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A Note on Partial Functional Differential Equations with State-dependent Delay

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Abstract

In this paper we study the existence of mild solutions for a class of abstract partial functional differential equation with state-dependent delay.

1 Introduction

In this note we establish the existence of mild solutions for a class of partial functional differential equation with state-dependent delay described in the form

$$x'(t) = Ax(t) + f(t, x_{\rho(t,x_t)}), \quad t \in I = [0, a], \tag{1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{2}$$

where A is the infinitesimal generator of a compact C_0-semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on a Banach space X; the function $x_s : (-\infty, 0] \to X$, defined by $x_s(\theta) = x(s+\theta)$, belongs to some abstract phase space $\mathcal{B}$ described axiomatically and $f : I \times \mathcal{B} \to X$, $\rho : I \times \mathcal{B} \to (-\infty, a]$ are appropriate functions.

The literature related to ordinary and partial functional differential equations with delay for which $\rho(t, \psi) = t$ is very extensive and respect this matter we cite the classical books Hale & Lunel [6] and Wu [15].

Functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received great attention in the last years, see for instance [1, 2, 3, 4, 5, 7, 8, 10] and the references therein. The literature devoted to this subject is concerned with functional differential equations which, in general, can be described by the abstract Cauchy problem

$$x'(t) = F(t, x(t - \rho(x_t))), \qquad t \in I, \tag{3}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{4}$$

where $x(t)$ belong to some finite dimensional space and the phase space $\mathcal{B}$ is usually $C([-r, 0] : X)$. Obviously, the abstract cauchy problem (3)-(4) does not permit the study of partial functional differential equations with state-dependent delay. This fact is the main motivation of our paper.

The paper has four sections. In Section 2 we introduce notations and the technical framework used in this paper. By using some well known fixed point theorem for condensing operators, in section 3 we establish the existence of mild solutions for the abstract Cauchy problem (1)-(2). The section 4 is reserved for examples.

2 Preliminaries

Throughout this paper, $A : D(A) \subset X \to X$ is the infinitesimal generator of a compact semigroup of linear operators $(T(t))_{t\geq 0}$ on a Banach space X and $\widetilde{M}$ is a constant such that $\| T(t) \| \leq \widetilde{M}$ for every $t \in I$. Related semigroup theory, we suggest the Pazy book [14].

In this work we will employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar at those introduced in [13]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\| \cdot \|_{\mathcal{B}}$ and satisfying the following axioms:

(A) If $x : (-\infty, b] \to X$, $b > 0$, is continuous on $[0, b]$ and $x_0 \in \mathcal{B}$, then for every $t \in [0, b]$ the following conditions hold:

(a) x_t is in $\mathcal{B}$.

(b) $\| x(t) \| \leq H \| x_t \|_{\mathcal{B}}$.

(c) $\| x_t \|_{\mathcal{B}} \leq M(t) \| x_0 \|_{\mathcal{B}} + K(t) \sup\{\| x(s) \| : 0 \leq s \leq t\}$,

where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, $K(\cdot)$ is continuous, $M(\cdot)$ is locally bounded and $H, K(\cdot), M(\cdot)$ are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ in **(A)**, x_t is a $\mathcal{B}$-valued continuous function on $[0, b]$.

(B) The space $\mathcal{B}$ is complete.

Remark 1 *Since $\rho(\cdot)$ is a $\mathbb{R}$-valued function, we consider necessary clarify the sense of the symbol $x_{\rho(t,\psi)}$. Let $\varphi \in \mathcal{B}$ and $t \leq 0$. The notation φ_t represent the function defined by $\varphi_t(\theta) = \varphi(t+\theta)$. Consequently, if the function $x(\cdot)$ in axiom* **A** *is such that $x_0 = \varphi$, then $x_t = \varphi_t$. It's interesting observe that φ_t is well defined for every $t < 0$ since the domain of φ is $(-\infty, 0]$.*

