UNIVERSIDADE DE SÃO PAULO

Instituto de Ciências Matemáticas e de Computação

ISSN 0103-2577

The Orbits O_1 and O_2

L. Kushner

Nº 209

NOTAS

Série Matemática

São Carlos – SP
Out./2004

The orbits $\mathcal{O}_1$ and $\mathcal{O}_2$

L. Kushner

Summary

In this paper we study the orbits of x^3, x^2y and $x(x^2-y^2)$ in the space of homogeneous polynomials of degree 3 in two variables.

Resumo

Neste artigo estudamos o espaço dos polinômios homogêneos de grau 3 em duas variáveis.

Descrevem-se as órbitas de: x^3, x^2y e $x(x^2-y^2)$.

Sendo g e h as parametrizações de x^2y e $x(x^2-y^2)$ respectivamente, obtém-se para g os resultados de [BKL] e para h a descrição das órbitas $x(x^2-y^2)$ e x^3, sendo esta última pontos de cúspide para h.

É interessante observar que no caso da função g, trabalhamos num toro e no caso de h em uma 3-esfera e num toro contido nela, que resultam os pontos de cúspide.

1 Introduction

Consider the real vector space V of homogeneous polynomials of degree 3 in two variables and $G = GL(2)$, the Group of isomorphisms in two variables. G acts on the right on V by composition and we have the orbits of 0, x^3, x^2y, $x(x^2-y^2)$ and $x(x^2+y^2)$ which will be denoted by; $\mathcal{O}_0$, $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_{1,2}$, and correspond to the factorization on irreducible terms: zero, one linear factor raised to the third power; two linearly independent linear factors, one raised to the second power; three linear factors, linearly independent any two of them and one linear factor times a quadratic irreducible factor.

Now, if we consider the diffeomorphism $\phi : \mathbb{R}^4 - \{\bar{0}\} \longrightarrow S^3 \times (0, \infty)$ given by $\phi(\bar{x}) = (\bar{x}/_{|\bar{x}|}, |\bar{x}|)$, it is clear that we only have to study the orbits on the sphere. On Zeeman's work [Z], it is shown that $(\mathcal{O}_1 \cup \mathcal{O}_2) \cap S^3$ is a umbilic brazalet with edges being simple cusps and corresponding to $\mathcal{O}_1$.

Another way of proving this result [BKL] is considering the parametrization of $\mathcal{O}_2$: $g(a,b,c,d) = (a^2c\,,\, a^2d + 2abc\,,\, b^2c + 2abd\,,\, b^2d)$. Its divided for the points (a,b,c,d) such that $ad-bc \neq 0$ and $ad-bc=0$ but $(a,b,c,d) \notin (\mathbb{R}^2 \times \{\bar{0}\}) \cup (\{\bar{0}\} \times \mathbb{R}^2)$.

2 The new orbit $\mathcal{O}_3$

We consider the parametrization of $\mathcal{O}_3$ given by $h(a,b,c,d) = (a^3 - ac^2\,,\, 3a^2b - 2acd - bc^2\,,\, 3ab^2 - 2bcd - ad^2\,,\, b^3 - bd^2)$. Its jabobian is $-12(ad-bc)^4$, then h is a local diffeomorphism on $\mathcal{U} = \{(a,b,c,d) \mid ad-bc \neq 0\}$ and in fact we get

Theorem 1. *If we consider $h : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ given by $h(a,b,c,d) = (a^3 - ac^2\,,\, 3a^2b - bc^2\,,\, 3ab^2 - 2bcd - ad^2\,,\, b^3 - bd^2)$, we get that*

a) $h| : \mathcal{U} \longrightarrow \mathbb{R}^4$ is a fibration six to one over the orbit $\mathcal{O}_3$.

b) rank $Dh(\bar{x})$ is two for $ad-bc=0$ but $(a,b,c,d) \neq \bar{0}$.

Proof

a) It can be shown that $h^{-1}(\alpha,\beta,\gamma,\delta)$ is

i) For $\alpha\delta \neq 0$, the points of $h^{-1}(\alpha,\beta,\gamma,\delta)$ are

$$\left(-\frac{br}{\alpha^{1/3}}\,,\, \frac{b}{\delta^{1/3}}\,,\, \pm\sqrt{\frac{b^3r^3+\alpha}{b\alpha^{2/3}}}\,,\, \pm\sqrt{\frac{b^3-\delta}{b\delta^{2/3}}}\right)$$

where $\delta r^3 + \gamma r^2 + \beta r + \alpha = 0$, and

$$b^3 = \frac{1}{4}\left(\frac{\gamma\delta r^2 + (\gamma^2 - \beta\gamma)r + 3}{\gamma r^2 + 2\beta r + 3}\right).$$

ii) For $\alpha = 0$ we get the six points $(\delta = 1)$ $\left(0, -\beta/_{\Delta^{1/3}}\,,\, \pm\Delta^{1/6}\,,\, \pm\frac{\gamma}{2\beta}\Delta^{1/6}\right)$ where $\Delta = \frac{4\beta^4}{\gamma^2 - 4\beta}$, $\left(a\,,\, ar\,,\, a\,,\, a\sqrt{\frac{ar^3-1}{ar}}\right)$ and $\left(a\,,\, ar\,,\, -a\,,\, -a\sqrt{\frac{ar^3-1}{ar}}\right)$ with $r = \frac{1}{2\beta}\left(\gamma \pm \sqrt{\gamma^2 - 4\beta}\right)$ and $a^3 = \frac{r\beta^2}{4(\gamma r - 2)}$.

