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The Space of Quasi-Homogeneous Maps in Two Variables of Weights (1/3, 1/6) and Degree 1

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The space of quasi-homogeneous maps in two variables of weights (1/3 , 1/6) and degree 1.

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Summary

In this paper we study the geometry of the space of quasi-homogeneous maps in two variables of weights $\left(\frac{1}{3}, \frac{1}{6}\right)$ and degree one. As in the case of homogeneous maps ([BKL],[Z]) the orbit of x^3 consists of simple cusps. We get 9 kind of orbits, some of them abbuting the orbits of y^6 and $-y^6$. As in a previous work we study a fundamental region, a torus.

Resumo

Continuando com a idéia de Zeeman ([2]), estudamos a geometria das funções quasi-homogeneas com duas variáveis de pesos $\left(\frac{1}{3}, \frac{1}{6}\right)$ e grau 1.

Analogamente ao resultado obtido em ([BKL]), o problema se restringe a órbitas em um toro e a órbita de x^3 resulta formada de cúspides. Neste caso, as órbitas de y^6 e $-y^6$ não pertencem à órbita de x^3, mas resultam interessantes no toro já que elas estão nos fechos topológicos das outras.

É interessante observar que neste caso as órbitas têm codimensão um ao menos, assim que temos uma infinidade de órbitas, mas só um número finito de modelos.

Introduction

In this work we study the geometry of the space quasi-homogeneus maps in two variables of weights $\left(\frac{1}{3}, \frac{1}{6}\right)$ and degree one. The group acting on this space is of dimension three, therefore the orbits are at most 3-dimentional. We shall prove that in fact we have nine models, two of them, parametrized by a real parameter.

We will define the stabilizers of these orbits and we will be interested in the ones that they are finite. Not all orbits of them are finitely determined on the right, so we have

*Apoio Parcial: PAPIIT IN110803-3 "Foliaciones Geométricas y Singularidades"

to introduce a concept of stability. Some parameters are very interesting since they are related with the cross-ratio of four lines in the plane.

Finally we study such orbits on a torus and prove that the points corresponding to the orbit of x^3 are simple cusps. We refer to the paper of Professor Show Han Man,([S]) edited by the Banach Center Publications, which refers also to the quasi-homogeneus map $\mathcal{J}_{10}$.

Definition. *We say that a polynomial p in two variables is quasi-homogeneous with weights (α, β), where α, β are positive rational numbers, and of degree one, if $p(t^\alpha x, t^\beta y) = t\,p(x,y)$.*

Example. The polynomials with weights: $(\frac{1}{3}, \frac{1}{6})$ and degree one is the real vector space Q of dimension four, generated by: x^3, x^2y^2, xy^4, y^6.

Definition. *We say that a polynomial p in n-variables is quasi-homogeneous with weights $(\alpha_1, \ldots, \alpha_n)$, where each α_i is a positive rational number, and of degree one, if $p(t^{\alpha_1}x_1, \ldots, t^{\alpha_n}x_n) = t\,p(x_1 \ldots, x_n)$.*

Lemma. *Consider G the set of of two by two triangular matrices with the product*

$$\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \cdot \begin{pmatrix} \alpha' & \beta' \\ 0 & \delta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' & \alpha\beta' + \beta\delta'^2 \\ 0 & \delta\delta' \end{pmatrix}$$

then G is a group.

Proof. Our operation is well defined, clearly the identity element (the identity matrix) is in G and

$$\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}^{-1} = \begin{pmatrix} 1/\alpha & -\beta/\alpha\delta^2 \\ 0 & 1/\delta \end{pmatrix}$$

$\square$

Consider Q, and the action of the diffeomorphisms of the form $h(x,y) = (\alpha x + \beta y^2\,,\, \delta y)$ with $\alpha\delta \neq 0$, defined by composition:

$$\begin{aligned}(ax^3 + bx^2y^2 + cxy^4 + dy^6) \circ (\alpha x + \beta y^2\,,\, \delta y) &= a\alpha^3 x^3 + (3\alpha^2\beta + \alpha\delta^2)x^2y^2 + \\ + \quad (3\alpha\beta^2 + 2\alpha\beta\delta^2 + \alpha\delta^4)xy^4 &+ (\beta^3 + \beta^2\delta^2 + \beta\delta^4 + \delta^6)y^6\,.\end{aligned}$$

Hence if $h'(x,y) = (\alpha' x + \beta' y^2\,,\, \delta' y)$, we get

i) $h \circ h'(x,y) = (\alpha\alpha' x + (\alpha\beta' + \beta\delta'^{\,2})y^2\,,\, \delta\delta' y)$

ii) $h^{-1}(x,y) = \left(\frac{1}{\alpha}x - \frac{\beta}{\alpha\delta^2}y^2\,,\,\frac{1}{\delta}\,y\right)$

Hence we have a motivation to study our triangular matrices with the above product. We will be speaking about the group G and this group of diffeomorphisms **indistinctively**.

We will denote by T^+, the group of invertible two by two triangular matrices with the usual product and $\delta > 0$, G^+ the subgroup of matrices of G with $\delta > 0$ and $h : T^+ \longrightarrow G^+$ the bijection given by

$$h\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & \delta^{1/2} \end{pmatrix}$$

We clearly have the following

Lemma. *Consider K the vector space of cubic in two variables under the action of T^+ and Q under the action of G^+. If ϕ is the natural isomorphism between K and Q and h is the bijection given above, hence we have*

$$\begin{aligned} &\phi((ax^3 + bx^2y + cxy^2 + dy^3) \circ (\alpha x + \beta y\,,\,\delta y)) \;= \\ = \;\; &\phi(ax^3 + bx^2y + cxy^2 + dy^3) \cdot h(\alpha x + \beta y\,,\,\delta y) \end{aligned}$$

Classification of quasi-homogeneous maps in Q

We want to give the representatives of the orbits of Q under the action of G. Let $p(x,y) = ax^3 + bx^2y^2 + cxy^4 + dy^6$, hence $p(x,y) = \prod_{i=1}^{3}(a_ix + b_iy^2)$ with a_i, b_i complex numbers.

First case: $p(x,y) = (ax + by^2)^3$, $a, b \in \mathbb{R}$.

1.1) If $a = 0$, $p(x,y) = b^3y^6 = \pm(|b|^{1/2}y)^6$. Hence we have two orbits: $\pm y^6$, $+$ for $b > 0$ and $-$ for $b < 0$.

1.2) If $a \neq 0$, $p(x,y) = (ax + by)^3$. Hence we have one orbit: x^3.