The terminology and notations are those generally used in functional analysis. In particular, for Banach spaces $(Z, \| \cdot \|_Z)$, $(W, \| \cdot \|_W)$, the notation $\mathcal{L}(Z, W)$ stands for the Banach space of bounded linear operators from Z into W. Moreover, $B_r(x, Z)$ denotes the closed ball with center at x and radius $r > 0$ in Z and for a bounded function $\xi : I \to [0, \infty)$ and $0 \leq t \leq a$ we employ the notation ξ_t for

$$\xi_t = \sup\{\xi(s) : s \in [0, t]\}. \tag{5}$$

To prove our results we shall make use of the classic Schauder fixed point and of the following result which is referred in the Literature as Leray Schauder Alternative Theorem, see [11, Theorem 6.5.4].

Theorem 1 *Let D be a convex subset of a Banach space X and assume that $0 \in D$. Let $G : D \to D$ be a completely continuous map. Then the map G has a fixed point in D or the set $\{x \in D : x = \lambda G(x),\ 0 < \lambda < 1\}$ is unbounded.*

3 Existence Results

In this section we establish the existence of mild solutions for the abstract Cauchy problem (1)-(2). To prove our results we always assume that $\rho : I \times \mathcal{B} \to (-\infty, a]$ is continuous and that $\varphi \in \mathcal{B}$ and $f(\cdot)$ verifies the next conditions.

$\mathbf{H_1}$ The function $f : I \times \mathcal{B} \to X$ satisfies the following properties.

(a) The function $f(\cdot, \psi) : I \to X$ is strongly measurable for every $\psi \in \mathcal{B}$.

(b) The function $f(t, \cdot) : \mathcal{B} \to X$ is continuous for each $t \in I$.

(c) There exist an integrable function $m : I \to [0, \infty)$ and a continuous nondecreasing function $W : [0, \infty) \to (0, \infty)$ such that

$$\| f(t, \psi) \| \leq m(t) W(\| \psi \|_{\mathcal{B}}), \qquad (t, \psi) \in I \times \mathcal{B}.$$

$\mathbf{H}_\varphi$ The function $t \to \varphi_t$ is well defined and continuous from the set $\mathcal{R}(\rho) = \{\rho(s, \psi) : (s, \psi) \in I \times \mathcal{B},\ \rho(s, \psi) \leq 0\}$ into $\mathcal{B}$ and there exists a continuous and bounded function $J^\varphi : \mathcal{R}(\rho) \to (0, \infty)$ such that $\| \varphi_t \|_{\mathcal{B}} \leq J^\varphi(t) \| \varphi \|_{\mathcal{B}}$ for every $t \in \mathcal{R}(\rho)$.

From Pazy [14], we adopt the following concept of mild solution of (1)-(2).

Definition 1 *A function $x : (-\infty, a] \to X$ is called a mild solution of the abstract Cauchy problem (1)-(2) if $x_0 = \varphi$, $x_{\rho(s, x_s)} \in \mathcal{B}$ for every $s \in I$ and*

$$x(t) = T(t)\varphi(0) + \int_0^t T(t-s) f(s, x_{\rho(s, x_s)}) ds, \quad t \in I.$$

To prove our first theorem we need the next preliminary result. In the next result, we will use the notation (5).

Lemma 1 *Let $\varphi \in \mathcal{B}$ and $J = (\alpha, 0]$ be such that $\varphi_t \in \mathcal{B}$ for every $t \in J$. Assume that there exists a locally bounded function $J^\varphi : J \to [0, \infty)$ such that $\| \varphi_t \|_{\mathcal{B}} \leq J^\varphi(t) \| \varphi \|_{\mathcal{B}}$ for every $t \in J$. If $x : (-\infty, a] \to X$ is continuous on I and $x_0 = \varphi$, then*

$$\| x_s \|_{\mathcal{B}} \leq (M_a + J^\varphi(\max\{\alpha, -|s|\}) \| \varphi \|_{\mathcal{B}} + K_a \| x \|_{\max\{0, s\}}, \quad s \in (\alpha, a].$$