iii) The case $\delta = 0$ is similar.

iv) For $\alpha = \delta = 0$ and if $\varepsilon = \frac{\gamma^{2/3}}{2^{2/3}\beta^{1/3}}$ and $\eta = \frac{2^{1/3}\beta^{2/3}}{\gamma^{1/3}}$ we get the six points $(0\,,\, -\varepsilon\,,\, \pm\eta\,,\, \pm\varepsilon)$, $(-\frac{\eta}{2}\,,\, 0\,,\, \pm\frac{\eta}{2}\,,\, \pm 2\varepsilon)$, $(2\eta\,,\, \varepsilon\,,\, \pm 2\eta\,,\, \pm\varepsilon)$.

Hence $h| : \mathcal{U} \longrightarrow \mathcal{O}_3$ is a six to one local diffeomorphism, hence a six to one fibration.

b) It is easy to prove. $\square$

Consider the map g defined above.

For $a \neq 0$ its clear that the kernel Dg is generated by $v_1 = \left(1, 0, \frac{-2c}{a}, 0\right)$ and $v_2 = \left(0, 1, 0, \frac{-2c}{a}\right)$. A linear combination $v_3 = \alpha v_1 + \beta v_2$ is tangent to the sphere if $a^2 - 2c^2 = 0$ or $a\alpha + b\beta = 0$. For the first case we get $P = \left(a, b, \frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ and $v_3 = (-b, a, \sqrt{2}b, -\sqrt{2}a)$. The curve $\gamma(t) = g(p + tv_3)$ is of the form:

$$\Big(\sqrt{2}b^3t^3 - \tfrac{3}{\sqrt{2}}\,ab^2t^2 + \tfrac{a^3}{\sqrt{2}},\ -3\sqrt{2}ab^2t^3 + 3\sqrt{2}(a^2b - b^3)t^2 + \tfrac{3}{\sqrt{2}}\,a^2b,\ 3\sqrt{2}ab^2t^3 \ + \\ + 3\sqrt{2}(ab^2 - a^3) + \tfrac{3}{\sqrt{2}}\,a^2b,\ -\sqrt{2}a^3t^3 - \tfrac{3}{\sqrt{2}}\,a^2bt^2 + \tfrac{b^3}{\sqrt{2}}\Big)\,.$$

In matrix notation

$$\begin{pmatrix} \sqrt{2}b^3 & -\frac{3}{\sqrt{2}}\,ab^2t & & \frac{a^3}{\sqrt{2}} \\ -3\sqrt{2}ab^2 & 3\sqrt{2}(a^2b - b^3) & & \frac{3}{\sqrt{2}}\,a^2b \\ 3\sqrt{2}a^2b & 3\sqrt{2}(ab^2 - a^3) & e_i & \frac{3}{\sqrt{2}}\,ab^2 \\ -\sqrt{2}a^3 & -\frac{3}{\sqrt{2}}\,a^2b & & \frac{b^3}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} t^3 \\ t^2 \\ 0 \\ 1 \end{pmatrix}$$

Where e_i is the canonical vector in $\mathbb{R}^4$ with 1 in the i-th place and zero in the others. We now consider:

i) for $b \neq 0$, the determinant for e_1 is $\sqrt{2}\,a^2b(3a^4 + 2a^2)$,

ii) for $b = 0$, the determinant for e_3 is also different of zero.

Hence $a^2 = 2c^2$ in $S^3 \cap \{ad - bc = 0\}$ are simple cusps for $g|$. In this case our curve is $a^2 = 2c^2$, $b^2 = 2d^2$ since $d = \frac{bc}{a}$. We now curve study carefully Dg on the tangent space of $S^3 \cap \{ad - bc = 0\}$, that is to say in the 2-dimensional vector space generated by the vectors $v_1 = (-c, -d, a, b)$ and $v_2 = (-b, a, -d, c)$.

Proposition 2. *Consider g as before, hence $Dg(a,b,c,d)(v_1)$ is parallel to $Dg(a,b,c,d)(v_2)$ if and only if (a,b,c,d) is of the form i) $(\cos\theta, \sin\theta, 0, 0)$, ii) $\left(\pm\frac{\sqrt{2}}{3}\cos\theta\,,\, \pm\frac{\sqrt{2}}{3}\sin\theta\,,\, \frac{\sqrt{1}}{3}\cos\theta\,,\, \frac{\sqrt{1}}{3}\sin\theta\right)$, iii) $(0\,,\,0\,,\,\cos\theta\,,\,\sin\theta)$.*

Proof

We get a 2×4 matrix

$$\begin{pmatrix} a^3 - 2ac^2 & 3a^2b - 6bc^2 & 3ab^2 - 6bcd & b^3 - 2bd^2 \\ -3abc & 3ac^2 - 6b^2c & 6abc - 3b^2d & 3b^2c \end{pmatrix}$$

It is clear that the minor M_{14} is zero if and only if $a = 0$, $b = 0$, $c = 0$ or $2c^2 - a^2 + 2d^2 + b^2 = 0$

i) $a = 0$, from $ad = bc$ we get $b = 0$ or $c = 0$. The case $a = b = 0$ is case iii) and $a = c = 0$ the matrix has rank 1 if and only if $b = 0$ or $d = 0$ or $b^2 = 2d^2$. These are included in our list.

ii) $b = 0$, hence $a = 0$ or $c = 0$. The case $b = c = 0$ is included in our list.

iii) $c = 0$, hence $a = 0$ or $d = 0$. The case $c = d = 0$ is included in our list.

iv) From $2c^2 - a^2 + 2d^2 + b^2 = 0$, we get $a^2 + b^2 = \frac{2}{3}$ and $c^2 + d^2 = \frac{1}{3}$.

