UNIVERSIDADE DE SÃO PAULO
Instituto de Ciências Matemáticas e de Computação
ISSN 0103-2577

Asymptocally Almost Periodic and Almost Periodic Solutions for Partial Neutral Differential Equations

Eduardo Hernández M
Mauricio L. Pelicer

Nº 203

NOTAS

Série Matemática

São Carlos – SP
Ago ./2004

SYSNO 1387895
DATA / /
ICMC - SBAB

Soluções Quase e Assintoticamente Quase Periódicas de Equações Diferenciais do Tipo Neutro.

Eduardo Hernández M & Maurício Pelicer

Departamento de Matemática - ICMC - USP.

Resumo

Neste artigo estudamos a existência de soluções quase periódicas e assintoticamente quase periódicas para equações diferenciais do tipo neutro modeladas na forma

$$\frac{d}{dt}(x(t) + g(t, x_t)) = Ax(t) + f(t, x_t), \quad t \in I = [0, a], \tag{1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{2}$$

sendo A o gerador infinitesimal de um C_0-semigrupo de operadores lineares limitados definido sobre um espaço de Banach X, $\mathcal{B}$ um espaço de fase definido de maneira axiomática e f, g funções apropriadas.

Asymptotically Almost Periodic and Almost Periodic Solutions for Partial Neutral Differential Equations.

Eduardo Hernández M & Mauricio L. Pelicer

AMS-Subject Classification: Primary 34K14, 35B15 , 35R10, 34K30, 34K40,

Keywords: Almost periodic solutions, Neutral differential equations, semigroup of linear operators.

Resumo

In this paper we study the existence of almost periodic and asymptotically almost periodic solutions for partial neutral functional differential equations with unbounded delay.

1 Introduction

The existence of asymptotically almost periodic and almost periodic solutions for partial abstract functional differential equations with delay has been treated in several works, see for instance [1, 3, 4]. In this work we study the existence of asymptotically almost periodic and almost periodic solutions for a class of neutral functional differential equation with unbounded delay modelled in the form

$$\frac{d}{dt}(x(t)+f(t,x_t)) = Ax(t)+g(t,x_t), \tag{1}$$

where A is the infinitesimal generator of an uniformly exponentially stable analytic semigroup of linear operators, $(T(t))_{t\geq 0}$, on a Banach space X; the history $x_t:(-\infty,0]\to X$, $x_t(\theta)=x(t+\theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically and $f(\cdot), g(\cdot)$ are appropriate functions.

Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received great attention in the last decades. The literature relative to ordinary neutral functional differential equations is very extensive and we refer the reader to [5] concerning this matter. Partial neutral functional differential equations have been studied in [6, 7, 8].

Throughout this paper, $A : D(A)\subset X\to X$ is the infinitesimal generator of an analytic semigroup of linear operators $(T(t))_{t\geq 0}$ on a Banach space X such that $0\in\rho(A)$, and $\widetilde{M},\delta$ are positive constants such that $\| T(t)\|\leq \widetilde{M}e^{-\delta t}$ for every $t\geq 0$. Under these conditions, it is possible to define the fractional power $(-A)^{\alpha}$, $0<\alpha\leq 1$, as a closed linear operator on its domain $D((-A)^{\alpha})$. If X_{α} is the space $D((-A)^{\alpha})$ endowed with the norm $\| x\|_{\alpha}=\|(-A)^{\alpha}x\|$, then the following properties hold, see [9] for details.

Lemma 1 *Let $0 < \gamma \leq \eta \leq 1$. Then X_η is a Banach space and $X_\eta \hookrightarrow X_\gamma$. Moreover, the function $s \to (-A)^\eta T(s)$ is continuous in the uniform operator topology on $(0,\infty)$ and there exists $C_\eta > 0$ such that $\| (-A)^\eta T(t) \| \leq \dfrac{C_\eta e^{-\delta t}}{t^\eta}$ for every $t > 0$.*

In this work, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ and verifying the following axioms:

(A) If $x : (-\infty, \sigma + a) \to X$, $a > 0, \sigma \in \mathbb{R}$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + a)$ the following conditions hold:

(i) x_t is in $\mathcal{B}$.

