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Images and varieties

J. W. BRUCE

W. L. MARAR

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J.W.Bruce and W.L.Marar

1 Introduction

Geometric objects in real or complex affine space are often described in two ways: in parametrised form, as the image of a mapping, and in implicit form as the set of zeros of some mapping. The connection between the two, in the case of a smooth manifold, is given by the implicit function theorem. When the object is singular (as images very often are) the interplay between the two representations is more interesting. In this paper we investigate the defining equations for the images of smooth mappings $\mathbf{C}^n, 0 \to \mathbf{C}^{n+1}, 0$. We recover some results of Piene ([14]), using an elementary lemma from linear algebra, and give some relevant examples from singularity theory. The authors are very grateful to David Mond for various helpful comments, and in particular for his assistance with Proposition 2.6 below. The second author is grateful to CNPq for financial support.

2 Maps $\mathbf{C}^n, 0 \to \mathbf{C}^{n+1}, 0$, $n > 1$.

It is usual in singularity theory to consider finite maps. For maps $\mathbf{C}^n, 0 \to \mathbf{C}^{n+1}, 0$ with $n \geq 2$ it is better to consider a certain subset.

Definition 2.1 A holomorphic map $f : \mathbf{C}^n, 0 \to \mathbf{C}^{n+1}, 0$ is *GIGIF* if it is finite, and the following holds true:

(i) for a proper analytic subset Σ of the source we have $f : \mathbf{C}^n \setminus \Sigma, 0 \to \mathbf{C}^{n+1}, 0$ injective (f is generically injective),

(ii) the set of critical points of f has codimension ≥ 2 (f is generically immersive).

Remark *Genericity.*

One can prove that if $n \geq 2$ all map-germs $\mathbf{C}^n, 0 \to \mathbf{C}^{n+1}, 0$ off a set of infinite codimension are GIGIF. The second condition can be deduced as follows. Let $J^1(n,n+1)$ be the set of 1-jets of maps $\mathbf{C}^n,0 \to \mathbf{C}^{n+1},0$; we can think of this as the set of $(n+1) \times n$ matrices. Inside this set we have the matrices of rank r which we denote by Σ^r; these are smooth submanifolds of $J^1(n,n+1)$, see [2]. Now for almost all maps $f : \mathbf{C}^n,0 \to \mathbf{C}^{n+1},0$ (all f off a set of infinite codimension) the jet-extension map $j^1f : \mathbf{C}^n,0 \to J^1(n,n+1)$ will be transverse to the manifolds Σ^r away from 0. But Σ^r has codimension $(n-r+1)(n-r) \geq 2$ for $r \geq 2$, so almost all f *are* immersions off a subset of codimension 2. The generic injective condition is proved in the same way; see [3].

We start our investigations proper with an elementary result from linear algebra.

Proposition 2.2 *Let B be an $n \times (n+1)$ matrix of maximal rank. Then its kernel is spanned by the vector $(\Delta_1, -\Delta_2, \ldots, (-1)^n \Delta_{n+1})$, where Δ_i is the minor obtained by deleting the i^{th} column from B.*

Proof Form an $(n+1) \times (n+1)$ matrix B_j by adding an initial row to B which is simply the j^{th} row of B. Clearly the resulting matrix is singular. Expanding the determinant of this matrix by the first row we find that $\sum(-1)^i b_{ij}\Delta_i = 0$ where Δ_i is the minor obtained by deleting the i^{th} column. Now since B has rank n one of these minors is non-zero, and so the given vector spans the kernel of the mapping. (This is just Cramer's rule.)

We denote the column vector $(\Delta_1, -\Delta_2, \ldots, (-1)^n \Delta_{n+1})^t$ by $\Delta(B)$, and derive some properties we will need later.

Proposition 2.3 *Let B be an $n \times (n+1)$, A an $(n+1) \times (n+1)$ and C an $n \times n$ matrix. Then $A(\Delta(BA)) = \det A\Delta(B)$, and $\Delta(CB) = \det C\Delta(B)$.*

Proof We first remark that the column vector $\Delta(B)$ is characterised by the fact that $u\Delta(B) = \det(\frac{u}{B})$ for all row vectors u with $n+1$ entries. One computes $u(A\Delta(BA)) = uA\Delta(BA) = \det(\frac{uA}{BA}) = \det((\frac{u}{B})A) = \det(\frac{u}{B})\det A = u\Delta(B)\det A$. For the second result let C^* denote the $(n+1) \times (n+1)$ matrix with C as its bottom right $n \times n$ block and a single additional non-zero entry 1 at the $(1,1)$ place. Then $u\Delta(CB) = \det(\frac{u}{CB}) = \det(C^*(\frac{u}{B})) = \det C^* \det(\frac{u}{B}) = u\Delta(B)\det C$.