Hence we only have to study the singularities $S^1 \times \{0\}$. These under the equivalence $(a, b, c, d) \sim (-a, -b, c, d)$ they are no longer singularities. □

Theorem 3. *Consider the torus* $T_2 = S^3 \cap \{ad - bc = 0\}$ *and the map* $g| : T_2 - \{(0, 0, \cos\theta, \sin\theta)\} \longrightarrow \mathcal{O}_1$. *Hence g is a one to one immersion except at the points of the form:* $\left(\frac{\sqrt{2}}{3}\cos\theta\,,\ \frac{\sqrt{2}}{3}\sin\theta\,,\ \frac{\sqrt{1}}{3}\cos\theta\,,\ \frac{\sqrt{2}}{3}\sin\theta\right)$ *which are simple cusps.*

Proof

The proof its already done in the previous proposition. We just have to notice that the two curves of simple cusps become one where $g|$ is one to one.

For the case $S^1 \times \{0\}$, which under our equivalence relation is $\mathbb{R}P^1 \times \{0\}$, we get that $\ker Dg$ is generated by $\langle e_3, e_4 \rangle$. For $a \neq 0$, let $v_3 = e_3$ and $g((a, b, 0, 0) + t(0, 0, 1, 0))$ is of the form: $(0, a^2t, 2abt, b^2t)$. □

We now return to the map h and we have the following

Proposition 4. *Consider $h|$ restricted to our torus, hence for the vector $v_1 = (-c, -d, a, b)$ and $v_2 = (-b, a, -d, c)$ we get that $Dh|(a, b, c, d)(v_1)$ is parallel to $Dh(a, b, c, d)(v_2)$ if and only if (a, b, c, d) is a point of the form*

i) $\left(\sqrt{\frac{1}{6}}\cos\theta\,,\ \sqrt{\frac{1}{6}}\sin\theta\,,\ \pm\sqrt{\frac{5}{6}}\cos\theta\,,\ \pm\sqrt{\frac{5}{6}}\sin\theta\right)$.

ii) $\left(\sqrt{\frac{1}{2}}\cos\theta\,,\ \sqrt{\frac{1}{2}}\sin\theta\,,\ \pm\sqrt{\frac{5}{6}}\cos\theta\,,\ \sqrt{\frac{1}{2}}\sin\theta\right)$.

iii) $S^1 \times \{0\}$.

iv) $\{0\} \times S^1$.

Remark The images of ii) and iv) are $\mathcal{O}_0$.

Proof

For $a = 0$ we get that $bc = 0$ hence

i) $a = 0,\ \ b = 0$ and $Dh(0,0,c,d)(v_1)$ is parallel to $Dh(0,0,c,d)(v_2)$ always since $Dh(0,0,c,d)(v_1) = 0$, this is (iv).

ii) $a = 0,\ \ c = 0$ then $Dh(0,b,0,d)(v_1) = (0,0,0,d(d^2-b^2))$ and $Dh(0,b,0,d)(v_2) = (0,0,-b(b^2-d^2),0)$ hence they are parallel if and only if $b = 0,\ d = 0$ or $b^2 = d^2$, equivalently $a = b = c = 0$, $a = c = d = 0$ or $a = c = 0$ and $b^2 = d^2$ which they are included in our list

For $a \neq 0$ and $d = \frac{bc}{a}$.

We get that $Dh(a,b,c,d)(v_1)$ is parallel to $Dh(a,b,c,d)(v_2)$ if and only if $(-5a^2 + c^2)(ac, 3abc, 3ab^2c, b^3c)$ is parallel to $(a^2-c^2)(-3ab, 3(u^3-2ab^2), 3ab(2a^2-b^2), 3a^2b^2)$, this is equivalent to

a) $c^2 = 5a^2$, which is case i).

b) $a^2 = c^2$, which is case ii).

c) $\operatorname{rank}\begin{pmatrix} ac & 3abc & 3ab^2c & b^3c \\ -3ab & 3(a^3-2ab^2) & 3ab(2a^2-b^2) & 3a^2b^2 \end{pmatrix} \leq 1$

This is equivalent to (a,b,c,d) of the form $S^1 \times \{0\}$ or one in our list. $\square$

Theorem 5. *Consider h restricted to our torus $S^3 \cap \{ad - bc = 0\}$. Hence we get that*

i) The points of the form $\left(\sqrt{\frac{1}{6}}\cos\theta\,,\ \sqrt{\frac{1}{6}}\sin\theta\,,\ \pm\sqrt{\frac{5}{6}}\cos\theta\,,\ \pm\sqrt{\frac{5}{6}}\sin\theta\right)$ *are simple cusps.*

ii) $h|$ sends the points of the form $\left(\sqrt{\frac{1}{2}}\cos\theta,\ \sqrt{\frac{1}{2}}\sin\theta\,,\ \pm\sqrt{\frac{1}{2}}\cos\theta\,,\ \pm\sqrt{\frac{1}{2}}\sin\theta\right)$ *and $\{0\} \times S^1$ to the origin.*

iii) When we make the identification $(a,b,c,d) \sim (-a,-b,c,d)$, $S^1 \times \{0\}$ are no longer singularities.