(ii) $\| x(t) \| \leq H \| x_t \|_{\mathcal{B}}$.

(iii) $\| x_t \|_{\mathcal{B}} \leq K(t-\sigma) \sup\{\| x(s) \| : \sigma \leq s \leq t\} + M(t-\sigma) \| x_\sigma \|_{\mathcal{B}}$,

where $H > 0$ is a constant; $K, M : [0,\infty) \to [1,\infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ in **(A)**, x_t is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma + a)$.

(B) The space $\mathcal{B}$ is complete.

(C2) Let $(\varphi^n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence in $C((-\infty, 0]; X)$ formed by functions with compact support. If $\varphi^n \to \varphi$ in the compact-open topology, then $\varphi \in \mathcal{B}$ and $\|\varphi^n - \varphi\|_{\mathcal{B}} \to 0$, as $n \to \infty$.

Definition 1 *Let $S(t) : \mathcal{B} \to \mathcal{B}$ be the C_0-semigroup defined by $S(t)\varphi(\theta) = \varphi(0)$ on $[-t, 0]$ and $S(t)\varphi(\theta) = \varphi(t+\theta)$ on $(-\infty, -t]$. The phase space $\mathcal{B}$ is called a fading memory if $\| S(t)\varphi \|_{\mathcal{B}} \to 0$ as $t \to \infty$ for every $\varphi \in \mathcal{B}$ such that $\varphi(0) = 0$.*

Remark 1 *In this paper, $\mathfrak{L} > 0$ is such that $\| \varphi \|_{\mathcal{B}} \leq \mathfrak{L} \sup_{\theta\leq 0} \| \varphi(\theta) \|$ for each $\varphi \in \mathcal{B}$ continuous and bounded, see [10, Proposition 7.1.1] for details. Moreover, for the case in which $\mathcal{B}$ is a fading memory, we will assume that $\max\{K(t), M(t)\} \leq \mathfrak{K}$ for $t \geq 0$, see [10, Proposition 7.1.5].*

In the next results and definitions, $(Z, \|\cdot\|_Z)$, $(W, \|\cdot\|_W)$ are Banach spaces and $C_0([0,\infty); Z)$ is the subspace of $C([0,\infty); Z)$ formed by the functions that vanishes at infinite.

Definition 2 *A function $F \in C(\mathbb{R}; Z)$ is almost periodic (a.p.) if for every $\varepsilon > 0$ there exists a relatively dense subset of $\mathbb{R}$, denoted by $\mathcal{H}(\varepsilon, F, Z)$, such that $\|F(t+\xi) - F(t)\|_Z < \varepsilon$ for every $t \in \mathbb{R}$ and all $\xi \in \mathcal{H}(\varepsilon, F, Z)$.*

Definition 3 *A function $F \in C([0,\infty); Z)$ is asymptotically almost periodic (a.a.p.) if there exists an almost periodic function g and $w \in C_0([0,\infty); Z)$ such that $F = g + w$.*

In this paper, $AP(Z)$ and $AAP(Z)$ are the spaces $AP(Z) = \{F \in C([0,\infty); Z) : F \text{ is a.p.}\}$ and $AAP(Z) = \{F \in C(\mathbb{R}; Z) : F \text{ is a.a.p.}\}$ endowed with the norms $||| u |||_Z = \sup_{s\in\mathbb{R}} \| u(s) \|$ and $\| u \|_Z = \sup_{s\geq 0} \| u(s) \|$ respectively. We know from [11] that $AP(Z)$ and $AAP(Z)$ are Banach spaces.

