the from the closed ball $\bar{B}(p_n u^*, \delta)$ into itself. First, we show that Φ_n is a contraction (uniformly in $n \in N$); that is,

$$\begin{aligned} \|\Phi_n(u_n) - \Phi_n(v_n)\|_{E_n^\alpha} &= \|(A_n + f'_n(p_n^\alpha u^*))^{-1} \left((f_n(u_n) - f_n(v_n) - f'_n(p_n^\alpha u^*)(u_n - v_n)) \right)\|_{E_n^\alpha} \\ &\leq \|(A_n + f'_n(p_n^\alpha u^*))^{-1}\|_{B(E_n, E_n^\alpha)} \|f_n(u_n) - f_n(v_n) - f'_n(p_n^\alpha u^*)(u_n - v_n)\|_{E_n} \\ &\leq M \left\| \left(f'_n(su_n + (1-s)v_n) - f'_n(p_n u^*) \right) (u_n - v_n) \right\|_{E_n} \\ &\leq M \|f'_n(su_n + (1-s)v_n) - f'_n(p_n^\alpha u^*)\|_{B(E_n^\alpha, E_n)} \|u_n - v_n\|_{E_n^\alpha} \\ &\leq \frac{1}{2C} \|u_n - v_n\|_{E_n^\alpha}, \end{aligned}$$

for some $0 \leq s \leq 1$ and $n \geq n_0$. To show that Φ_n maps $B(p_n u^*, \delta)$ into itself we observe that if $u_n \in B(p_n u^*, \delta)$, then

$$\begin{aligned} \|\Phi_n(u_n) - p_n u^*\|_{E_n^\alpha} &\leq \|\Phi_n(u_n) - \Phi_n(p_n u^*)\|_{E_n^\alpha} + \|\Phi_n(p_n u^*) - p_n u^*\|_{E_n^\alpha} \\ &\leq \frac{\delta}{2} + \|\Phi_n(p_n u^*) - p_n u^*\|_{E_n^\alpha}. \end{aligned}$$

It follows from (5.2) that there is a n_0 such that

$$\|\Phi_n(u_n) - p_n u^*\|_{E_n^\alpha} \leq \frac{3\delta}{4} \text{ as } n \geq n_0.$$

From that we obtain that if $u_n \in \bar{B}(p_n^\alpha u^*, \delta)$, then

$$\|\Phi_n(u_n) - p_n^\alpha u^*\|_{E_n^\alpha} \leq \delta$$

and therefore $\Phi_n : B(p_n u^*, \delta) \rightarrow B(p_n u^*, \delta)$ is a contraction for all $n \geq n_0$. This shows that there is a unique fixed point of (5.1) in the ball $B(p_n^\alpha u^*, \delta)$.

To show that $u_n^* \xrightarrow{\mathcal{P}^\alpha} u^*$ we proceed in the following manner

$$\|p_n^\alpha u^* - u_n^*\|_{E_n^\alpha} = \|\Phi_n(u_n^*) - p_n u^*\|_{E_n^\alpha} \leq \|\Phi_n(u_n^*) - \Phi_n(p_n u^*)\|_{E_n^\alpha} + \|\Phi_n(p_n u^*) - p_n u^*\|_{E_n^\alpha}.$$

From the fact that Φ_n is a uniform contraction in E_n^α we have that

$$\|p_n^\alpha u^* - u_n^*\|_{E_n^\alpha} \leq 2\|\Phi_n(p_n^\alpha u^*) - p_n^\alpha u^*\|_{E_n^\alpha}.$$

From (5.2) we have the desired convergence and the result follows. $\square$

As an immediate consequence of proposition 5.3, we have

Theorem 5.2. *Assume moreover that problem (2.4) has exactly m solutions $u^1, \dots, u^m$ and that all of them are hyperbolic. Assume that for any point u^i and any $\epsilon > 0$ there exists $\delta > 0$ such that $\|f'_n(u_n) - f'_n(p_n^\alpha u^i)\|_{E_n} \leq \epsilon$, whenever $\|u_n - p_n^\alpha u^i\|_{E_n^\alpha} \leq \delta$. Then, there exists n_0 such that for all $n \geq n_0$ problems (5.1) has exactly m solutions $u_n^1, \dots, u_n^m$. Moreover, we have*

$$\|u_n^k - p_n^\alpha u^k\|_{E_n^\alpha} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for any } k = 1, \dots, m.$$

Proof: By Proposition 5.1 we have that for any solution u_n^k of (5.1) for $n \rightarrow \infty$ lies in a neighborhood of the set of equilibria (2.4). But by Proposition 5.3, in a neighborhood of u^k there is only one solution of (5.1) which converges to u^k . This proves the result.

5.3. Approximation of Unstable Manifolds.

Recall that we are assuming that all equilibrium solutions of (1.2) are hyperbolic; that is, assume that each solution u^* of (2.4) is such that the operator $A + f'(u^*)$ has spectrum $\sigma(A + f'(u^*))$ disjoint from the imaginary axis.