Stabilizers (over $\mathbb{C}$)

1.1) $\alpha \neq 0$ and $\delta^6 = 1$.

1.2) $\alpha^3 = 1$, $\beta = 0$ and $\delta \neq 0$.

Second case: $p(x,y) = (ax+by^2)^2(cx+dy^2)$ such that $ad-bc \neq 0$.

2.1) $a \neq 0$ and $c = 0$ $(d \neq 0)$.

If $h(x,y) = (ax+by^2\,,\,|d|^{1/2}y)$, hence $\pm x^2y^2 \circ h(x,y) = \pm(ax+by^2)^2|d|y^2$, and we have two orbits: $\pm x^2y^2$, $+$ for $d>0$ and $-$ for $d<0$.

2.2) $a \neq 0$ and $c \neq 0$.

We can choose α and β non zero, such that $h(x,y) = \left(\frac{ax+by^2}{\alpha}\,,\,\frac{y}{\beta}\right)$ is a diffeomorphism and

$$x^2(x \pm y^2) \circ h(x,y) = (ax+by^2)^2(cx+dy)\,,$$

$+$ for $(ad-bc)a > 0$ and $(ad-bc)a < 0$.

2.3) $a = 0$.

If $h(x,y) = (cx+dy^2\,,\,|b|^{1/2}y)$, hence $xy^4 \circ h(x,y) = (cx+dy^2)b^2y^4 = p(x,y)$

Stabilizers (over $\mathbb{C}$)

2.1) $\beta = 0$ and $\alpha^2\delta^2 = 1$.

2.2) $\left\{\begin{pmatrix}1 & 0\\ 0 & \pm 1\end{pmatrix}, \begin{pmatrix}\xi & 0\\ 0 & \pm\xi^2\end{pmatrix}, \begin{pmatrix}\xi^2 & 0\\ 0 & \pm\xi\end{pmatrix}\right\}$ where $\xi^2+\xi+1=0$.

2.3) $\beta = 0$ and $\alpha\delta^4 = 1$.

Remark: For the case 2.2), the stabilizer is isomorphic to Z_6.

Third case: $p(x,y) = (ax+by^2)(cx+dy^2)(ex+dy^2)$ such that any two of the vectors: (a,b), (c,d) and (e,f) are linearly independent.

We can start with $ac \neq 0$ and using 2.2:

$p(x,y)$ is in the orbit of $x(x\pm y^2)(gx+hy^2)$, where $g = e/_{a^{2/3}c^{1/3}}$ and $h = \frac{(af-be)c^{2/3}}{(ad-bc)a^{2/3}}$.

3.1) If $e=0$, $p(x,y)$ is in the orbit of: $\pm(x^2y^2+xy^4)$.

3.2) If $e \neq 0$, $p(x,y)$ is in the orbit of: $\pm x(x+y^2)(x+\varepsilon y^2)$ $(\varepsilon \neq 0,\ \varepsilon \neq 1)$.

Stabilizers (over $\mathbb{C}$)

3.1) Using our notation of matrices:

$$\left\{\begin{pmatrix}\delta^2 & 0\\ 0 & \delta\end{pmatrix} \mid \delta^6=1\right\} \cup \left\{\begin{pmatrix}-\delta^2 & -\delta^2\\ 0 & \delta\end{pmatrix} \mid \delta^6=1\right\}$$

which is clearly an abelian group of order 12 isomorphic to $Z_2 \times Z_6$, since it does not have elements of order 12.

3.2) This case will be done step by step.

We want α, β, δ such that:

$$(x^3+(\varepsilon+1)x^2y^2+\varepsilon xy^4)(\alpha x+\beta y^2\,,\,\delta y) = x^3+(\varepsilon+1)x^2y^2+\varepsilon xy^4$$

or the following system:

$$\begin{aligned} \alpha^3 &= 1\\ 3\alpha^2\beta+(\varepsilon+1)\alpha^2\delta^2 &= \varepsilon+1\\ 3\alpha\beta^2+2(\varepsilon+1)\alpha\beta\delta^2+\varepsilon\alpha\delta^4 &= \varepsilon\\ \beta^3+(\varepsilon+1)\beta^2\delta^2+\varepsilon\beta\delta^4 &= 0 \end{aligned}$$

From the last equation.

$$\text{If } \ \beta=0,\ \alpha^3=1,\ (\varepsilon+1)\alpha^2\delta^2=\varepsilon+1 \ \text{ and } \ \varepsilon\alpha\delta^4=\varepsilon$$

i) If $\varepsilon=-1$: $\alpha^3=1$ and $\alpha\delta^4=1$. Therefore $\varepsilon=-1$, $\alpha^3=1$ and $\delta=\pm\alpha^2$ or $\delta=\pm i\alpha^2$.

ii) If $\varepsilon\neq -1$: $\alpha^3=1$, $\alpha^2\delta^2=1$ and $\alpha\delta^4=1$. Hence $\alpha^3=1$ and $\delta=\pm\alpha^{-1}$.

If $\beta\neq 0$, hence $\beta=\frac{(\varepsilon+1)(\alpha-\delta^2)}{3}$ and $(\varepsilon^2-\varepsilon+1)(\alpha^2-\delta^4)=0$.

iii) If $\varepsilon^2-\varepsilon+1\neq 0$, we get

a) $\alpha^3=1$, $\beta=2\alpha$, $\delta^2=-\alpha$ and $\varepsilon=2$ or

b) $\alpha^3=1$, $\beta=\alpha$, $\delta^2=-\alpha$ and $\varepsilon=1/2$.

iv) If $\varepsilon^2-\varepsilon+1=0$, hence $\varepsilon=\xi+1$ or $\varepsilon=-\xi$, where $\xi^2+\xi+1=0$.

Remark: If we consider the product: $(x+\varepsilon y)xy(x+y)$, hence in the plane: (y,x), the lines

$$x+\varepsilon y=0,\ y=0,\ x=0 \ \text{ and } \ x+y=0$$

have as slopes: ε , ∞ , 0 and -1 respectively.