If $J^\varphi(\cdot)$ is bounded on J and $\widetilde{J^\varphi} = \sup_{s \in J} J^\varphi(s)$, then

$$\| x_s \|_{\mathcal{B}} \leq (M_a + \widetilde{J^\varphi}) \| \varphi \|_{\mathcal{B}} + K_a \| x \|_{\max\{0, s\}}, \qquad s \in (\alpha, a].$$

Proof. From our assumption follow that

$$\begin{aligned} \| x_s \|_{\mathcal{B}} &= \| \varphi_s \|_{\mathcal{B}} \leq J^{\varphi}(s) \| \varphi \|_{\mathcal{B}}, \qquad s \in J, \\ \| x_s \|_{\mathcal{B}} &\leq M(s) \| \varphi \|_{\mathcal{B}} + K(s) \| x \|_s, \quad s \in I, \end{aligned}$$

and hence

$$\| x_s \|_{\mathcal{B}} \leq (M_a + J^{\varphi}(\max\{\alpha, -|s|\})) \| \varphi \|_{\mathcal{B}} + K_a \| x \|_{\max\{0,s\}}, \quad s \in (\alpha, a].$$

The inequality for the case in which $J^{\varphi}(\cdot)$ is bounded is obvious. This proof is ended. ■

Now, we can prove our first existence result.

Theorem 2 *Assume conditions* $\mathbf{H}_1, \mathbf{H}_{\varphi}$ *be satisfied. If*

$$1 < \widetilde{M} K_a \liminf_{\xi \to \infty^+} \frac{W(\xi)}{\xi} \int_0^a m(s)ds,$$

then there exists a mild solution of (1)-(2).

Proof. Let $Y = \{ u \in C(I : X) : u(0) = \varphi(0) \}$ endowed with the uniform convergence topology and $\Gamma : Y \to Y$ be the operator defined by

$$\Gamma x(t) = T(t)\varphi(0) + \int_0^t T(t-s) f(s, \bar{x}_{\rho(s,\bar{x}_s)})ds, \qquad s \in I,$$

where $\bar{x} : (-\infty, a] \to X$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on I. From the phase space axiom **A**, the strong continuity of $(T(t))_{t \geq 0}$ and our assumptions on φ, we infer that $\Gamma x(\cdot)$ is well defined and continuous.

Let $\bar{\varphi} : (-\infty, a] \to X$ be the extension of φ to $(-\infty, a]$ such that $\bar{\varphi}(\theta) = \varphi(0)$ on I. We affirm that there exists $r > 0$ such that $\Gamma(B_r(\bar{\varphi}|_I, Y)) \subset B_r(\bar{\varphi}|_I, Y)$. If we assume that this property is false, then for every $r > 0$ there exist $x^r \in B_r(\bar{\varphi}|_I, Y)$ and $t^r \in I$ such that $r < \| \Gamma x^r(t^r) - \varphi(0) \|$. Then, from Lemma 1 we find that

$$\begin{aligned} & r < \| \Gamma x^r(t^r) - \varphi(0) \| \\ & \leq \| T(t^r)\varphi(0) - \varphi(0) \| + \int_0^{t^r} \| T(t^r - s) \| \| f(s, \overline{x^r}_{\rho(s,\overline{x^r_s})}) \| ds \\ & \leq (\widetilde{M} + 1)H \| \varphi \|_{\mathcal{B}} + \widetilde{M} \int_0^{t^r} m(s) W(\| \overline{x^r}_{\rho(s,\overline{x^r_s})} \|_{\mathcal{B}})ds \\ & \leq (\widetilde{M} + 1)H \| \varphi \|_{\mathcal{B}} + \widetilde{M} \int_0^{t^r} m(s) W\left((M_a + \widetilde{J^{\varphi}}) \| \varphi \|_{\mathcal{B}} + K_a \| \overline{x^r} \|_a \right) ds \\ & \leq (\widetilde{M} + 1)H \| \varphi \|_{\mathcal{B}} + \widetilde{M} W\left((M_a + \widetilde{J^{\varphi}}) \| \varphi \|_{\mathcal{B}} + K_a (r + \| \varphi(0) \|) \right) \int_0^a m(s)ds, \end{aligned}$$

and hence

$$1 \leq \widetilde{M} K_a \liminf_{\xi \to \infty} \frac{W(\xi)}{\xi} \int_0^a m(s)ds,$$

which is contrary to our assumption.