Proof

We only have to prove i) and iii). For i) consider the points of the form $p = (a, b, \pm\sqrt{5}\,a, \pm\sqrt{5}\,b)$ with $a^2+b^2 = 1/6$. It is clear that $\ker Dh$ in these points is generated by $v_1 = (-\sqrt{5}, 0, 1, 0)$ and $v_2 = (0, -\sqrt{5}, 0, 1)$. For any α, β real $\alpha v_1 + \beta v_2$ are tangent to S^3, hence consider $v_3 = -bv_1 + av_2$ which is transversal to $\{ad - bc = 0\}$ and $h(p + tv_3)$ will be of the form (mod constants)

$$\left(\begin{array}{cccc} 1 & ab^2 & & 1 \\ -3 & b^3 - 2a^2b & & 3 \\ & & e_i & \\ 3 & a^3 - 2ab^2 & & 3 \\ -1 & a^2b & & 1 \end{array} \right) \begin{pmatrix} t^3 \\ t^2 \\ 0 \\ 1 \end{pmatrix}$$

The determinant of this matrix will be $6b(5a^2 - b^2)$ for $i = 1$, $2a(a^2 - 5b^2)$ for $i = 2$, $2b(b^2 - 5a^2)$ for $i = 3$ and $6a(5b^2 - a^2)$ for $i = 4$, hence since $a^2 + b^2 = 1/6$ one has to be non zero.

Consider the points of the form $p = (a, b, 0, 0)$ it is clear that $\ker h$ is generated by e_3 and e_4. If we consider $v_3 = -be_3 + ae_4$, v_3 is transversal to $ad - bc = 0$ and $h((a, b, 0, 0) + t(0, 0, -b, a)) = h(a, b, -bt, at) = (a^3 - ab^2t^2, 3a^2b + (2a^2b - b^3)t^2, 3ab^2 + (2ab^2 - a^3)t^2, b^3 - a^2bt^2)$.

After the identification $(a, b, c, d) \sim (a, b, -c, -d)$ we see that $S^1 \times \{0\}$ are no longer singularities. $\square$

We give the following notation for our special curves for h:

1. $C_1 = S^1 \times \{0\}$, $C_3 = \left\{\left(\sqrt{\tfrac{1}{6}}\cos\theta\,,\ \sqrt{\tfrac{1}{6}}\sin\theta\,,\ \sqrt{\tfrac{5}{6}}\cos\theta\,,\ \sqrt{\tfrac{5}{6}}\sin\theta\right)\right\}$ and $C_5 = \left\{\left(\sqrt{\tfrac{1}{6}}\cos\theta\,,\ \sqrt{\tfrac{1}{6}}\sin\theta\,,\ -\sqrt{\tfrac{5}{6}}\cos\theta\,,\ -\sqrt{\tfrac{5}{6}}\sin\theta\right)\right\}$.

2. $C_2 = \{0\} \times S^1$, $C_4 = \left\{\left(\sqrt{\tfrac{1}{2}}\cos\theta\,,\ \sqrt{\tfrac{1}{2}}\sin\theta\,,\ \sqrt{\tfrac{1}{2}}\cos\theta\,,\ \sqrt{\tfrac{1}{2}}\sin\theta\right)\right\}$ and $C_6 = \left\{\left(\sqrt{\tfrac{1}{2}}\cos\theta\,,\ \sqrt{\tfrac{1}{2}}\sin\theta\,,\ -\sqrt{\tfrac{1}{2}}\cos\theta\,,\ -\sqrt{\tfrac{1}{2}}\sin\theta\right)\right\}$.

Proposition 6. *With the above notation and $R_{=0} = \{ad - bc = 0\}$ we get*

1. *The set $R_{=0} - C_1$ is path-connected in S^3.*

2. *The curves C_i, $1 \le i \le 6$ inside $R = 0$ are homotopic in S^3.*

Proof

For i), pick a point $p = (a,b,c,d)$ in $R_{=0} - C$ and consider $\gamma : [0,1] \longrightarrow R_{=0} - C_1$ defined by $\gamma(t) = \left(ta, tb, \sqrt{\frac{1-t^2(a^2+b^2)}{1-(a^2+b^2)}}\, c\,,\ \sqrt{\frac{1-t^2(a^2+b^2)}{1-(a^2+b^2)}}\, d\right)$, since $a^2+b^2 < 1$, $ad-bc = 0$ and $a^2+b^2+c^2+d^2 = 1$ it is well defined. We get $\gamma(1) = (a,b,c,d)$ and $\gamma(0) \in \{0\} \times S^1$. Hence any point p in $R_{=0} - C_1$ is connected to a point in $\{0\} \times S^1$, which itself is path-connected.