Define $\lambda_1 = \infty$, $\lambda_2 = -1$, $\lambda_3 = 0$ and $\lambda_4 = -\varepsilon$ and the cross ratio:

$$\frac{\lambda_4 - \lambda_2}{\lambda_4 - \lambda_1} \; \frac{\lambda_3 - \lambda_1}{\lambda_3 - \lambda_2} \, .$$

Under the action of S_4 , the permutation group, we get the complex numbers:

$$\left\{ 1 - \varepsilon \, , \, \frac{1}{1 - \varepsilon} \, , \, \frac{\varepsilon}{\varepsilon - 1} \, , \, \frac{\varepsilon - 1}{\varepsilon} \, , \, \varepsilon \, , \, \frac{1}{\varepsilon} \right\}$$

And we get repetitions only when:

$$\varepsilon \in \left\{ 0 \, , \, \frac{1}{2} \, , \, 2 \, , \, \pm 1 \, , \, \frac{1 \pm \sqrt{3}\, i}{2} \right\} .$$

Stabilizers (over $\mathbb{C}$)

Case (i): $\varepsilon \notin \left\{ -1 \, , \, \frac{1}{2} \, , \, 2 \, , \, \frac{1 \pm \sqrt{3}\, i}{2} \right\}$
Hence $\alpha^3 = 1$, $\beta = 0$ and $\delta = \pm \alpha^2$, therefore as in 2.2, the stabilizer is isomorphic to Z_6 .

Case (ii): $\alpha^3 = 1$, $\beta = 0$, $\delta = \pm \alpha^2$ or $\delta = \pm i \alpha^2$ and $\varepsilon = -1$, we get:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} , \begin{pmatrix} 1 & 0 \\ 0 & \pm i \end{pmatrix} , \begin{pmatrix} \xi & 0 \\ 0 & \pm \xi^2 \end{pmatrix} , \begin{pmatrix} \xi & 0 \\ 0 & \pm i \xi^2 \end{pmatrix} , \begin{pmatrix} \xi^2 & 0 \\ 0 & \pm \xi \end{pmatrix} , \begin{pmatrix} \xi^2 & 0 \\ 0 & \pm i \xi \end{pmatrix} \right\} .$$

Since $\circ \begin{pmatrix} \xi^2 & 0 \\ 0 & i\xi \end{pmatrix} = 12$, the stabilizer is isomorphic to Z_{12}, (Remember our product).

Case (iii)(a): $\alpha^3 = 1$, $\beta = 2\alpha$, $\delta^2 = -\alpha$ and $\varepsilon = 2$, we get:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} , \begin{pmatrix} \xi & 0 \\ 0 & \pm \xi^2 \end{pmatrix} , \begin{pmatrix} \xi^2 & 0 \\ 0 & \pm \xi \end{pmatrix} , \begin{pmatrix} 1 & 2 \\ 0 & \pm i \end{pmatrix} , \begin{pmatrix} \xi & 2\xi \\ 0 & \pm i \xi^2 \end{pmatrix} , \begin{pmatrix} \xi^2 & 2\xi^2 \\ 0 & \pm i \xi \end{pmatrix} \right\} .$$

Since $\circ \begin{pmatrix} \xi & 2\xi \\ 0 & i\xi^2 \end{pmatrix} = 12$, the stabilizer is isomorphic to Z_{12}.

Case (iii)(b): $\alpha^3 = 1$, $\beta = \alpha$, $\delta^2 = -\alpha$ and $\varepsilon = 1/2$, we get:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} , \begin{pmatrix} \xi & 0 \\ 0 & \pm \xi^2 \end{pmatrix} , \begin{pmatrix} \xi^2 & 0 \\ 0 & \pm \xi \end{pmatrix} , \begin{pmatrix} 1 & 1 \\ 0 & \pm i \end{pmatrix} , \begin{pmatrix} \xi & \xi \\ 0 & \pm i \xi^2 \end{pmatrix} , \begin{pmatrix} \xi^2 & \xi^2 \\ 0 & \pm i \xi \end{pmatrix} \right\} .$$

Since $\circ \begin{pmatrix} \xi & \xi \\ 0 & i\xi^2 \end{pmatrix} = 12$, the stabilizer is isomorphic to Z_{12}.

Case (iv):

a) $\varepsilon = \xi + 1$, we get

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} \xi & 0 \\ 0 & \pm\xi^2 \end{pmatrix}, \begin{pmatrix} \xi^2 & 0 \\ 0 & \pm\xi \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & \pm\xi^2 \end{pmatrix}, \begin{pmatrix} 1 & -\xi^2 \\ 0 & \pm\xi \end{pmatrix}, \begin{pmatrix} \xi & -1 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} \xi & \xi \\ 0 & \pm\xi \end{pmatrix}, \begin{pmatrix} \xi^2 & \xi^2 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} \xi^2 & -\xi \\ 0 & \pm\xi^2 \end{pmatrix} \right\}.$$

b) $\varepsilon = -\xi$, we get

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} \xi & 0 \\ 0 & \pm\xi^2 \end{pmatrix}, \begin{pmatrix} \xi^2 & 0 \\ 0 & \pm\xi \end{pmatrix}, \begin{pmatrix} 1 & -\xi \\ 0 & \pm\xi^2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & \pm\xi \end{pmatrix}, \begin{pmatrix} \xi & \xi \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} \xi & -\xi^2 \\ 0 & \pm\xi \end{pmatrix}, \begin{pmatrix} \xi^2 & -1 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} \xi^2 & \xi^2 \\ 0 & \pm\xi^2 \end{pmatrix} \right\}.$$

In both cases, we have groups of order 18 with: one element of order 1, one element of order 2, eight elements of order three and eight elements of order six.

Hence both are isomorphic to $Z_3 \times Z_6$.

Fourth Case: $p(x, y) = (ax + by^2)(cx + dxy^2 + ey^4)$ with $d^2 - 4ce < 0$. Under change of coordinates $p(x, y) = c\left(ax + \left(\frac{4c^2}{4ce-d^2}\right)^{1/2}\left(\frac{2bc-ad}{2c}\right) y^2\right)(x^2 + y^4)$.

3.3) $a = 0$, hence f is in the orbit of $\pm y^2(x^2 \pm y^4)$.

3.4) $a \neq 0$, hence f is in the orbit of $\pm(x + \varepsilon y^2)(x^2 + y^4)$.

Stabilizers (over $\mathbb{C}$)

3.3) We get $\alpha^2\delta^2 = 1$ and $\delta^6 = 1$, hence in our matrix notation:

$$\left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} \pm\xi & 0 \\ 0 & \pm\xi^2 \end{pmatrix}, \begin{pmatrix} \pm\xi^2 & 0 \\ 0 & \pm\xi^2 \end{pmatrix} \right\}.$$

It is an abelian group of order 12, but not cyclic, hence isomorphic to $Z_2 \times Z_6$.