Lemma 2 *[11, Theorem 5] A function $F \in C([0,\infty); Z)$ is asymptotically almost periodic if and only if, for every $\varepsilon > 0$ there exists $L(\varepsilon, F, Z) > 0$ and a relatively dense subset of $[0,\infty)$, denoted by $\mathcal{T}(\varepsilon, F, Z)$, such that $\|F(t+\xi) - F(t)\|_Z < \varepsilon$ for every $t \geq L(\varepsilon, F, Z)$ and every $\xi \in \mathcal{T}(\varepsilon, F, Z)$.*

Definition 4 *Let Ω be an open subset of W.*

(a) *A continuous function $F : \mathbb{R}\times\Omega \to Z$ (resp. $F : [0,\infty)\times\Omega \to Z$) is called pointwise almost periodic (p.a.p.), (resp. pointwise asymptotically almost periodic (p.a.a.p.)) if $F(\cdot, x) \in AP(Z)$ (resp. $F(\cdot, x) \in AAP(Z)$) for every $x \in \Omega$.*

(b) *A continuous function $F : \mathbb{R}\times\Omega \to Z$ is called uniformly almost periodic (u.a.p.), if for every $\varepsilon > 0$ and every compact $K \subset \Omega$ there exists a relatively dense subset of $\mathbb{R}$, denoted by $\mathcal{H}(\varepsilon, F, K, Z)$, such that $\|F(t+\xi, y) - F(t, y)\|_Z \leq \varepsilon$ for every $(t, \xi, y) \in \mathbb{R}\times\mathcal{H}(\varepsilon, F, K, Z)\times K$.*

(c) *A continuous function $F : [0,\infty)\times\Omega \to Z$ is called uniformly asymptotically almost periodic (u.a.a.p.), if for every $\varepsilon > 0$ and every compact $K \subset \Omega$ there exists a relatively dense subset of $[0,\infty)$, denoted by $\mathcal{T}(\varepsilon, F, K, Z)$, and $L(\varepsilon, F, K, Z) > 0$ such that $\|F(t+\xi, y) - F(t, y)\|_Z \leq \varepsilon$ for every $t \geq L(\varepsilon, F, K, Z)$ and every $(\xi, y) \in \mathcal{T}(\varepsilon, F, K, Z)\times K$.*

Lemma 3 *[12, Theorem 1.2.7] Let $\Omega \subset W$ be an open set. Then the following properties hold.*

(a) *If $F \in C(\mathbb{R}\times\Omega; Z)$ (resp. $F \in C([0,\infty)\times\Omega; Z)$) is p.a.p. (resp. p.a.a.p.) and satisfies a local Lipschitz condition at $x \in \Omega$, uniformly at t, then F is u.a.p (resp. u.a.a.p)*

(b) *If $x \in AP(X)$, then $t \to x_t \in AP(\mathcal{B})$. Moreover, if $\mathcal{B}$ is a fading memory space and $z \in C(\mathbb{R}; X)$ is such that $z_0 \in \mathcal{B}$ and $z \in AAP(X)$, then $t \to z_t \in AAP(\mathcal{B})$.*

(c) *If $F \in C(\mathbb{R}\times\Omega; Z)$ (resp. $F \in C([0,\infty)\times\Omega; Z)$) is u.a.p. (resp. is u.a.a.p.) and $y \in AP(W)$ (resp. $y \in AAP(W)$) is such that $\overline{\{y(t) : t \in \mathbb{R}\}}^W \subset \Omega$, then $F(t, y(t)) \in AP(Z)$ (resp. $F(t, y(t)) \in AAP(Z)$).*

2 Existence Results

To establish our results of existence we need the next technical lemma.

Lemma 4 *Let $\mu \in (0,1]$ and $v \in AAP(X_\mu)$. If $u(\cdot) : [0,\infty) \to X$ is the function defined by $u(t) = \int_0^t AT(t-s)v(s)ds$, $t \geq 0$, then $u(\cdot) \in AAP(X)$.*

Proof: To prove this result we use Lemma 2. Let $\varepsilon > 0$ be given and $\mathcal{T}(\varepsilon, v, X_\mu)$, $L = L(\varepsilon, v, X_\mu)$ be as in Lemma 2. If $t \geq L = L(\varepsilon, v, X_\mu) + 1$ and $\xi \in \mathcal{T}(\varepsilon, v, X_\mu)$, we get

$$\|u(t+\xi) - u(t)\| \leq \int_0^\xi \|(-A)^{1-\mu}T(t+\xi-s)(-A)^\mu v(s)\|ds$$