Let $r > 0$ be such that $\Gamma(B_r(\bar{\varphi}|_I, Y)) \subset B_r(\bar{\varphi}|_I, Y)$. Next, we will prove that $\Gamma(\cdot)$ is completely continuous from $B_r(\bar{\varphi}|_I, Y)$ into $B_r(\bar{\varphi}|_I, Y)$.

Step 1 The set $\Gamma(B_r(\bar{\varphi}|_I, Y))(t) = \{\Gamma x(t) : x \in B_r(\bar{\varphi}|_I, Y)\}$ is relatively compact in X for every $t \in I$.

The case $t = 0$ is obvious. Let $0 < \varepsilon < t \leq a$. If $x \in B_r(\bar{\varphi}|_I, Y)$, from Lemma 1 follows that

$$\| \bar{x}_{\rho(t,\bar{x}_t)} \|_{\mathcal{B}} \leq r^* := (M_a + \widetilde{J}^{\varphi}) \| \varphi \|_{\mathcal{B}} + K_a(r + \| \varphi(0) \|)$$

and so that

$$\left\| \int_0^{\tau} T(\tau - s) f(s, \bar{x}_{\rho(s,\bar{x}_s)}) ds \right\| \leq r^{**} := \widetilde{M} W(r^*) \int_0^a m(s) ds, \qquad \tau \in I. \tag{6}$$

If $x \in B_r(\bar{\varphi}|_I, Y)$, from (6) we get

$$\begin{aligned} \Gamma x(t) &= T(t)\varphi(0) + T(\varepsilon) \int_0^{t-\varepsilon} T(t - \varepsilon - s) f(s, \bar{x}_{\rho(s,\bar{x}_s)}) ds \\ &\quad + \int_{t-\varepsilon}^{t} T(t - s) f(s, \bar{x}_{\rho(s,\bar{x}_s)}) ds \\ &\in \{T(t)\varphi(0)\} \; + \; T(\varepsilon) B_{r^{**}}(0, X) \; + \; C_{\varepsilon}, \end{aligned}$$

$diam(C_{\varepsilon}) \leq 2\widetilde{M} W(r^*) \int_{t-\varepsilon}^{t} m(s) ds$. Since $T(\varepsilon)$ is compact and ε arbitrary, this prove that $\Gamma(B_r(\bar{\varphi}|_I, Y))(t)$ is totaly bounded and hence relatively compact in X.

Step 2 The set of functions $\Gamma(B_r(\bar{\varphi}|_I, Y))$ is equicontinuous on I.

Let $0 < t < a$. From step 1 and the strong continuity of $(T(t))_{t \geq 0}$, for every $\varepsilon > 0$ we can choose $0 < \delta \leq a - t$ such that

$$\| T(h)x - x \| < \varepsilon, \qquad x \in \Gamma(B_r(\bar{\varphi}|_I, Y))(t),$$

when $0 < h < \delta$. Under these conditions, for $x \in B_r(\bar{\varphi}|_I, Y)$ and $0 < h < \delta$ we get

$$\begin{aligned} \| \Gamma x(t+h) - \Gamma x(t) \| &\leq \| T(t+h)\varphi(0) - T(t)\varphi(0) \| + \| (T(h) - I)\Gamma x(t) \| \\ &\quad + \widetilde{M} \int_t^{t+h} m(s) W(r^*) ds \\ &\leq \widetilde{M} \| (T(h) - I)\varphi(0) \| + \varepsilon + \widetilde{M} W(r^*) \int_t^{t+h} m(s) ds, \end{aligned}$$

where $r^* := (M_a + \widetilde{J}^{\varphi}) \| \varphi \|_{\mathcal{B}} + K_a(r + \| \varphi(0) \|)$. This proves that $\Gamma(B_r(\bar{\varphi}|_I, Y))$ is right equicontinuous at $t \in (0, a)$. A analogous procedure shows that $\Gamma(B_r(\bar{\varphi}|_I, Y))$ is right equicontinuous at zero and left equicontinuous at $t \in (0, a]$. Thus, the set of functions $\Gamma(B_r(\bar{\varphi}|_I, Y))$ is equicontinuos on I.