For 2), consider the homotopy $H : [0,\sqrt{6}] \times S^1 \longrightarrow R_{=0}$ defined by $H(t,\theta) = \left(\frac{t}{\sqrt{6}}\cos\theta, \frac{t}{\sqrt{6}}\sin\theta, \sqrt{\frac{6-t^2}{6}}\cos\theta, \sqrt{\frac{6-t^2}{6}}\sin\theta\right)$. H is clear that: $H(0,\theta) = C_2$, $H(1,\theta) = C_3$, $H(\sqrt{3},\theta) = C_4$ and $H(\sqrt{6},\theta) = C_1$. Hence the curves C_1, C_2, C_3, C_4 are homotopic in $R_{=0}$. $\square$

For C_5 and C_6 the proof is similar. We can join the above theorem and proposition to get

Theorem 7. *Consider the map $h| : R_{=0} \cap S^3 \longrightarrow \mathbb{R}^4$, hence*

1. *The images of C_2, C_4 and C_6 are the origin.*

2. *The curves C_i, $1 \le i \le 6$ are homotopic and not contractible in $R_{=0} \cap S^3$.*

3. *The points of the curves C_3 and C_5 are simple cusps.*

We will denote by

i) $R_{>0} = \{(a,b,c,d) \mid ad - bc > 0\}$

ii) $R_{=0} = \{(a,b,c,d) \mid ad - bc = 0\}$

iii) $R_{<0} = \{(a,b,c,d) \mid ad - bc < 0\}$

If we see a point $(a,b,c,d) \in \mathbb{R}^4$ as a 2 by 2 matrix, this regions mean where the determinant is positive, zero and negative respectively.

We get the following

Lemma 8. *Consider the 3-sphere S^3 in $\mathbb{R}^4$, then*

i) $R_{>0} \cap S^3$ and $R_{<0} \cap S^3$ are diffeomorphic.

ii) If we consider the orthogonal linear map $\ell(a,b,c,d) = \sqrt{\frac{1}{2}}(a-d\,, b+c\,, a+d\,, b-c)$ hence ℓ maps $R_{=0} \cap S^3$ to $S_{\sqrt{1/2}} \times S_{\sqrt{1/2}}$.

Proof

For i) consider the map which takes (a, b, c, d) to $(a, b, -c, -d)$.

For ii) is an easy calculation. $\square$

The topology of $R_{>0} \cap S^3$ and $R_{<0} \cap S^3$ comes from the following

Theorem 9. *Let $\phi : R_{>0} \cap S^3 \longrightarrow S^1$ defined by $\phi(a, b, c, d) = \frac{1}{\sqrt{a^2+b^2}}(a, b)$, hence ϕ is a fiber-bundle with fiber an open disc.*

Proof

Let $\mathcal{U} = \phi^{-1}\{(a, b) \in S^1 \mid a > 0\}$ and $\psi : \mathcal{U} \longrightarrow \{(a, b) \in S^1 \mid a > 0\} \times \{(\gamma, \delta, \varepsilon) \in S^2 \mid \gamma > 0, \varepsilon > 0\}$ given by

$$\psi(a, b, c, d) = \left(\left(\frac{a}{\xi}, \frac{b}{\xi}\right), \left(\xi, \frac{1}{\xi}(ac + bd), \frac{1}{\xi}(ad - bc)\right)\right)$$

where $\xi = \sqrt{a^2 + b^2}$. H is clear that ψ is well defined since $\xi > 0$ and $ad - bc > 0$, also

$$\xi^2 + \frac{1}{\xi^2}(ac + bd)^2 + \frac{1}{\xi^2}(ad - bc)^2 = \xi^2 + \frac{1}{\xi^2}(a^2 + b^2)(c^2 + d^2) = \xi^2 + \frac{1}{\xi^2}(\xi^2(1 - \xi^2)) = 1$$

and

$$\psi^{-1}((\alpha, \beta), (\gamma, \delta, \varepsilon)) = (\alpha\gamma, \beta\gamma, \alpha\delta - \beta\varepsilon, p\delta + \alpha\varepsilon).$$ $\square$

3 An algebraic viewpoint

Consider our parametrization h and the following system for θ_{x^3}:

$$\begin{aligned} a^3 - ac^2 &= \alpha^3 \\ 3a^2b - 2acd - bd^2 &= 3\alpha^2\beta \\ 3ab^2 - 2bcd - ad^2 &= 3\alpha\beta^2 \\ b^3 - bd^2 &= \beta^3 \end{aligned} \qquad (*)$$

We solve it in $S^3 \cap \{ad - bc = 0\}$.

If we consider $a \neq 0$, $d = \frac{bc}{a}$ and $b^2 = a^2\left(\frac{1-(a^2+c^2)}{a^2+c^2}\right)$ hence

$$\begin{aligned} (\alpha^2 + \beta^2)^3 &= (a^2 - c^2)^2\left(a^2 + 3b^2 + \frac{3b^4}{a^2} + \frac{b^6}{a^4}\right) \\ &= (a^2 - c^2)^2\frac{1}{a^4}(a^2 + b^2)^3 = \frac{(a^2 - c^2)^2 a^2}{(a^2 + c^2)^3} \end{aligned} \qquad (**)$$

Also for $\alpha \neq 0$, $\beta \neq 0$, a has to satisfy $a^3 - \frac{\alpha^2}{2(\alpha^2+\beta^2)}\ a - \frac{\alpha^3}{2} = 0$ and $\Delta = \frac{\alpha^6}{4(\alpha^2+\beta^2)^3}(2 - 27(\alpha^2+\beta^2)^3)$ hence for $\Delta = 0$ we get from $(**)$: $27a^2(a^2-c^2)^2 = 2(a^2+c^2)^3$ and solutions of the form $a^2 + b^2 = 1/6$ and $a^2 + b^2 = 2/3$.

