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Asymptotically Almost Periodic and Almost Periodic Solutions for Neutral Integrodifferential Equations

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Abstract

In this paper we study the existence of asymptotically almost periodic and almost periodic solutions for a class of partial neutral functional integrodifferential equation with unbounded delay.

1 Introduction

In this work we study the existence of asymptotically almost periodic and almost periodic solutions for a class of partial neutral integrodifferential equation with unbounded delay modelled in the form

$$\frac{d}{dt}D(t,u_t) = AD(t,u_t) + \int_0^t B(t-s)D(s,u_s)ds + g(t,u_t), \tag{1}$$

where $A : D(A) \subset X \to X$, $B(t) : D(B(t)) \subset X \to X$, $t \geq 0$, are linear closed and densely defined operators on a Banach space X; $D(B(t)) \supset D(A)$ for every $t \geq 0$; the history $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$, belongs to some abstract phase space $\mathcal{B}$ described axiomatically; $D(t,\varphi) = \varphi(0) + f(t,\varphi)$ and $f(\cdot), g(\cdot)$ are appropriate functions.

The existence of almost periodic and asymptotically almost periodic solutions is one of the most attracting topics in the qualitative theory of differential equations due to their significance in physical sciences. For the cases of partial functional differential equations, $f \equiv 0$, and ordinary neutral differential equations, this problem has been treated recently in [1, 2, 3] and [4, 5] respectively. The existence of these type of solution for " partial " retarded neutral integrodifferential equations and, in particular, for " partial " neutral differential equations with delay is an untreated topic and it's the main motivation of our work.

Neutral differential equations arise in many areas of applied mathematics and for this reason, this type of equation has received much attention in recent years. The literature relative to ordinary neutral differential equations is very extensive, we suggest the reader the Hale & Lunel book [6] concerning this matter. Referring to partial neutral functional differential equations, we cite the pioneer Hale paper [7] and Wu [8, 9, 10] for finite delay equations and Hernández & Henriquez [11, 12], Hernández [13] for the unbounded delay one.

Partial neutral integrodifferential equations arise, for example, in the theory of viscoelastic materials with memory. In this type of problems, the constitutive law of these materials incorporate the history of the strain rate and this phenomena can be satisfactory modellated by mean of a partial neutral differential or a partial neutral integrodifferential equation with delay. In fact, in [14] was observed that the abstract differential equation

$$u'(t) = A\left(u(t) + \int_{-\infty}^{t} F(t-s)u(s)ds\right) + \int_{-\infty}^{t} K(t-s)u(s)ds, \qquad (2)$$

can be regarded as the abstract formulation of the model studied by Coleman and Gurtin [15] for the heat flow in a rigid, isotropic and homogeneous material and for a model in thermoviscoelasticity studied by Leugering in [16]. If $F = F_1 + F_2$ and F_i, $i = 1, 2$, are differentiable, equation (2) can be transformed into the abstract partial neutral integrodifferential equation with unbounded delay

$$\begin{aligned}
&[u(t) + \int_{-\infty}^{t} F_1(t-s)u(s)ds + \int_{-\infty}^{0} F_2(t-s)u(s)ds]' = \\
&\qquad A\left(u(t) + \int_{-\infty}^{t} F_1(t-s)u(s)ds + \int_{-\infty}^{0} F_2(t-s)u(s)ds\right) \\
&\qquad + \int_{0}^{t} AF_2(t-s)u(s)ds \\
&\qquad + F_1(0)u(t) + \int_{-\infty}^{t} [K(t-s) + F_1'(t-s)]u(s)ds + \int_{-\infty}^{0} F_2'(t-s)u(s)ds.
\end{aligned}$$

For additional details about this matter, see Coleman & Guttin [15], Grimmer [14, 17] and the references therein.

The paper has four sections. In section 2 we mention a few results and notations referents resolvent of operators, asymptotically almost periodic and almost periodic functions needed to establish our results. In section 3, we discuss the existence of asymptotically almost periodic and almost periodic solutions for system (1). Section 4 is reserved for examples.

2 Preliminaries

In this paper, $(X, \|\cdot\|)$ is an abstract Banach space; $A : D(A) \subset X \to X$ and $B(t) : D(B(t)) \subset X \to X$, $t \geq 0$, are linear, closed and densely defined operator on X with $D(B(t)) \supset D(A)$ for each $t \geq 0$. To obtain our results we always assume that the integrodifferential abstract Cauchy problem

$$x'(t) = Ax(t) + \int_{0}^{t} B(t-s)x(s)ds, \qquad t \geq 0, \qquad (3)$$

$$x(0) = x_0 \in X, \qquad (4)$$

has associated a resolvent family of bounded linear operators $(R(t))_{t\geq 0}$ on X.

