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Bifurcação de Hopf no infinito para campos vetoriais planares.

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Resumo

Estudamos, de um novo ponto de vista, familias de campos vetoriais planares sem singularidades $\{X_\mu : -\varepsilon < \mu < \varepsilon\}$ definidos no complemento de uma bola aberta de centro na origem tal que o infinito muda de atrator a repulsor, ou viceversa, em $\mu = 0$. Também estudamos um tipo de estabilidade local, ao redor do infinito, de alguns campos vetoriais planares de classe C^1.

Hopf Bifurcation at Infinity for Planar Vector Fields

B. Alarcón [*], V. Guíñez [†] and C. Gutierrez [‡]

Abstract. We study, from a new point of view, families of planar vector fields without singularities $\{X_\mu : -\varepsilon < \mu < \varepsilon\}$ defined in the complement of an open ball centered at the origin such that infinity changes from repellor to attractor, or vice versa, at $\mu = 0$. We also study a sort of local stability of some C^1 planar vector fields around infinity.

1 Introduction

Let $r > 0$, and let $B_r^n = \{x \in \mathbb{R}^n, \|x\| < r\}$. We denote the Alexandroff compactification of $(\mathbb{R}^n - B_r^n)$ with the point ∞ (at infinity) by $(\mathbb{R}^n - B_r^n) \cup \{\infty\}$.

A C^1−vector field $X : \mathbb{R}^n - B_r^n \to \mathbb{R}^n - \{0\}$ (without singularities) can be extended to the map

$$\tilde{X} : ((\mathbb{R}^n - B_r^n) \cup \{\infty\}, \infty) \to (\mathbb{R}^n, 0)$$

(which takes ∞ to 0). We shall consider the map $\tilde{X}$ as if it were a vector field having ∞ as an isolated singularity. In this manner, all questions concerning the local theory of isolated singularities of vector fields on $\mathbb{R}^n$ can be formulated and studied in the case of the vector field $\tilde{X}$. Observe that the phase portrait of $\tilde{X}$ is always a well defined object. Therefore in the specific case of a planar vector field X, we may talk about either hyperbolic, or parabolic, or elliptic sectors at ∞ of both X and $\tilde{X}$. For instance, if the origin is a global attractor of X, then ∞ is a repellor of X, that is, of $\tilde{X}$. In this context, when ∞ is an isolated singularity of X, we may also talk about the Poincaré index of X at ∞.

Here we discuss the appearing of a Hopf bifurcation at infinity of a one–parameter family of C^1 planar vector fields defined outside of a ball centered at the origin. This subject has been studied from two different points of view. The first one deals with families which have finitely many parameters and are made

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up of continuous perturbations of linear systems. The second point of view deals with polynomial families. In the former point of view, the strong domination imposed by the linear part of the vector field is used. (See [2],[3],[9],[10], [12].) In the latter the study, thanks to the Poincaré compactification, is carried out using the local theory of planar vector fields. (See [1],[4],[11].) The papers [5] and [7] develop a theory that provides us with the means for considering yet another point of view that makes possible the study of the Hopf bifurcation for certain families of class C^1. Though these families are not perturbations of linear systems, they somehow contain them.

In order to state the main results of this paper, we need the following concepts.

We will say that the C^1−vector field $X : \mathbb{R}^n - B_r^n \to \mathbb{R}^n$ has a **repellor** (resp. **attractor**) at ∞ if the map extending X to ∞

$$\tilde{X} : ((\mathbb{R}^n - B_r^n) \cup \{\infty\}, \infty) \to (\mathbb{R}^n, 0)$$

is locally topologically equivalent at ∞ to the vector field $x \to x$ (resp. $x \to -x$) at the origin.

We will say that the family of C^1−vector fields $\{X_\mu : \mathbb{R}^n - B_r^n \to \mathbb{R}^n;\ -\varepsilon < \mu < \varepsilon\}$ has at $\mu = 0$ a **Hopf bifurcation at** ∞ if two conditions are satisfied:

(1) For $\mu < 0$ (resp. $\mu > 0$), the vector field X_μ has a repellor at ∞, and for $\mu > 0$ (resp. $\mu < 0$), the vector field X_μ has an attractor at ∞.

(2) The vector field X_μ has no singularities in $\mathbb{R}^n - B_s^n$, for $-\varepsilon < \mu < \varepsilon$ and some $s \geq r$.