If $a = 0$ then $\alpha = 0$ and $b^2 + d^2 = 1$.

If $b = 0$ then $\beta = 0$ and $a^2 + c^2 = 1$.

In the previous cases we get $\{(0.\cos\theta, 0, \sin\theta)\}$ and $\{(\cos\theta, 0, \sin\theta, 0)\}$ respectively.

The cases $\left\{\left(\sqrt{\frac{1}{\theta}}\cos\theta\,,\ \sqrt{\frac{1}{2}}\sin\theta\,,\ \pm\sqrt{\frac{1}{2}}\cos\theta\,,\ \pm\sqrt{\frac{1}{2}}\sin\theta\right)\right\}$ and $\{(0,0,\sin\theta,\cos\theta)\}$ represent the cubic zero . In the case $S^1 \times \{0\} = (\sin\theta, \cos\theta, 0, 0)$ we get $\Delta = -25$.

We then have the following

Theorem 10. *Consider the system* $(*)$ *then*

i) The discriminant Δ *vanishes in the curves*

a) $C_i\,,\quad 2 \leq i \leq 6$

b) $C_7 = \left\{\left(\sqrt{\frac{2}{3}}\sin\theta\,,\ \sqrt{\frac{2}{3}}\cos\theta\,,\ \sqrt{\frac{1}{3}}\sin\theta\,,\ \sqrt{\frac{1}{3}}\cos\theta\right)\right\}$

$C_8 = \left\{\left(\sqrt{\frac{2}{3}}\sin\theta\,,\ \sqrt{\frac{2}{3}}\cos\theta\,,\ -\sqrt{\frac{1}{3}}\sin\theta\,,\ -\sqrt{\frac{1}{3}}\cos\theta\right)\right\}$

c) $C_9 = \{(\sin\theta\,,\ 0\,,\ \cos\theta\,,\ 0)\}$

$C_{10} = \{(0\,,\ \sin\theta\,,\ 0\,,\ \cos\theta)\}$

and they are all homotopic in $S^3 \cap \{ad - bc = 0\}$.

ii) The curves C_9 *and* C_{10} *intersect orthogonally each curve* $C_i\,,\ 2 \leq i \leq 8$ *in two points.*

iii) With the orthogonal isomorphism $\ell(a,b,c,d) = \frac{1}{\sqrt{2}}(a-d\,,\ b+c\,,\ a+d\,,\ b-c)$, *the image of the curves* $C_i\,,\ 1 \leq i \leq 8$ *and* $C_i\,,\ 9 \leq i \leq 10$ *in the rectangle* $[-\pi,\pi] \times [-\pi,\pi]$ *have slope 1 and -1 respectively.*

iv) In the above rectangle, the discriminant changes of sign when we cross the corresponding segments of C_7 *and* C_8 *and does not change crossing* C_3 *and* C_5. *The discriminant is not defined at the segments corresponding to* C_2, C_4 *and* C_6.

Proof

For i) consider the map $H : [0,2\pi] \times [0,1] \longrightarrow S^3 \cap \{ad - bc = 0\}$ given by $H(\theta,t) = \frac{1}{t^2+(1-t)^2}(t\cos\theta\,,\ (1-t)\cos\theta\,,\ t\sin\theta\,,\ (1-t)\sin\theta)$. Then $H(\theta,0) = (0, \cos\theta, 0, \sin\theta)$

and $H(\theta,1)=(\cos\theta,0,\sin\theta,0)$. Using our isomorphism ℓ we get $\ell(C_9)=\frac{1}{\sqrt{2}}(\cos\theta,\sin\theta,\cos\theta,-\sin\theta)$ and $\ell(C_1)=\frac{1}{\sqrt{2}}(\cos\theta,\sin\theta,\cos\theta,\sin\theta)$

Define

$$F(t,\theta)=\frac{1}{\sqrt{2}}(\cos\theta,\sin\theta,\cos 2t\theta\cos\theta+\sin 2t\theta\sin\theta\,,\,-\sin 2t\theta\cos\theta+\cos 2t\theta\ \sin\theta)\,.$$

It is clear that $F(t,\theta)\in S_{1/\sqrt{2}}\times S_{1/\sqrt{2}}$ and that $F(t,\theta)$ comes from an orthogonal 4×4 matrix applied to $(\cos\theta,\sin\theta,\cos\theta,\sin\theta)$ for each t fixed. Also $F(0,\theta)=\frac{1}{\sqrt{2}}(\cos\theta,\sin\theta,\cos\theta,\sin\theta)$ and $F(1,\theta)=\frac{1}{\sqrt{2}}(\cos\theta,\sin\theta,\cos\theta,-\sin\theta)$. Hence $\ell(C_1)$ is homotopic to $\ell(C_9)$ in $S_{1/\sqrt{2}}\times S_{1/\sqrt{2}}$ and so are C_1 and C_9 in $S^3\cap\{ad-bc=0\}$.