$$
\begin{aligned}
& + \int_0^t \|(-A)^{1-\mu}T(t-s)\left((-A)^{\mu}v(s+\xi)-(-A)^{\mu}v(s)\right)\|ds \\
\leq\ & e^{-\delta t}\|v\|_{X_\mu}\int_0^{\xi}\frac{C_{1-\mu}e^{-\delta(\xi-s)}}{(\xi-s)^{1-\mu}}ds \\
& + \int_0^{L+1} \|(-A)^{1-\mu}T(t-s)\left((-A)^{\mu}v(s+\xi)-(-A)^{\mu}v(s)\right)\|ds \\
& + \int_{L+1}^{t} \|(-A)^{1-\mu}T(t-s)\|\|(-A)^{\mu}v(s+\xi)-(-A)^{\mu}v(s)\|ds \\
\leq\ & d_1e^{-\delta t}+2\|v\|_{X_\mu}e^{-\delta(t-L-1)}\int_0^{L+1}\frac{C_{1-\mu}e^{-\delta(L+1-s)}}{(L+1-s)^{1-\mu}}ds \\
& +\varepsilon\int_0^{\infty}\frac{C_{1-\mu}e^{-\delta s}}{s^{1-\mu}}ds
\end{aligned}
$$

where $d_1 > 0$ is independent of t and ξ. Thus, there exist positive constants d_2, d_3 independents of t and ξ such that

$$
\|u(t+\xi)-u(t)\| \leq d_2e^{\delta(L(\varepsilon,v,X_\mu)+1)}e^{-\delta t}+\varepsilon d_3, \quad t \geq L(\varepsilon,v,X_\mu)+1,\ \xi\in\mathcal{T}(\varepsilon,v,X_\mu).
$$

From this inequality we infer that $u(\cdot)$ verifies the conditions of Lemma 2. The proof is complete. ■

To prove our results we always assume that the next condition is satisfied.

H_1 The functions $f, g : \mathbb{R}\times\mathcal{B}\to X$ are continuous and there exist $\beta\in(0,1)$ and continuous functions $L_g, L_f : [0,\infty)\to[0,\infty)$ such that $f(\cdot)$ is X_β-valued, $(-A)^{\beta}f : \mathbb{R}\times\mathcal{B}\to X$ is continuous and

$$
\begin{aligned}
\|(-A)^{\beta}f(t,\psi_1)-(-A)^{\beta}f(t,\psi_2)\| &\leq L_f(r)\|\psi_1-\psi_2\|_{\mathcal{B}}, \\
\|g(t,\psi_1)-g(t,\psi_2)\| &\leq L_g(r)\|\psi_1-\psi_2\|_{\mathcal{B}},
\end{aligned}
$$

for every $t\in\mathbb{R}$ and every $\psi_i\in\mathcal{B}$ such that $\|\psi_i\|_{\mathcal{B}}\leq r$.

From Hernández & Henríquez [7] we adopt the next concept of mild solution of (1).

Definition 5 *A function $u : (-\infty,\sigma+a)\to X$, $a>0$, is a mild solution of system (1) on $[\sigma,\sigma+a)$, if $u_\sigma\in\mathcal{B}$, $u\,|_{[\sigma,\sigma+a)}$ is continuous; the function $s\to AT(t-s)f(s,u_s)$ is integrable on $[0,t)$ for every $\sigma<t<\sigma+a$ and*

$$
\begin{aligned}
u(t) &= T(t)(\varphi(\sigma)+f(\sigma,\varphi))-f(t,u_t)-\int_{\sigma}^{t}AT(t-s)f(s,u_s)ds \\
&\quad+\int_{\sigma}^{t}T(t-s)g(s,u_s)ds, \quad t\in[\sigma,\sigma+a).
\end{aligned}
$$

Theorem 1 *Assume that $\mathcal{B}$ is a fading memory space and that $(-A)^{\beta}f(\cdot)$ and $g(\cdot)$ are p.a.a.p. If $L_f(0)=L_g(0)=0$ and $(-A)^{\beta}f(t,0)=g(t,0)=0$ for every $t\in\mathbb{R}$, then there exists $\varepsilon>0$ such that for each $\varphi\in B_\varepsilon(0,\mathcal{B})$ there exists a mild solution, $u(\cdot,\varphi)$, of (1) on $[0,\infty)$ such that $u(\cdot,\varphi)\in AAP(X)$ and $u_0(\cdot,\varphi)=\varphi$.*