Let A be a Lebesgue measurable subset of $\mathbb{R}^n$, and let $f : A \to \mathbb{R}$ be a measurable function. We define as usual

$$f^+(x) = \max\{f(x), 0\} \quad \text{and} \quad f^-(x) = \max\{-f(x), 0\}\ .$$

We will say that $f : A \to \mathbb{R}$ is **Lebesgue integrable** if

$$\min\left\{\int_A f^+ \, d\lambda\,,\ \int_A f^- \, d\lambda\right\} < \infty\,,$$

in which case we define

$$\int_A f \, d\lambda = \int_A f^+ \, d\lambda \ - \int_A f^- \, d\lambda\,,$$

which is a well defined value of the extended real line $[-\infty, \infty]$.

Given $r > 0$ and $k, n \in \mathbb{N}$, let $\mathcal{A}^k(n, r)$ be the set of C^k−vector fields $X : \mathbb{R}^n - B_r^n \to \mathbb{R}^n$ such that $\operatorname{div}(X) : \mathbb{R}^n - B_r^n \to \mathbb{R}$ is Lebesgue integrable. Suppose $\Gamma = \{\Gamma_\rho\ ;\ a \leq \rho < \infty\}$ is an arbitrary lamination of a neighborhood U of ∞ in $(\mathbb{R}^n - B_r^n) \cup \{\infty\}$ such that:

(1) each Γ_ρ is a leaf of Γ, and is a smooth manifold diffeomorphic to $\{x \in \mathbb{R}^n : \|x\| = 1\}$,

(2) if $\rho > \sigma$, then $\Gamma_\rho \cap \Gamma_\sigma = \emptyset$, and Γ_ρ is closer to ∞ than Γ_σ is.

Given $X \in \mathcal{A}^k(n,r)$, with $r > 0$ and $k, n \in \mathbb{N}$, define

$$\mathcal{I}(X) \;=\; \lim_{\rho\to\infty} \int_{\Gamma_\rho} X(x) \cdot \frac{x}{||x||}\, ds \;=\; \int_{\mathbb{R}^n} \operatorname{div}(\hat{X})$$

where $\hat{X} : \mathbb{R}^n \to \mathbb{R}^n$ is an arbitrary C^1−vector field which coincides with X in the complement of a ball centered at the origin. In Section 3 we shall show that $\mathcal{I}(X)$ is well defined. Let $\mathcal{A}^1(2,r,\infty)$ be the set consisting of the vector fields of $\mathcal{A}^1(2,r)$ such that

$$\int_r^\infty \Upsilon(s)\, ds \;=\; \infty \,,$$

where

$$\Upsilon(s) \;=\; \inf\{||X(p)|| : ||p|| = s\}\,.$$

The following are our two main results, which will be proved in Section 2.

Theorem 1.1. *Let $\{X_\mu : -\varepsilon < \mu < \varepsilon\}$ be a family of vector fields in $\mathcal{A}^1(2,r,\infty)$ such that:*

(1) we have $\mu \cdot \mathcal{I}(X_\mu) > 0$, if $\mu \neq 0$,

(2) for all $-\varepsilon < \mu < \varepsilon$, the vector field X_μ has no singularities, and the Poincaré index at ∞ of the vector field

$$\tilde{X}_\mu : ((\mathbb{R}^2 - B_r^2) \cup \{\infty\}, \infty) \to (\mathbb{R}^2, 0)$$

(which extends X_μ to ∞) is less than or equal to 1.

Then $\mu = 0$ is a Hopf bifurcation at ∞ of the family $\{X_\mu\}$.

The following surprised us.

Theorem 1.2. *Let $X, Y \in \mathcal{A}^1(2,r,\infty)$ be vector fields such that their Poincaré index at ∞ is the same integer $k \geq 2$. Then the vector fields X and Y are locally topologically equivalent at ∞.*

2 Proof of Main Results

In the next Proposition " $\cdot$ " denotes the usual inner product, and ds denotes the arc length element in the metric associated to " $\cdot$ ".