Proposition 2.4 *Let $f : \mathbf{C}^n,0 \to \mathbf{C}^{n+1},0$ be a finite mapping, so that the image of f is a hypersurface with defining equation $F : \mathbf{C}^{n+1},0 \to \mathbf{C},0$ say. Suppose further that f is an immersion off a subset of $\mathbf{C}^n$ of codimension 2. Then for some holomorphic $\lambda : \mathbf{C}^n,0 \to \mathbf{C}$ we have $\lambda\Delta_i = \pm\partial F/\partial x_i(f)$*

where Δ_i is the determinant of the matrix obtained by deleting the i^{th} row from the Jacobian matrix of f.

Proof Let $F(x_1, \ldots, x_{n+1}) = 0$ be a reduced defining equation for the image $f(\mathbf{C}^n)$, so $F \circ f$ is identically zero. Differentiating we find that

$$\sum_{i=1}^{n+1} \partial F/\partial x_i(f(u))\partial f_i/\partial u_j(u) \equiv 0.$$

Consequently the column vector $(\partial F/\partial x_1, \ldots, \partial F/\partial x_{n+1})^t$ lies in the kernel of the transpose of the Jacobian matrix of f. However we can deduce from the above result that at points of the domain of f where it is an immersion the column vector $(\Delta_1, -\Delta_2, \ldots, (-1)^n\Delta_{n+1})^t$ spans the kernel of the transpose of the Jacobian matrix of f. So on this set we can find a holomorphic function λ with

$$(\partial F/\partial x_1, \ldots, \partial F/\partial x_{n+1}) = \lambda(\Delta_1, -\Delta_2, \ldots, (-1)^n\Delta_{n+1}).$$

We can now apply Hartogs' Theorem [6] to deduce that λ extends to a holomorphic function defined on the source.

Note that we can run into problems when f fails to be generically injective.

Non-example.

Consider the map $f : \mathbf{C}^2, 0 \to \mathbf{C}^3, 0$ defined by $f(u_1, u_2) = (u_1^2, u_2, 0)$ so $F = x_3$ and $\partial F/\partial x_3 = 1$, while the corresponding minor is $2u_1$, so λ is meromorphic. Clearly this is rather an artificial counterexample, for the defining equation for the image of f should be $x_3^2 = 0$ not $x_3 = 0$.

Of course the above expression for λ is co-ordinate dependent. We next show how it is altered by a change of co-ordinates.

Proposition 2.5 *Let $\psi : \mathbf{C}^n, 0 \to \mathbf{C}^n, 0$, $\phi : \mathbf{C}^{n+1}, 0 \to \mathbf{C}^{n+1}, 0$ be germs of diffeomorphisms, $\alpha : \mathbf{C}^{n+1}, 0 \to \mathbf{C}, 0$ a non-zero function. Then replacing f, F by $\phi \circ f$, $F \circ \phi^{-1}$ (respectively f by $f \circ \psi$ or F by αF) we replace λ by $(\det(d\phi)) \circ f\lambda$ (respectively $\det(d\psi)\lambda \circ \psi$ and $(\alpha \circ f)\lambda$).*

Proof These all follow from the chain rule and Proposition 2.3.

We next turn to the singularities of the image of f.

Proposition 2.6 *Let $f : \mathbf{C}^n, 0 \to \mathbf{C}^{n+1}, 0$ be a finite germ, with $n \geq 2$, which is an injective immersion off a set of codimension 2. Then f is an immersion.*

Proof Any finite $f : \mathbf{C}^n, 0 \to \mathbf{C}^{n+1}, 0$ yields (in the terminology of [8]) an analytic cover. Moreover our hypotheses imply that it is one-sheeted; indeed this would follow from the fact that f is generically injective. As a consequence the map f is a normalisation of its image. On the other hand the image is a hypersurface, and hence Cohen-Macaulay, and this together with the fact that it is smooth in codimension 1 implies that the image is also normal by Serre's S_2 criterion (see [10] Theorem 23.8).

An alternative proof can be deduced from the work of Mond and Pellikaan. In [12] they consider a presentation matrix of the push forward $f^*(\mathcal{O}_n)$, and prove that the first Fitting ideal of $f^*(\mathcal{O}_n)$ is generated by the minors of the matrix obtained by deleting the row in this matrix corresponding to the generator 1. Now by a standard result of Eagon-Northcott ([10] page 103) the codimension of the support of the first Fitting ideal can be no greater than 2 in the target. So the singular locus of the image is of dimension at least $n-1$, unless the map is an isomorphism onto its image, which given the hypotheses must be the case.