Proof: Let $r > 0$ and $0 < \lambda < 1$ be such that

$$\begin{aligned}\Theta &= \widetilde{M}\lambda[H+ \| (-A)^{-\beta} \| L_f(\lambda r)] \\ &+ \left[L_f(r^*)\left(\| (-A)^{-\beta} \| +C_{1-\beta}(\frac{1}{\delta}+\frac{1}{\beta})\right)+\frac{\widetilde{M}L_g(r^*)}{\delta}\right](\lambda+1)\mathfrak{K} < 1,\end{aligned}$$

where $\mathfrak{K}$ is the number in Remark 1 and $r^* = (\lambda+1)\mathfrak{K}r$. We affirm that the assertion holds for $\varepsilon = \lambda r$. Let $\varphi \in B_\varepsilon(0, \mathcal{B})$. On the space

$$\mathfrak{D} = \{x \in AAP(X) : x(0) = \varphi(0), \| x(t) \| \leq r, \, t \geq 0\}$$

endowed with the metric $d(u, v) = \| u - v \|_X$, we define the operator $\Gamma : \mathfrak{D} \to C([0, \infty); X)$ by

$$\begin{aligned}\Gamma u(t) &= T(t)(\varphi(0) + f(0, \varphi)) - f(t, \widetilde{u}_t) - \int_0^t AT(t-s)f(s, \widetilde{u}_s)ds \\ &+ \int_0^t T(t-s)g(s, \widetilde{u}_s)ds, \quad t \geq 0,\end{aligned}$$

where $\widetilde{u} : \mathbb{R} \to X$ is such that $(\widetilde{u})_0 = \varphi$ and $\widetilde{u} = u$ on $[0, \infty)$. From the continuity of $s \to AT(t-s)$ and $s \to T(t-s)$ in the uniform operator topology on $[0, t)$ and the estimate

$$\| AT(t-s)f(s, \widetilde{u}_s) \| = \| (-A)^{1-\beta}T(t-s)(-A)^{\beta}f(s, \widetilde{u}_s) \| \leq \frac{C_{1-\beta}L_f(r^*)r^*}{(t-s)^{1-\beta}},$$

it follows that $s \to AT(t-s)f(s, \widetilde{u}_s)$ and $s \to T(t-s)g(s, \widetilde{u}_s)$ are integrable on $[0, t)$ for every $t > 0$ and so that $\Gamma u \in C([0, \infty); X)$. Moreover, from Lemmas 3 and 4 we infer that $\Gamma u \in AAP(X)$.

In the sequel, we prove that $\Gamma(\cdot)$ is a contraction from $\mathfrak{D}$ into $\mathfrak{D}$. If $u \in \mathfrak{D}$ we get

$$\begin{aligned}\| \Gamma u(t) \| &\leq \widetilde{M}H\lambda r + \widetilde{M} \| (-A)^{-\beta} \| L_f(\lambda r)\lambda r + \| (-A)^{-\beta} \| L_f(r^*)(\lambda+1)\mathfrak{K}r \\ &+ C_{1-\beta}\int_0^t \frac{e^{-\delta(t-s)}}{(t-s)^{1-\beta}}L_f(r^*)(\lambda+1)\mathfrak{K}rds + \int_0^t \widetilde{M}e^{-\delta(t-s)}L_g(r^*)(\lambda+1)\mathfrak{K}rds \\ &\leq \widetilde{M}H\lambda r + \widetilde{M} \| (-A)^{-\beta} \| L_f(\lambda r)\lambda + \| (-A)^{-\beta} \| L_f(r^*)(\lambda+1)\mathfrak{K}r \\ &+ C_{1-\beta}L_f(r^*)(\frac{1}{\delta}+\frac{1}{\beta})(\lambda+1)\mathfrak{K}r + \frac{\widetilde{M}}{\delta}L_g(r^*)(\lambda+1)\mathfrak{K}r \\ &\leq \Theta r,\end{aligned}$$