Definition 1 *A one parameter family $(R(t))_{t\geq 0}$ of bounded linear operators from X into X is called a strongly continuous resolvent operator for (3)-(4) if the following conditions are verified.*

(i) *$R(0) = I_d$ and the function $R(t)x$ is continuous on $[0, \infty)$ for every $x \in X$,*

(ii) *$R(t)D(A) \subset D(A)$ for all $t \geq 0$ and for $x \in D(A)$, $AR(t)x$ is continuous on $[0, \infty)$ and $R(t)x$ is continuously differentiable on $[0, \infty)$.*

(iii) *For $x \in D(A)$, the next resolvent equations are verified,*

$$R'(t)x = AR(t)x + \int_0^t B(t-s)R(s)xds, \quad t \geq 0, \tag{5}$$

$$R'(t)x = R(t)Ax + \int_0^t R(t-s)B(s)xds. \quad t \geq 0. \tag{6}$$

In this paper, we always assume that the resolvent family $(R(t))_{t\geq 0}$ is uniformly exponentially stable and that $\widetilde{M}, \delta$ are positive constants such that $\| R(t) \| \leq \widetilde{M}e^{-\delta t}$ for every $t \geq 0$.

For a complementary literature related partial integrodifferential equations and the theory of resolvent of operators, we cite the papers [14, 17, 18].

Throughout this work, $\mathcal{B}$ is a phase space defined axiomatically. Specifically, $\mathcal{B}$ is a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\| \cdot \|_{\mathcal{B}}$ and verifying the following axioms:

(A) If $x : (-\infty, \sigma + a) \to X$, $a > 0, \sigma \in \mathbb{R}$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + a)$ the following conditions hold:

(i) x_t is in $\mathcal{B}$.

(ii) $\| x(t) \| \leq H \| x_t \|_{\mathcal{B}}$.

(iii) $\| x_t \|_{\mathcal{B}} \leq K(t-\sigma) \sup\{\| x(s) \| : \sigma \leq s \leq t\} + M(t-\sigma) \| x_\sigma \|_{\mathcal{B}}$,

where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ in **(A)**, x_t is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma + a)$.

(B) The space $\mathcal{B}$ is complete.

(C2) If $(\varphi^n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence in $C((-\infty, 0]; X)$ formed by functions with compact support and $\varphi^n \to \varphi$ in the compact-open topology, then $\varphi \in \mathcal{B}$ and $\|\varphi^n - \varphi\|_{\mathcal{B}} \to 0$ as $n \to \infty$.

Definition 2 *Let $S(t) : \mathcal{B} \to \mathcal{B}$ be the C_0-semigroup defined by $S(t)\varphi(\theta) = \varphi(0)$ for $\theta \in [-t, 0]$ and $S(t)\varphi(\theta) = \varphi(t + \theta)$ for $\theta \in (-\infty, -t]$. The phase space $\mathcal{B}$ is called a fading memory if $\| S(t)\varphi \|_{\mathcal{B}} \to 0$ as $t \to \infty$ for every $\varphi \in \mathcal{B}$ such that $\varphi(0) = 0$.*

Remark 1 *In this paper, $\mathfrak{L} > 0$ is such that $\| \varphi \|_{\mathcal{B}} \leq \mathfrak{L} \sup_{\theta \leq 0} \| \varphi(\theta) \|$ for each $\varphi \in \mathcal{B}$ continuous and bounded. Moreover, for the case in which $\mathcal{B}$ is a fading memory, we will assume that $\mathfrak{K}$ is a constant such that $\max\{K(t), M(t)\} \leq \mathfrak{K}$ for every $t \geq 0$. For details respect these assumptions see [19, Proposition 7.1.1] and [19, Proposition 7.1.5].*

For details respect phase space we refer the reader to the book Hino, Murakami & Naito [19].

Next we mention a few results, definitions and notations related to asymptotically almost periodic and almost periodic functions. Next, $(Z, \| \cdot \|_Z)$, $(W, \| \cdot \|_W)$ are Banach spaces and $C_0([0,\infty); Z)$ is the subspace of $C([0,\infty); Z)$ formed by the functions that vanishes at infinity.

Definition 3 *A function $F \in C(\mathbb{R}; Z)$ is called almost periodic (a.p.) if for every $\varepsilon > 0$ there exists a relatively dense subset of $\mathbb{R}$, denoted by $\mathcal{H}(\varepsilon, F, Z)$, such that*

$$\|F(t+\xi) - F(t)\|_Z < \varepsilon, \qquad t \in \mathbb{R},\ \xi \in \mathcal{H}(\varepsilon, F, Z).$$

Definition 4 *A function $F \in C([0,\infty); Z)$ is called asymptotically almost periodic (a.a.p.) if there exists an almost periodic function $g(\cdot)$ and $w \in C_0([0,\infty); Z)$ such that $F(\cdot) = g(\cdot) + w(\cdot)$.*

The next Lemmas are useful characterizations of a.p and a.a.p functions.