Proposition 5.4. *Let A be a sectorial operator with compact resolvent and let $f'(u^*) \in L(E^\alpha, E)$. Then the operator $(A + f'(u^*))$ is sectorial with compact resolvent.*

Proof: The result follows immediately from Theorems 1.3.2 and 1.4.8 in [48] or [79]. $\square$

From the above proposition, $\sigma^+ = \sigma((A + f'(u^*))) \cap \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\}$ consists of a finite number (possibly zero) of eigenvalues with finite multiplicity and there is a $\beta > 0$ and a finite rank projection $Q(\sigma^+)$ such that (2.10) holds.

Proposition 5.5. *Let $u_n^* \xrightarrow{P^\alpha} u^*$ with u_n^* be solution of (5.1) and u^* be solution of (2.4). Assume that $(\lambda I_n - A_n)^{-1} \xrightarrow{PP} (\lambda I - A)^{-1}$ compactly. Then there is a $n \geq n_0$ such that $\sigma(A_n + f'_n(u_n^*))$ does not intersect the imaginary axis for $n \geq n_0$ and $\|(A_n + f'_n(u_n^*))^{-1}\| \leq C$ with C independent of n . Furthermore, if $Q_n(\sigma_n^+)$ denotes the projection defined by the spectral set $\sigma_n^+ = \{\lambda \in \sigma(A_n + f'_n(u_n^*)) : \operatorname{Re} \lambda > 0\}$, then $Q_n(\sigma_n^+)$ converges compactly to $Q(\sigma^+)$ as $n \rightarrow \infty$ (hence, $\dim(Q_n(\sigma_n^+)) = \dim(Q(\sigma^+))$ for $n \geq n_0$) and the family of sets $\sigma_n^+ \rightarrow \sigma^+$ in the sense of (i) – (iii) of Theorem 3.1 and (iv) – (v) of Theorem 3.2.*

Proof. It follows from Lemma 6.6 and Theorem 3.2.

Proposition 5.6. *Assume that $(\lambda I_n - A_n)^{-1} \xrightarrow{PP} (\lambda I - A)^{-1}$ compactly. Then for n large enough the equilibrium u_n^* of (5.1) is hyperbolic and there is a $\beta > 0$ such that if $Q_n(\sigma_n^+)$ is the projection determined by σ_n^+ , then*

$$(5.3) \quad \begin{aligned} & \left\| \exp \left(t(A_n + f'_n(u_n^*)) \right) Q_n(\sigma_n^+) \right\|_{B(E_n)} \leq M e^{\beta t}, \quad t \leq 0, \\ & \left\| \exp \left(t(A_n + f'_n(u_n^*)) \right) (I_n - Q_n(\sigma_n^+)) \right\|_{B(E_n, E_n^\alpha)} \leq M t^{-\alpha} e^{-\beta t}, \quad t \geq 0, \alpha \geq 0. \end{aligned}$$

Proof. It is exactly Theorem 6.5.

Theorem 5.3. Let $\Delta_{cc} \neq \emptyset$ and u^* be hyperbolic point of equilibrium of (1.2). Then for sufficiently small $\delta > 0$ equations (2.15) has a solution $\zeta^*(\cdot)$ such that equation $\zeta_n(t) = \Theta_n(\eta_n, \zeta_n(t))$ have solutions $\zeta_n^*(\cdot)$ for $n \geq n_0$ and $\zeta_n^*(t) \rightarrow \zeta^*(t)$ uniformly in $t \in (-\infty, 0]$ as $n \rightarrow \infty$.

Proof. We know that $|\text{ind}(\zeta_0, I - \Theta)| = 1$ for $\zeta_0 = 0$. According to Theorem 54.1 [53] there exist $\rho, \delta > 0$ such that $|\text{ind}(\zeta, I - \Theta)| = 1$ for $\|\eta - 0\| \leq \delta, \|\zeta - 0\| \leq \rho$. By Theorem 4.4 operators $\Theta_n \rightarrow \Theta$ compactly, so $\gamma(I_n - \Theta_n, \partial\Omega_n^{M,\omega}) = \text{ind}(\zeta, I - \Theta)$ and therefore equation $\zeta_n(t) = \Theta_n(\eta_n, \zeta_n(t))$ has at least one solution ζ_n^* at any set $\Omega_n^{M,\omega}$ for $n \geq n_0$. This sequence $\{\zeta_n^*\}$ is discretely compact and $\zeta_n^* \rightarrow \zeta^*$ as $n \rightarrow \infty$.

Proposition 5.7. Assume also that u^* is a hyperbolic equilibrium for (1.2) and that zero is not in the spectrum of the operator $A + f'(u^*) : D(A) \subset E \rightarrow E$. Assume that $\Delta_{cc} \neq \emptyset$, $\|f'_n(u_n) - f'_n(p_n^\alpha u^*)\|$ uniformly in n vanished, i.e. converge to zero as $\|u_n - p_n^\alpha u^*\| \rightarrow 0$. Then, there exist $\delta, n_0 > 0$ such that u_n^* has a local unstable manifold $W_{loc}^u(u_n^*) \subset E_n^\alpha$ for $n \geq n_0$ and if we denote by

$$W_{n,\delta}^u(u^*) = \{w_n \in W_{loc}^u(u_n^*) : \|w_n - u_n^*\|_{E_n^\alpha} < \delta\}, \quad n \geq n_0,$$

then $W_{n,\delta}^u(u^*)$ converges to $W_\delta^u(u^*)$ as $n \rightarrow \infty$, that is

$$\sup_{w_n \in W_{n,\delta}^u(u_n^*)} \inf_{w \in W_\delta^u(u^*)} \|w_n - w\|_{E^\alpha} + \sup_{w \in W_\delta^u(u^*)} \inf_{w_n \in W_{n,\delta}^u(u_n^*)} \|w_n - w\|_{E_n^\alpha} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Proof: Notice that by Proposition 5.3, equations (5.1) have a unique solutions, u_n^* , near u^* such that $\|u_n^* - p_n^\alpha u^*\|_{E_n^\alpha} \rightarrow 0$ as $n \rightarrow \infty$. This implies by Proposition 5.5 that the spectra of $A_n + f'_n(u_n^*)$ behaves continuously as $n \rightarrow \infty$.