To prove (ii) $\to$ (iv) we note that

I) $\ell(\cos\theta,0,\sin\theta,0)=\sqrt{\frac{1}{2}}(\cos\theta,\sin\theta,\cos\theta,-\sin\theta)=(z,\bar{z})$

II) $\ell(0,\sin\theta,0,\cos\theta)=\sqrt{\frac{1}{2}}(-\cos\theta,\sin\theta,\cos\theta,\sin\theta)=(z,-\bar{z})$

III) $\ell\left(\frac{1}{\sqrt{6}}\cos\theta,\frac{1}{\sqrt{6}}\sin\theta,\sqrt{\frac{5}{6}}\cos\theta,\sqrt{\frac{5}{6}}\sin\theta\right)=$
$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{6}}(\cos\theta-\sqrt{5}\sin\theta)\right),\frac{1}{\sqrt{6}}(\sin\theta+\sqrt{5}\cos\theta),\frac{1}{\sqrt{6}}(\cos\theta+\sqrt{5}\sin\theta),\frac{1}{\sqrt{6}}(\sin\theta-\sqrt{5}\cos\theta)$
hence we get that if $z=(\cos\theta,\sin\theta)$ and A is the orthogonal linear transformation given by the first two entries, this set is of the form the graph of B, where $B=(A^t)^2$

IV) $\ell\left(\frac{\cos\theta}{\sqrt{6}},\frac{\sin\theta}{\sqrt{6}},-\sqrt{\frac{5}{6}}\cos\theta,-\sqrt{\frac{5}{6}}\sin\theta\right)$, V) $\ell\left(\sqrt{\frac{2}{3}}\cos\theta\,,\,\sqrt{\frac{2}{3}}\sin\theta\,,\,\sqrt{\frac{1}{3}}\cos\theta\,,\,\sqrt{\frac{1}{3}}\sin\theta\right)$ and VI) $\ell\left(\sqrt{\frac{2}{3}}\cos\theta\,,\,\sqrt{\frac{2}{3}}\sin\theta\,,\,-\sqrt{\frac{1}{3}}\cos\theta\,,\,-\sqrt{\frac{1}{3}}\sin\theta\right)$ are similar to the case III), for a suitable B.

For VII) $\ell(0,0,\cos\theta,\sin\theta)$, VIII) $\ell\left(\frac{\cos\theta}{\sqrt{2}},\frac{\sin\theta}{\sqrt{2}},\frac{\cos\theta}{\sqrt{2}},\frac{\sin\theta}{\sqrt{2}}\right)$ and IX) $\ell\left(\frac{\cos\theta}{\sqrt{2}},\frac{\sin\theta}{\sqrt{2}},-\frac{\cos\theta}{\sqrt{2}},-\frac{\sin\theta}{\sqrt{2}}\right)$, we get the graphs of $z\to -z$, $z\to iz$ and $z\to -iz$ respectively.

The cases I) $\to$ IX) correspond to the graphs in $[-\pi,\pi]\times[-\pi,\pi]$ (in (θ,φ)).

I) $\theta\to-\theta$, II) $\theta\to-\theta+\pi$, III) $\theta\to\theta+\theta_0$ with $\tan\theta_0=\sqrt{\frac{5}{2}}$ and $\theta_0\in(-\pi/2\,,\,\pi)$,

IV) $\theta\to\theta+\theta_0$ with $\tan\theta_0=-\sqrt{\frac{5}{2}}$ and $\theta_0\in(\pi/2\,,\,\pi)$,

V) $\theta\to\theta+\theta_0$ with $\tan\theta_0=-2\sqrt{2}$ and $\theta_0\in(-\pi/2\,,\,0)$,

VI) $\theta\to\theta+\theta_0$ with $\tan\theta_0=2\sqrt{2}$ and $\theta_0\in(0\,,\,\pi/2)$,

VII) $\theta\to\theta+\pi$, VIII) $\theta\to\theta+\pi/2$ and IX) $\theta\to\theta-\pi/2$.

Finally for iv) we have the following

Lemma 11. *Considering the 2^d quadrant of the rectangle $[-\pi,\pi]\times[-\pi,\pi]$ in coordinates (θ,φ), the change of sign od Δ in $(0,\varphi)$ occurs when $\varphi = 2\tan^{-1}\frac{1}{\sqrt{2}}$ and in $\varphi = \pi - 2\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$ and we do not have change of sign in $\varphi = \pi - 2\tan^{-1}\sqrt{5}$ and in $2\tan^{-1}\sqrt{5}$.*

Proof

In the case $\beta = 0$ our discriminant becomes $\Delta_1 = \frac{1}{4}(2-27\alpha^6)$ from the cubic $a^3 - \frac{a}{2} - \frac{\alpha^3}{2} = 0$, which has double root for $\alpha = \pm\frac{2^{1/6}}{3^{1/2}}$, we get:

i) for $a^3 - \frac{a}{2} - \frac{1}{2^{1/2}3^{3/2}}$ we get a local maximum equal to zero at $-\sqrt{\frac{1}{6}}$ and a local minimum at $\frac{1}{\sqrt{6}}$ and the value zero at $\frac{2}{\sqrt{6}}$, that is to say, the graph non-positive in $(-\infty, \frac{2}{\sqrt{6}})$ and positive in $(\frac{2}{\sqrt{6}}, \infty)$

ii) for $a^3 - \frac{a}{2} + \frac{1}{2^{1/2}3^{3/2}}$ we get a local maximum at $-\sqrt{\frac{1}{6}}$ and a local minimum at $\sqrt{\frac{1}{6}}$ equal to zero and the value zero at $-\frac{2}{\sqrt{6}}$, that is to say, the graph non-negative in $\left(-\infty, \sqrt{\frac{1}{6}}\right)$ and positive in $\left(\sqrt{\frac{1}{6}}, \infty\right)$.