which implies that $\Gamma(\mathfrak{D}) \subset \mathfrak{D}$. Moreover, for $u, v \in \mathfrak{D}$ we get

$$\begin{aligned}&\| \Gamma u(t) - \Gamma v(t) \| \\ &\leq \| f(t, \widetilde{u}_t) - f(t, \widetilde{v}_t) \| + \int_0^t \left[\frac{C_{1-\beta}L_f(r^*)}{(t-s)^{1-\beta}} + \widetilde{M}L_g(r^*)\right] e^{-\delta(t-s)} \| \widetilde{u}_s - \widetilde{v}_s \|_{\mathcal{B}}\, ds \\ &\leq \left[L_f(r^*)\left(\| (-A)^{-\beta} \| +C_{1-\beta}(\frac{1}{\delta}+\frac{1}{\beta})\right)+\frac{\widetilde{M}L_g(r^*)}{\delta}\right]\mathfrak{K} \| u - v \|_X,\end{aligned}$$

which shows that $\Gamma(\cdot)$ is a contraction and completes the proof of this result. ■

The next result is proved using the steps in the proof of the previous result. We will omit details.

Theorem 2 *If $\mathcal{B}$ is a fading memory space; $(-A)^{\beta}f(\cdot)$ and $g(\cdot)$ are p.a.a.p; $L_f(t) = L_f$ and $L_g(t) = L_g$ for every $t \geq 0$ and $\left[L_f\left(\| (-A)^{-\beta} \| + C_{1-\beta}(\frac{1}{\delta}+\frac{1}{\beta})\right) + \frac{\widetilde{M}L_g}{\delta}\right]\mathfrak{K} < 1$, then for every $\varphi \in \mathcal{B}$ there exists a unique mild solution, $u(\cdot,\varphi)$, of (1) on $[0,\infty)$ such that $u(\cdot,\varphi) \in AAP(X)$ and $u_0(\cdot,\varphi) = \varphi$.*

In the next result we establish the existence of an almost periodic solution.

Theorem 3 *Assume that $(-A)^{\beta}f(\cdot)$ and $g(\cdot)$ are p.a.p. If $L_f(0) = L_g(0) = 0$ and $(-A)^{\beta}f(t,0) = g(t,0) = 0$ for every $t \geq 0$, then there exists $u \in AP(X)$ such that $u(\cdot)$ is a mild solution of (1) on every interval $[\sigma, \sigma + a)$.*

Proof: Let $\Gamma : AP(X) \to C(\mathbb{R}; X)$ be the map defined by

$$\Gamma u(t) \;=\; -f(t,u_t) - \int_{-\infty}^{t} AT(t-s)f(s,u_s)ds + \int_{-\infty}^{t} T(t-s)g(s,u_s)ds, \qquad t \in \mathbb{R}.$$

From the proof of Theorem 1, it follows that $\Gamma u(\cdot)$ is a continuous function. From Lemma 3 and Zaidman [11, pp. 30] we infer that $v(t) = ((-A)^{\beta}f(t,u_t), g(t,u_t)) \in AP(X \times X)$. If $t \in \mathbb{R}$ and $\xi \in \mathcal{H}(\varepsilon, v, X \times X)$ we get

$$\begin{aligned}
&\| \Gamma u(t+\xi) - \Gamma u(t) \\
\| \;\leq\; & \| f(t+\xi, u_{t+\xi}) - f(t,u_t) \| \\
&+ \int_{-\infty}^{t} \frac{C_{1-\beta}e^{-\delta(t-s)}}{(t-s)^{1-\beta}} \| (-A)^{\beta}f(s+\xi, u_{s+\xi}) - (-A)^{\beta}f(s,u_s) \| ds \\
&+ \int_{-\infty}^{t} \widetilde{M}e^{-\delta(t-s)} \| g(s+\xi, u_{s+\xi}) - g(s,u_s) \| ds \\
\leq\; & \varepsilon(\| (-A)^{-\beta} \| + C_{1-\beta}(\frac{1}{\delta}+\frac{1}{\beta}) + \frac{\widetilde{M}}{\delta}),
\end{aligned}$$

which proves that $\Gamma u \in AP(X)$. Thus, $\Gamma(\cdot)$ is well defined and with values in $AP(X)$.