3.4) As in 3.2) we get the next system of equations.

$$\begin{aligned}
\alpha^3 &= 1\\
3\beta + \delta^2\varepsilon &= \varepsilon\,\alpha\\
3\beta^2 + 2\beta\delta^2\varepsilon + \delta^4 &= \alpha^2\\
\beta^3 + \beta^2\delta^2\varepsilon + \beta\delta^4 + \varepsilon\delta^6 &= \varepsilon
\end{aligned}$$

If $\beta = 0$, we get

i) $\alpha^3 = 1\,,\ \ \alpha^2 = \delta^4$ and $\varepsilon = 0$

ii) $\alpha^3 = 1\,,\ \ \delta^6 = 1\,,\ \ \alpha = \delta^2$ and $\varepsilon \neq 0$

In matrix notation

i) $\left\{\begin{pmatrix}1 & 0\\ 0 & \pm 1\end{pmatrix}, \begin{pmatrix}\xi & 0\\ 0 & \pm\xi^2\end{pmatrix}, \begin{pmatrix}\xi^2 & 0\\ 0 & \pm\xi\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & \pm i\end{pmatrix}, \begin{pmatrix}\xi & 0\\ 0 & \pm i\xi^2\end{pmatrix}, \begin{pmatrix}\xi^2 & 0\\ 0 & \pm i\xi\end{pmatrix}\right\}$

Since $\circ\begin{pmatrix}\xi & 0\\ 0 & i\,\xi^2\end{pmatrix} = 12\,,$ this group is isomorphic to Z_{12}:

ii) $\left\{\begin{pmatrix}1 & 0\\ 0 & \pm 1\end{pmatrix}, \begin{pmatrix}\xi & 0\\ 0 & \pm\xi^2\end{pmatrix}, \begin{pmatrix}\xi^2 & 0\\ 0 & \pm\xi\end{pmatrix}\right\}$, isomorphic to Z_6.

If $\beta \neq 0$, hence $\beta = \frac{\varepsilon(\alpha-\delta^2)}{3}$ and we have

iii) $\alpha^3 = 1\,,\ \ \beta = \frac{\alpha-\delta^2}{\sqrt{3}}\,,\ \delta^6 = 1$ and $\varepsilon = \sqrt{3}$.

iii') $\alpha^3 = 1\,,\ \ \beta = -\frac{\alpha-\delta^2}{\sqrt{3}}\,,\ \delta^6 = 1$ and $\varepsilon = -\sqrt{3}$.

In both cases we get a group of order 18 isomorphic to $Z_3 \times Z_6$.

iv) $\alpha^3 = 1\,,\ \ \alpha = -\delta^2\,,\ \beta = -2\,\delta^2 i$ and $\varepsilon = 3i$.

iv') $\alpha^3 = 1\,,\ \ \alpha = -\delta^2\,,\ \beta = 2\,\delta^2 i$ and $\varepsilon = -3i$.

In both cases we get a group isomorphic to Z_{12}.

Remark:

For $\varepsilon = \sqrt{3}$, the stabilizer is

$$\left\{\begin{pmatrix}1 & 0\\ 0 & \pm 1\end{pmatrix}, \begin{pmatrix}\xi & 0\\ 0 & \pm\xi^2\end{pmatrix}, \begin{pmatrix}\xi^2 & 0\\ 0 & \pm\xi\end{pmatrix}, \begin{pmatrix}1 & \frac{1-\xi}{\sqrt{3}}\\ 0 & \pm\xi^2\end{pmatrix}, \begin{pmatrix}\xi & \frac{\xi-1}{\sqrt{3}}\\ 0 & \pm 1\end{pmatrix}, \begin{pmatrix}\xi & \frac{2\xi+1}{\sqrt{3}}\\ 0 & \pm\xi\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}\xi^2 & -\frac{(2+\xi)}{\sqrt{3}}\\ 0 & \pm 1\end{pmatrix}, \begin{pmatrix}1 & \frac{2+\xi}{\sqrt{3}}\\ 0 & \pm\xi\end{pmatrix}, \begin{pmatrix}\xi^2 & -\frac{(2+\xi)}{\sqrt{3}}\\ 0 & \pm\xi^2\end{pmatrix}\right\}.$$

For $\varepsilon = 3i$

$$\left\{\begin{pmatrix}1 & 0\\ 0 & \pm 1\end{pmatrix}, \begin{pmatrix}\xi & 0\\ 0 & \pm\xi^2\end{pmatrix}, \begin{pmatrix}\xi^2 & 0\\ 0 & \pm\xi\end{pmatrix}, \begin{pmatrix}1 & 2i\\ 0 & \pm i\end{pmatrix}, \begin{pmatrix}\xi & 2\xi i\\ 0 & \pm i\xi^2\end{pmatrix}, \begin{pmatrix}\xi^2 & 2\xi^2 i\\ 0 & \pm i\xi\end{pmatrix}\right\}.$$

The cases $\varepsilon = -\sqrt{3}$ and $\varepsilon = -3i$, we put a negative sign in the place $(1,2)$.

If we consider $(x+\varepsilon y)y(x+iy)(x-iy)$ and as before the slopes in the (y,x) plane: $\lambda_1 = \infty$, $\lambda_2 = -i$, $\lambda_3 = i$ and $\lambda_4 = -\varepsilon$, under the action of S_4, the permutation group, the cross ratios are

$$\left\{\frac{2i}{-\varepsilon+i}, \frac{-\varepsilon+i}{2i}, \frac{-\varepsilon-i}{-\varepsilon+i}, \frac{-\varepsilon+i}{-\varepsilon-i}, \frac{2i}{\varepsilon+i}, \frac{\varepsilon+i}{2i}\right\}$$

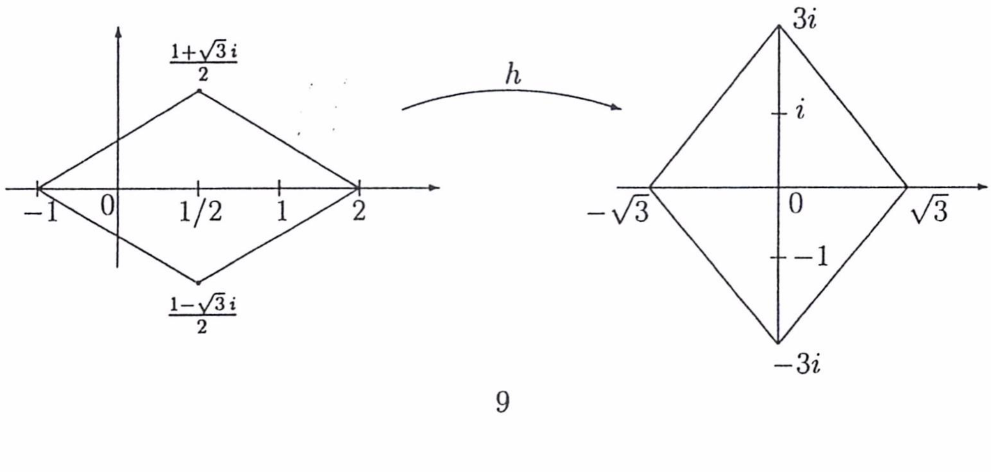
And we get repetitions only when

$$\varepsilon \in \left\{\pm i,\ 0,\ \pm\sqrt{3},\ \pm 3i\right\}$$

In the real case we had

$$\varepsilon \in \left\{0,\ \frac{1}{2},\ 2,\ \pm 1,\ \frac{1\pm\sqrt{3}\,i}{2}\right\}$$