Step 3 The map $\Gamma(\cdot)$ is continuous on $B_r(\bar{\varphi}|_I, Y)$.

Let $(x^n)_{n\in N}$ be a sequence in $B_r(\bar{\varphi}|_I, Y)$ and $x \in B_r(\bar{\varphi}|_I, Y)$ such that $x^n \to x$ in Y. At first, we study the convergence of the sequences $(\overline{x^n}_{\rho(s,\overline{x^n_s})})_{n\in N}$, $s \in I$. If $s \in I$ is such that $\rho(s, \overline{x}_s) > 0$, we can fix $N \in I\!N$ such that $\rho(s, \overline{x^n_s}) > 0$ for every $n > N$. In this case, for $n > N$ we see that

$$\begin{aligned}
\| \overline{x^n}_{\rho(s,\overline{x^n_s})} - \overline{x}_{\rho(s,\overline{x}_s)} \|_{\mathcal{B}} &\leq \| \overline{x^n}_{\rho(s,\overline{x^n_s})} - \overline{x}_{\rho(s,\overline{x^n_s})} \|_{\mathcal{B}} + \| \overline{x}_{\rho(s,\overline{x^n_s})} - \overline{x}_{\rho(s,\overline{x}_s)} \|_{\mathcal{B}} \\
&\leq K_a \| x^n(\theta) - x(\theta) \|_{\rho(s,\overline{x^n_s})} + \| \overline{x}_{\rho(s,\overline{x^n_s})} - \overline{x}_{\rho(s,\overline{x}_s)} \|_{\mathcal{B}} \\
&\leq K_a \| x^n - x \|_a + \| \overline{x}_{\rho(s,\overline{x^n_s})} - \overline{x}_{\rho(s,\overline{x}_s)} \|_{\mathcal{B}},
\end{aligned}$$

which proves that $\overline{x^n}_{\rho(s,\overline{x^n_s})} \to \overline{x}_{\rho(s,\overline{x}_s)}$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in I$ such that $\rho(s, \overline{x}_s) > 0$. Similarly, if $\rho(s, \overline{x}_s) < 0$ and $N \in I\!N$ is such that $\rho(s, \overline{x^n_s}) < 0$ for every $n > N$, we get

$$\| \overline{x^n}_{\rho(s,\overline{x^n_s})} - \overline{x}_{\rho(s,\overline{x}_s)} \|_{\mathcal{B}} = \| \varphi_{\rho(s,\overline{x^n_s})} - \varphi_{\rho(s,\overline{x}_s)} \|_{\mathcal{B}},$$

which also shows that $\overline{x^n}_{\rho(s,\overline{x^n_s})} \to \overline{x}_{\rho(s,\overline{x}_s)}$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in I$ such that $\rho(s, \overline{x}_s) < 0$.

Combining the previous arguments, we can prove that $\overline{x^n}_{\rho(s,\overline{x^n_s})} \to \varphi$ for every $s \in I$ such that $\rho(s, \overline{x}_s) = 0$.

From the previous remarks, $f(s, \overline{x^n}_{\rho(s,\overline{x^n_s})}) \to f(s, \overline{x}_{\rho(s,\overline{x}_s)})$ for every $s \in [0, a]$. Now, condition $\mathbf{H_1}$ and the Lebesgue Dominated Convergence Theorem permit to assert that $\Gamma x^n \to \Gamma x$ in Y. Thus, $\Gamma(\cdot)$ is continuous.

The existence of a mild solution for (1)-(2) is now a consequence of the Schauder Fixed Point Theorem. The proof is completed.