Also, for some positive M, β , independent of n according to Proposition 5.6 one has

$$(5.4) \quad \begin{aligned} \|\exp(\tilde{A}_n t) z_n\|_{E_n^\alpha} &\leq M e^{-\beta t} \|z_n\|_{E_n^\alpha}, \quad t \geq 0, \\ \|\exp(\tilde{A}_n t) z_n\|_{E_n^\alpha} &\leq M t^{-\alpha} e^{-\beta t} \|z_n\|_{E_n}, \quad t \geq 0, \\ \|\exp(B_n t) v_n\|_{W_n(\sigma_n^+)} &\leq M e^{\beta t} \|v_n\|_{W_n(\sigma_n^+)}, \quad t \leq 0. \end{aligned}$$

Now we will show that for a suitably small $\rho > 0$, there is an unstable manifold for u_n^*

$$W_n^u = \{(v_n(\cdot), z_n(\cdot)) : z_n(\cdot) \in (I_n - Q_n(\sigma_n^+))E_n^\alpha, v_n(\cdot) \in W_n(\sigma_n^+)\}$$

Introducing the same way as in Theorem 2.2 operators Θ_n and vector $\zeta_n(t) = \begin{pmatrix} v_n(t) \\ z_n(t) \end{pmatrix}$, one can see from (4.4) the following. There is a positive δ such that operators $\Theta_n(\eta_n, \cdot)$ are Fréchet differentiable in the balls $B(p_n^\alpha \zeta^*, \delta)$ and for any $\epsilon > 0$ there is $\delta_\epsilon > 0$ such that $\|\Theta'_n(\eta_n, \zeta_n) - \Theta'_n(\eta_n, p_n^\alpha \zeta^*)\| \leq \epsilon$, whenever $\|\zeta_n(\cdot) - p_n^\alpha \zeta^*(\cdot)\|_{\Gamma_n} \leq \delta_\epsilon$. It is clear also that

conditions $\|p_n^\alpha \zeta^*(\cdot) - \Theta_n(\eta_n, p_n^\alpha \zeta^*(\cdot))\|_{\mathcal{R}_n} \rightarrow 0$, $\|(I_n - \Theta'_n(\eta_n, p_n^\alpha \zeta^*(\cdot)))^{-1}\| \leq \text{constant}$ as $n \rightarrow \infty$ are satisfied. Then by Theorem 2 from [75] it follows that there exist $n_0 \in \mathbb{N}$ and $0 < \delta_0 \leq \delta$ such that the equations (4.5) have for $n \geq n_0$ a unique solution $\zeta_n^*(\cdot)$ in the ball $B(p_n^\alpha \zeta^*, \delta_0)$ and $\zeta_n^* \rightarrow \zeta^*$ as $n \rightarrow \infty$.

As an immediate consequence of this proposition, we have

Theorem 5.4. *Assume the conditions of Proposition 5.7 hold, that problem (2.4) has exactly m solutions $u^1, \dots, u^m$ and that all of them are hyperbolic. Then there exist $\delta > 0$ small enough such that problem (5.1) has exactly m solutions and their local unstable manifolds $W_\delta^u(u_n^k)$, $k = 1, \dots, m$, behave continuously in E_n^α as $n \rightarrow \infty$.*

5.4. Lower Semicontinuity of Attractors.

Theorem 5.5. *Let $\mathcal{A}$, $\mathcal{A}_n$ be the attractors for (1.2) and (1.14), respectively. Assume that the attractor $\mathcal{A}$ of (1.2) can be written in the form (1.12) and that all points in $\mathcal{E}$ are hyperbolic equilibrium. If $x^* \in \mathcal{A}$, then there is a sequence of points $\{x_n\}$, with $x_n^* \in \mathcal{A}_n$, such that $x_n^* \xrightarrow{\mathcal{P}^\alpha} x^*$ and the $\{\mathcal{A}_n\}$ is $\mathcal{P}^\alpha$ -lower semicontinuous at infinity.*

Proof. Assume that $x^* \in \mathcal{E}$. By Proposition 5.2 the statement of Theorem follows. Assume now that x^* is on the unstable manifold. Then follow back in time by some trajectory from the point x^* one can come to the point y from the neighborhood of some equilibrium point, which as we know by Theorem 5.3 is approximated by points $y_n \in \mathcal{A}_n$. Starting now from the points y_n according to Theorem 4.6 we obtain the statement of Theorem.

