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Finite Determination, Finite Relative Determination and the Artin-Rees Lemma

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Finite Determination, Finite Relative Determination and the Artin-Rees Lemma

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Summary

If we consider the 4-dimensional vector space of quasi-homogeneous maps in two variables of weights $(\frac{1}{3}, \frac{1}{6})$ and degree 1, we get two 1-parameter families $f_\epsilon(x,y) = x^3 + (\epsilon+1)x^2y^2 + \epsilon xy^4$ and $g_\epsilon(x,y) = x^3 + \epsilon x^2y^2 + xy^4 + \epsilon y^6$. We are interested in comparing the usual finite determination with a suitable group of diffeomorphisms. We prove analogous theorems of finite determination for such group. This work is a continuation of [K]

Resumo

Continuando um trabalho anterior [K] onde estudamos V, o espaço vetorial de funções quasi-homogêneas em duas variáveis de pesos $(\frac{1}{3}, \frac{1}{6})$ e grau 1, agora estamos interessados na determinação com um grupo de difeomorfismos G apropriado. Estudamos os dois modelos 1-parametrizados em V, e damos para G os teorema apropriados de determinação finita, encontramos diferenças com a determinação do grupo usual de difeomorfismos e nosso grupo G.

Introduction

Following our work [K], we consider the quasi-homogeneous maps in two variables of weights $(\frac{1}{3}, \frac{1}{6})$ and degree 1. The group of germes of diffeomorphisms considered there are of form $h(x,y) = (\alpha x + \beta y^2, \delta y)$. We then study G, a subgroup of the group of diffeomorphisms of the form $h(x,y) = (\alpha x + \beta y^2 + h_1, \delta y + h_2)$ where $h_1 \in m(2)^3$ and $h_2 \in m(2)^2$. We study the concepts of finite determination on the right and finite relative determination for two models, obtaining different numbers. We finish with a version of Artin-Rees Lema in $\mathcal{E}(n)$, the algebra of smooth germs.

Consider $\mathcal{E}(n)$ the algebra of smooth germs of functions from the n-dimensional euclidean space to the real numbers.

Theorem 1 (Stefan). *A germ f is k-determined on the right if and only if for each germ $\mu \in m(n)^{k+1}$, we have that*

$$m(n)^{k+1} \subseteq m(n)\langle \frac{\partial(f+\mu)}{\partial x_i} \rangle + m(n)^{k+2}.$$

■

Remark 2. *The theorem is valid if we test all homogeneous polinomials μ of degree $k+1$.*

We first work with the germ $f_\epsilon(x,y) = x^3 + (\epsilon+1)x^2y^2 + \epsilon xy^4$, for $\epsilon \neq 0$, $\epsilon \neq 1$. We want to find the minimal k which satisfies

$$m(2)^{k+1} \subseteq m(2)\langle 3x^2 + 2(\epsilon+1)xy^2 + \epsilon y^4 + \frac{\partial \mu}{\partial x}, 2(\epsilon+1)x^2y + 4\epsilon xy^3 + \frac{\partial \mu}{\partial y} \rangle$$

for all μ homogeneous polynomials of degree $k+1$.

It is clear that the contention is true for

i) The monomials factoring x^2

ii) Since we have the equality

$$2(\epsilon+1)y\frac{\partial f_\epsilon}{\partial x} - 3\frac{\partial f_\epsilon}{\partial y} = 4(\epsilon^2-\epsilon+1)xy^3 + 2\epsilon(\epsilon+1)y^5 + 2(\epsilon+1)y\frac{\partial \mu}{\partial x} - 3\frac{\partial \mu}{\partial y}$$

hence the contention is true for xy^3y^s if $s \geq k-3$ and $s \geq 2$.

iii) Now consider the equation

$$axy\frac{\partial f_\epsilon}{\partial x} + by^3\frac{\partial f_\epsilon}{\partial x} + cx\frac{\partial f_\epsilon}{\partial y} + dy^2\frac{\partial f_\epsilon}{\partial y} =$$

$$= (3a + 2c(\epsilon+1))x^3y + (2a(\epsilon+1) + 3b + 4c\epsilon + 2d(\epsilon+1))x^2y^3$$

$$+ (a\epsilon + 2b(\epsilon+1) + 4d\epsilon)xy^5 + b\epsilon y^7 + h$$

where h is a term in $m(2)^{k+2}$ plus a term in $m(2)^{k+1}$ factorizing x. Using i) and ii): It's clear that there exist a,b,c,d real numbers with $b \neq 0$ such that our equality will be $b\epsilon y^7$. We conclude that

$$m(2)^{6+1} \subseteq m(2)\langle \frac{\partial(f_\epsilon + \mu)}{\partial x}, \frac{\partial(f_\epsilon + \mu)}{\partial y} \rangle + m(2)^8$$

for all $\mu \in m(2)^7$. Therefore f_ϵ is 6-determined on the right.

Remark 3. *The germ f_ϵ (for $\epsilon \neq 0, \epsilon \neq 1$) is not 5-determined on the right, since the following equality is impossible*

$$y^6 = h_1(x,y)(3x^2 + 2(\epsilon+1)xy^2 + \epsilon y^4) + h_2(x,y)(2(\epsilon+1)x^2y + 4\epsilon xy^3)$$

with h_1 and h_2 in $m(2)$. This equality implies that $h_1 \in m(2)^2$ and if its quadratic part is $ax^2 + bxy + cy^2$, hence the factor of xy^4 on the right is $2c(\frac{\epsilon^2-\epsilon+1}{\epsilon+1})$ which is zero only when $c = 0$.

Therefore the equality is impossible and we conclude that f_ϵ is 6-determined on the right.

Remark 4. *The previous proof for $\epsilon = -1$ is trivial.*

Definition 5. *Let A be an $\mathbb{R}$-algebra and B a vector subspace. If I is an ideal of A we say that B is an I-module if for some k,*

$$I^k B \subseteq B$$

Remark 6. *If B and B' subspaces are I-submodules of A, hence $B + B'$ is an I-submodule.*

Next we give a version of Nakayma 's Lemma.