On the other hand, for $u, v \in B_r = \{u \in AP(X) : ||| u |||_X \leq r\}$ we find that

$$\begin{aligned}
&\| \Gamma u(t) - \Gamma v(t) \| \\
&\leq \| (-A)^{-\beta} \| L_f(\mathfrak{L}r) \| u_t - v_t \|_{\mathcal{B}} \\
&\quad + \int_{-\infty}^{t} \frac{C_{1-\beta}e^{-\delta(t-s)}}{(t-s)^{1-\beta}} L_f(\mathfrak{L}r) \| u_s - v_s \|_{\mathcal{B}}\, ds \\
&\quad + \int_{-\infty}^{t} \widetilde{M}e^{-\delta(t-s)} L_g(\mathfrak{L}r) \| u_s - v_s \|_{\mathcal{B}}\, ds \\
&\leq \mathfrak{L}\left[L_f(\mathfrak{L}r)\left(\| (-A)^{-\beta} \| + C_{1-\beta}(\frac{1}{\delta}+\frac{1}{\beta})\right) + \frac{\widetilde{M}L_g(\mathfrak{L}r)}{\delta}\right] ||| u - v |||_X,
\end{aligned}$$

where $\mathfrak{L}$ is the constant in Remark 1. This inequality and the fact that $\Gamma 0 \equiv 0$ allows us to infer that $\Gamma(\cdot)$ is a contraction from B_r into B_r for $r > 0$ small enough, which proves the assertion. ■

Using similar arguments we can prove the next result.

Theorem 4 *Assume that $(-A)^{\beta}f(\cdot)$ and $g(\cdot)$ are p.a.p. functions. If $L_f(t) = L_f$ and $L_g(t) = L_g$ for all $t \geq 0$ and $\mathfrak{L}\Big[L_f\Big(\| (-A)^{-\beta} \| + C_{1-\beta}(\frac{1}{\delta}+\frac{1}{\beta})\Big) + \frac{\widetilde{M}L_g}{\delta}\Big] < 1$, then there exists a unique $u \in AP(X)$ such that $u(\cdot)$ is a mild solution of (1) on every interval $[\sigma, \sigma + a)$.*

2.1 Example

Consider the first order neutral equation with unbounded delay

$$\begin{aligned} \frac{\partial}{\partial t}[u(t,\xi) + \int_{-\infty}^{t}\int_{0}^{\pi} b(t-s,\eta,\xi)u(s,\eta)d\eta ds] &= \frac{\partial^2}{\partial \xi^2}u(t,\xi) + a_0(\xi)u(t,\xi) \\ &\quad + \int_{-\infty}^{t} a(t-s)u(s,\xi)ds + a_1(t,\xi), \\ &\qquad \xi \in I = [0,\pi], \qquad (2) \\ u(t,0) &= u(t,\pi) = 0, \qquad (3) \\ u(\tau+\sigma,\xi) &= \varphi(\tau,\xi), \quad \tau \leq 0, \quad \xi \in [0,\pi]. \qquad (4) \end{aligned}$$

To study this system, we adopt the notations and technical conditions in [7, Example 4.1]. In particular, $X = L^2([0,\pi])$, $\mathcal{B} = L^2(h; X)$ and $h(\cdot)$ is a continuous function satisfying the conditions (g-5)-(g-7) of [10, Theorem 1.3.8]. Under these conditions, $\mathcal{B}$ is a fading memory space, see [10, Example 7.1.8] for details. Next, $\mathcal{A} : \mathbb{R} \to X$, (resp. $\mathcal{A} : [0,\infty) \to X$) is the map defined by $\mathcal{A}(t)(\xi) = a(t,\xi)$. The followings result is a consequence of Theorems 2 and 4.

Proposition 1 *If $\mathcal{A}(\cdot)$ is asymptotically almost periodic (resp. almost periodic) then there exists a mild solution $u(\cdot) \in AAP(X)$ (resp. $u(\cdot) \in AP(X)$) of (2)-(3).*

Referências

[1] Henríquez, Hernán R.; Vásquez, Carlos H. Almost periodic solutions of abstract retarded functional-differential equations with unbounded delay. *Acta Appl. Math.* 57 (1999), no. 2, 105–132.