Lemma 1 *[20, p. 25] A function $f \in C(\mathbb{R}; Z)$ is almost periodic if, and only if, the set of functions $\{H_t f : t \in \mathbb{R}\}$, where $(H_t f)(s) = f(s+t)$, is relatively compact in $C(\mathbb{R}; Z)$.*

Lemma 2 *[20, Theorem 5.5] A function $F \in C([0,\infty); Z)$ is asymptotically almost periodic if and only if, for every $\varepsilon > 0$ there exists $L(\varepsilon, F, Z) > 0$ and a relatively dense subset of $[0,\infty)$, denoted by $\mathcal{T}(\varepsilon, F, Z)$, such that*

$$\|F(t+\xi) - F(t)\|_Z < \varepsilon, \qquad t \geq L(\varepsilon, F, Z),\ \xi \in \mathcal{T}(\varepsilon, F, Z).$$

In this paper, $AP(Z)$ and $AAP(Z)$ are the spaces

$$\begin{aligned} AP(Z) &= \{F \in C(\mathbb{R}; Z) : F \text{ is a.p.}\}, \\ AAP(Z) &= \{F \in C([0,\infty); Z) : F \text{ is a.a.p.}\}, \end{aligned}$$

endowed with the norms $||| \ u \ |||_Z = \sup_{s \in \mathbb{R}} \| \ u(s) \ \|$ and $\| \ u \ \|_Z = \sup_{s \geq 0} \| \ u(s) \ \|$ respectively. We know from [20] that $AP(Z)$ and $AAP(Z)$ are Banach spaces.

Definition 5 *Let Ω be an open subset of W.*

(a) *A function $F \in C(\mathbb{R} \times \Omega; Z)$ is called pointwise almost periodic (p.a.p.) if $F(\cdot, x) \in AP(Z)$ for every $x \in \Omega$.*

(b) *A function $F \in C([0,\infty) \times \Omega; Z)$ is called pointwise asymptotically almost periodic (p.a.a.p.) if $F(\cdot, x) \in AAP(Z)$ for every $x \in \Omega$.*

(c) *A function* $F \in C(\mathbb{R} \times \Omega; Z)$ *is called uniformly almost periodic (u.a.p.), if for every* $\varepsilon > 0$ *and every compact* $K \subset \Omega$ *there exists a relatively dense subset of* $\mathbb{R}$, *denoted by* $\mathcal{H}(\varepsilon, F, K, Z)$, *such that*

$$\|F(t+\xi, y) - F(t, y)\|_Z \leq \varepsilon \qquad (t, \xi, y) \in \mathbb{R} \times \mathcal{H}(\varepsilon, F, K, Z) \times K.$$

(d) *A function* $F : C([0,\infty) \times \Omega; Z)$ *is called uniformly asymptotically almost periodic (u.a.a.p.), if for every* $\varepsilon > 0$ *and every compact* $K \subset \Omega$ *there exists a relatively dense subset of* $[0,\infty)$, *denoted by* $\mathcal{T}(\varepsilon, F, K, Z)$, *and* $L(\varepsilon, F, K, Z) > 0$ *such that*

$$\|F(t+\xi, y) - F(t, y)\|_Z \leq \varepsilon, \qquad t \geq L(\varepsilon, F, K, Z), \;\; (\xi, y) \in \mathcal{T}(\varepsilon, F, K, Z) \times K.$$

Next lemma summarizes some properties which are fundamental to obtain our results. This results can be obtained from [21, Theorem 1.2.7] and Hino [19, Proposition 7.1.3].

Lemma 3 *Let* $\Omega \subset W$ *be an open set. Then the following properties hold.*

(a) *If* $F \in C(\mathbb{R} \times \Omega; Z)$ *is p.a.p. and satisfies a local Lipschitz condition at* $x \in \Omega$, *uniformly at* t, *then* F *is u.a.p.*

(b) *If* $F \in C([0,\infty) \times \Omega; Z)$ *is p.a.a.p. and satisfies a local Lipschitz condition at* $x \in \Omega$, *uniformly at* t, *then* F *is u.a.a.p.*

(c) *If* $x \in AP(X)$, *then* $t \to x_t \in AP(\mathcal{B})$. *Moreover, if* $\mathcal{B}$ *is a fading memory space and* $z \in C(\mathbb{R}; X)$ *is such that* $z_0 \in \mathcal{B}$ *and* $z|_{[0,\infty)} \in AAP(X)$, *then* $t \to z_t \in AAP(\mathcal{B})$.

(d) *If* $F \in C(\mathbb{R} \times \Omega; Z)$ *is u.a.p. and* $y \in AP(W)$ *is such that* $\overline{\{y(t) : t \in \mathbb{R}\}}^W \subset \Omega$, *then* $F(t, y(t)) \in AP(Z)$.

(e) *If* $F \in C([0,\infty) \times \Omega; Z)$ *is u.a.a.p and* $y \in AAP(W)$ *is such that* $\overline{\{y(t) : t \in \mathbb{R}\}}^W \subset \Omega$, *then* $F(t, y(t)) \in AAP(Z)$.

The terminology and notations are those generally used in functional analysis. In particular, the notation $\mathcal{L}(Z, W)$ stands for the Banach space of bounded linear operators from Z into W and we abbreviate this notation to $\mathcal{L}(Z)$ when $Z = W$. Moreover $B_r(x, Z)$ denotes the closed ball with center at x and radius $r > 0$ in Z.

3 Existence Results

In this section we study the existence of asymptotically almost periodic and almost periodic solutions of (1). Next result is proved using the ideas and estimates in [20, Example 2.2] and we include the proof by completeness.