Proposition 2.1. *Let $X \in \mathcal{A}^1(n,r)$, and let $\tilde{X} : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1−vector field such that, for some $s > r$, we have*

$$\tilde{X}|_{(\mathbb{R}^n - B_s^n)} \;=\; X|_{(\mathbb{R}^n - B_s^n)}\,.$$

Then

$$\mathcal{I}(X) \;=\; \lim_{\rho\to\infty} \int_{\Gamma_\rho} X(x) \cdot \frac{x}{||x||}\, ds \;=\; \int_{\mathbb{R}^n} div(\tilde{X})$$

is a well defined number of the extended real line $[-\infty, \infty]$ which does not depend on either the particular lamination Γ or the particular extension $\tilde{X}$ of X.

Proof. Since $\operatorname{div}(X)$ is integrable, we have that $\operatorname{div}(\tilde{X})$ (defined in $\mathbb{R}^n$) is integrable. Let Σ_ρ be the closure of the bounded connected component of $\mathbb{R}^n - \Gamma_\rho$. According to the Divergence Theorem,

$$\int_{\Sigma_\rho} \operatorname{div}(\tilde{X})\, dV = \int_{\Gamma_\rho} \tilde{X}(x) \cdot \frac{x}{\|x\|}\, dA\,,$$

which implies

$$\begin{aligned}\int_{\mathbb{R}^n} \operatorname{div}(\tilde{X})\, dV &= \lim_{\rho\to\infty} \int_{\Sigma_\rho} \operatorname{div}(\tilde{X})\, dV = \lim_{\rho\to\infty} \int_{\Gamma_\rho} \tilde{X}(x) \cdot \frac{x}{\|x\|}\, dA \\ &= \lim_{\rho\to\infty} \int_{\Gamma_\rho} X(x) \cdot \frac{x}{\|x\|}\, dA\,.\end{aligned}$$

The equality

$$\int_{\mathbb{R}^n} \operatorname{div}(\tilde{X})\, dV = \lim_{\rho\to\infty} \int_{\Gamma_\rho} X(x) \cdot \frac{x}{\|x\|}\, dA$$

on the one hand means that

$$\lim_{\rho\to\infty} \int_{\Gamma_\rho} X(x) \cdot \frac{x}{\|x\|}\, dA$$

does not depend on the particular lamination Γ. On the other hand, it means that

$$\lim_{\rho\to\infty} \int_{\Gamma_\rho} X(x) \cdot \frac{x}{\|x\|}\, dA$$

is a well defined number of $[-\infty, \infty]$. Therefore, neither does

$$\int_{\mathbb{R}^n} \operatorname{div}(\tilde{X})\, dV$$

depend on the particular extension $\tilde{X}$ of X. □

Proposition 2.2. *Any $X \in \mathcal{A}^1(2, r, \infty)$ has no hyperbolic sectors at ∞ (that is, the vector field $\tilde{X} : \left((\mathbb{R}^2 - B_r^2) \cup \{\infty\}, \infty\right) \to (\mathbb{R}^2, 0)$ has no hyperbolic sectors at ∞).*

Proof. Suppose that X had a hyperbolic sector at ∞. Consider only the case $\mathcal{I}(X) \in [-\infty, \infty)$. Then we would find a rectangle R whose boundary is made up of two arcs of trajectory of X, say $[p_2, \infty]$, $[p_1, q_1]$, and two arcs of trajectory of $X^* = (-g, f)$, say $[p_1, p_2]^*$, $[q_1, \infty]^*$, as is shown in Figure 1.

Since $X \in \mathcal{A}^1(2, r, \infty)$, we must have $L(q_1, \infty) = \infty$ and $L(p_1, p_2) < \infty$, where

$$L(a, b) = \left| \int_{[a,b]^*} \|X(x)\|\, ds \right|, \quad \text{while} \quad [a, b]^* \in \{[p_1, p_2]^*, [q_1, \infty]^*\}\,.$$

According to the Divergence Theorem,

$$\infty = L(q_1, \infty) - L(p_1, p_2) = \int_R \operatorname{div}(X) = K \in [-\infty, \infty)\,,$$

which is impossible. □

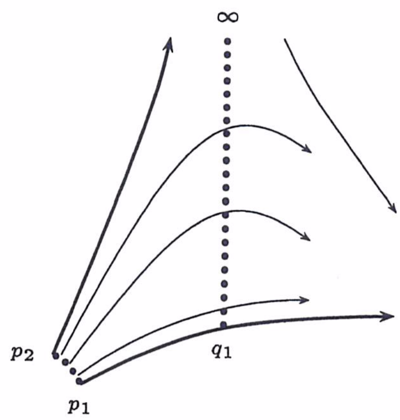

Figure 1. Arcs of trajectories of Proposition 2.2.