Let $h : \mathbb{C} \to \mathbb{C}$ given by $h(z) = -2iz + i$, then $h(0) = i$, $h(\frac{1}{2}) = 0$, $h(2) = -3i$, $h(1) = -i$, $h(-1) = 3i$, $h\left(\frac{1}{2} + \frac{\sqrt{3}i}{2}\right) = \sqrt{3}$ and $h\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -\sqrt{3}$.



type	representative	stabilizer	codim.
1.1	$\pm y^6$	$\delta^6 = 1$	2
1.2	x^3	$\alpha^3 = 1\ ,\ \beta = 0$	1
2.1	$\pm x^2 y^2$	$\beta = 0\ ,\ \alpha^2\delta^2 = 1$	1
2.2	$x^2(x \pm y^2)$	$\mathbb{Z}_6$	0
2.3	xy^4	$\beta = 0\ ,\ \alpha\delta^4 = 1$	1
3.1	$\pm(x^2y^2 + y^4)$	$\mathbb{Z}_2 \times \mathbb{Z}_6$	0
3.2	$\pm x(x + y^2)(x + \varepsilon y^2)$	Z_6 $\varepsilon \notin \left\{-1, \frac{1}{2}, 2, \frac{1 \pm \sqrt{3}\,i}{2}\right\}$	0
		Z_{12} $\varepsilon \in \{-1, 2, \frac{1}{2}\}$	0
		$Z_3 \times Z_6$ $\varepsilon \in \left\{\frac{1 \pm \sqrt{3}\,i}{2}\right\}$	0
3.3	$\pm y^2(x^2 + y^4)$	$\mathbb{Z}_2 \times \mathbb{Z}_6$	0
3.4	$\pm(x + \varepsilon y^2)(x^2 + y^4)$	Z_6 $\varepsilon \notin \{0, \pm\sqrt{3}, \pm 3i\}$	0
		Z_{12} $\varepsilon \in \{0, 3i, -3i\}$	0
		$Z_3 \times Z_6$ $\varepsilon \in \{\sqrt{3}, -\sqrt{3}\}$	0

Definition. *We say that a quasi-homogeneous germ η of weights $(\frac{1}{3}, \frac{1}{6})$ of degree 1 is stable if there exists a neighborhood V of η such that all the germs in V are of the same type.*

Theorem. *The quasi-homogeneous germs of weights $(\frac{1}{3}, \frac{1}{6})$ satisfy*

i) If the stabilizer is infinite or isomorphic to $Z_2 \times Z_6$, they are not finitely determined on the right neither stable.

ii) If the stabilizer is isomorphic to Z_{12} or $Z_3 \times Z_6$ they are finitely determined.

iii) If the stabilizer of η is isomorphic to Z_6, hence η is stable if and only if η is finitely determined.

Topology of the fibers of x^2y

We consider the map $h : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ given by $h(a,b,c,d) = (a^2c, a^2d + 2abc, b^2c + 2abd, b^2d)$. We recall that the counter domain is identified with the four dimensional space of the quasi-homogeneous polynomials of weights $(\frac{1}{3}, \frac{1}{6})$ and degree 1.

Examples

i) $h(0,1,0,\pm 1) = \pm y^6$

ii) $h(1,0,1,0) = x^3$

iii) $h(1,0,0,\pm 1) = \pm x^2y^2$

iv) $h(1,0,1,\pm 1) = x^3 \pm x^2y^2$

v) $h(0,1,1,0) = xy^4$

We shall denote by: $\mathcal{O}^+_{1.1}$, $\mathcal{O}^-_{1.1}$, $\mathcal{O}_{1,2}$, $\mathcal{O}^+_{2.1}$, $\mathcal{O}^-_{2.1}$, $\mathcal{O}^+_{2,2}$, $\mathcal{O}^-_{2,2}$, $\mathcal{O}_{2,3}$, the orbits of y^6, $-y^6$, x^3, x^2y^2, $-x^2y^2$, $x^3 + x^2y^2$, $x^3 - x^2y^2$ and xy^4 respectively.

Remark. $h(\mathbb{R}^2 \times \{\bar{0}\}) = h(\{\bar{0}\} \times \mathbb{R}^2) = \bar{0}$ or more precisely: $h^{-1}(\bar{0}) = \mathbb{R}^2 \times \{\bar{0}\} \cup \{\bar{0}\} \times \mathbb{R}^2$.

We have the following cases

i) $h^{-1}h(\alpha,\beta,\gamma,\delta) = \left\{ \left(a\,, \frac{\beta}{\alpha} a\,, \frac{\alpha^2\gamma}{a^2}\,, \frac{\gamma^2\delta}{a^2} \right) \mid a \neq 0 \right\}$

ii) $h^{-1}h(\alpha,\beta,\gamma,\delta) = \left\{ \left(0\,, b\,, \frac{\beta^2\gamma}{b^2}\,, \frac{\beta^2\delta}{b^2} \right) \mid a = 0 \right\}$

The nearest points to the origin in both cases are

$$\left(\pm\Gamma\alpha\,, \pm\Gamma\beta\,, \frac{\gamma}{\Gamma^2}\,, \frac{\delta}{\Gamma^2} \right), \quad \text{where } \Gamma = \left(\frac{2(\gamma^2+\delta^2)}{\alpha^2+\beta^2} \right)^{1/6}$$

and if we intersect with the 3-sphere, we get

$$(\alpha^2+\beta^2)^2(\gamma^2+\delta^2) = \frac{4}{27}\,.$$

Therefore, if $\alpha = r\cos\theta$, $\beta = r\,\mathrm{sen}\theta$, $\gamma = s\cos\varphi$ and $\delta = s\,\mathrm{sen}\varphi$, $r^4s^2 = 4$ and $\Gamma = (\frac{2}{3})^{1/2}\frac{1}{r}$.