Lemma 4 *Let* $v \in AAP(X)$ *and* $u : [0,\infty) \to X$ *be the function defined by*

$$u(t) = \int_0^t R(t-s)v(s)ds, \quad t \geq 0.$$

Then $u \in AAP(X)$.

Proof: It's clear that $u(\cdot)$ is well defined and continuous. Let $\varepsilon > 0$ be given and $\eta = \int_0^\infty \widetilde{M}e^{-\delta s}ds$. Let $\mathcal{T}(\frac{\varepsilon}{3}\eta^{-1}, v, X_\mu)$, $L = L(\frac{\varepsilon}{3}\eta^{-1}, v, X)$ be as in Lemma 2 and $L_1 > 0$ be such that $2 \parallel v\|_X \eta e^{\delta L} e^{-\delta L_1} \leq \frac{\varepsilon}{3}$. For $t \geq L + L_1$ and $\xi \in \mathcal{T}(\frac{\varepsilon}{3}\eta^{-1}, v, X)$ we get

$$\begin{aligned}
\|u(t+\xi) - u(t)\| &\leq \int_0^\xi \|R(t+\xi-s)v(s)\|ds + \int_0^t \|R(t-s)(v(s+\xi) - v(s))\|ds \\
&\leq e^{-\delta t}\|v\|_X \int_0^\xi \widetilde{M}e^{-\delta(\xi-s)}ds + \int_0^L \|R(t-s)\| \|\, v(s+\xi) - v(s)\|ds \\
&+ \int_L^t \|R(t-s)\|\|v(s+\xi) - v(s)\|ds \\
&\leq e^{-\delta t}\|v\|_X\eta + 2\|v\|_X e^{-\delta(t-L)} \int_0^L \widetilde{M}e^{-\delta(L-s)}ds + \frac{\varepsilon}{3\eta}\int_0^\infty \widetilde{M}e^{-\delta s}ds \\
&\leq e^{-\delta t}\|v\|_X\eta + 2\|v\|_X e^{\delta L}\eta e^{-\delta t} + \frac{\varepsilon}{3},
\end{aligned}$$

which implies that

$$\|u(t+\xi) - u(t)\| \leq \varepsilon, \qquad t \geq L(\frac{\varepsilon}{3}\eta^{-1}, v, X) + L_1, \;\; \xi \in \mathcal{T}(\frac{\varepsilon}{3}\eta^{-1}, v, X).$$

This inequality and Lemma 2 permit to conclude that $u(\cdot)$ is a.a.p. The proof is now completed. ■

To prove our existence results we always assume that the next condition holds.

$\mathbf{H_1}$ The functions $f, g : \mathbb{R} \times \mathcal{B} \to X$ are continuous and there are continuous and nondecreasing functions $L_g, L_f : [0, \infty) \to [0, \infty)$ such that

$$\begin{aligned}
\| f(t,\psi_1) - f(t,\psi_2) \| &\leq L_f(r) \| \psi_1 - \psi_2 \|_{\mathcal{B}}, \\
\| g(t,\psi_1) - g(t,\psi_2) \| &\leq L_g(r) \| \psi_1 - \psi_2 \|_{\mathcal{B}},
\end{aligned}$$

for every $t \in \mathbb{R}$ and every $\psi_i \in \mathcal{B}$ such that $\| \psi_i \|_{\mathcal{B}} \leq r$.

From the theory of resolvent operators, see [18], we introduce the following concept of mild solution of (1).

Definition 6 *A function $u : (-\infty, \sigma + a) \to X$, $a > 0$, is a mild solution of the neutral integrodifferential system (1) on $[\sigma, \sigma + a)$, if $u_\sigma \in \mathcal{B}$, $u|_{[\sigma,\sigma+a)}$ is continuous and*

$$u(t) = R(t)(\varphi(0) + f(\sigma, \varphi)) - f(t, u_t) + \int_\sigma^t R(t-s)g(s, u_s)ds, \qquad t \in [\sigma, \sigma + a).$$

Now, we can establish our first existence result.

Theorem 1 *Assume that $\mathcal{B}$ is a fading memory space and that $f(\cdot)$ and $g(\cdot)$ are p.a.a.p. If $L_f(0) = L_g(0) = 0$ and $f(t,0) = g(t,0) = 0$ for every $t \in \mathbb{R}$, then there exists $\varepsilon > 0$ such that for each $\varphi \in B_\varepsilon(0, \mathcal{B})$ there exists a mild solution $u(\cdot, \varphi)$ of (1) on $[0, \infty)$ such that $u(\cdot, \varphi) \in AAP(X)$ and $u_0(\cdot, \varphi) = \varphi$.*