Lemma 7. *Let A be an $\mathbb{R}$-algebra and B a subspace of A. Suppose I is an ideal contained in the Jacobson radical of A and such that B is finitely generated as I-module (there exist $v_1, \cdots, v_m$ in B such that any element of B is written as a linear combination of the v_i with coefficients in I^k). If $B \subseteq I^k B$, hence $B = 0$.*

Proof. Let $\{v_1, \cdots, v_m\}$ a minimal set of generators of the I-module B, hence $v_1 = a_1v_1 + \ldots + anv_n$ with $a_i \in I^k$ then $(1 - a_1)v_1 = a_2v_2 + \ldots + a_nv_n$ and $\{v_2, \ldots, v_n\}$ also generates B as I-module. Therefore $B = \{0\}$. ■

Corollary 8. *Let A be an $\mathbb{R}$-algebra, B and B' I-submodules of A. Suppose I is an ideal contained in the Jacobian radical of A. If $(B+B')/B'$ is finitely generated as I-module and $I^kB' + I^{k+1}B \supseteq I^kB$, hence $I^kB' \supseteq I^kB$.*

Proof. We have that

$$I\frac{I^k(B+B')}{I^kB'} = \frac{I^{k+1}B + I^kB'}{I^kB'} \supseteq \frac{I^kB + I^kB'}{I^kB'}.$$

Hence, by the previous lemma, $\frac{I^kB+I^kB'}{I^kB'} = 0$ and therefore $I^kB \subseteq I^kB'$. ■

Let G be the group of germs of diffeomorphism of the form

$$h(x,y) = (\alpha x + \beta y^2 + h_1(x,y), \delta y + h_2(x,y))$$

where $\alpha \cdot \delta \neq 0$, $h_1 \in m(2)^3$ and $h_2 \in m(2)^2$.

Definition 9. *We say that the germ f is k-determined relative to G if given a germ g such that $j^kf(0) = j^kg(0)$, there exists $h \in G$ such that $g = f \circ h$.*

Theorem 10 (A). *A germ f is $(k+2)$-determined on the right relative to G if*

$$m(2)^{k+1} \subseteq m(2)\{\frac{\partial f}{\partial x}(\alpha x + \beta y^2 + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\} + m(2)^{k+2}$$

where α, β and δ range in the real numbers, $h_1 \in m(2)^3$ and $h_2 \in m(2)^2$.

Theorem 11 (B). *If the germ f is k-determined on the right relative to G hence*

$$m(2)^{k+1} \subseteq \{\frac{\partial f}{\partial x}(\alpha x + \beta y^2 + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\} + m(2)^{k+2}$$

where α, β and δ range in the real numbers, $h_1 \in m(2)^3$ and $h_2 \in m(2)^2$.

Corollary 12. *The germ* $f_\epsilon(x,y) = x^3 + (\epsilon+1)x^2y^2 + \epsilon xy^4$ *is* 9*-determined relative to* G.

Proof. We want to prove that

$$m(2)^{7+1} \subseteq m(2)\{(3x^2 + 2(\epsilon+1)xy^2 + \epsilon y^4)(\alpha x + \beta y^2 + h_1)$$

$$+(2(\epsilon+1)x^2y + 4\epsilon xy^3)(\delta y + h_2)\} + m(2)^{7+2}$$

It is clear for monomials factoring x^3 or x^2. Now from ii) in our previous remarks, xy^7 is also in our inclusion. Finally multiplying by y iii) we get y^8. Therefore using Theorem 10 we get that f_ϵ is 9-determined. ■

Consider the germ of the curve in G

$$\gamma_{(x,y)}(t) = (x + t(\alpha x + \beta y^2 + h_1), y + t(\delta y + h_2)) \tag{1}$$

with $(1+t\alpha)(1+t\delta) \neq 0$.

If $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a smooth map, with $\overline{0} \in U$, we get

$$\frac{d}{dt}(f \circ \gamma_{(x,y)}(t))(0) = \frac{\partial f}{\partial x}(x,y)(\alpha x + \beta y^2 + h_1) + \frac{\partial f}{\partial y}(x,y)(\delta y + h_2)$$

Proposition 13. *Let* $\pi_n : m(2) \to \frac{m(2)}{m(2)^{n+1}}$ *the canonical projection,* Q *be the orbit of* z*, where* z *is the* k*-jet of* f*. Then we get the equality*

$$T_z(\pi_n(Q)) = \pi_n\{\frac{\partial f}{\partial x}(\alpha x + \beta y^2 + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\}$$

where α, β *and* δ *range in the real numbers* $h_1 \in m(2)^3$ *and* $h_2 \in m(2)^2$.

The proof is a consequence of a classical theorem in Lie groups. Let G be a Lie group acting smoothly on a manifold M. For a fixed point x_0 in M, consider the orbit x_0G. We denote by $est(x_0)$, the estabilizer of x_0 and consider the canonical projection $\pi : G \to G/est(x_0)$.

Theorem 14. *Let* $c : (-\epsilon, \epsilon) \to x_0G$ *a curve with* $c(0) = x_0$. *Then there exists a curve* $C : (-\delta, \delta) \to G$ *with* $C(0) = Id$ *and* $c'(0) = \frac{d}{dt}(x_0C(t))(0)$.

Proof. If $\rho : G \to x_0G$ is defined by $\rho(g) = x_0g$ we have the canonical map $p : G \to G/est(x_0)$ and a unique map $\widetilde{\rho} : G/est(x_0) \to x_0G$, such that $\rho = \widetilde{\rho} \circ p$. We know that p is a submersion and $\widetilde{\rho}$ an one to one immersion, hence at each component of G, ρ has constant rank. Therefore, there exist U and V open neighborhoods of the identity in G and of x_0 in x_0G respectively, $\phi = (\phi_1, \phi_2)$ and $\psi = (\psi_1, \psi_2)$ charts, $\phi : U \to U_1 \times V_1$, $\psi : V \to U_1 \times V_2$, such that $\psi \circ \rho \circ \phi^{-1}(x, y) = (x, \overline{0})$. Define the curve $C(t) = \phi^{-1} \circ i \circ \psi \circ c(t)$ where $i : U_1 \times V_2 \to U_1 \times V_1$ is defined by $i(x, z) = (x, \overline{0})$. It is clear that

$$\begin{aligned}\psi \circ \rho \circ C(t) &= (\psi \circ \rho \circ \phi^{-1}) \circ i \circ (\psi \circ c(t)) \\ &= (\psi \circ \rho \circ \phi^{-1}) \circ (\psi_1(c(t)), \overline{0}) \\ &= (\psi_1(c(t)), \overline{0})\end{aligned}$$

Hence $\psi_2(\rho \circ C(t)) \equiv 0$ and on the other hand $\psi_2 \circ c(t) = 0$ for $t \in (-\delta, \delta)$ (for small δ). Since $(\psi_1, \psi_2)(c(t)) = (\psi_1, \psi_2) \circ \rho \circ C(t)$, for $t \in (-\delta, \delta)$, therefore $c(t) = \rho \circ C(t)$, and we conclude our assertion.