Theorem 3 *Let $\mathbf{H}_\varphi, \mathbf{H_1}$ be satisfied. If $\rho(t, \psi) \leq t$ for every $(t, \psi) \in I \times \mathcal{B}$ and*

$$K_a \widetilde{M} \int_0^a m(s)ds < \int_C^\infty \frac{ds}{W(s)},$$

where $C = (K_a \widetilde{M} H + M_a + \widetilde{J}^\varphi)$ and $\widetilde{J}^\varphi = \sup_{s\in\mathcal{R}(\rho)} J^\varphi(s)$, then there exists a mild solution of (1)-(2).

Proof. Let $\Gamma(\cdot)$ be the map defined in the proof of Theorem 2. In order to use Leray Schauder Alternative Theorem, next we will shall *a priori* estimates for the solutions of the integral equation $z = \lambda \Gamma z$, $\lambda \in (0, 1)$. If $x^\lambda = \lambda \Gamma x^\lambda$, $\lambda \in (0, 1)$, then from Lemma 1 we have that

$$\begin{aligned}
&\| x^\lambda(t) \| \\
&\leq \widetilde{M} H \| \varphi \|_{\mathcal{B}} + \int_0^t \widetilde{M} \| f(s, \overline{x^\lambda}_{\rho(s,\overline{x}^\lambda_s)}) \| ds \\
&\leq \widetilde{M} H \| \varphi \|_{\mathcal{B}} + \widetilde{M} \int_0^t m(s) W\left((M_a + \widetilde{J}^\varphi) \| \varphi \|_{\mathcal{B}} + K_a \| x^\lambda \|_{\max\{0, \rho(s,\overline{x^\lambda_s})\}} \right) ds \\
&\leq \widetilde{M} H \| \varphi \|_{\mathcal{B}} + \widetilde{M} \int_0^t m(s) W\left((M_a + \widetilde{J}^\varphi) \| \varphi \|_{\mathcal{B}} + K_a \| x^\lambda \|_s \right) ds,
\end{aligned}$$

since $\rho(s, \overline{x^\lambda_s}) \leq s$ for every $s \in I$. If $\xi^\lambda(t) = (M_a + \widetilde{J}^\varphi) \| \varphi \|_{\mathcal{B}} + K_a \| x^\lambda \|_t$, we obtain that

$$\xi^\lambda(t) \leq (K_a \widetilde{M} H + M_a + \widetilde{J}^\varphi) \| \varphi \|_{\mathcal{B}} + K_a \widetilde{M} \int_0^t m(s) W(\xi^\lambda(s)) ds. \tag{7}$$

Denoting by $\beta_\lambda(t)$ the right hand side of (7), follows that

$$\beta'_\lambda(t) \leq K_a \widetilde{M} m(t) W(\beta_\lambda(t))$$

and hence

$$\int_{\beta_\lambda(0)=C}^{\beta_\lambda(t)} \frac{ds}{W(s)} \leq K_a \widetilde{M} \int_0^a m(s)ds < \int_C^\infty \frac{ds}{W(s)},$$

which implies that the set of functions $\{\beta_\lambda(\cdot) : \lambda \in (0,1)\}$ is bounded in $C(I : \mathbb{R})$ and consequently that $\{x^\lambda(\cdot) : \lambda \in (0,1)\}$ is also bounded in $C(I : X)$.

Arguing as in the proof of Theorem 2 we can prove that $\Gamma(\cdot)$ is completely continuous.

These remarks and Theorem 1 shows the existence of mild solution for (1)-(2). The proof is now completed.

4 Examples

In this section we discuss briefly a pair of examples of partial functional differential equations with state dependent delay. At first, and by completeness, we introduce the phase space which will be used to describe these systems.