Proof: Let $r > 0$ and $0 < \lambda < 1$ be such that

$$\Theta = \widetilde{M}(H + L_f(\lambda r))\lambda + \left[L_f((\lambda+1)\mathfrak{K}r) + \frac{\widetilde{M}L_g((\lambda+1)\mathfrak{K}r)}{\delta} \right](\lambda+1)\mathfrak{K} < 1,$$

where $\mathfrak{K}$ is the constant introduced in Remark 1. We affirm that the assertion holds for $\varepsilon = \lambda r$. Let $\varphi \in B_\varepsilon(0, \mathcal{B})$. On the space

$$\mathfrak{D} = \{ u \in AAP(X) : u(0) = \varphi(0), \parallel u(t) \parallel \leq r, \, t \geq 0 \}$$

endowed with the metric $d(u, v) = \parallel u - v \parallel_X$, we define the operator $\Gamma : \mathfrak{D} \to C([0, \infty); X)$ by

$$\Gamma u(t) = R(t)(\varphi(0) + f(0, \varphi)) - f(t, \widetilde{u}_t) + \int_0^t R(t-s) g(s, \widetilde{u}_s) ds, \quad t \geq 0,$$

where $\widetilde{u} : \mathbb{R} \to X$ is such that $\widetilde{u}_0 = \varphi$ on $(-\infty, 0)$ and $\widetilde{u} = u$ on $[0, \infty)$. From the properties of $(R(t))_{t \geq 0}$ and $f(\cdot), g(\cdot)$, we infer that $\Gamma u(\cdot)$ is well defined and that $\Gamma u \in C([0, \infty); X)$. Moreover, from Lemmas 3 and 4 it follows that $\Gamma u \in AAP(X)$.

Next, we prove that $\Gamma(\cdot)$ is a contraction from $\mathfrak{D}$ into $\mathfrak{D}$. Let $u \in \mathfrak{D}$ and $t \geq 0$. From the inequality $\parallel \widetilde{u}_t \parallel \leq (\lambda + 1)\mathfrak{K}r$, which follows from the space phase axioms, we get

$$\begin{aligned}
\parallel \Gamma u(t) \parallel &\leq \widetilde{M}(H\lambda r + L_f(\lambda r)\lambda r) + L_f((\lambda+1)\mathfrak{K}r)(\lambda+1)\mathfrak{K}r \\
&\quad + \int_0^t \widetilde{M} e^{-\delta(t-s)} L_g((\lambda+1)\mathfrak{K}r)(\lambda+1)\mathfrak{K}r ds \\
&\leq \widetilde{M}(H + L_f(\lambda r))\lambda r + L_f((\lambda+1)\mathfrak{K}r)(\lambda+1)\mathfrak{K}r \\
&\quad + \frac{\widetilde{M}L_g((\lambda+1)\mathfrak{K}r)(\lambda+1)\mathfrak{K}r}{\delta} \\
&\leq \Theta r,
\end{aligned}$$

which implies that $\Gamma(\mathfrak{D}) \subset \mathfrak{D}$. On the other hand, for $u, v \in \mathfrak{D}$ we see that

$$\begin{aligned}
&\parallel \Gamma u(t) - \Gamma v(t) \parallel \\
&\quad \leq \parallel f(t, \widetilde{u}_t) - f(t, \widetilde{v}_t) \parallel + \int_0^t \widetilde{M} \parallel g(s, \widetilde{u}_s) - g(s, \widetilde{v}_s) \parallel ds \\
&\quad \leq L_f((\lambda+1)\mathfrak{K}r) \parallel \tilde{u}_t - \tilde{v}_t \parallel_X + \int_0^t \widetilde{M} L_g((\lambda+1)\mathfrak{K}r) e^{-\delta(t-s)} \parallel \widetilde{u}_s - \widetilde{v}_s \parallel_{\mathcal{B}} ds \\
&\quad \leq \left[L_f((\lambda+1)\mathfrak{K}r) + \frac{\widetilde{M}L_g((\lambda+1)\mathfrak{K}r)}{\delta} \right] \mathfrak{K} \parallel u - v \parallel_X,
\end{aligned}$$

which shows that $\Gamma(\cdot)$ is a contraction from $\mathfrak{D}$ into $\mathfrak{D}$.

The existence of a mild solution with the required properties is now a consequence of the contraction mapping principle. The proof is completed. ■

The next result is proved using the ideas and estimates in the proof of the previous theorem and because of this we choose to omit the proof.

Theorem 2 *If $\mathcal{B}$ is a fading memory space; $f(\cdot)$ and $g(\cdot)$ are p.a.a.p; $L_f(t) = L_f$ and $L_g(t) = L_g$ for all $t \geq 0$ and $[L_f + \frac{\widetilde{M}L_g}{\delta}]\mathfrak{K} < 1$, then for every $\varphi \in \mathcal{B}$ there exists a unique mild solution, $u(\cdot, \varphi)$, of (1) on $[0, \infty)$ such that $u(\cdot, \varphi) \in AAP(X)$ and $u_0(\cdot, \varphi) = \varphi$.*

In the remaining, we discuss the existence of almost periodic solution for (1).