Remark 15. *Our lemma implies that we can use curves in the Lie group instead of curves in the orbit.*

We now prove

Theorem B. (11) *If the germ $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is k-determined relative to G, then we have*

$$m(2)^{k+1} \subseteq \{\frac{\partial f}{\partial x}(\alpha x + \beta y^2 + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\} + m(2)^{k+2}$$

where α, β and δ range in the real numbers, $h_1 \in m(2)^3$ and $h_2 \in m(2)^2$.

Proof. Let z be the k-jet of f. since f is k-determined relative to G, we have that $z + m(2)^{k+1} \subseteq Q$, Q the G-orbit of z. Therefore $\pi_{k+1}(z + m(2)^{k+1}) \subseteq \pi_{k+1}(Q)$. Taking the tangent spaces at z, we get from our previous proposition that

$$\pi_{k+1}(m(2)^{k+1}) \subseteq \pi_{k+1}\{\frac{\partial f}{\partial x}(\alpha x + \beta y^2 + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\}$$

and therefore

$$m(2)^{k+1} \subseteq \{\frac{\partial f}{\partial x}(\alpha x + \beta y^2 + h_1) + \{\frac{\partial f}{\partial y}(\delta y + h_2)\} + m(2)^{k+2}$$

■

Remark 16. *By corollary 8, since* $\{\frac{\partial f}{\partial x}(\alpha x + \beta y^2 + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\}$ *is an* $m(2)^2$*-module and* $m(2)^{k+2}$ *is a finitely generated* $m(2)^2$*-module, we get that*

$$m(2)^{k+3} \subseteq \{\frac{\partial f}{\partial x}(\alpha x + \beta y^2 + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\}.$$

Theorem A. (10) *A germ f is $(k+2)$-determined relative to G if*

$$m(n)^{k+1} \subseteq m(2)\{\frac{\partial f}{\partial x}(\alpha x + \beta y^2 + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\}$$

Let g be a germ such that $j^{k+2}g(\overline{0}) = j^{k+2}f(\overline{0})$ and $\phi(t,(x,y)) = tg(x,y) + (1-t)f(x,y)$. For $t_0 \in [0,1]$ and t near t_0, we want to find a homotopy of diffeomorphisms Γ^t, such that $\Gamma^{t_0} = Id$, $\Gamma^t(\overline{0}) = \overline{0}$ and $\phi^t \circ \Gamma^t = \phi^{t_0}$, where $\phi^t(x,y) \equiv \phi(t,(x,y))$ and t near t_0. This assertion will be proved by sequence of lemmas. Since the interval $[0,1]$ is compact and connected, this proves Theorem A (10).

Our assertion can be written as

Lemma 17. *There exist* $\Gamma : (\mathbb{R} \times \mathbb{R}^2, (t_0, \overline{0})) \to \mathbb{R}^2$ *such that*

a) $\Gamma(t_0,(x,y)) = (x,y)$.

b) $\Gamma(t,\overline{0}) = \overline{0}$, *for t near t_0.*

c) $\phi(t,\Gamma(t,(x,y))) = \phi(t_0,(x,y))$.

This lemma is equivalent to the following

Lemma 18. *There exists* $\Gamma : (\mathbb{R} \times \mathbb{R}^2, (t_0, \overline{0})) \to \mathbb{R}^2$ *such that*

a) $\Gamma(t_0,(x,y)) = (x,y)$.

b) $\Gamma(t,\overline{0}) = 0$, *for t near t_0.*

c') $\frac{\partial\phi}{\partial x}(t,\Gamma(t,(x,y))\frac{\partial\Gamma_1}{\partial t}(t,(x,y))+\frac{\partial\phi}{\partial y}(t,\Gamma(t,(x,y))\frac{\partial\Gamma_2}{\partial t}(t,(x,y))+\frac{\partial\phi}{\partial t}(t,\Gamma(t,(x,y))=0.$

Let A be the $\mathbb{R}$-algebra of germs of functions around $(t_0,\overline{0})$ and $\pi:\mathbb{R}^3\to\mathbb{R}^2$ the projection $\pi(t,(x,y))=(x,y)$. Consider the induced monomorphism $\pi^*:\mathcal{E}(2)\to A$, hence A has the structure of an $\mathcal{E}(2)$-module.

Lemma 19. *If we have*

$$m(2)^{k+1}\subseteq m(2)\{\frac{\partial f}{\partial x}(\alpha x+\beta y^2+h_1)+\frac{\partial f}{\partial y}(\delta y+h_2)\}+m(2)^{k+2}$$

hence its also true that

$$m(2)^{k+1}A\subseteq m(2)\{\frac{\partial\phi}{\partial x}(\alpha x+\beta y^2+h_1)+\frac{\partial\phi}{\partial y}(\delta y+h_2)\}+m(2)^{k+2}$$

and by Nakayma 's Lemma we get

$$m(2)^{k+3}A\subseteq m(2)^3\{\frac{\partial\phi}{\partial x}(\alpha x+\beta y^2+h_1)+\frac{\partial\phi}{\partial y}(\delta y+h_2)\}$$

If we suppose above lemma true, since $\frac{\partial\phi}{\partial t}=g-f\in m(2)^{k+3}$,

$$\frac{\partial\phi}{\partial t}=h_1(x,y)(\alpha x+\beta y^2+h_1)\frac{\partial\phi}{\partial x}+h_2(x,y)(\delta y+h_2)\frac{\partial\phi}{\partial y}\tag{2}$$

Define the new functions

$$\psi_1(x,y)=h_1(x,y)(\alpha x+\beta y^2+h_1)$$

$$\psi_2(x,y)=h_2(x,y)(\delta y+h_2)$$

hence $\psi_1(\overline{0})=\psi_2(\overline{0})=0$.