Our nearest points in the 3-sphere are

$$\left(\pm(\frac{2}{3})^{1/2}\cos\theta\,, \pm(\frac{2}{3})^{1/2}\,\mathrm{sen}\theta\,, (\frac{1}{3})^{1/2}\cos\varphi\,, (\frac{1}{3})^{1/2}\,\mathrm{sen}\varphi \right)$$

which is a point in $S^1_{\sqrt{2/3}} \times S^1_{\sqrt{1/3}}$.

Remark 1. If $\gamma = \lambda\alpha$ and $\delta = \lambda\beta$, hence $\Gamma = 2^{1/6}\lambda^{1/3}$, and for $\lambda = 1$, our nearest points are

$$\left(\pm(\frac{2}{3})^{1/2}\cos\theta\,,\ \pm(\frac{2}{3})^{1/2}\,\mathrm{sen}\theta\,,\ (\frac{1}{3})^{1/2}\cos\theta\,,\ (\frac{1}{3})^{1/2}\,\mathrm{sen}\theta\right)$$

Remark 2. **i)** If we consider (for $a \neq 0$) the parametrization of the fiber intersected with the 3-sphere, we get ($a = t$)

$$t^6 - \left(\frac{\alpha^2}{\alpha^2+\beta^2}\right)t^4 + \alpha^6\left(\frac{\gamma^2+\delta^2}{\alpha^2+\beta^2}\right) = 0 \qquad (*)$$

If we denote $z = t^2$, we get

$$z^3 - \left(\frac{\alpha^2}{\alpha^2+\beta^2}\right)z^2 + \frac{\alpha^6(1-(\alpha^2+\beta^2))}{\alpha^2+\beta^2} = 0 \qquad (**)$$

and its discriminant is zero if and only if

$$(\alpha^2+\beta^2)^2(\gamma^2+\delta^2) = \frac{4}{27}$$

or

$$(\alpha^2+\beta^2)^2(1-(\alpha^2+\beta^2)) = \frac{4}{27}\,.$$

If we denote $\xi = \alpha^2+\beta^2$, we get

$$\xi^3 - \xi^2 + \frac{4}{27} = 0\,.$$

Therefore $\xi = \frac{2}{3}$ (twice), hence

$$\alpha^2+\beta^2 = \frac{2}{3} \quad \text{and} \quad \gamma^2+\delta^2 = \frac{1}{3}\,.$$

Therefore $(**)$ remains

$$z^3 - \frac{3}{2}\alpha^2 z^2 + \frac{1}{2}\alpha^6 = 0\,.$$

The roots are clearly α^2 (twice) and $-\frac{\alpha^2}{2}$,

ii) For the case $a = 0$, our sextic will be

$$t^6 - t^4 + \beta^4(\gamma^2+\delta^2).$$

The discriminant is zero if and only if $\beta^6 - \beta^4 + \frac{4}{27} = 0$.
Therefore $\beta^2 = \frac{2}{3}$ or $\beta = \pm\sqrt{\frac{2}{3}}$ and we get

i) For $t = \pm\alpha$, $\{(\alpha,\beta,\gamma,\delta) \mid \alpha^2+\beta^2 = \frac{2}{3}\,,\ \gamma^2+\delta^2 = \frac{1}{3}\}$

ii) For $t = \pm\sqrt{\frac{2}{3}}$, $\left\{(0, \pm\sqrt{\frac{2}{3}}, \gamma, \delta) \mid \gamma^2 + \delta^2 = \frac{1}{3}\right\}$.

Hence we have the following

Theorem. *Let $h : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by $h(a, b, c, d) = (a^2c, a^2d + 2abc, b^2c + 2abd, b^2d)$, hence*

i) The pre-images of a point of the form $(\alpha^2\gamma\,,\ \alpha^2\delta + 2\alpha\beta\gamma\,,\ \beta^2\gamma + 2\alpha\beta\delta\,,\ \beta^2\delta)$ are

a) $\left\{\left(t\,,\ \frac{\beta}{\alpha}t\,,\ \frac{\alpha^2\gamma}{t^2}\,,\ \frac{\alpha^2\delta}{t^2}\right) \mid t \neq 0\right\}$ *if* $\alpha \neq 0$

b) $\left\{\left(0\,,\ t\,,\ \frac{\beta^2\gamma}{t^2}\,,\ \frac{\beta^2\delta}{t^2}\right) \mid t \neq 0\right\}$ *if* $\alpha = 0$

ii) The points of smallest norm in each fiber are of the form

a) $(\pm\Gamma\alpha\,,\ \pm\Gamma\beta\,,\ \frac{\gamma}{\Gamma^2}\,,\ \frac{\delta}{\Gamma^2})$, *where* $\Gamma = \left(\frac{2(\gamma^2+\delta^2)}{\alpha^2+\beta^2}\right)^{1/6}$

b) $(0\,,\ \pm\Gamma\beta\,,\ \frac{\gamma}{\Gamma^2}\,,\ \frac{\delta}{\Gamma^2})$, *where* $\Gamma = \left(\frac{2(\gamma^2+\delta^2)}{\beta^2}\right)^{1/6}$

iii) The points of smallest norm which are of norm one in fact are in the torus

$$S^1_{\sqrt{2/3}} \times S^1_{\sqrt{1/3}}\,.$$

iv) The points in a fiber which minimize the distance come from a cubic. The case this cubic has discriminant zero is the same of case iii).

We will continue to work with our map h restricted to the torus $S^1_{\sqrt{2/3}} \times S^1_{\sqrt{1/3}}$.

Lemma. *We have the following fibers*

1.1) $h^{-1}(0, 0, 0, \pm\delta^6) = \left\{\left(0\,,\ \pm\left(\frac{2}{3}\right)^{1/2}\,,\ 0\,,\ \pm\left(\frac{1}{3}\right)^{1/2}\right)\right\}$

1.2) $h^{-1}\left\{(\alpha^3, 3\alpha^2\beta^2, 3\alpha\beta^4, \beta^6)\right\} =$

$$= \left\{\left(\pm\left(\tfrac{2}{3}\right)^{1/2}\cos\theta\,,\ \pm\left(\tfrac{2}{3}\right)^{1/2} sen\theta\,,\ \left(\tfrac{1}{3}\right)^{1/2}\cos\theta\,,\ \left(\tfrac{1}{3}\right)^{1/2} sen\theta\right) \mid \theta \neq \tfrac{\pi}{2}\,,\ \theta \neq \tfrac{3}{2}\pi\right\}$$