Theorem 3 *Assume that $f(\cdot)$ and $g(\cdot)$ are p.a.p. If $L_f(0) = L_g(0) = 0$ and $f(t,0) = g(t,0) = 0$ for every $t \geq 0$, then there exists $u \in AP(X)$ such that $u(\cdot)$ is a mild solution of (1) on every interval $[\sigma, \sigma + a)$.*

Proof: Let $\Gamma : AP(X) \to C(\mathbb{R}; X)$ be the map defined by

$$\Gamma u(t) \; = \; -f(t, u_t) + \int_{-\infty}^{t} R(t-s) g(s, u_s) ds, \qquad t \in \mathbb{R}.$$

From the assumption, it is easy to see that $\Gamma u(\cdot)$ is continuous and from Lemmas 1 and 3 we infer that $v(t) = (f(t,u_t), \, g(t,u_t)) \in AP(X \times X)$. If $t \in \mathbb{R}$ and $\xi \in \mathcal{H}(\varepsilon, v, X \times X)$ we get

$$\begin{aligned} \| \Gamma u(t+\xi) - \Gamma u(t) \| &\leq \| f(t+\xi, u_{t+\xi}) - f(t,u_t) \| \\ &\quad + \int_{-\infty}^{t} \widetilde{M} e^{-\delta(t-s)} \| g(s+\xi, u_{s+\xi}) - g(s,u_s) \| \, ds \\ &\leq \varepsilon + \int_{-\infty}^{t} \widetilde{M} e^{-\delta(t-s)} \varepsilon \; ds \\ &\leq \varepsilon (1 + \frac{\widetilde{M}}{\delta}), \end{aligned}$$

which implies that $\Gamma u \in AP(X)$. Thus, $\Gamma(\cdot)$ is well defined and with values in $AP(X)$. Moreover, if $\mathfrak{L}$ is the constant in Remark 1 and $u, v \in B_r = \{u \in AP(X) : ||| \, u \, |||_X \leq r\}$, $r > 0$, we see that

$$\begin{aligned} \| \Gamma u(t) - \Gamma v(t) \| &\leq L_f(\mathfrak{L} r) \| u_t - v_t \|_{\mathcal{B}} + \int_{-\infty}^{t} \widetilde{M} e^{-\delta(t-s)} L_g(\mathfrak{L} r) \| u_s - v_s \|_{\mathcal{B}} \, ds \\ &\leq \mathfrak{L} \left[L_f(\mathfrak{L} r) + \frac{\widetilde{M} L_g(\mathfrak{L} r)}{\delta} \right] ||| \, u - v \, |||_X, \end{aligned}$$

which permits conclude that $\Gamma(\cdot)$ is a contraction from B_r into B_r for $r > 0$ small enough since $L_f(0) = L_g(0) = 0$ and $\Gamma 0 \equiv 0$.

These remarks and the contraction mapping principle permit to conclude the proof. ∎

In a similar manner we can prove the next result.

Theorem 4 *Assume that $f(\cdot)$ and $g(\cdot)$ are p.a.p. functions. If $L_f(t) = L_f$ and $L_g(t) = L_g$ for all $t \geq 0$ and $\mathfrak{L}[L_f + \frac{\widetilde{M} L_g}{\delta}] < 1$, where $\mathfrak{L}$ is the constant in Remark 1, then there exists a unique $u \in AP(X)$ such that $u(\cdot)$ is a mild solution of (1) on every interval $[\sigma, \sigma + a)$.*

Remark 2 *When $B(t) = 0$ for every $t \geq 0$, the resolvent $(R(t))_{t \geq 0}$ is a uniformly exponential stable C_0-semigroup of bounded linear operator on X and our results are valid for the partial neutral functional differential equation with unbounded delay*

$$\frac{d}{dt} D(t, u_t) \; = \; AD(t, u_t) + g(t, u_t).$$

4 Examples

In this section we apply our result to establish the existence of a.a.p. and a.p solutions for some partial functional differential equation with unbounded delay. We have already introduced the required technical framework in Section 3.

Let $h : (-\infty, -r) \to \mathbb{R}$ be a positive Lebesgue integrable function and assume that there exists a non-negative and bounded function γ on $(-\infty, 0]$ such that $h(\xi + \theta) \leq \gamma(\xi)h(\theta)$, for all $\xi \leq 0$ and every $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. The space $\mathcal{B} = C_r \times L^p(h; X)$ consists of all classes of functions $\varphi : (-\infty, 0] \to X$ such that φ is continuous on $[-r, 0]$, Lebesgue-measurable and $h \parallel \varphi \parallel^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $\mathcal{B}$, denoted by $\parallel \cdot \parallel_{\mathcal{B}}$, is defined by

$$\parallel \varphi \parallel_{\mathcal{B}} := \sup\{\parallel \varphi(\theta) \parallel : -r \leq \theta \leq 0\} + \left(\int_{-\infty}^{-r} h(\theta) \parallel \varphi(\theta) \parallel^p d\theta \right)^{1/p}.$$

Assume that $h(\cdot)$ verifies the conditions (h-5), (h-6) and (h-7) in the nomenclature of [19]. In this conditions, $\mathcal{B}$ is a fading memory space which verifies axioms **(A), (A-1), (B)** and **(C2),** see [19, Theorem 1.3.8] and [19, Example 7.1.7] for details. Moreover, when $r = 0$ and $p = 2$, we have that $H = 1$, $M(t) = \gamma(-t)^{1/2}$, $K(t) = 1 + \left(\int_{-t}^{0} h(\theta)\, d\theta\right)^{1/2}$ for $t \geq 0$, $\mathfrak{L} = \left(\int_{-\infty}^{0} h(s)ds\right)^{\frac{1}{2}}$ and $\mathfrak{K} = \left(\sup_{s\leq 0} \mid \gamma(s)^{1/2} \mid + (1 + (\int_{-\infty}^{0} h(\theta)d\theta\,)^{1/2}\right)$.