Consider the ordinary differential equation

$$\frac{\partial\Gamma}{\partial t}(t,(x,y))=(-\psi_1,-\psi_2)\circ\Gamma(t,(x,y))$$

with initial condition $\Gamma(t_0,(x,y))=(x,y)$. Since $\psi_1(\overline{0})=\psi_2(\overline{0})=0$, we get $\Gamma(t,\overline{0})=\overline{0}$. Hence our equation 2 can be written as $\frac{\partial\phi}{\partial t}=\psi_1\frac{\partial\phi}{\partial x}+\psi_2\frac{\partial\phi}{\partial y}$, equivalently

$$\frac{\partial\phi}{\partial t}\circ\Gamma^t+\frac{\partial\Gamma_1^t}{\partial t}(\frac{\partial\phi}{\partial x}\circ\Gamma^t)+\frac{\partial\Gamma_2^t}{\partial t}(\frac{\partial\phi}{\partial y}\circ\Gamma^t)=0$$

with $\Gamma^{t_0}(x,y) = (x,y)$ and $\Gamma^t(\overline{0}) = \overline{0}$ for t near t_0.

Proof of the previous lemma. Let ϕ be defined as $\phi(t,(x,y)) = tg(x,y) + (1-t)f(x,y)$, hence we have the following equalities

$$\frac{\partial \phi}{\partial x} = \frac{\partial f}{\partial x} + t\frac{\partial (g-f)}{\partial x}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial f}{\partial y} + t\frac{\partial (g-f)}{\partial y}$$

Clearly

$$(\alpha x + \beta y^2 + h_1)\frac{\partial f}{\partial x} \in (\alpha x + \beta y^2 + h_1)\frac{\partial \phi}{\partial x} + A\ m(2)^{k+1}, \text{and}$$

$$(\delta y + h_2)\frac{\partial f}{\partial y} \in (\delta y + h_2)\frac{\partial \phi}{\partial y} + A\ m(2)^{k+1}$$

Therefore our hypothesis implies that

$$m(2)^{k+1}A \subseteq m(2)\{\frac{\partial \phi}{\partial x}(\alpha x + \beta y^2 + h_1) + \frac{\partial \phi}{\partial y}(\delta y + h_2)\} + A\ m(2)^{k+2}$$

■

<u>Complex case</u>

We consider $g_\epsilon(x,y) = x^3 + \epsilon x^2y^2 + xy^4 + \epsilon y^6$. We want to find the minimal k that

$$m(2)^{k+1} \subseteq \langle 3x^2 + 2\epsilon xy^2 + y^4 + \frac{\partial \mu}{\partial x}, 2\epsilon x^2 y + 4xy^3 + 6\epsilon y^5 + \frac{\partial \mu}{\partial y}\rangle + m(2)^{k+2},$$

for all homogeneous polynomials μ of degree $k+1$. As before, we divide our inclusion in three cases

i) The monomials factorizing x^2

ii) Consider

$$2\epsilon y(3x^2 + 2\epsilon xy^2 + y^4 + \frac{\partial \mu}{\partial x}) - 3(2\epsilon x^2 y + 4xy^3 + 6\epsilon y^5 + \frac{\partial \mu}{\partial y})$$

$$= (4\epsilon^2 - 12)xy^3 - 16\epsilon y^5 + 2\epsilon y\frac{\partial \mu}{\partial x} - 3\frac{\partial \mu}{\partial y}$$

a) If $\epsilon^2 \neq 3$ we get xy^3y^s, $s \geq k-3, s \geq 2$

b) If $\epsilon^2 = 3$ we get xy^5y^s and y^6y^s, $s \geq 1$.

iii) Consider the equation

$$axy\frac{\partial g_\epsilon}{\partial x} + by^3\frac{\partial g_\epsilon}{\partial x} + cx\frac{\partial g_\epsilon}{\partial y} + dy^2\frac{\partial g_\epsilon}{\partial y}$$

$= x^3y(3a+2\epsilon c) + x^2y^3(2\epsilon a + 3b + 4c + 2\epsilon d) + xy^5(a + 2\epsilon b + 6\epsilon c + 4d) + y^7(b + 6\epsilon d)$

If $b = -6\epsilon d$ we get a system of three equation in variables a, c and d with determinant $48(\epsilon^4 + 2\epsilon^2 + 1)$ which is never zero on the real numbers.

Hence as before, we have a, b, c and d , with $b + 6\epsilon d \neq 0$ such that y^7 is in our ideal. We conclude that

$$m(2)^{6+1} \subseteq m(2)\langle\frac{\partial(g_\epsilon + \mu)}{\partial x}, \frac{\partial(g_\epsilon + \mu)}{\partial y}\rangle + m(2)^8$$

for all $\mu \in m(2)^7$. Therefore g_ϵ is 6-determined on the right.

Remark 20. *The germ g_ϵ is not 5-determined on the right, since the following equality is impossible*

$$y^6 = h_1(x,y)(3x^2 + 2\epsilon xy^2 + y^4 + \frac{\partial\mu}{\partial x} + h_2(x,y)(2\epsilon x^2y + 4xy^3 + 6\epsilon y^5 + \frac{\partial\mu}{\partial y})$$

with h_1 and h_2 in $m(2)$. Clearly $h_1 \in m(2)^2$ and let its quadratic part be $ax^2 + bxy + cy^2$. As in the real case, we get a contradiction in the 5-jet for the monomial xy^4 except for $\epsilon^2 = 3$.

<u>The case $\epsilon^2 = 3$</u> It is not hard to show that $y^6 \notin m(2)\langle 3x^2 + 2\epsilon xy^2 + y^4 + \frac{\partial\mu}{\partial x}, 2\epsilon x^2y + 4xy^3 + 6\epsilon y^5 + \frac{\partial\mu}{\partial y}\rangle$ for $\mu = \frac{-8\epsilon}{9}y^6$.