6. APPENDIX

In this section we present some basic results on compact convergence and its consequences

6.1. Basic Results on Compact Convergence.

Recall that $B \in B(E)$ is called a Fredholm operator if $\mathcal{R}(B)$ is closed, $\dim \mathcal{N}(B) < \infty$ and $\text{codim } \mathcal{R}(B) < \infty$, the index of B is defined as $\text{ind } B = \dim \mathcal{N}(B) - \text{codim } \mathcal{R}(B)$.

Theorem 6.1. [75] *For $B_n \in B(E_n)$ and $B \in B(E)$ the following conditions are equivalent:*

- (i) $B_n \xrightarrow{\mathcal{PP}} B$ regularly, B_n are Fredholm operators of index 0 and $\mathcal{N}(B) = \{0\}$;
- (ii) $B_n \rightarrow B$ stably and $\mathcal{R}(B) = E$;
- (iii) $B_n \rightarrow B$ stably and regularly;
- (iv) if one of conditions (i)–(iii) holds, then there exist $B_n^{-1} \in B(E_n)$, $B^{-1} \in B(E)$, and $B_n^{-1} \xrightarrow{\mathcal{PP}} B^{-1}$ regularly and stably.

This theorem admits an extension to the case of closed operators $B \in \mathcal{C}(E)$, $B_n \in \mathcal{C}(E_n)$ [77].

Lemma 6.1. Assume that $B_n \xrightarrow{\mathcal{PP}} B$ compactly and operators B, B_n are compact. Assume also that $\mathcal{N}(I + B) = \{0\}$. Then there exist an $n_0 > 0$ and $M > 0$ such that

$$(6.1) \quad \|(I_n + B_n)^{-1}\|_{B(E_n)} \leq M, \quad \text{for } n \geq n_0.$$

Proof: Since the compact convergence $B_n \xrightarrow{\mathcal{PP}} B$ implies that $I_n + B_n \xrightarrow{\mathcal{PP}} I + B$ regularly, the result follows from Theorem 6.1.

Lemma 6.2. Let $\Delta_{cc} \neq \emptyset$. Then for any $\lambda \in \rho(A)$, there is a $n_\lambda > 0$ such that $\lambda \in \rho(A_n)$ for all $n \geq n_\lambda$ and there is a constant $M_\lambda > 0$ such that

$$\|(\lambda I_n - A_n)^{-1}\| \leq M_\lambda, \quad \text{for } n \geq n_\lambda.$$

Furthermore, $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$ compactly.

Proof. The result follows from Lemma 6.1 and Corollary 3.1.

Theorem 6.2. [72] Let $B \in B(E), B_n \in B(E_n)$. The following conditions are equivalent:

- (i) operators $B_n \xrightarrow{\mathcal{PP}} B$ as $n \rightarrow \infty$;
- (ii) $\|B_n\| \leq \text{constant}, n \in N$, and $\|B_n p_n x - p_n Bx\| \rightarrow 0$ for any $x \in E$ as $n \rightarrow \infty$;
- (iii) $\|B_n\| \leq \text{constant}, n \in N$, and $\|B_n x_n - p_n Bx\| \rightarrow 0$ for any $x \in E$ as $n \rightarrow \infty$, whenever $x_n \rightarrow x \in E$.

Lemma 6.3. Assume that $\Delta_{cc} \neq \emptyset$ and resolvents $(\lambda I - A)^{-1}, (\lambda I_n - A_n)^{-1}$ are compact. Let Λ be a compact subset of $\rho(A)$. Then there is a constant $n_\Lambda > 0$ such that $\Lambda \subset \rho(A_n)$ for all $n \geq n_\Lambda$ and

$$(6.2) \quad \sup_{\substack{\lambda \in \Lambda \\ n \geq n_\Lambda}} \|(\lambda I_n - A_n)^{-1}\| < \infty.$$

Furthermore, for any $u \in E$

$$(6.3) \quad \sup_{\lambda \in \Lambda} \|(\lambda I_n - A_n)^{-1} p_n u - p_n (\lambda I - A)^{-1} u\| \rightarrow 0.$$

Proof. Let us first prove that there is a $n_\Lambda > 0$ such that $\Lambda \subset \rho(A_n)$ for all $n \geq n_\Lambda$. Suppose that this is not the case, then there are sequences $k_n \rightarrow \infty, \{\lambda_{k_n}\} \in \Lambda$ such that λ_{k_n} is an eigenvalue of A_{k_n} . Since Λ is compact we may assume that there is a $\bar{\lambda} \in \Lambda$ such that $\lambda_{k_n} \rightarrow \bar{\lambda}$. It follows from Theorem 3.1 (ii), that $\bar{\lambda} \in \sigma(A)$ which is a contradiction.

To prove (6.2), it is enough to prove that

$$\sup_{\substack{\lambda \in \Lambda \\ n \geq n_\Lambda}} \|(I - \lambda A_n^{-1})^{-1}\| < \infty.$$

Assuming in contradiction that this is not the case; that is, assume that there are sequences $k_n \rightarrow \infty, \lambda_{k_n} \in \Lambda$ (which we may assume convergent to $\bar{\lambda} \in \Lambda$) such that

$$\|(I - \lambda_{k_n} A_{k_n}^{-1})^{-1}\| \rightarrow \infty.$$

Since $\lambda_{k_n} A_{h_n}^{-1}$ converges compactly to $\bar{\lambda} A^{-1}$ this is in contradiction with Lemma 6.1.