4.1 A general abstract system

Consider the integrodifferential system (2) and assume that $K(\cdot), F(\cdot) : [0, \infty) \to X$ are strongly continuous families of bounded linear operators. Suppose, furthermore, that $F = F_1 + F_2$ where $F_i : [0, \infty) \to \mathcal{L}(X)$, $i = 1, 2$, are functions of class C^1 on $[0, \infty)$ and that

$$\begin{aligned} L_i^{(j)} &= \left(\int_{-\infty}^{0} \frac{\parallel F_i^{(j)}(s) \parallel^2_{\mathcal{L}(X)}}{h(s)} ds \right)^{\frac{1}{2}} < \infty, \qquad i = 1, 2,\ j = 0, 1 \\ L_3 &= \left(\int_{-\infty}^{0} \frac{\parallel K(s) \parallel^2_{\mathcal{L}(X)}}{h(s)} ds \right)^{\frac{1}{2}} < \infty. \end{aligned}$$

Under these assumptions, equation (7) can be transformed into the abstract partial neutral integrodifferential equation with unbounded delay

$$\begin{aligned} &[u(t) + \int_{-\infty}^{t} F_1(t-s)u(s)ds + \int_{-\infty}^{0} F_2(t-s)u(s)ds]' = \\ &\qquad A\left(u(t) + \int_{-\infty}^{t} F_1(t-s)u(s)ds + \int_{-\infty}^{0} F_2(t-s)u(s)ds\right) \\ &\qquad + \int_{0}^{t} AF_2(t-s)u(s)ds \\ &\qquad + F_1(0)u(t) + \int_{-\infty}^{t} [K(t-s) + F_1'(t-s)]u(s)ds + \int_{-\infty}^{0} F_2'(t-s)u(s)ds, \end{aligned}$$

and by defining the operator $D, f, g : \mathbb{R} \times \mathcal{B} \to X$ by

$$\begin{aligned} f(t,\varphi)(\xi) &= \int_{-\infty}^{0} F_1(-s)\varphi(s)(\xi)ds + \int_{-\infty}^{-t} F_2(-s)\varphi(s)(\xi)ds, \\ D(t,\varphi)(\xi) &= \varphi(0)(\xi) + f(t,\varphi)(\xi), \\ g(t,\varphi)(\xi) &= F_1(0)\varphi(0)(\xi) + \int_{-\infty}^{0} [K(-s) + F_1'(-s)]\varphi(s)(\xi)ds + \int_{-\infty}^{-t} F_2'(-s)\varphi(s)(\xi)ds, \end{aligned}$$

the equation can be described by system (1). Moreover, $f(t,\cdot), g(t,\cdot)$ are bounded linear operators, $\| f(t,\cdot) \|_{\mathcal{L}(\mathcal{B},X)} \leq L_1^{(0)} + L_2^{(0)}$ and $\| g(t,\cdot) \|_{\mathcal{L}(\mathcal{B},X)} \leq \| F_1(0) \|_{\mathcal{L}(X)} + L_3 + L_1^{(1)} + L_2^{(1)}$ for every $t \geq 0$.

In the next result, which is a direct consequence of Theorems 2 and 4, we will assume that the system

$$u'(t) = Au(t) + \int_0^t AF_2(t-s)u(s)ds,$$

has associated a resolvent family $(R(t))_{t\geq 0}$ and the existence of positive numbers $\widetilde{M}, \delta$ such that $\| R(t) \| \leq \widetilde{M}e^{-\delta t}$ for every $t \geq 0$.

Theorem 5 *Assume that the previous conditions are verified. If*

$$\left(L_1^{(0)} + L_2^{(0)} + \frac{\widetilde{M}}{\gamma}(L_3 + L_1^{(1)} + L_2^{(1)}) \right) \left(\sup_{s\leq 0} | \gamma(s)^{1/2} | + (1 + (\int_{-\infty}^{0} h(\theta)d\theta)^{1/2} \right) < 1,$$

then there exist $u \in AAP(X)$ and $v \in AP(X)$ which are mild solutions of (2) on $\mathbb{R}$ and $[0,\infty)$ respectively.

4.2 An integrodifferential equation

Let $X = H_0^1(\Omega) \times L^2(\Omega)$, where $\Omega \subset \mathbb{R}^3$ is an open set with smooth boundary of class C^∞; $\alpha(\cdot), \beta(\cdot)$ are $\mathbb{R}$-valued functions of class C^2 on $[0,\infty)$ with $\alpha(0) > 0$, $\beta(0) > 0$ and $A : D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \to X$ be the operator defined by

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ \alpha(0)x'' - \beta(0)y \end{pmatrix}.$$

We know from Chen [22], that A is the infinitesimal generator of a uniformly exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$ on X. In the sequel, we will assume that $\widetilde{M}, \gamma$ are positive constants such that $\| T(t) \| \leq \widetilde{M}e^{-\gamma t}$ for all $t \geq 0$.