For our group G we get $m(2)^{7+1} \subseteq m(2)\{(3x^2 + 2\epsilon xy^2 + y^4)(\alpha x + \beta y^2 + h_1) + (2\epsilon x^2y + 4xy^3 + 6\epsilon y^5)(\delta y + h_2)\} + m(2)^9$

i) The monomials of degree 8 factorizing x^3 or x^2y^2 they are clearly in the RHS of our contention.

ii) We get the equality

$$2\epsilon y^2(3x^2 + 2\epsilon xy^2 + y^4) - 3y(2\epsilon x^2 y + 4xy^3 + 6\epsilon y^5)$$

$$= (4\epsilon^2 - 12)xy^4 + (-16)\epsilon y^6$$

a) If $\epsilon^2 \neq 3$, we get xy^7.

b) If $\epsilon^2 = 3$, we get xy^6 and y^7.

iii) For $\epsilon^2 \neq 3$ consider the sum

$$axy^2\frac{\partial g_\epsilon}{\partial x} + by^4\frac{\partial g_\epsilon}{\partial x} + cxy\frac{\partial g_\epsilon}{\partial y} + dy^3\frac{\partial g_\epsilon}{\partial y}$$

As before, for suitable constants a, b, c and d, we get y^8.

Therefore using Theorem B (11) we get

Theorem 21. *Consider the germ* $g_\epsilon(x, y) = x^3 + \epsilon x^2y^2 + xy^4 + \epsilon y^6$, *then*

a) If $\epsilon^2 \neq 3$, g_ϵ *is* 9*-determined relative a* G.

b) If $\epsilon^2 = 3$, g_ϵ *is* 8*-determined relative to* G.

We state the following

Theorem 22. *The germ* $f \in m(2)$ *is finitely determined on the right if and only if there exists* k *such that* $m(2)^k \subseteq m(2)\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle$.

We also have from Theorems 10 and 11

Theorem 23. *The germ* f *is finitely determined relative to* G *if and only if there exists* k *such that*

$$m(2)^k \subseteq m(2)\{\frac{\partial f}{\partial x}(\alpha x + \beta y^2 + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\} + m(2)^{k+1}$$

These two theorems give a relation between both concepts of determination.

Theorem 24. *The germ f is finitely determined on the right if and only if is finitely determined relative to G.*

Proof. If $m(2)^k \subseteq m(2)\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$, therefore

$$m(2)^{k+3} \subseteq m(2)\{\frac{\partial f}{\partial x}\widetilde{h_1} + \frac{\partial f}{\partial y}\widetilde{h_2} \mid \widetilde{h_1}, \widetilde{h_2} \in m(2)^3\}$$

hence

$$m(2)^{(k+2)+1} \subseteq m(2)\{\frac{\partial f}{\partial x}(\alpha x + \beta y^2 + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\}$$

with $h_1 \in m(2)^3$ and $h_2 \in m(2)^2$. Therefore f is $(k+3)$-determined relative to G.

Now if f is k-determined relative to G, hence

$$m(n)^{k+1} \subseteq m(2)\{\frac{\partial f}{\partial x}(\alpha x + \beta y^2 + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\}$$

and

$$m(n)^{k+1} \subseteq m(n)^2\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle.$$

Therefore f is k-determined on the right. ∎

Artin-Rees Lemma

(Following [A])

We shall work in the local algebra $\mathcal{E}(n)$ of germs of smooths maps. We will denote by $\mathbb{R}[[x_1, \ldots, x_n]]$ the algebra of formal power series in n variables over the real numbers.

Lemma 25 (Borel). *Consider the canonical map $\pi : \mathcal{E}(n) \to \mathbb{R}[[x_1, \ldots, x_n]]$ given by the infinite Taylor series, then π is onto and its kernel is $m(n)^\infty$, the ideal of germs $\overline{f}$, such that $\overline{f}$ and all its derivatives vanish at the origin.*

Remarks 26.

1) *By Hilbert Basis Theorem $\mathbb{R}[[x_1, \ldots, x_n]]$ is noetherian.*

2) *The ideal $m(n)^\infty$ is not finitely generated since $m(n)^\infty = m(n)^k m(n)^\infty$ for $1 \leq k \leq \infty$ and $m(n)^\infty$ is not trivial. Therefore $\mathcal{E}(n)$ is not noetherian.*

Definition 27.

a) *Let X and Y subsets of $\mathbb{R}^n$ containing the origin. We say that they are equivalent if there exists a neighborhood of the origin U such that $U \cap X = U \cap Y$. the germ $\mathcal{X}$ will denote the equivalent class of X.*

b) *A subset X of $\mathbb{R}^n$ is an analytic subset if there exist $f_1, \ldots, f_m$ analytic maps such that their common zeros coincide with X.*

Notation If $f_1, \ldots, f_n$ are analytic and I denotes the ideal generated by them, we will denote $X = V(I)$, the common zeros of I. Such ideal can be seen in $\mathcal{E}(n)$ or in $\mathcal{O}_n$, the algebra of analytic germs.

Remark 28. *For $f_1, \ldots, f_n$ analytic maps we get*

$$V(\langle f_1, \ldots, f_n \rangle) = V(f_1^2 + \ldots + f_n^2)$$

We state the following

Lemma 29.

1) *The sets ϕ and $\mathbb{R}^n$ are analytic.*

2) *Finite intersection and finite union of analytic sets are also analytic sets.*

Definition 30. *let $\mathcal{X}$ be a germ of an analytic set X, then $\mathcal{I}(\mathcal{X})$ is the ideal of germs vanishing at $\mathcal{X}$. This coincides with the ideal of germs $\overline{f}$ such that given f a representative, there exist U neighborhood of the origin such that f vanishes at $X \cap U$.*

Remark 31. *If X' is another representative of $\mathcal{X}$ there exists a neighborhood V such that $V \cap X = V \cap X'$, hence our f vanishes at $(V \cap U) \cap X'$. Therefore our definition is independent of the representative of $\mathcal{X}$.*

Definition 32. *let I be the ideal of $\mathcal{E}(n)$ generated by $\overline{f}_1, \ldots, \overline{f}_m$. We denote by $\mathcal{V}(I)$ the germ of the common zeros of representatives $f_1, \ldots, f_m$*

Remark 33.