2.1) $h^{-1}\left\{(0, \pm\alpha^2\delta^2, \pm2\alpha\beta\delta^2, \pm\beta^2\delta^2)\right\} = \left\{\left(\pm\sqrt{\frac{2}{3}}\cos\theta\,,\ \pm\sqrt{\frac{2}{3}}\, sen\theta\,,\ 0\,,\ \pm\sqrt{\frac{1}{3}}\right)\right\}$

2.3) $h^{-1}\left\{(0, 0, \alpha\delta^4, \beta\delta^4)\right\} = \left\{\left(0\,,\ \pm\sqrt{\frac{2}{3}}\right)\right\} \times S^1_{\sqrt{1/3}} - \left\{\left(0\,,\ \pm\frac{1}{\sqrt{3}}\right)\right\}$

2.2) $h^{-1}\left(\theta^+_{2,2}\right) \cup h^{-1}\left(\theta^-_{2,2}\right)$ *is the complement.* □

Remark.

a) $h^{-1}\left(\alpha^3, 3\alpha^2\beta-\alpha^2\delta^2, 3\alpha\beta^2-2\alpha\beta\delta^2, \beta^3-\beta^2\delta^2\right) =$

$$= \left(\pm\sqrt{\tfrac{2}{3}}\cos\theta\,,\ \pm\sqrt{\tfrac{2}{3}}\operatorname{sen}\theta\,,\ \sqrt{\tfrac{1}{3}}\cos\varphi\,,\ \sqrt{\tfrac{1}{3}}\operatorname{sen}\varphi\right) \text{ with } \tan\theta = \tfrac{\beta}{\alpha} \text{ and } \tan\varphi = \tfrac{\beta-\delta^2}{\alpha}.$$

b) $h^{-1}\left(\alpha^3, 3\alpha^2\beta+\alpha^2\delta^2, 3\alpha\beta^2+2\alpha\beta\delta^2, \beta^3+\beta^2\delta^2\right) =$

$$= \left(\pm\sqrt{\tfrac{2}{3}}\cos\theta\,,\ \pm\sqrt{\tfrac{2}{3}}\operatorname{sen}\theta\,,\ \sqrt{\tfrac{1}{3}}\cos\psi\,,\ \sqrt{\tfrac{1}{3}}\operatorname{sen}\psi\right) \text{ with } \tan\theta = \tfrac{\beta}{\alpha} \text{ and } \tan\psi = \tfrac{\beta+\delta^2}{\alpha}.$$

In terms of coordinates (θ, φ) we get

$$\begin{aligned} h(\theta,\varphi) \;=\; & \left(\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right)^{1/2}\left(\cos^2\theta\cos\varphi\,,\ \cos^2\theta\operatorname{sen}\varphi\,+\right. \\ & \left. +\,2\cos\theta\operatorname{sen}\theta\cos\varphi,\ \operatorname{sen}^2\theta\cos\varphi+2\cos\theta\operatorname{sen}\theta\operatorname{sen}\varphi,\ \operatorname{sen}^2\theta\operatorname{sen}\varphi\right) \end{aligned}$$

Since $h(\theta,\varphi) = h(\theta+\pi\,,\,\varphi)$, we get the following diagram:

$$\begin{tikzcd} S^1_{\sqrt{\frac{2}{3}}} \times S^1_{\sqrt{\frac{1}{3}}} \arrow[r, "h"] \arrow[d, "\pi\times Id"] & \mathbb{R}^4 \\ \mathbb{R}P^1_{\sqrt{\frac{2}{3}}} \times S^1_{\sqrt{\frac{1}{3}}} \arrow[ur, "\widetilde{h}"'] & \end{tikzcd}$$

Where $\widetilde{h}$ is s differentiable homeomorphism to its image.

Since $\pi : S^1_{\sqrt{\frac{2}{3}}} \longrightarrow \mathbb{R}P^1_{\sqrt{\frac{2}{3}}}$ is a diffeomorphism with $\pi^{-1}(\theta) = \frac{\theta}{2}$, we get

$$\begin{aligned} \widetilde{h}(\theta,\varphi) \;=\; & \left(\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right)^{1/2}\left(\cos^2\tfrac{\theta}{2}\cos\varphi\,,\ \cos^2\tfrac{\theta}{2}\operatorname{sen}\varphi\,+\right. \\ & \left. +\,2\cos\tfrac{\theta}{2}\operatorname{sen}\tfrac{\theta}{2}\cos\varphi,\ \operatorname{sen}^2\tfrac{\theta}{2}\cos\varphi+2\cos\tfrac{\theta}{2}\operatorname{sen}\tfrac{\theta}{2}\operatorname{sen}\varphi,\ \operatorname{sen}^2\tfrac{\theta}{2}\operatorname{sen}\varphi\right) \end{aligned}$$

Lemma. *Consider our map* $\widetilde{h} : S^1_{\sqrt{\frac{2}{3}}} \times S^1_{\sqrt{\frac{1}{3}}} \longrightarrow \mathbb{R}^4$, *as a map* $\widetilde{h} : [-\pi,\pi]\times[-\pi,\pi] \longrightarrow \mathbb{R}^4$, *hence*

i) rank $d\widetilde{h}(\theta,\varphi)$ *is one if and only if* $\varphi = \frac{\theta}{2}$ *or* $\varphi = \frac{\theta}{2}+\pi$.

ii) for such points, $\ker d\widetilde{h}$ *is generated by* $v = (1,-1)$.

Proof. It is clear that the columns of $d\widetilde{h}(\theta,\varphi)$ are parallel if and only if $\varphi = \frac{\theta}{2}$ or $\varphi = \frac{\theta}{2}+\pi$.

Therefore our singularities in our torus are the image of the segments

a) $\left\{\left(\theta,\frac{\theta}{2}\right) \mid -\pi \le \theta \le \pi\right\}$

b) $\left\{\left(\theta,\frac{\theta}{2}+\pi\right) \mid -\pi \le \theta \le 0\right\} \cup \left\{\left(\theta,\frac{\theta}{2}-\pi\right) \mid 0 \le \theta \le \pi\right\}$

In fact the columns of $d\widetilde{h}$ in such points are the same.