It remains to prove (6.3). Once again, we prove it by contradiction. Assume that there are sequences $k_n \rightarrow \infty$, $\Lambda \ni \lambda_{k_n} \rightarrow \bar{\lambda} \in \Lambda$ and $\epsilon > 0$ such that

$$(6.4) \quad \|(\lambda_{k_n} I_{k_n} - A_{k_n})^{-1} p_{k_n} u - p_{k_n} (\lambda_{k_n} I - A)^{-1} u\| \geq \epsilon.$$

But this contradicts according to Theorem 6.2 to $\|(\lambda I_n - A_n)^{-1} p_n u - p_n (\lambda I - A)^{-1} u\| \rightarrow 0$ for $\lambda = \bar{\lambda}$ and estimate (6.2).

Next we show that the compact convergence of A_n^{-1} to A^{-1} ensures that the resolvent of A_n contains a fixed sector for almost all n . We observe that no information about bounds on the resolvent are provided. Let us denote $\Sigma_{\phi, \bar{\omega}} = \{\lambda \in \mathbb{C} : |\arg(\lambda - \bar{\omega})| < \pi - \phi\}$.

Lemma 6.4. *Assume that $\Delta_{cc} \neq \emptyset$ and conditions (1.1), (B_1) are satisfied. Then there are a sector $\Sigma_{\bar{\phi}, -\omega}$, $\phi < \bar{\phi} < \pi/2$, $0 < \omega < \bar{\omega}$, a constant $n_\omega > 0$ and a constant M_ω such that*

$$(6.5) \quad \|(\lambda I_n - A_n)^{-1}\| \leq \frac{M_\omega}{|\lambda - \omega|}, \quad \forall \lambda \in \Sigma_{\bar{\phi}, -\omega} \text{ as } n \geq n_\omega.$$

Proof. Let $\omega < \omega_1 < \bar{\omega}$ and consider the region $\Lambda = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq -\omega_1 \text{ and } \lambda \notin \Sigma_{\phi, \bar{\omega}}\}$. Then Λ (see Figure 2) is a compact set and $\Lambda \subset \rho(A)$. It follows from Lemma 6.3 that there is a constant $n_\Lambda > 0$ such that $\Lambda \subset \rho(A_n)$ for all $n \geq n_\Lambda$ and

$$(6.6) \quad \sup_{\substack{\lambda \in \Lambda \\ n \geq n_\Lambda}} \|(\lambda I_n - A_n)^{-1}\| < \infty.$$

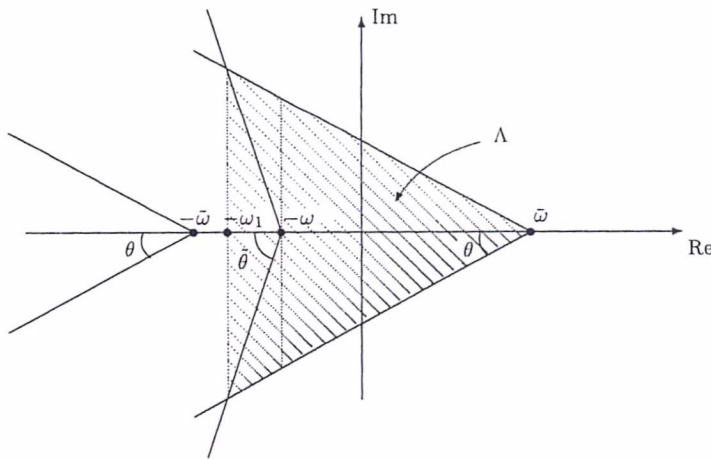


Figure 2

If we take $\bar{\phi} < \pi/2$ such that $\Sigma_{\bar{\phi}, \omega} \cap \{\lambda \in \Sigma_{\phi, \bar{\omega}} : \operatorname{Re} \lambda < \omega_1\} = \emptyset$ we have that $\Sigma_{\bar{\phi}, \omega} \subset \rho(A_n)$ for all $n \geq n_\Lambda$. The estimate (6.5) follows easily from (6.6) and from estimate (B_1) . $\square$

Proposition 6.1. *Let conditions (A) and (B₁) be satisfied. Then the compact convergence of resolvents $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$ is equivalent to the compact convergence of analytic C_0 -semigroups $\exp(tA_n) \xrightarrow{\mathcal{PP}} \exp(tA)$ for any $t > 0$.*

Proof: First, let us show that the compact convergence of resolvents $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$ is equivalent to the compact convergence of C_0 -semigroups $\exp(tA_n) \rightarrow \exp(tA)$ for any $t > 0$. Let $\|x_n\| = O(1)$. Then from the estimate $\|A_n \exp(tA_n)\| \leq \frac{M}{t} e^{\omega t}$, we obtain the boundedness of the sequence $\{(A_n - \lambda I_n) \exp(tA_n)x_n\}$. Because of Theorem 3.4 and the compact convergence of resolvents, we obtain the compactness of the sequence $\{\exp(tA_n)x_n\}$.