Definition 2.7 *Let f be finite. We let $D^2(f)$ denote the double point set of f, that is the closure of the set of points u_1 in the source for which there is some $u_2 \neq u_1$ with $f(u_1) = f(u_2)$.*

One consequence of the result above is the following:

Proposition 2.8 *(i) Let $f : \mathbf{C}^n, 0 \to \mathbf{C}^{n+1}, 0$ be finite, and an immersion off a set of codimension 2 (again $n \geq 2$). Every critical point of f is in the closure of the double point set of f.*

(ii) Let $f : \mathbf{C}^n, 0 \to \mathbf{C}^{n+1}, 0$ be finite and an immersion off a set of codimension 2, and set $X = f^{-1}(Sing(Im\ f))$, that is, the inverse image of the singular set of the image hypersurface. Then $X = D^2(f)$ and is the set $\lambda = 0$.

Proof (i) Suppose this was not the case. Then we could find such an f with $0 \in \mathbf{C}^n$ a critical point not in the closure of the double point set. But this would imply that f was an injective immersion off a set of codimension 2, and so by the previous result that f is an immersion, a contradiction.
(ii) The fact that $X = D^2(f)$ follows from (i). Clearly if $\lambda(u) = 0$ then

grad $F(f(u))$ vanishes, by the previous result; that is $u \in X$. Suppose, on the other hand, that u is an immersive point for f with $f(u) = x \in f(X)$ say. Then there are one or more distinct points u' in the source with $f(u') = x$, and so near u the set X is a hypersurface. Now since $\partial F/\partial x_i(f(u)) = 0$ we deduce that $\lambda(u)\Delta_i(u) = 0$. But one of the Δ_i does not vanish, so λ vanishes on X near u. In other words λ vanishes on a dense subset of X, and hence vanishes on X.

Proposition 2.9 *If f fails to be an immersion with normal crossings on a set of codimension ≥ 2 then $\lambda = 0$ is a reduced defining equation for X.*

Proof Let u be a generic point of $D^2(f)$, so that for some (unique) u' we have $f^{-1}(f(u)) = \{u, u'\}$ and f is immersive at u and u'. At $p = f(u) = f(u')$ the set $\{F = 0\}$ is diffeomorphic to the germ $\{x_1x_2 = 0\}$. Indeed we may replace u, u' by $0 \in \mathbf{C}^n$, f at u by $(u_1, \ldots, u_n) \mapsto (0, u_1, \ldots, u_n)$, f at u' by $(u_1, \ldots, u_n) \mapsto (u_1, 0, u_2, \ldots, u_n)$, and F by x_1x_2. (See Proposition 2.5 above.) Calculation now shows that at u_1 the function $\lambda(u) = u_1$. Clearly $u_1 = 0$ is a reduced defining equation for$D^2(f)$ near u_1 and the result follows.

Proposition 2.10 *Assuming $n \geq 2$ a germ $f : \mathbf{C}^n, 0 \to \mathbf{C}^{n+1}, 0$ which is finitely-$\mathcal{A}$-determined fails to be an immersion with normal crossings on a set of codimension ≥ 2. This property also holds true for a set of germs whose complement is of infinite codimension.*

Proof The fact that finitely $\mathcal{A}$-determined germs are immersions off a subset of codimension 2 follows from the argument given after 2.2, and Gaffney's geometric characterisation of finite -$\mathcal{A}$-determinacy, discussed in [16]. The fact that we have normal crossings off a set of codimension 2 follows from similar arguments involving multi-jet spaces. One simply checks that the pairs of 1-jets of immersions with the same target, and non-transverse images have codimension ≥ 2 (resp. that triples of 0-jets with the same target have codimension ≥ 2). The second result follows from theorems in the paper [7] (see also [3]).

Remark 2.11 We shall produce a modified version of Proposition 2.9 for a wider class of germs in §4.

Example 2.12 *Maps* $\mathbf{C}^2, 0 \to \mathbf{C}^3, 0$.

Given a map-germ $\mathbf{C}^n, 0 \to \mathbf{C}^{n+1}, 0$ one can use the algorithm of Mond and Pellikan to determine the corresponding defining equation for the image (see [12]). We provide an alternative in the case when $n = 2$. So suppose $f(x,y) = (X(x,y), Y(x,y), Z(x,y))$ and denote the defining equation by $F(X,Y,Z) = 0$. We shall suppose that when we compose f with the projection to the (X,Y)-plane we obtain a finite map-germ. This will have multiplicity d say, where $d = dim_{\mathbb{C}}\mathcal{O}_{x,y}/\langle X(x,y), Y(x,y)\rangle$. It follows that $F(0,0,Z)$ has initial part cZ^d for some $c \neq 0$, and this is what we shall use below. We can deduce from our results above that

$$\lambda(x,y) = \partial F/\partial Z(f(x,y))/|\partial(X,Y)/\partial(x,y)|$$

$$= -\partial F/\partial Y(f(x,y))/|\partial(X,Z)/\partial(x,y)| = \partial F/\partial X(f(x,y))/|\partial(Y,Z)/\partial(x,y)| = 0$$

is the defining equation of the double point curve $D^2(f)$.