4.1 The space $\mathbf{C_r \times L^p(g; X)}$

Let $g : (-\infty, -r) \to \mathbb{R}$ be a positive Lebesgue integrable function and assume that there exists a non-negative and locally bounded function γ on $(-\infty, 0]$ such that $g(\xi + \theta) \leq \gamma(\xi) g(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. The space $C_r \times L^p(g; X)$ consists of all classes of functions $\varphi : (-\infty, 0] \to X$ such that φ is continuous on $[-r, 0]$, Lebesgue-measurable and $g \parallel \varphi \parallel^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(g : X)$ is defined by

$$\parallel \varphi \parallel_{\mathcal{B}} := \sup\{\parallel \varphi(\theta) \parallel : -r \leq \theta \leq 0\} + \left(\int_{-\infty}^{-r} g(\theta) \parallel \varphi(\theta) \parallel^p d\theta \right)^{1/p}.$$

Assume that $g(\cdot)$ verifies the conditions (g-5), (g-6) and (g-7) in [13]. In this conditions, $\mathcal{B}$ verifies axioms **(A)**, **(A-1)**, **(B)** see [13, Theorem 1.3.8] for details. Moreover, when $r = 0$ and $p = 2$ we have that $H = 1$, $M(t) = \gamma(-t)^{1/2}$ and $K(t) = 1 + \left(\int_{-t}^0 g(\theta)\, d\theta \right)^{1/2}$ for $t \geq 0$.

4.2 The space $\mathbf{C_g^0(X)}$

Let $g : (-\infty, 0] \to [0, \infty)$ be a continuous positive function such that $g(0) = 1$ and assume that $g(\cdot)$ verifies conditions (g-1) and (g-2) in [13]. Let $C_g^0(X)$ be the space consisting of all continuous functions $\varphi : (-\infty, 0] \to X$ such that $\dfrac{\|\varphi(\theta)\|}{g(\theta)} \to 0$ as $\theta \to -\infty$ endowed with the norm

$$\|\varphi\|_g := \sup_{-\infty < \theta \leq 0} \frac{\|\varphi(\theta)\|}{g(\theta)}.$$

Then, C_g^0 satisfies axioms **(A)**, **(A-1)**, **(B)**, see [13, Theorem 1.3.2]. Moreover, $H = 1$, $K(t) = \sup\{g^{-1}(\theta) : -t \leq \theta \leq 0\}$ and $M(t) = G(t) = e^{\varepsilon}e^{-\gamma(\varepsilon)t}$ for $t \geq 0$, see [13, Theorem 1.3.6].

Remark 2 *The previous spaces verifies axiom C_2 in the nomenclature of [13]. In this case, there exists a constant $\mathrm{L} > 0$ such that $\| \varphi \|_{\mathcal{B}} \leq \mathrm{L} \sup_{\theta \leq 0} \| \varphi(\theta) \|$ for each $\varphi \in \mathcal{B}$ continuous and bounded, see [13, p.10] and [13, Proposition 7.1.1]. In particular, we observe that $\| \varphi_t \|_{\mathcal{B}} \leq L \frac{\sup_{\theta\leq 0}\|\varphi(\theta)\|}{\|\varphi\|_{\mathcal{B}}} \| \varphi \|_{\mathcal{B}}$ for every continuous and bounded function $\varphi \in \mathcal{B}$ and every $t \leq 0$.*

Let $X = L^2([0, \pi])$ and the operator $Af = f''$ with domain

$$D(A) := \{f \in L^2([0, \pi]) : f'' \in L^2([0, \pi]),\ f(0) = f(\pi) = 0\}.$$

It is well known that A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X. Moreover, A has discrete spectrum, the eigenvalues are $-n^2, n \in I\!N$, with corresponding normalized eigenvectors $z_n(\xi) := \left(\frac{2}{\pi}\right)^{1/2} \sin(n\xi)$ and the following properties hold:

1. $\{z_n : n \in I\!N\}$ is an orthonormal basis of X.
2. For $f \in X$, $T(t)f = \sum_{n=1}^{\infty} e^{-n^2 t} < f, z_n > z_n$ which implies that the semigroup is compact.
3. For $f \in D(A)$, $Af = -\sum_{n=1}^{\infty} n^2 < f, z_n > z_n$.