The necessity will be proved if for the measure of noncompactness $\mu(\cdot)$ we establish that $\mu(\{(\lambda I_n - A_n)^{-1}x_n\}) = 0$ for $\|x_n\| = O(1)$. We have

$$\begin{aligned} \mu(\{(\lambda I_n - A_n)^{-1}x_n\}) &= \mu\left(\left\{\int_0^\infty e^{-\lambda t} \exp(tA_n)x_n dt\right\}\right) \leq \mu\left(\left\{\int_0^q e^{-\lambda t} \exp(tA_n)x_n dt\right\}\right) + \\ &\quad \mu\left(\left\{\int_q^\infty e^{-\lambda t} \exp(tA_n)x_n dt\right\}\right) + \mu\left(\left\{\exp(\epsilon A_n) \int_q^Q e^{-\lambda t} \exp((t - \epsilon)A_n)x_n dt\right\}\right). \end{aligned}$$

Two first terms can be made less than ϵ by the choice of q, Q . The last term is equal to zero because of the compact convergence of semigroups $\exp(\epsilon A_n) \xrightarrow{\mathcal{PP}} \exp(\epsilon A)$ for any $0 < \epsilon < q$ and boundedness of the sequence $\{\int_q^Q e^{-\lambda t} \exp((t - \epsilon)A_n)x_n dt\}$. $\square$

6.2. Compact Convergence of Fractional Powers.

Here we obtain the compact convergence of negative fractional powers assuming compact convergence of resolvents. Such results are essential to obtain the $\mathcal{P}^\alpha$ -compactness of the union of the attractors in Lemma 5.2.

Lemma 6.5. *Let A_n, A be such that conditions (1.1), (B₁) and $\Delta_{cc} \neq \emptyset$ are satisfied. Then for any $0 < \theta < 1$ one has $A_n^{-\theta} \xrightarrow{\mathcal{PP}} A^{-\theta}$ compactly.*

Proof. Let $\Sigma_{\bar{\phi}, -\omega}$ be as in Lemma 6.4 and Ω be the boundary of $\Sigma_{\bar{\phi}, -\omega}$. Since

$$(-A_n)^{-\theta} = \frac{1}{2\pi i} \int_{\Omega} (-\lambda)^{-\theta} (\lambda I_n - A_n)^{-1} d\lambda.$$

From Lemma 6.4, the above integral is absolutely convergent uniformly for $n \geq n_\omega$. Therefore, given $\epsilon > 0$, one can divide contour $\Omega = \Omega_1^\epsilon \cup \Omega_2^\epsilon$ in such way that Ω_1^ϵ is bounded and the integral

$$\frac{1}{2\pi i} \int_{\Omega_2^\epsilon} \|(-\lambda)^{-\theta} (\lambda I_n - A_n)^{-1}\| d\lambda \leq \epsilon, \quad n \geq n_\omega.$$

As for the integral over Ω_1^ϵ we rewrite it as

$$B_n := \frac{A_n^{-1}}{2\pi i} \int_{\Omega_1^\epsilon} (-\lambda)^{-\theta} \lambda (\lambda I_n - A_n)^{-1} d\lambda$$

we observe that $\|(-\lambda)^{-\theta} \lambda (\lambda I_n - A_n)^{-1}\|$ is bounded uniformly for $n \geq n_\omega$. From the fact that $A_n^{-1} \rightarrow A^{-1}$ compactly it follows that $B_n \rightarrow B$ compactly. Let μ be a measure of non-compactness. Now, taking any sequences $n \rightarrow \infty$ and $\{u_n\}$, $u_n \in E_n$, $\|u_n\| = 1$, we obtain that

$$\mu(\{(-A_n)^{-\theta} u_n\}) \leq \mu(\{B_n u_n\}) + \mu(\{\frac{1}{2\pi i} \int_{\Omega_2^\epsilon} (-\lambda)^{-\theta} (\lambda I_n - A_n)^{-1} d\lambda u_n\}) \leq \epsilon.$$

It follows that, given $\{u_n\}$, with $u_n \in E_n$, $n \rightarrow \infty$, $\|u_n\| = 1$, $\mu(\{(-A_n)^{-\theta} u_n\}) = 0$ and that $(-A_n)^{-\theta} \rightarrow (-A)^{-\theta}$ compactly.

To see that $(-A_n)^{-\theta} \xrightarrow{\mathcal{PP}} (-A)^{-\theta}$ we use that $(\lambda I_n - A_n)^{-1} \xrightarrow{h \rightarrow 0} (\lambda I - A)^{-1}$ for all $\lambda \in \Omega$ and the Dominated Convergence Theorem. $\square$

Now we consider a relatively bounded perturbation of A_n and A in the following manner. We consider perturbations satisfying

(L) $D_n \in B(E_n^\alpha, E_n)$, $D \in B(E^\alpha, E)$ such that D_n, D are consistent.

We assume the following hyperbolicity condition

(H) $\sigma(A + D) \cap \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda = 0\} = \emptyset$.

It is clear that if resolvent of A is compact, then $A + D$ has compact resolvent.