[2] Hino, Yoshiyuki; Murakami, Satoru Limiting equations and some stability properties for asymptotically almost periodic functional differential equations with infinite delay. Tohoku Math. J. (2) 54 (2002), no. 2, 239–257.

[3] Furumochi, Tetsuo; Naito, Toshiki; Minh, Nguyen Van Boundedness and almost periodicity of solutions of partial functional differential equations. *J. Differential Equations* 180 (2002), no. 1, 125–152.

[4] Hino, Yoshiyuki; Murakami, Satoru; Yoshizawa, Taro Existence of almost periodic solutions of some functional-differential equations with infinite delay in a Banach space. *Tohoku Math. J.* (2) 49 (1997), no. 1, 133–147.

[5] Hale, Jack K.; Verduyn Lunel, Sjoerd M. Introduction to functional-differential equations. Applied Mathematical Sciences, 99. Springer-Verlag, New York, 1993.

[6] Adimy, M.; Ezzinbi, K. Strict solutions of nonlinear hyperbolic neutral differential equations. Appl. Math. Lett. 12 (1999), no. 1, 107–112.

[7] Hernández, E. and Henríquez, H. R., Existence results for partial neutral functional differential equations with unbounded delay. *J. Math. Anal. Appl.* 221 (1998), no. 2, 452–475.

[8] Hernández, E. and Henríquez, H. R., Existence of periodic solutions of partial neutral functional differential equations with unbounded delay. *J. Math. Anal. Appl.* 221 (1998), no. 2, 499–522.

[9] A. Pazy, Semigroups of linear operators and applications to partial differential equations. Springer-Verlag, New-York, (1983).

[10] Hino, Y., Murakami, S. and Naito, T., Functional-differential equations with infinite delay. Lecture Notes in Mathematics, 1473. Springer-Verlag, Berlin, 1991.

[11] Zaidman, S. Almost-periodic functions in abstract spaces. Research Notes in Mathematics, 126. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[12] Yoshizawa, T. Stability theory and the existence of periodic solutions and almost periodic solutions. Applied Mathematical Sciences, Vol. 14. Springer-Verlag, New York-Heidelberg, 1975.

Eduardo Hernández M & Mauricio pellicer.
Departamento de Matemática
Instituto de Ciências Matemáticas e de Computação
Universidade de São Paulo - Campus de São Carlos
Caixa Postal 668
13560-970 São Carlos, SP. Brazil
E-mail : lalohm@icmc.sc.usp.br

NOTAS DO ICMC

SÉRIE MATEMÁTICA

203/2004 MENEGATTO, V.A.; OLIVEIRA, C.P.; PERON, A.P. – Strictly positive definite Kernels on subsets of the complex plane.

201/2004 NASCIMENTO, A.S.; CREMA, J. – On role of the equal-area condition in internal transition layer stationary solutions to a class of reaction-diffusion systems.

200/2004 PRILEPKO, A.; PISCAREV, S.; SHAW, S.Y – Control element approximation for parabolic differential equations with final observation.

199/2004 OLIVEIRA, R.D.S.; TEIXEIRA, M.A. – On pairs of polynomial planar foliations.

198/2004 CARBINATTO, M.C. ; RYBAKOWSKI, K.P. - Continuation of the connection matrix in singular perturbation problems.

197/2004 CARVALHO, A. N.; PISKAREV, S. – A general approximation scheme for attractors of abstract parabolic problems with hyperbolic equilibrium points.

196/2003 CARVALHO, A. N.; CHOLEWA, J.W. – Continuation and asymptotics of solutions to semilinear parabolic equations with critical nonlinearities.

195/2004 SOARES JR., C.H.; SAIA, M.J. – C^1-G-triviality of Newton non degenerate map germs, G=R,C and K.

194/2004 SOARES, L.M.F.; SAIA, M.J. A-topological triviality of map germs and Newton filtrations.

193/2004 ARAUJO,V.; TAHZIBI, A. – Stochastic stability at the boundary of expanding maps.