Hence we have change of sign in the case i) at $\frac{2}{\sqrt{6}}$ but we do not have it in $-\frac{1}{\sqrt{6}}$. In the case ii) we have change of sign in $-\frac{1}{\sqrt{6}}$. In the case ii) we have change of sign in $-\frac{2}{\sqrt{6}}$ lent we don't have it in $\frac{1}{\sqrt{6}}$.

For the case i) we have

i-1) The point $p = \left(\frac{2}{\sqrt{6}}, 0, \frac{1}{\sqrt{3}}, 0\right)$ and $\ell(p) = \frac{1}{\sqrt{2}}\left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{3}}\right)$

i-2) The point $p = \left(\frac{2}{\sqrt{6}}, 0, -\frac{1}{\sqrt{3}}, 0\right)$ and $\ell(p) = \frac{1}{\sqrt{2}}\left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{3}}\right)$

i-3) The point $p = \left(-\frac{1}{\sqrt{6}}, 0, \sqrt{\frac{5}{6}}, 0\right)$ and $\ell(p) = \frac{1}{\sqrt{2}}\left(-\sqrt{\frac{1}{6}}, \sqrt{\frac{5}{6}}, -\sqrt{\frac{1}{6}}, -\sqrt{\frac{5}{6}}\right)$

i-4) The point $p = \left(-\frac{1}{\sqrt{6}}, 0, -\sqrt{\frac{5}{6}}, 0\right)$ and $\ell(p) = \frac{1}{\sqrt{2}}\left(-\sqrt{\frac{1}{6}}, -\sqrt{\frac{5}{6}}, -\sqrt{\frac{1}{6}}, \sqrt{\frac{5}{6}}\right)$

For the case ii) we have

ii-1) The point $p = \left(-\frac{2}{\sqrt{6}}, 0, \frac{1}{\sqrt{3}}, 0\right)$ and $\ell(p) = \frac{1}{\sqrt{2}}\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{3}}\right)$

ii-2) The point $p = \left(-\frac{2}{\sqrt{6}}, 0, -\frac{1}{\sqrt{3}}, 0\right)$ and $\ell(p) = \frac{1}{\sqrt{2}}\left(-\frac{2}{\sqrt{6}}, -\sqrt{\frac{1}{3}}, -\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{3}}\right)$

ii-3) The point $p = \left(\frac{1}{\sqrt{6}}, 0, \sqrt{\frac{5}{6}}, 0\right)$ and $\ell(p) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{6}}, \sqrt{\frac{5}{6}}, \sqrt{\frac{1}{6}}, -\sqrt{\frac{5}{6}}\right)$

ii-4) The point $p = \left(\frac{1}{\sqrt{6}}, 0, -\sqrt{\frac{5}{6}}, 0\right)$ and $\ell(p) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{6}}, -\sqrt{\frac{5}{6}}, \sqrt{\frac{1}{6}}, \sqrt{\frac{5}{6}}\right)$

Its clear that in cases i) and ii), the first and the third are not in the 2^d-quadrant, also the second of each case changes of sign and the fourth does not. $\square$

References

[BKL] Bulajich R., Kushner L. and López de Mediano S., "Cubics in $\mathbb{R}$ and $\mathbb{C}$". Proceedings of the 6^{th} Workshop on Real and Complex Singularities. Lecture Notes in Pure and Applied Mathematics, by Marcell Decker Inc., 2003.

[Z] Zeeman, E.C., "The Umbilic Brazalet and the Double Cusp Catastrophe, in Catastrophe Theory", Select Paper 1972-77, Addison-Wesley, 1977, pp. 563-601.

León Kushner

Departamento de Matemáticas
Facultad de Ciencias
UNAM
México D.F. 04510 - MEXICO

and

Departamento de Matemática
Instituto de Ciências Matemáticas de São Carlos
Universidade de São Paulo - Campus de São Carlos
Caixa Postal, 668
13560-970 - São Carlos - SP

NOTAS DO ICMC

SÉRIE MATEMÁTICA

208/2004 CARBINATTO, M.C. ; RYBAKOWSKI, K.P. – Homology index braids in infinite-dimensional Conley index theory.

207/2004 PÉREZ, V.H.; RIZZIOLLI, E.C.; SAIA, M.J. – Whitney equisingularity and invariants of map germs from C^n to C^3, n>3.

206/2004 PÁDUA, J.C.; SILVA, E.A.B.; SOARES, S.H.M. – Positive solutions of critical semilinear problems involving a sublinear term at the origin.

205/2004 AROCA, F. – Valuations compatible with a projection.

204/2004 SNOUSSI, J. – The Nash modification and hiperplane sections on surface.

203/2004 HERNÁNDEZ, M., E. ; PELICER, M. – Asymptotically almost periodic and almost periodic solutions for partial neutral differential equations.

202/2004 MENEGATTO, V.A.; OLIVEIRA, C.P.; PERON, A.P. – Strictly positive definite Kernels on subsets of the complex plane.

201/2004 NASCIMENTO, A.S.; CREMA, J. – On role of the equal-area condition in internal transition layer stationary solutions to a class of reaction-diffusion systems.

200/2004 PRILEPKO, A.; PISCAREV, S.; SHAW, S.Y – Control element approximation for parabolic differential equations with final observation.

199/2004 OLIVEIRA, R.D.S.; TEIXEIRA, M.A. – On pairs of polynomial planar foliations.