Lemma 6.6. *Assume that $\Delta_{cc} \neq \emptyset$ and conditions (1.1), (B_1) are satisfied. Then, for any $0 \leq \theta < 1$ operators $D_n \in B(E_n^\alpha, E_n)$, such that $D_n \rightarrow D$ in the sense of the spaces E_n^α, E_n and E^α, E , then one has $(-A_n)^\theta (A_n + D_n)^{-1} \xrightarrow{\mathcal{PP}} (-A)^\theta (A + D)^{-1}$ compactly.*

Proof: To prove the result chose positive numbers r_1 and r_2 such that $1 > \alpha + r_1 \geq \theta$ and $r_2 = 1 - \alpha - r_1$ and note that

$$(-A_n)^\theta (A_n + D_n)^{-1} = (-A_n)^{-(\alpha+r_1-\theta)} (-I_n + (-A_n)^{-r_2} D_n (-A_n)^{-\alpha-r_1})^{-1} (-A_n)^{-r_2}$$

Since $\{(-A_n)^{-r_2} D_n (-A_n)^{-\alpha-r_1}\}$ converge compactly, the result now follows from Lemma 6.5 and from Lemma 6.1. $\square$

6.3. Convergence of Linearized Semigroups.

The next result ensures a stability condition.

Lemma 6.7. *Let A_n, A be such that conditions (1.1), (B_1) , $\Delta_{cc} \neq \emptyset$, (L) and (H) are satisfied. Then, there is a sector $\Sigma_{\phi_d, \omega_d}$, $\phi_d < \pi/2$, $\omega_d \in \mathbb{R}$, a constant $n_d > 0$ and a constant M_d such that*

$$(6.7) \quad \|(\lambda I_n - (A_n + f'_n(u_n^*)))^{-1}\| \leq \frac{M_d}{|\lambda - \omega_d|}, \text{ for any } \lambda \in \Sigma_{\phi_d, \omega_d} \text{ and for all } n \geq n_d.$$

Proof. It follows from Lemma 6.4 and from the Moment Inequality that for $\lambda \in \Sigma_{\bar{\phi},0}$

$$\|(-A_n)^\alpha(\lambda I_n - A_n)^{-1}\| \leq \frac{M}{|\lambda|^{1-\alpha}}, \quad n \geq n_\omega.$$

Since $D_n(-A_n)^{-\alpha}$ is bounded uniformly in n and

$$(\lambda I_n - A_n - D_n) = (I_n - D_n(-A_n)^{-\alpha}(-A_n)^\alpha(\lambda I_n - A_n)^{-1})(\lambda I_n - A_n)$$

we have that there is a $R > 0$ such that for $|\lambda| > R$, $\lambda \in \Sigma_{\bar{\phi},0}$, we have that the operator $(I_n - D_n(-A_n)^{-\alpha}(-A_n)^\alpha(\lambda I_n - A_n)^{-1})$ is invertible and

$$\|(I_n - D_n(-A_n)^{-\alpha}(-A_n)^\alpha(\lambda I_n - A_n)^{-1})^{-1}\| \leq 2.$$

From that we obtain the desired result (see Figure 3) for any $\omega_d > R$ with θ_d suitably chosen such that $\Sigma_{\theta_d, \omega_d} \subset \{\lambda \in \Sigma_{\bar{\theta},0} : |\lambda| > R\}$. $\square$

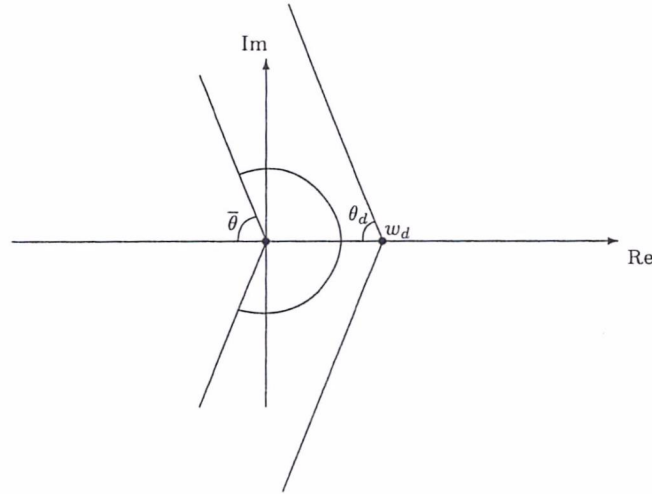


Figure 3

Observing that, from Lemma 6.6, $(A_n + f'_n(u_n^*))^{-1}$ converges compactly to $(A + f'(u^*))^{-1}$ and proceeding exactly as in Lemma 6.3 we obtain the following result

Lemma 6.8. *Let A_n, A be such that conditions (1.1), (B_1) , $\Delta_{cc} \neq \emptyset$, (L) and (H) are satisfied. Let Λ be a compact subset of $\rho((A + f'(u^*)))$. Then, there is a constant $n_\Lambda > 0$ such that $\Lambda \subset \rho((A_n + f'_n(u_n^*)))$ for all $n \geq n_\Lambda$ and*

$$(6.8) \quad \sup_{\substack{\lambda \in \Lambda \\ n \geq n_\Lambda}} \|(\lambda I_n - (A_n + f'_n(u_n^*)))^{-1}\| < \infty.$$

Furthermore, for any $u \in E$

$$(6.9) \quad \sup_{\lambda \in \Lambda} \|(\lambda I_n - (A_n + f'_n(u_n^*)))^{-1} p_n u - (\lambda I - (A + f'(u^*)))^{-1} u\| \rightarrow 0.$$

The following result can be proved exactly as Theorem 3.2 and Lemma 6.4.