(1) $f(x,y) = (x,y,g(x,y))$.

Here $F = Z - G(X,Y)$ and $D^2(f) = \emptyset$.

(2) $f(x,y) = (x,y^2,g(x,y))$.

After changes of co-ordinates in the target we may suppose that g is in the form $yp(x,y^2)$. So $F(X,Y,Z) = Z^2 - Yp(X,Y)^2$, and $\lambda(x,y) = 2yp(x,y^2)/2y = p(x,y^2)$.

(3) $f(x,y) = (x,y^3,g(x,y))$.

After changes of co-ordinates in the target we may suppose that g is in the form

$$g(x,y) = yp_1(x,y^3) + y^2p_2(x,y^2).$$

It follows that

$$Z^3 = Yp_1(X,Y)^3 + 3yYp_1^2p_2 + 3y^2Yp_1p_2^2 + Y^2p_2^3$$
$$= Yp_1^3 + 3Yp_1p_2(yp_1 + y^2p_2) + Y^2p_2^3.$$

So,

$$F(X,Y,Z) = Z^3 - 3ZYp_1p_2 - Yp_1^3 - Y^2p_2^3,$$

where $p_i = p_i(X,Y)$. Then $\partial F/\partial Z = 3Z^2 - 3Yp_1p_2$ and

$$\lambda(x,y) = \partial F/\partial Z(f(x,y))/|\partial(X,Y)/\partial(x,y)| = (3(yp_1+y^2p_2)^2 - 3y^3p_1p_2)/3y^2,$$

ie

$$\lambda(x,y) = p_1(x,y^3)^2 + yp_1(x,y^3)p_2(x,y^3) + y^2p_2(x,y^3)^2.$$

In the particular case when $f(x,y) = (x,y^3,x^ry + y^{3s-1})$ we deduce that $\lambda(x,y) = x^{2r} + x^ry^{3s-2} + y^{6s-4}$. Note that in this case λ has an isolated singularity (so f is finitely $\mathcal{A}$-determined by [11]), and it is weighted homogeneous. Using the Milnor/Orlik formula ([9]) we deduce that λ has Milnor number $(6s-5)(2r-1)$. (Quite generally if a germ $f: \mathbf{C}^n, 0 \to \mathbf{C}^{n+1}, 0$ is weighted homogeneous when weights $w_1, \ldots, w_n$ are assigned to the variables $u_1, \ldots, u_n$, and the weight of the i^{th} component is d_i, then the defining equation for the image is also weighted homogeneous of degree d say, and the defining equation of the double point curve is then weighted homogeneous of degree $d - \sum_{i=1}^{n+1} d_i + \sum_{i=1}^{n} w_i$. In this example the degree d can be deduced merely using the fact that F contains a Z^3 term.)

(4) $f(x,y) = (x, xy+y^3, g(x,y))$.

After a change of co-ordinates in the target we can reduce f to the form

$$f(x,y) = (x, xy+y^3, yp_1(x, xy+y^3) + y^2 p_2(x, xy+y^3)).$$

Then $Z = yp_1 + y^2 p_2$, $Z^2 = y^2 p_1^2 + 2(Y-X)p_1p_2 + (Y-Xy)yp_2^2$ and $Z^3 = XYp_1p_2^2 + Yp_1^3 + Yp_2^3 - 2Xp_2Z^2 - Z(Xp_1^2 + X^2p_2^2 - 3Yp_1p_2)$. So

$$F(X,Y,Z) = Z^3 + 2Z^2 q_2 + Zq_1 + q_0,$$

where $q_2 = Xp_2$, $q_1 Xp_1^2 + X^2p_2^2 - 3Yp_1p_2$ and $q_0 XYp_1p_2^2 - Yp_1^3 - Yp_2^3$. It follows that

$$\lambda(x,y) = \partial F/\partial Z(f(x,y))/|\partial(X,Y)/\partial(x,y)| = p_1^2 + yp_1p_2 + (x+y^2)p_2^2.$$

In the case $f(x,y) = (x, xy+y^3, xy^2+cy^4)$ we deduce that

$$\lambda(x,y) = (1-c)^2x^3 + (1-c+c^2)x^2y^2 + (c^2+c)xy^4 + c^2y^6.$$

(5) $f(x,y) = (x^2, y^2, g(x,y))$.