Proposition. *The singularities of* $\widetilde{h} : [-\pi, \pi] \times [-\pi, \pi] \rightarrow \mathbb{R}^4$ *are cusps.*

Proof. We will consider the map

$$g(t) = \widetilde{h}\left(\left(t_0\,,\,\frac{t_0}{2}\right) + t(1\,,\,-1)\right) = \widetilde{h}\left(t_0 + t\,,\,\frac{t_0}{2} - t\right).$$

1^st coordinate: $\cos^2\left(\frac{t_0+t}{2}\right)\cos\left(\frac{t_0}{2}-t\right).$

It's 3-jet will be

$$\cos^3\frac{t_0}{2} + \left(\frac{-3}{4}\right)\left(\cos^3\frac{t_0}{2} + \operatorname{sen}^2\frac{t_0}{2}\cos\frac{t_0}{2}\right)t^2 + \frac{1}{4}\left(\operatorname{sen}\frac{t_0}{2}\cos^2\frac{t_0}{2} - \operatorname{sen}^3\frac{t_0}{2}\right)t^3.$$

2^d coordinate: $\cos^2\left(\frac{t_0+t}{2}\right)\operatorname{sen}\left(\frac{t_0}{2}-t\right) + 2\cos\left(\frac{t_0+t}{2}\right)\operatorname{sen}\left(\frac{t_0+t}{2}\right)\cos\left(\frac{t_0}{2}-t\right).$

It's 3-jet will be

$$3\operatorname{sen}\frac{t_0}{2}\cos^2\frac{t_0}{2} + \left(\frac{-3}{4}\right)\left(\cos^2\frac{t_0}{2}\operatorname{sen}\frac{t_0}{2} + \operatorname{sen}^3\frac{t_0}{2}\right)t^2 + \left(-\frac{1}{4}\right)\left(\cos^3\frac{t_0}{2} - \operatorname{sen}^2\frac{t_0}{2}\cos\frac{t_0}{2}\right)t^3.$$

3^d coordinate: $\operatorname{sen}^2\left(\frac{t_0+t}{2}\right)\cos\left(\frac{t_0}{2}-t\right) + 2\cos\left(\frac{t_0+t}{2}\right)\operatorname{sen}\left(\frac{t_0+t}{2}\right)\operatorname{sen}\left(\frac{t_0}{2}-t\right).$

It's 3-jet will be

$$3\operatorname{sen}^2\frac{t_0}{2}\cos\frac{t_0}{2} + \left(\frac{-3}{4}\right)\left(\cos^3\frac{t_0}{2} + \operatorname{sen}^2\frac{t_0}{2}\cos\frac{t_0}{2}\right)t^2 + \frac{1}{4}\left(\operatorname{sen}^3\frac{t_0}{2} + \operatorname{sen}\frac{t_0}{2}\cos^2\frac{t_0}{2}\right)t^3.$$

4^d coordinate: $\operatorname{sen}\left(\frac{t_0+t}{2}\right)\operatorname{sen}\left(\frac{t_0}{2}-t\right).$

It's 3-jet will be

$$\operatorname{sen}^3\frac{t_0}{2} + \left(\frac{-3}{4}\right)\left(\operatorname{sen}^3\frac{t_0}{2} + \operatorname{sen}\frac{t_0}{2}\cos^2\frac{t_0}{2}\right)t^2 + \left(-\frac{1}{4}\right)\left(\operatorname{sen}^2\frac{t_0}{2}\cos\frac{t_0}{2} + \cos^3\frac{t_0}{2}\right)t^3.$$

It can be written in the following way

$$\left(\begin{array}{cccc}
\frac{1}{4}\left(\operatorname{sen}\frac{t_0}{2}\cos^2\frac{t_0}{2} + \operatorname{sen}^3\frac{t_0}{2}\right) & -\frac{3}{4}\left(\operatorname{sen}^2\frac{t_0}{2}\cos\frac{t_0}{2} + \cos^3\frac{t_0}{2}\right) & & \cos^3\frac{t_0}{2} \\
-\frac{1}{4}\left(\cos^3\frac{t_0}{2} + \operatorname{sen}^2\frac{t_0}{2}\cos\frac{t_0}{2}\right) & -\frac{3}{4}\left(\cos^2\frac{t_0}{2}\operatorname{sen}\frac{t_0}{2} + \operatorname{sen}^3\frac{t_0}{2}\right) & & 3\cos^2\frac{t_0}{2}\operatorname{sen}\frac{t_0}{2} \\
 & & e_i & \\
\frac{1}{4}\left(\operatorname{sen}^3\frac{t_0}{2} + \operatorname{sen}\frac{t_0}{2}\cos^2\frac{t_0}{2}\right) & -\frac{3}{4}\left(\cos^3\frac{t_0}{2} + \operatorname{sen}^2\frac{t_0}{2}\cos\frac{t_0}{2}\right) & & 3\operatorname{sen}^2\frac{t_0}{2}\cos\frac{t_0}{2} \\
-\frac{1}{4}\left(\operatorname{sen}^2\frac{t_0}{2}\cos\frac{t_0}{2} + \cos^3\frac{t_0}{2}\right) & -\frac{3}{4}\left(\operatorname{sen}^3\frac{t_0}{2} + \operatorname{sen}\frac{t_0}{2}\cos^2\frac{t_0}{2}\right) & & \operatorname{sen}^3\frac{t_0}{2}
\end{array}\right)\left(\begin{array}{c} t^3 \\ t^2 \\ 0 \\ 1 \end{array}\right)$$

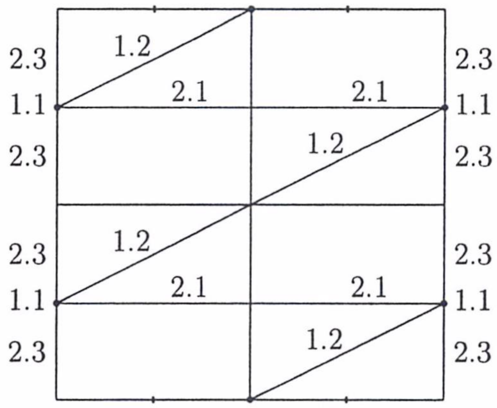
We conclude with the next theorem

Theorem. *Let e_i the column with 1 in the i-place and zero in all others, then*

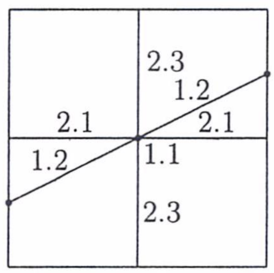
i) Given t_0, there exists i such that the above matrix is invertible.

ii) All the singular points of $\widetilde{h}$ are cusps.

One way of looking at our points in the torus $\mathbb{R}P^1_{\sqrt{\frac{2}{3}}} \times S^1_{\sqrt{\frac{1}{3}}}$ is in the rectangle $[-\pi, \pi] \times [-\pi, \pi]$, and we get the following picture



Our orbits around 1.1 will look like the following image where 1.2 are simple cusps and the complement is 2.2.



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