1) *The above definition does not depend of the representatives. Suppose that the ideal I is also generated by germs $\overline{g}_1, \ldots, \overline{g}_m$ with $\overline{g}_i = \overline{f}_i$, hence there exist neighborhoods $U_1, \ldots, U_m$ such that $f_i(x) = g_i(x)$, $\forall x \in U_i$. Therefore the common zeroes of the set $\{f_1, \ldots, f_m\}$ coincide with ones of the set $\{g_1, \ldots, g_m\}$ at $U_1 \cap \ldots \cap U_m$.*

2) *Moreover, if $I = \langle \overline{f}_1, \ldots, \overline{f}_m \rangle = \langle \overline{g}_1, \ldots, \overline{g}_l \rangle$ we get that $\overline{g}_k = \sum_{i=1}^{m} \overline{h}_{ik} \overline{f}_i$ and therefore $\mathcal{V}(\overline{f}_1, \ldots, \overline{f}_m) \subseteq \mathcal{V}(\overline{g}_1, \ldots, \overline{g}_l)$. The other inclusion is proved similarly*

Lemma 34. *Let I and J be finitely generated ideals of $\mathcal{E}(n)$, hence we have the following properties*

1) *$\mathcal{I}(\mathcal{V}(I)) \supseteq I$.*

2) *If $I \subseteq J$ then $\mathcal{V}(I) \supseteq \mathcal{V}(J)$*

3) *If $\mathcal{V}(I) \subseteq \mathcal{V}(J)$ then $\mathcal{I}(\mathcal{V}(I)) \supseteq \mathcal{I}(\mathcal{V}(J))$.*

4) *$\mathcal{V}(\mathcal{I}(\mathcal{V}(I))) = \mathcal{V}(I)$.*

Proof. We only prove 4). Since $\mathcal{I}(\mathcal{V}(I)) \supseteq I$ then $\mathcal{V}(\mathcal{I}(\mathcal{V}(I))) \subseteq \mathcal{V}(I)$. Let $x \in \mathcal{V}(I)$, then $f_i(x) = 0$ for $1 \leq i \leq m$, with $\overline{f}_1, \ldots, \overline{f}_m$ generators of I. Hence $f(x) = 0$, $\forall \overline{f} \in \mathcal{I}(\mathcal{V}(I))$ and $x \in \mathcal{V}(\mathcal{I}(\mathcal{V}(I)))$. ■

Definition 35. *A germ $\mathcal{X}$ of an analytic set is irreducible if we have $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$, where $\mathcal{X}_1$ and $\mathcal{X}_2$ are proper analytic subsets of $\mathcal{X}$, hence $\mathcal{X}_1 = \emptyset$ or $\mathcal{X}_2 = \emptyset$. A germ $\mathcal{X}$ of an analytic set is reducible if it is not irreducible.*

Proposition 36. *A germ $\mathcal{X}$ of an analytic set is irreducible if and only if $\mathcal{I}(\mathcal{X})$ is a prime ideal.*

Proof. Let $\overline{f}$ and $\overline{g}$ germs not in $\mathcal{I}(\mathcal{X})$ such that $\overline{fg}$ is in $\mathcal{I}(\mathcal{X})$. Consider the ideals $I_1 = \mathcal{I}(\mathcal{X}) + \langle \overline{f} \rangle$ and $I_2 = \mathcal{I}(\mathcal{X}) + \langle \overline{g} \rangle$. It is clear that $\mathcal{X}_1 = \mathcal{V}(I_1)$ and $\mathcal{X}_2 = \mathcal{V}(I_2)$ are proper analytic sets of $\mathcal{X}$. Now if $x \in \mathcal{X}$ and $x \notin \mathcal{X}_1$ we get $f(x) \neq 0$ but $f(x)g(x) = 0$, then $g(x) = 0$ and $x \in \mathcal{X}_2$. Hence $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$ and $\mathcal{X}_1 \neq \emptyset$ and $\mathcal{X}_2 \neq \emptyset$, so $\mathcal{X}$ is reducible. Now if $\mathcal{X}$ is reducible, let $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$ with $\mathcal{X}_1 \subsetneq \mathcal{X}$ and $\mathcal{X}_2 \subsetneq \mathcal{X}$, hence $\mathcal{I}(\mathcal{X}_1) \supsetneq \mathcal{I}(\mathcal{X})$ and $\mathcal{I}(\mathcal{X}_2) \supsetneq \mathcal{I}(\mathcal{X})$. Let $\overline{f} \in \mathcal{I}(\mathcal{X}_1) - \mathcal{I}(\mathcal{X})$ and $\overline{g} \in \mathcal{I}(\mathcal{X}_2) - \mathcal{I}(\mathcal{X})$. it's clear that $\overline{fg} \in \mathcal{I}(\mathcal{X})$ but $\overline{f}$ and $\overline{g}$ are not elements of $\mathcal{I}(\mathcal{X})$, then $\mathcal{I}(\mathcal{X})$ is not a prime ideal. ■

Example 37. *We clearly have that $\overline{f}(x, y) = \overline{x^2 + y^2}$ is irreducible and that $\mathcal{I}(\mathcal{V}(\langle \overline{f} \rangle)) = m(2)$.*

Definitions 38.

a) Let I ideal of $\mathcal{E}(n)$, then the radical of I is $\mathcal{I}(\mathcal{V}(I))$, also denoted by $rad(I)$.

b) An ideal I is radical if $I = rad(I)$.

c) Let U be an open subset of $\mathbb{R}^n$ and X an analytic set in U. We say that X is coherent at a point x in X if there exist W neighborhood of x and $f_1, \ldots, f_m$ analytic functions in W such that for each $y \in W \cap X$ we have that $\mathcal{I}(W \cap X) = \langle \overline{f}_1^y, \ldots, \overline{f}_m^y \rangle$, where $\overline{f}_i^y$ is the germ of f_i at the point y.

c) A germ of a set $\mathcal{X}$ is coherent if a representant X is coherent at $\overline{0}$

Theorem 39 (Malgrange-Tougeron)**.** *Let $\mathcal{X}_0$ be a germ of an analytic set at the origin, $\mathcal{I}(\mathcal{X}_0)$ as above and $\mathcal{I}_*(\mathcal{X}_0)$ the ideal in $\mathcal{O}_n$ of analytic maps vanishing at $\mathcal{X}_0$. Then the following assertions are equivalent.*

i) $\mathcal{I}_*(\mathcal{X}_0)\mathcal{E}(n) = \mathcal{I}(\mathcal{X}_0)$

ii) $\mathcal{X}_0$ *is coherent*

Definition 40. *An ideal I of $\mathcal{E}(n)$ is a Malgrange-ideal if*

i) I is finitely generated by analytic germs.

ii) $\mathcal{V}(I)$ is coherent.