After a change of co-ordinates in the target we can reduce f to the form

$$f(x,y) = (x^2, y^2, xp_1(x^2,y^2) + yp_2(x^2,y^2) + xyp_3(x^2,y^2)).$$

Similar calculations to those above show that

$$F(X,Y,Z) = Z^4 - 2Z^2q_2 - 8Zq_1 + q_0,$$

where

$$q_0 = (Yp_2^2 - XYp_3^2)^2 + X^2p_1^4 - 6XYp_1^2(p_2^2 + Xp_3^2),$$

$$q_1 = XYp_1p_2p_3,$$
$$q_2 = Xp_1^2 + Yp_2^2 + XYp_3^2.$$

It follows that

$$\lambda(x,y) = p_3(xp_1 + yp_2)^2 + xp_2(p_1 - yp_3)^2 + yp_1(p_2 + xp_3)^2.$$

(a) In the case when $f(x,y) = (x^2, y^2, x^{2r+1} + xy + y^{2s+1})$ we find that

$$\lambda(x,y) = xy^{2s+2} + x^{2r+2}y + (x^{2r+1} + y^{2s+1})^2 + x^{2r}y^{4s+1} + x^{4r+1}y^{2s}.$$

This is semi-weighted homogeneous with principal part $xy^{2s+2} + x^{2r+2}y$, and hence Milnor number $(2r+2)(2s+2)$. In particular the germ is finitely-$\mathcal{A}$-determined.

(b) When $f(x,y) = (x^2, y^2, x(x^2+y^2) + y(x^2-y^2))$ we have $p_1 = x^2 + y^2$, $p_2 = x^2 - y^2$ and $p_3 = 0$. A calculation shows that $\lambda(x,y) = (x-y)(x(x^2+xy+y^2) - y^3)(x(x^2+xy+y^2) + y^3)$. Clearly this has an isolated singularity at the origin, and indeed its Milnor number is 36. Consequently the germ f is finitely-$\mathcal{A}$-determined.

(c) If p_1 or p_2 is zero then the germs of this type are not finitely-$\mathcal{A}$-determined. For if $p_1 = 0$ (respectively $p_2 = 0$) the double point curve is given by $y^2p_2p_3(p_2 + xp_3) = 0$ (respectively $x^2p_1p_3(p_1 + yp_3) = 0$) and in both cases we have a non-isolated singularity at the origin.

3 Plane Curves

The condition $n > 1$ in section 2 excludes the germs $\mathbf{C}, 0 \to \mathbf{C}^2, 0$. This case is worth analysing, because it will be important when extending the results of section 2 to some non-generic, but nevertheless very natural, mappings.

So let $f : \mathbf{C}, 0 \to \mathbf{C}^2, 0$ be the parametrisation of an irreducible plane curve (equivalently f is $\mathcal{L}$- or $\mathcal{A}$-finite). The defining equation for its image we denote by $F = 0$. As before $F \circ f(u) \equiv 0$ so differentiating we find

$$F_x(f(u))f_1'(u) + F_y(f(u))f_2'(u) \equiv 0$$

so that $F_x(f(u))/f_1'(u) = -F_y(f(u))/f_2'(u)$.

We now seek an interpretation of the orders of these quotients.

Proposition 3.1 *The above order is 2δ where δ is the delta invariant of the plane curve singularity discussed in [15], or [9]. (In the latter reference 2δ is shown to be μ, where μ is the Milnor number of the singularity.)*

Proof There are a number of possible approaches but we give a geometric proof using deformations. So deform f via a family f^t with f^t having δ double points for t small and non-zero. We can also deform the defining equation F to F^t so that $F^t = 0$ and the image of f^t coincide for t small. (See [4].) Now $F_x(f(u))$ measures the contact between $F_x = 0$ and the image of f. So for small t the equation $F_x^t(f^t(u)) = 0$ has roots corresponding to the singular points of the image of f, or smooth points wth horizontal tangent. Moreover each double point, which we can assume has non-horizontal tangent lines, corresponds to a pair of roots. On the other hand the points with horizontal tangent are given by $(f^t)'(u) = 0$, and the result now follows.

Example 3.2 *The A_{2k} singularities.*

Consider the A_{2k} singularity $f(u) = (u^2, u^{2k+1})$, with $F(x,y) = y^2 - x^{2k+1}$. We can choose $f^t(u) = (u^2, u^{2k+1} + tu)$. Note that for $t \neq 0$ this is an immersion with k ordinary double points, determined by the roots of the equation $u^{2k} + t = 0$; for given any root r we have $f(r) = f(-r)$ a double point. Now the corresponding defining equation is $F^t(x,y) = y^2 - x(x^k + t)^2$ and $F_x^t(f^t(u)) = -(u^{2k} + t)((2k+1)u^{2k} + t)$. We have k ordinary double points from the first bracket, and $2k$ horizontal tangents from the second (the derivative $(f_2^t)'(u) = (2k+1)u^{2k} + t$). On the other hand $F_y^t(f^t(u)) = 2u(u^{2k} + t)$. The first factor corresponds to the one horizontal tangent, the other to the k double points.