Consider the differential system

$$\begin{aligned}\frac{\partial u(t,\xi)}{\partial t} &= \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + \int_{-\infty}^{t} a_1(s-t)u(s - \rho_1(t)\rho_2(\int_0^{\pi} a_2(\theta) \mid u(t,\theta) \mid^2 d\theta), \xi)ds \\ & \qquad t \in I = [0, a], \xi \in [0, \pi], \end{aligned} \tag{8}$$

submitted to the initial conditions

$$u(t,0) = u(t,\pi) = 0, \qquad t \geq 0, \tag{9}$$

$$u(\tau,\xi) = \varphi(\tau,\xi), \qquad \tau \leq 0, \quad 0 \leq \xi \leq \pi, \tag{10}$$

where $\varphi \in \mathcal{B} = C_0 \times L^2(g; X)$, the functions $a_i : I\!R \to I\!R$, $\rho_i : [0, \infty) \to [0, \infty)$, $i = 1, 2$, are continuous, functions $a_2(\cdot)$ is positive and $L_1 = (\int_0^{\infty} \frac{a_1^2(s)}{g(s)} ds)^{\frac{1}{2}} < 1$.

By defining the operators $f : I \times \mathcal{B} \to X$, $\rho : I \times \mathcal{B} \to I\!R$ by

$$\begin{aligned} f(t,\psi)(\xi) &= \int_{-\infty}^{0} a_1(s)\psi(s,\xi)ds, \\ \rho(s,\psi) &= s - \rho_1(s)\rho_2\left(\int_0^{\pi} a_2(\theta) \mid \psi(0,\xi) \mid^2 d\theta\right), \end{aligned}$$

we can transform system (8)-(10) into the abstract Cauchy problem (1)-(2). Moreover, $f(\cdot)$ is a continuous linear operator and $\| f \|_{\mathcal{L}(\mathcal{B},X)} \leq L_1$.

The next result is a consequence of Theorem 3 and Remark 2.

Theorem 4 *Let $\varphi \in \mathcal{B}$ be continuous and bounded. Then there exists a mild solution of (8)-(10) on I.*

Now, we consider briefly the existence of mild solution for the differential system

$$\frac{\partial u(t,\xi)}{\partial t} = \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + a(t)b(u(t-\sigma(u(t,0)),\xi), \tag{11}$$

$$u(t,0) = u(t,\pi) = 0, \tag{12}$$

$$u(\tau,\xi) = \varphi(\tau,\xi), \qquad \tau \leq 0, \tag{13}$$

for $t \in I = [0,a]$ and $\xi \in [0,\pi]$. Here, $\varphi \in \mathcal{B} = C_g^0(X)$, the functions $a : I \to I\!R$, $b : I\!R \times J \to I\!R$, $\sigma : I\!R \to I\!R^+$ are continuous and we assume the existence of positive constants b_1, b_2 such that $\mid b(t) \mid \leq b_1 \mid t \mid + b_2$ for every $t \in I\!R$.

By defining the maps $f(t,\psi)(\xi) = a(t)b(\psi(0,\xi))$ and $\rho(t,\psi) = t - \sigma(\psi(0,0))$, we represent system (11)-(12) by the abstract Cauchy problem (1)-(2). Moreover, a simples estimate show that $\| f(t,\psi) \| \leq a(t)[b_1 \| \psi \|_{\mathcal{B}} + b_2\pi^{\frac{1}{2}}]$ for all $(t,\psi) \in I \times \mathcal{B}$.

The next result, is a consequence of Theorem 3 and Remark 2.

Theorem 5 *For every continuous and bounded function $\varphi \in \mathcal{B}$, there exits a mild solution of (11)-(13).*

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Uma Nota Sobre Equações Parciais com Retardamento Dependendo do Estado.

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Resumo

Neste artigo estudamos a existência de soluções para uma clase de equações funcionais com retardo dependendo do estado que podem ser modeladas pelo problema de Cauchy abstrato

$$x'(t) = Ax(t) + f(t, x_{\rho(t,x_t)}), \quad t \in I = [0, a], \tag{1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{2}$$

sendo A o gerador infinitesimal de um C_0-semigrupo de operadores lineares limitados definido sobre um espaço de Banach X, $\mathcal{B}$ um espaço de fase definido de maneira axiomática e $f : [0, a] \times \mathcal{B} \to X$, $\rho : \mathbb{R} \times \mathcal{B} \to \mathbb{R}$ funções apropriadas.

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