Let $B(t) = AF(t)$ where $F(t) : X \to X$, $t \geq 0$, is defined by

$$F = (F_{ij}) = \begin{bmatrix} 0 & 0 \\ -\beta'(t) + \beta(0)\dfrac{\alpha'(t)}{\alpha(0)} & \dfrac{\alpha'(t)}{\alpha(0)} \end{bmatrix}.$$

Assume functions $\alpha^{(i)}(\cdot)$, $\beta^{(i)}(\cdot)$, $i=1,2$, be bounded, uniformly continuous and that

$$\begin{aligned}
\max\{\| F_{22}(t) \|, \| F_{21}(t) \|\} &\leq \frac{\gamma e^{-\gamma t}}{2M}, \qquad t \geq 0,\\
\max\{\| F_{22}'(t) \|, \| F_{21}'(t) \|\} &\leq \frac{\gamma^2 e^{-\gamma t}}{4M^2}, \qquad t \geq 0.
\end{aligned}$$

Under these conditions, the abstract integrodifferential system

$$x'(t) = Ax(t) + \int_0^t AF(t-s)x(s)ds,$$

has associated a resolvent of operator $(R(t))_{t\geq 0}$ on X such that $\| R(t) \|\leq \widetilde{M}e^{\frac{-\gamma t}{2}}$ for $t \geq 0$, see Grimmer [17, p. 343] for details.

Motivated by the abstract equation studied in Grimmer [14, p. 343], we consider the neutral integrodifferential system

$$\begin{aligned}
\left[u(t,\xi) + \int_{-\infty}^t a_1(t-s)u(s,\xi)ds\right]' &= A\left(u(t,\xi) + \int_{-\infty}^t a_1(t-s)u(s,\xi)ds\right)\\
&+ \int_0^t AF(t-s)u(s,\xi)ds\\
&+ \int_{-\infty}^t a_2(t-s)u(s,\xi)ds,
\end{aligned}$$

where $a_i : \mathbb{R} \to \mathbb{R}, i = 1,2$, are continuous and $L_i = (\int_{-\infty}^0 \frac{a_i^2(s)}{h(s)}ds)^{\frac{1}{2}} < \infty$ for $i = 1,2$. If $\mathcal{B} := C_0 \times L^p(h;X)$ and $f, g : \mathbb{R} \times \mathcal{B} \to X$ are the operators defined by

$$\begin{aligned}
f(t,\varphi)(\xi) &= \int_{-\infty}^0 a_1(-s)\varphi(s)(\xi)ds,\\
D(t,\varphi)(\xi) &= \varphi(0)(\xi) + f(t,\varphi)(\xi),\\
g(t,\varphi)(\xi) &= \int_{-\infty}^0 a_2(-s)\varphi(s)(\xi)ds,
\end{aligned}$$

system (7) can be modelled by the neutral integrodifferential equation (1). Moreover, $f(\cdot), g(\cdot)$ are bounded linear operator, $\| f(t,\cdot) \|_{\mathcal{L}(\mathcal{B},X)}\leq L_1$ and $\| g(t,\cdot) \|_{\mathcal{L}(\mathcal{B},X)}\leq L_2$ for every $t \geq 0$.

The next result follows from Theorems 1 and 3. We will omit the proof.

Theorem 6 *Assume that the previous conditions are verified. If*

$$\left[(\int_{-\infty}^0 \frac{a_1^2(s)}{h(s)}ds)^{\frac{1}{2}} + \frac{\widetilde{M}}{\gamma}(\int_{-\infty}^0 \frac{a_2^2(s)}{h(s)}ds)^{\frac{1}{2}}\right]\left(\sup_{s\leq 0} | \gamma(s)^{1/2} | + (1 + (\int_{-\infty}^0 h(\theta)d\theta)^{1/2}\right) < 1,$$

then there exist $u \in AAP(X)$ and $v \in AP(X)$ which are mild solutions of (7) on $\mathbb{R}$ and $[0,\infty)$ respectively.

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Soluções Quase e Assintoticamente Quase Periódicas de Equações Integrodiferenciais do Tipo Neutro

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Resumo

Neste artigo estudamos a existência de soluções quase e assintoticamente quase periódicas para equações integrodiferenciais do tipo neutro modeladas na forma

$$\frac{d}{dt}D(t,u_t) = AD(t,u_t) + \int_0^t B(t-s)D(s,u_s)ds + g(t,u_t), \tag{1}$$

$$x_\sigma = \varphi \in \mathcal{B}, \tag{2}$$

sendo A o gerador infinitesimal de um C_0-semigrupo de operadores lineares limitados definido sobre um espaço de Banach X, $(B(t))_{\geq 0}$ uma familia de operadores lineares, fechados e não limitados, $\mathcal{B}$ um espaço de fase definido de maneira axiomática e $g : \mathbb{R} \times \mathcal{B} \to X$ uma função apropriada.

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