Example 3.3 *Cuspidal edges.*

Given a plane curve parametrisation $f : \mathbf{C}, 0 \to \mathbf{C}^2, 0$, consider now a map $g : \mathbf{C}^n, 0 \to \mathbf{C}^{n+1}, 0$ of the form $g(u, v_1, \ldots, v_{n-1}) = (f(u), v_1, \ldots, v_{n-1})$. The defining equation for the image of g is exactly the same as that for f, and we deduce that if we define λ as usual, it is holomorphic, a function of u only of course, and has order equal to twice the delta invariant of f. In other words we get the defining equation of the singular set raised to the power 2δ.

The approach for curves works for multi-germs $f : \mathbf{C}, S \to \mathbf{C}^2, 0$, where S is a finite set, and F is the defining equation for the plane curve which is the union of the images of the germs making up f. Indeed we can establish the following result in the same way.

Proposition 3.4 *At each point s of S the order of the quotient is*

$$2\delta_s + \sum_{s'} m(s,s'),$$

where δ_s is the delta invariant associated to $f : \mathbf{C}, s \to \mathbf{C}^2, 0$, and $m(s,s')$ is the intersection multiplicity of the branch at s with that at s'. The summation is over all $s' \in S$ with $s \neq s'$. (Note that summing over all $s \in S$ we obtain twice the delta invariant for the image of f.) We denote this sum by $\epsilon(s)$.

Example 3.5 *A bi-germ.*

Consider the case of a bi-germ, with component germs $f(u) = (u^2, u^3)$ and $g(u) = (u^3, u^2)$. The defining equation is simply $H(x,y) = (x^2 - y^3)(y^2 - x^3)$. We deform f as $f^t(u) = (u^2, u^3 + tu)$, and g as $g^t(u) = (u^3 + tu, u^2)$. The corresponding defining equation is $H^t(x,y) = (y^2 - x(x+t)^2)(x^2 - y(y+t)^2) = F^t(x,y)G^t(x,y)$. Now $H^t_x = F^t_x G + F^t G_x$, so $H^t_x(f^t(u)) = F^t_x(f^t(x))G^t(f^t(u))$. The number of 'small'roots of the first factor is the same as that for the individual germ f, namely 4. The number of 'small'roots of the second is that of $u^4 - (u^3 + tu)(u^3 + tu + t)^2$ which is clearly the intersection multiplicity of the two curves, in this case 4.

Similar results hold for a family of curves. So let $f : \mathbf{C}^n, 0 \to \mathbf{C}^{n+1}, 0$ be of the form $f(u,y) = (g(u,y), y)$ where $(u,y) \in \mathbf{C} \times \mathbf{C}^{n-1}$, $g(0,y) \equiv 0 \in \mathbf{C}^2$, and for all small y the germs $g_y : \mathbf{C}, 0 \to \mathbf{C}^2, 0$ all have the same δ invariant (in particular they are all topologically equivalent). Let $F(x,y) = 0$ be the defining equation for the image of f.

Proposition 3.6 *For such an f we have $\lambda(u,y) = u^{2\delta}\mu(u,y)$ with $\mu(0,0) \neq 0$, and where δ is the delta invariant of all/any of the g_y's.*

Proof Note first that the multiplicity is also constant for the family. Suppose without loss of generality that the first component of g_0 is u^r where r is this multiplicity. Then $\lambda(u,y) = \frac{\partial F}{\partial x_2}(g(u,y),y)/\frac{\partial g_1}{\partial u}(u,y)$. But $\frac{\partial g_1}{\partial u}(u,y_0)$ has order at least $r-1$ for small y_0, and hence order exactly $r-1$, so we can write $\frac{\partial g_1}{\partial u}(u,y) = u^{r-1}\alpha(u,y)$ with $\alpha(0,0) \neq 0$. So for all small y_0 the germ $\frac{\partial F}{\partial x_2}(g(u,y_0),y_0)$ has order $2\delta+r-1$, so we can write $\frac{\partial F}{\partial x_2}(g(u,y),y)$ as $u^{2\delta+r-1}\beta(u,y)$ with $\beta(0,0)\neq 0$ and $\mu=\beta/\alpha$.

A similar statement and proof holds for multi-germs. We leave the details to the reader.

4 Less Generic Germs

In §2 we dealt with a large class of map-germs $\mathbf{C}^n,0\to\mathbf{C}^{n+1},0$. However other types of mapping do arise naturally in singularity theory; here is one source.