Remark 2.3. *If $X \in \mathcal{A}^1(2,r)$ is such that, for all $s \geq r$, we have $\Upsilon(s) = \inf\{||X(p)|| : ||p|| = s\} \geq \frac{c}{s}$, then $X \in \mathcal{A}^1(2,r,\infty)$. In fact, we have*

$$\int_r^{\infty} \Upsilon(s) \geq \int_r^{\infty} \frac{c}{s}\, ds = \infty .$$

Proof of Theorem 1.1. By Proposition 2.2, each $\tilde{X}_\mu$ has no hyperbolic sectors at ∞. It follows from this fact and condition (2) that the Poincaré index of each $\tilde{X}_\mu$ at ∞ equals 1. Hence $\tilde{X}_\mu$ has no elliptic sectors at ∞ either. Therefore, condition (2) implies that, given $\mu \neq 0$, the vector field $\tilde{X}_\mu$ has no closed trajectories that accumulate at ∞, and ∞ is either an attractor or a repellor of $\tilde{X}_\mu$. Moreover, for μ fixed, we may find a circle $\Gamma_\mu \subset \mathbb{R}^2$ near ∞ which is transversal to $\tilde{X}_\mu$, and is such that μ, $\mathcal{I}(X_\mu)$, and

$$\int_{\Gamma_\mu} X_\mu(x) \cdot \frac{x}{||x||}\, ds$$

have the same sign. Consequently, if $\mu > 0$ (resp. $\mu < 0$), then $\{\tilde{X}_\mu\}$ has an attractor (resp. repellor) at ∞. $\square$

Proof of Theorem 1.2. The vector fields X and Y have no hyperbolic sectors at ∞. Therefore, in terms of the local–sector–classification of vector fields around singularities (see [8]), X and Y have the same number of elliptic sectors. Moreover, for each pair of consecutive elliptic sectors there is a parabolic sector that separates them. In other words, there is a circle $\gamma \subset \mathbb{R}^2$ near and enclosing ∞ such that:

a) there are exactly $2k-2$ points of γ, say $p_i(X)$ (resp. $p_i(Y)$), of internal (with respect to ∞) quadratic type tangency of γ with the flow generated by X (resp. Y),

b) the vector field X (resp. Y) is transversal to $\gamma - \{p_1(X), \cdots p_{2k-2}(X)\}$ (resp. $\gamma - \{p_1(Y), \cdots p_{2k-2}(Y)\}$).

Using standard arguments, we can establish from these considerations a topological equivalence between $X|_U$ and $Y|_V$, where U and V are appropriate neighborhoods of ∞. (See [6].) $\square$

3 Examples

In this section, we give some examples that illustrate our results. Most of them are vector fields of the form (1). The computations of Remark 3.1 help us to decide whether or not the vector fields belong to $\mathcal{A}^1(2, r, \infty)$.

Remark 3.1. *Let $X : \mathbb{R}^2 - B_1^2 \to \mathbb{R}^2$ be a C^1-vector field of the form*

$$X(x,y) \;=\; \frac{1}{r^k}\left[G(x,y)\,(-y,x) \;+\; F(x,y)\,(x,y)\right], \tag{1}$$

where $r = \sqrt{x^2+y^2}$, and $F, G : \mathbb{R}^2 - B_1^2 \to \mathbb{R}$. Then:

a) $\|X(x,y)\| \;=\; \frac{1}{r^{k-1}}\sqrt{G(x,y)^2 + F(x,y)^2}\,.$

b) $div(X) \;=\; \frac{1}{r^k}\left[(2-k)\,F \;+\; x\,\frac{\partial F}{\partial x} \;+\; y\,\frac{\partial F}{\partial y} \;-\; y\,\frac{\partial G}{\partial x} \;+\; x\,\frac{\partial G}{\partial y}\right].$

c) $< X(x,y)\,,\, \frac{1}{\|x\|}(x,y) > \;=\; \frac{1}{r^{k-1}}\,F(x,y).$

Example 3.2. *We obtain the vector field*

$$X(x,y) = X_{b_0,b_1}(x,y) \;=\; (-y,x) \;+\; \frac{1}{r^2}\,(b_0 + b_1\,r)\,(x,y)$$