Let $Q(\sigma^+)$ be as in Definition 2.9 and denote by $W(\sigma^+, A + f'(u^*))$ the subspace $Q(\sigma^+)E^\alpha$. Similarly, if $Q_n(\sigma_n^+)$ is as in Proposition 5.5, we denote by $W_n(\sigma_n^+, A_n + f'_n(u_n^*))$ the subspace $Q_n(\sigma_n^+)E_n^\alpha$,

Theorem 6.3. *Let A_n, A be such that conditions (1.1), (B_1) , $\Delta_{cc} \neq \emptyset$, (L) and (H) are satisfied. Then the following statements hold :*

- (i) *there exists $n_0 > 0$ such that $\dim W_n(\sigma_n^+, (A_n + f'_n(u_n^*))) = \dim W(\sigma^+, (A + f'(u^*)))$ for all $n \geq n_0$;*
- (ii) *for any $u \in W(\sigma^+, (A + f'(u^*)))$, there exists a sequence $\{u_n\}$, $u_n \in W_n(\sigma_n^+, (A_n + f'_n(u_n^*)))$, such that $u_n \xrightarrow{\mathcal{P}^\alpha} u$;*
- (iii) *any sequence $n \rightarrow \infty$, $\{u_n\}$, $u_n \in W_n(\sigma_n^+, (A_n + f'_n(u_n^*)))$, $n \in \mathbb{N}$, with $\|u_n\|_{E_n} = 1$ has a convergent subsequence and any limit point of this sequence belongs to $W(\sigma^+, (A + f'(u^*)))$;*
- (iv) *there is a $\beta > 0$ and $n_\beta > 0$ such that $\sigma((A_n + f'_n(u_n^*))) \cap \{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| > \beta\} = \emptyset$, $n \geq n_\beta$;*
- (v) *there are $\pi/2 > \phi_\beta > 0$, $\omega_\beta > 0$ and $n_\beta > 0$ such that $\Sigma_{\phi_\beta, -\omega_\beta} \subset \rho(A_n)$, $n \geq n_\beta$ and*

$$(6.10) \quad \|(\lambda - (A_n + f'_n(u_n^*)))^{-1}(I - Q_n(\sigma_n^+, A_n + f'_n(u_n^*)))\| \leq \frac{M_\beta}{|\lambda - \omega_\beta|}, \quad \forall \lambda \in \Sigma_{\phi_\beta, -\omega_\beta}.$$

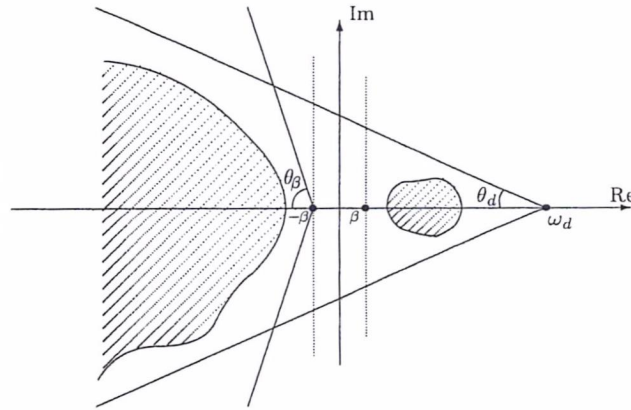


Figure 4

6.4. Some Uniform Estimates.

It is well known that, if condition (1.1) is satisfied, for each $\omega < \bar{\omega}$ there is a constant $M_\omega \geq 1$ such that

$$\|t^\alpha \exp(tA)\|_{B(E, E^\alpha)} \leq M_\omega e^{-\omega t}, \quad \forall t \geq 0.$$

From Lemma 6.4, the estimate in (1.1) holds uniformly for the semigroups generated by A_n , for n large enough. From this we obtain the following result

Theorem 6.4. *Assume that $\Delta_{cc} \neq \emptyset$ and conditions (1.1), (B_1) , (L) , (H) are satisfied. Then, for each $\omega < \tilde{\omega}$, there are constants, n_0 and M_ω such that*

$$\|t^\alpha \exp(tA_n)\|_{B(E_n, E_n^\alpha)} \leq M_\omega e^{-\omega t}, \quad \forall t \geq 0, \quad 0 \leq \alpha \leq 1, \quad n \geq n_0.$$

As a consequence of Theorem 6.3 we have that

Theorem 6.5. *Assume that the conditions Assume that $\Delta_{cc} \neq \emptyset$ and conditions (1.1), (B_1) , (L) , (H) are satisfied. Then, for each $\omega_\beta < \beta$, there are constants, n_{ω_β} and M_{ω_β} such that*

$$\|t^\alpha e^{t(A_n + f'_n(u_n^*))} (I_n - Q_n(\sigma_n^+, A_n + f'_n(u_n^*)))\|_{B(E_n, E_n^\alpha)} \leq M_{\omega_\beta} e^{-\omega_\beta t}, \quad \forall t \geq 0, \quad 0 \leq \alpha \leq 1,$$

$$\|e^{t(A_n + f'_n(u_n^*))} Q_n(\sigma_n^+, A_n + f'_n(u_n^*))\|_{B(E_n)} \leq M_{\omega_\beta} e^{\omega_\beta t}, \quad \forall t \leq 0,$$

$$n \geq n_{\omega_\beta}.$$

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