Let $G:\mathbf{C}^m,0\to\mathbf{C}^n,0$, with $m\geq n$, be holomorphic, and suppose that

(i) the map G has rank $n-1$ at 0;

(ii) the set of critical points of G, ΣG is smooth;

(iii) G is finitely-$\mathcal{A}$-determined.

The first condition means that we can write G in the form

$$(x_1,\ldots,x_{n-1},y_1,\ldots,y_{m-n+1})=(x,y)\mapsto(x,g(x,y))$$

where g has zero 1-jet. Further reduction simply depends on the rank of the matrix $(\partial^2 g/\partial x_i\partial x_j(0))$. If this has rank k we can use the Morse lemma with parameters [1] to reduce to the form $g(x,y)=\sum_{i=1}^k y_i^2+h(x,y_{k+1},\ldots,y_{m-n+1})$. Condition (ii) now implies that the matrix

$$(\partial^2 h/\partial x_i\partial y_j(0)),\ 1\leq i\leq n-1,\ k+1\leq j\leq m-n+1$$

has rank $m-n-k$. A straightforward change of co-ordinates reduces the 2-jet of f to

$$(x_1,\ldots,x_{n-1},\sum_{i=1}^k y_i^2+\sum_{j=k+1}^{m-n+1}x_jy_j).$$

Note that we need $m-2n+1\leq k\leq m-n+1$. (See [5].)

Proposition 4.1 *Let G be as above. Then the set of critical values of G, the discriminant $\Delta(G)\subset\mathbf{C}^n,0$ is a hypersurface. For $n\geq 3$ its singularities in codimension 1 are locally diffeomorphic to products of ordinary double points and cusps with affine spaces. (The corresponding germs are pairs of transverse folds and cusp mappings.)*

Proof This just uses the geometric criterion for finite determinacy (again) [16] and the fact that the stable germs of corank 1 of the simplest types are

(i) submersion with folds: $(x,y) \mapsto (x, \sum_{i=1}^{m-n+1} y_i^2)$;

(ii) cusps: $(x,y) \mapsto (x, \sum_{i=1}^{m-n} y_i^2 + y_{m-n+1}^3 + x_1 y_{m-n+1})$.

The simplest bi-germ on the other hand is a pair of submersions with folds, whose corresponding critical values are transverse.

Now for our new germs f we consider the restriction of G to its critical set ΣG. This of course is no longer an immersion outside a set of codimension 2. However we can now show that an analogue of earlier results holds for finite mappings which are injective off a set of codimension 1. Let $f : \mathbf{C}^n, 0 \to \mathbf{C}^{n+1}, 0$ be such a mapping, and consider the singular part of the image of f. Then this is an analytic subset of codimension ≥ 2 in $\mathbf{C}^{n+1}$. Denote its irreducible components of dimension $n-1$ by $Y_1, \ldots, Y_r$. Its inverse image under f, that is $X = f^{-1}(Sing(Im\ f))$ is an analytic subset of $\mathbf{C}^n$ of codimension ≥ 1, and we label the irreducible components of $f^{-1}Y_i$ by X_{ij} where $1 \leq j \leq m(i)$ say. Now the map f is an immersion when restricted to an open subset (of the regular points) of each component. For otherwise its image would have dimension $\leq n-1$. So at each point of this open set the restriction can be written in the form $(g_{ij}(u,y), y) \in \mathbf{C}^2 \times \mathbf{C}^{n-1}$ where the components are locally given by $y = 0$ in $\mathbf{C}^{n+1}$, and the image Y_i of the X_{ij} for a fixed i, is given locally as $(0,y) \in \mathbf{C}^2 \times \mathbf{C}^{n-1}$. It is easy to see that the ϵ-invariant associated to each germ in the curve multigerm $g_{ij}(u,y)$ (fixed i and y) depends only on the map f (and y of course) and not on the choice of co-ordinates. The set of regular points where this fails to achieve its minimum value is a proper analytic subset. Let this minimum for the component X_{ij} be denoted by $\epsilon(ij)$.

Theorem 4.2 *The germ λ defined in section 2 is holomorphic, and can be written $\lambda = \Pi_{ij} h_{ij}^{\epsilon(ij)}$ where the h_{ij} are defining equations for the components X_{ij} and $\epsilon(ij)$ is the corresponding epsilon invariant. In the particular case of the restriction of the map G to its critical set ΣG the function λ is $g_D g_C^2$ where $g_D = 0$ and $g_C = 0$ are reduced defining equations for the double point and cusp set of f.*

Proof The proof of Theorem 2.4 shows that λ is holomorphic off the X_{ij}. On the other hand on an open subset of this set λ is also holomorphic, from results in the previous section. So λ is holomorphic off a subset of codimension 2 and hence holomorphic by Hartogs' theorem. Its expression in terms of the defining equations of the X_{ij} also follows from previous results concerning families of curves.