Remark 41. *For such ideals, by the previous theorem $\mathcal{I}(\mathcal{X}_0)$ is finitely generated.*

Proposition 42. *Consider the sets*

$$X = \{I \textit{is a Malgrange-ideal and also radical}\}$$

$$Y = \{\mathcal{V}(I) | \mathcal{V}(I) \textit{is a coherent germ} \}.$$

Hence $\mathcal{I} : Y \to X$ is well defined and $\mathcal{V} : X \to Y$ is its inverse.

Proof. Since I is a Malgrange ideal and also radical, $\mathcal{V}(I)$ is coherent and $\mathcal{I}(\mathcal{V}(I)) = I$. Also if $\mathcal{V}(I)$ is in Y, $\mathcal{I}(\mathcal{V}(I))$ is finitely generated and $\mathcal{V}(\mathcal{I}(\mathcal{V}(I)))$ $= \mathcal{V}(I)$ is coherent. Moreover $\mathcal{I}(\mathcal{V}(\mathcal{I}(\mathcal{V}(I))) = \mathcal{I}(\mathcal{V}(I))$. Therefore $\mathcal{I}$ and $\mathcal{V}$ are bijective and one is the inverse of the other. ∎

Proposition 43. *Let $I_1, \ldots, I_m$ radical ideals of $\mathcal{E}(n)$, hence their intersection is also radical.*

Proof. Since $\bigcap_{j=1}^m I_j \subseteq I_i \ \forall i$, we get $rad(\bigcap_{j=1}^m I_j) \subseteq rad(I_i) \ \forall i$, therefore $rad(\bigcap_{j=1}^m I_j) \subseteq \bigcap_{j=1}^m rad(I_j)$. Now, since $\bigcap_{k=1}^m I_k \subseteq rad(\bigcap_{k=1}^m I_k)$, we get by hypothesis $\bigcap_{k=1}^m rad(I_k) \subseteq rad(\bigcap_{k=1}^m I_k)$. ∎

Proposition 44. *Let I be a radical ideal of $\mathcal{E}(n)$, then we get $I \cap m(n)^\infty = Im(n)^\infty$.*

Proof. Let $\overline{f} \in I \cap m(n)^{\infty}$, then $f \in m(n)^{\infty}$ and $\overline{f} = \overline{g}\overline{h}$, where $\overline{g}$ and $\overline{h}$ are in $m(n)^{\infty}$ and $\overline{h}(x) > 0$, $\forall x \neq 0$. Now since $\overline{f}$ vanishes at $V(I)$, then $\overline{g}$ too and $\overline{g} \in \mathcal{I} = I$. Therefore $\overline{f} \in m(n)^{\infty} I$.

Remark 45. *Since $\pi : \mathcal{E}(n) \to \mathbb{R}[[x_1, \ldots, x_n]]$ is an epimorphism between local algebras, we get $\pi^{-1}(\mathcal{M}(n)) = m(n)$, where $\mathcal{M}(n)$ is the maximal ideal of $\mathbb{R}[[x_1, \ldots, x_n]]$. We state Artin-Rees Lemma for $\mathbb{R}[[x_1, \ldots, x_n]]$.*

Lemma 46 (Artin-Rees). *Let J ideal, then there exist a natural number k_0 such that for each natural number k we get*

$$\mathcal{M}(n)^{k+k_0} \cap J = \mathcal{M}(n)^k (\mathcal{M}(n)^{k_0} \cap J)$$

We now give a version of this lemma in the algebra $\mathcal{E}(n)$.

Theorem 47. *Let I be a radical ideal of $\mathcal{E}(n)$, then there exist a natural number k_0 such that for each natural number k we get*

$$m(n)^{k+k_0} \cap I = m(n)^k (m(n)^{k_0} \cap I).$$

Proof. Consider the ideal $\pi(I)$ in $\mathbb{R}[[x_1, \ldots, x_n]]$, hence there exists a natural number k_0 such that for each natural number k we get

$$\mathcal{M}(n)^{k+k_0} \cap \pi(I) = \mathcal{M}(n)^k (\mathcal{M}(n)^{k_0} \cap \pi(I))$$

therefore

$$m(n)^{k+k_0} \cap (I + m(n)^{\infty}) = m(n)^k (m(n)^{k_0} \cap (I + m(n)^{\infty}))$$

and

$$m(n)^{k+k_0} \cap I + m(n)^{\infty} = m(n)^k (m(n)^{k_0} \cap I) + m(n)^{\infty}$$

Again, intersecting with I and using the modular law

$$m(n)^{k+k_0} \cap I = m(n)^k (m(n)^{k_0} \cap I) + (m(n)^{\infty} \cap I) \tag{3}$$

Since I is radical, $I \cap m(n)^\infty = Im(n)^\infty = (Im(n)^\infty)m(n)^\infty = (I \cap m(n)^\infty)m(n)^\infty$, and the equality 3 can be written as

$$m(n)^{k+k_0} \cap I = m(n)^k(m(n)^{k_0} \cap I)$$

■

Lemma 48. *Let I be an ideal of $\mathcal{E}(n)$, then there exist $g_1, \dots, g_m$ germs of smooths functions in $\mathcal{E}(n)$ such that $I = \langle g_1, \dots g_m \rangle + I \cap m(n)^\infty$*

Proof. Let $h_1, \dots, h_m$ be generators of $\pi(I)$ and $g_i \in I$ such that $\pi(g_i) = h_i$, for $1 \leq i \leq m$. Hence $I + m(n)^\infty = \langle g_1, \dots, g_m \rangle + m(n)^\infty$, if we intersect with I each side of the equality we get

$$I = \langle g_1, \dots, g_m \rangle + I \cap m(n)^\infty$$

■

Remark 49. *With the above notation $I = \langle g_1, \dots, g_m \rangle$ if only if $I \cap m(n)^\infty \subseteq \langle g_1, \dots, g_m \rangle$.*

Proposition 50. *Let I be an ideal of $\mathcal{E}(n)$, $\mathcal{I}$ its radical and suppose that $\mathcal{I}$ is finitely generated. If $\pi(I) = \pi(\mathcal{I})$ then $I = \mathcal{I}$.*

Proof. Since $\pi(I) = \pi(\mathcal{I})$ we get that $I + m(n)^\infty = \mathcal{I} + m(n)^\infty$, if we intersect both sides of the equality with $\mathcal{I}$ we get $I + m(n)^\infty \cap \mathcal{I} = \mathcal{I}$ or $I + m(n)^\infty \mathcal{I} = \mathcal{I}$. Using Nakayama 's Lemma we get $I = \mathcal{I}$. ■

Remark 51. *Let I be an ideal of $\mathcal{E}(n)$ and $\mathcal{I}$ its radical. We have the following two questions*

a) If $\mathcal{I}$ is finitely generated, is I finitely generated?

b) If I is finitely generated, is $\mathcal{I}$ finitely generated?