Example 4.3 *Swallowtail*

Consider the germ $G : \mathbf{C}^3, 0 \to \mathbf{C}^3, 0$ defined by

$$G(x, u, v) = (x^4 + ux^2 + vx, u, v),$$

The singular set of G is given by $4x^3 + 2ux + v = 0$, so that f is the map $\mathbf{C}^2, 0 \to \mathbf{C}^2, 0$ given by $(x, u) \mapsto (-3x^4 - ux^2, u, -4x^3 - 2ux)$. Now suppose that we use (t, u, v) co-ordinates in the target; we seek the equation of the discriminant of f, so we seek to eliminate x from the equations

$$3x^4 + ux^2 + t = 4x^3 + 2ux + v = 0.$$

This yields (using Maple) $F = 4u^3v^2 + 27v^4 + 16tu^4 + 128t^2u^2 + 144uv^2t + 256t^3$.

However the cuspidal edge set is given by adding the condition $6x^2 + u = 0$ to the above two conditions, while for the double point set we need to add $2x^2 + u = 0$. It is not difficult to check that up to constants we have

$$\partial F/\partial t(f(x, u)) = (6x^2 + u)^3(2x^2 + u),$$

$$\partial F/\partial u(f(x, u)) = (6x^2 + u)^3(2x^2 + u)x^2,$$

$$\partial F/\partial v(f(x, u)) = (6x^2 + u)^3(2x^2 + u)x.$$

One the other hand the corresponding minors are (up to constants again) $\Delta_1 = 6x^2 + u$, $\Delta_2 = (6x^2 + u)x^2$, $\Delta_3 = (6x^2 + u)x$. This verifies the result, and also shows that λ need not be the gcd of the Δ_i.

References

[1] V.I.Arnold, 'Wave front evolution and equivariant Morse lemma', *Commun. Pure and Appl. Math.* 29 (1976), 557-582.

[2] Th. Brocker and L.C.Lander, Differentiable Germs and Catastrophes, London Mathematical Society Lecture Note Series 17, Cambridge University Press, 1975.

[3] J.W.Bruce, Weak transversality in jet spaces, To appear.

[4] J.W.Bruce and T.J.Gaffney, Simple singularities of mappings from $\mathbf{C}$ to $\mathbf{C}^2$, *J.London Math. Soc.*, 24 (1982), 465-474.

[5] A.du Plessis and L.C.Wilson, On right-equivalence, *Math.Z.* 190, (1985), 163-205.

[6] M.Field, Several Complex Variables and Complex Manifolds I, London Mathematical Society Lecture Note Series 65, Cambridge University Press, 1982.

[7] T.Fukuda, Local topological properties of differentiable mappings, *Invent. math.* 65, (1981), 227-250.

[8] H.Grauert and R.Remmert, Coherent Analytic Sheaves, Grundleheren der mathematischen Wissenschaften 265, Springer-Verlag, Berlin, 1984.

[9] J.W.Milnor, Singular Points of Complex Hypersurfaces, Annals of Mathematics Studies, 61, Princeton University Press, 1968.

[10] H.Matsumura, Commutative Ring Theory, Cambridge Studies in Advanced Mathematics 8, Cambridge University Press, 1986.

[11] W.L.Marar and D.M.Q.Mond, Multiple point schemes for corank 1 maps, *J.London Math. Soc.* (2), 39(1989), 553-567.

[12] D.M.Q.Mond, Some remarks on the geometry and classification of germs of maps from surfaces to 3-space, *Topology* 26 (1987), 361-383.

[13] D.M.Q.Mond and R.Pellikaan, Fitting deals and multiple points of analytic mappings, in Algebraic Geometry and Complex Analysis, Springer Lecture Notes in Mathematics 1414, Springer-Verlag, Berlin, 1990.

[14] R.Piene, Ideals associated to a desingularisation, Springer Lecture Notes in Mathematics 732, 503-517, Springer-Verlag, Berlin.

[15] J.-P.Serre, Groupes Algébriques et Corps de Classes, Hermann, Paris, 1959.

[16] C.T.C.Wall, Finite determinacy of smooth map germs, *Bull. Lond. Math. Soc.*13(1981), 481-539.

J. W. Bruce
Department of Pure Mathematics
University of Liverpool
P O Box 147
Liverpool L69 3BX
England

W. L. Marar
Instituto de Ciências Matemáticas de São Carlos
Universidade de São Paulo
Caixa Postal 668
13560-970 São Carlos (SP)
Brazil

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