For a) Let $I = m(n)^\infty$, hence $\mathcal{V}(I) = \{\bar{0}\}$ and $\mathcal{I} = \mathcal{I}(\mathcal{V}(I)) = m(n)$.

For b) In the special case that I is finitely generated by analytic germs and $\mathcal{V}(I)$ is coherent, then $\mathcal{I}$ is finitely generated.

We know that in an arbitrary commutative ring A with unitary element the following assertion is not true in general: Let I and J finitely generated ideals in A, hence $I \cap J$ is finitely generated. In our algebra $\mathcal{E}(n)$ we get the following result.

Proposition 52. *Let I be a finitely generated ideal of $\mathcal{E}(n)$, then for each k, $I \cap m(n)^{k+1}$ is also finitely generated.*

Proof. Let $f_1, \ldots, f_s$ generators of I and $f \in I$, hence $f = h_1 f_1 + \ldots + h_s f_s$. We write $h_i = h_i^k + \widetilde{h}_i^k$, where h_i^k is the k-jet of h_i and $\widetilde{h}_i^k = h_i - h_i^k$. Therefore

$$f = h_1^k f_1 + \ldots + h_s^k f_s + \widetilde{h}_1 f_1 + \ldots + \widetilde{h}_s f_s.$$

Consider J the vector space generated by $\{x^I f_j\}$ where $1 \leq j \leq s$ and $|I| \leq k$, hence as vector spaces we get $I = J + I\ m(n)^{k+1}$ and therefore $I \cap m(n)^{k+1} = J \cap m(n)^{k+1} + I\ m(n)^{k+1}$. Hence a set of generators for $I \cap m(n)^{k+1}$ can be formed by a basis for the vector space $J \cap m(n)^{k+1}$ and $\{f_i x^I\}$ where $1 \leq i \leq s$ and $|I| = k+1$ ∎

Remark 53. *We have a one to one correspondence between ideals of $\mathcal{E}(n)$ containing $m(n)^\infty$ and ideals of $\mathbb{R}[[x_1, \ldots, x_n]]$. It makes sense to define for ideals I containing $m(n)^\infty$ its real radical as $\mathbb{R} - rad(I) = \cap P$ where P are all the real prime ideals of $\mathcal{E}(n)$ containing $m(n)^\infty$ and I.*

In general, for an ideal I of $\mathcal{E}(n)$ we can also define $\mathbb{R} - rad(I) = \cap P$, where P are the real prime ideals containing I. We note that $m(n)$ and $m(n)^\infty$ are themselves real prime ideals.

Consider I be a real radical ideal of $\mathcal{E}(n)$, hence $\mathcal{I}(\mathcal{V}(I)) = I$. Also $\mathbb{R} - rad(I)$ is the smallest real ideal containing I, hence $\mathbb{R} - rad(I) = I$ and both concepts coincide.

Properties of $\mathbb{R} - rad$.

i) $\mathbb{R} - rad(I) \supseteq I$

ii) If $I \subseteq J$ hence $\mathbb{R} - rad(I) \supseteq \mathbb{R} - rad(J)$

iii) $\mathcal{V}(\mathbb{R} - rad(I)) = \mathcal{V}(I)$

For iii) If is clear that $\mathcal{V}(\mathbb{R} - rad(I)) \subseteq \mathcal{V}(I)$. If $x_0 \in \mathcal{V}(I) - \mathcal{V}(\mathbb{R} - rad(I))$ hence there exist $f_0 \in \mathbb{R} - rad(I)$ such that $f_0(x_0) \neq 0$ and $f(x_0) = 0, \forall f \in I$. On the another side, since $f_0 \in \mathbb{R} - rad(I)$ there exist $m \geq 0$ natural number and σ a sum of squares in $\mathcal{E}(n)$ such that $f_0^{2m} + \sigma \in I$. We then conclude our equality.

Remark 54. *We also have $\mathbb{R} - rad(I) \subseteq \mathcal{I}(\mathcal{V}(I))$. Let $f_0 \in \mathbb{R} - rad(I)$, then as above we have $f_0^{2m} + \sigma \in I$ and $f(x_0) = 0$, $\forall x \in \mathcal{V}(I)$. Therefore $f_0 \in \mathcal{I}(\mathcal{V}(I))$. In the case $I = m(n)^\infty$, $\mathcal{I}(\mathcal{V}(I)) = m(n)$ and $\mathbb{R} - rad(I) = m(n)^\infty$*

A special case: Consider the ring $\mathbb{R}[[x_1, \ldots, x_n]]$ and real ideals $I_1, \ldots, I_k$ which are coprime. Define $I = I_1^{s_1} \cdots I_k^{s_k}$. Hence I is real if only if $s_1 = \ldots = s_k = 1$. If $s_1 > 1$ hence $2(s_1 - 1) \geq s_1$ and we get $I_1^{2(s_1-1)} I_2^{2s_2} \cdots I_k^{2s_k} \subseteq I$, therefore $I_1^{(s_1-1)} I_2^{s_2} \cdots I_k^{s_k} = I_1^{s_1} I_2^{s_2} \cdots I_k^{s_k}$.

Conversely, since $I_1, \ldots, I_k$ are coprime we have that $\dfrac{A}{I} \cong \displaystyle\prod_{j=1}^{s} \dfrac{A}{I_j}$, therefore we get an embedding from $\dfrac{A}{I}$ to the product of the field of fractions of each domain $\dfrac{A}{I_j}$, it is real and so is $\dfrac{A}{I}$, therefore I is real.

In some sense, such ideal must be square free product of real coprime ideals.

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