by setting $k=2$, $G(x,y) = G(r) = r^2$, and $F(x,y) = F(r) = b_0 + b_1\,r$ in (1). Observe that the Poincaré index of X_{b_0,b_1} at ∞ is 1. Moreover, we have

$$div(X_{b_0,b_1}) \;=\; \frac{b_1}{r}\,,$$

which implies that $div(X_{b_0,b_1})$ is Lebesgue integrable on $\mathbb{R}^2 - B_1^2$.
Since $\Upsilon(s) \;=\; \inf\{\|X_{b_0,b_1}(p)\| : \|p\| = s\} \;=\; s\sqrt{1 + \frac{1}{s^4}\,(b_0 + b_1\,s)^2} \;\geq\; s$, we also have that

$$\int_1^{\infty} \Upsilon(s)\; ds \;=\; \infty\,.$$

Therefore, $X_{b_0,b_1} \in \mathcal{A}^1(2,1,\infty)$ and

$$\begin{aligned}
\mathcal{I}(X_{b_0,b_1}) \;&=\; \lim_{r\to\infty}\int_{\|(x,y)\|=r} < X(x,y), \frac{1}{\|(x,y)\|}(x,y) > ds \\
&=\; \lim_{r\to\infty}\int_0^{2\pi} \frac{1}{r}\,(b_0 + b_1\,r)\,r\,dr \\
&=\; 2\,\pi\,\lim_{r\to\infty}\,(b_0 + b_1\,r) \;=\; \begin{cases} 2\,\pi\,b_0 & \text{if}\;\; b_1 = 0\,, \\ +\infty & \text{if}\;\; b_1 > 0\,, \\ -\infty & \text{if}\;\; b_1 < 0\,. \end{cases}
\end{aligned}$$

Then, for each b_0 fixed, the family $X(b_0,\mu)$ has a Hopf bifurcation at $\mu = 0$, with $\mathcal{I}(X(b_0,\mu)) \;=\; 2\,\pi\,b_0$. Moreover, for $b_0\cdot\mu < 0$, the vector field $X(b_0,\mu)$ has a limit cycle which is close to infinity for μ small.

Also, the family $X(b_0,0)$ has a Hopf bifurcation at $b_0 = 0$, with $|\mathcal{I}(X(b_0,0))| < \infty$ for all b_0.

Example 3.3. *In $\mathbb{R}^2 - B_1^2$ consider the vector field*

$$X(x,y) \;=\; (-y,x) \;+\; \frac{1}{r^2}\left[a_0 + \frac{a_1}{r} \;+\; \frac{a_2}{r^2} \;+\; \cdots \;+\; \frac{a_n}{r^n}\right]\,(x,y)\,.$$

Observe that this vector field is of the form (1), with $k=2$, $G(x,y)=G(r)=r^2$ and $F(x,y)=F(r)=a_0+\frac{a_1}{r} \;+\; \frac{a_2}{r^2} \;+\; \cdots \;+\; \frac{a_n}{r^n}$, and that the Poincaré index of X at ∞ is 1.
Since

$$\|X(x,y)\| \;=\; r\sqrt{1+\frac{1}{r^4}\,F(r)^2} \;>\; r\,,$$

we have

$$\int_1^{\infty} \Upsilon(s)\,ds \;=\; \infty\,.$$

Now since

$$div(X) \;=\; -\frac{1}{r^{n+2}}\,\left[a_1\,r^{n-1} \;+\; 2a_2\,r^{n-2} \;+\; \cdots \;+\; na_n\right]\,,$$

we have

$$div(X)^+ \;\le\; \frac{1}{r^{n+2}}\,\left[|a_1|\,r^{n-1} \;+\; 2\,|a_2|\,r^{n-2} \;+\; \cdots \;+\; n\,|a_n|\right]\,.$$

Therefore,

$$\begin{aligned}\int\!\!\int_{B_R^2-B_1^2} div(X)^+ \;&\le\; 2\pi\int_1^R \frac{1}{r^{n+1}}\,\left[|a_1|\,r^{n-1} \;+\; \cdots \;+\; n\,|a_n|\right]\,dr\\ &\le\; 2\pi\,M\int_1^R \frac{1}{r^2}\,dr \;=\; 2\pi\,M\left[-\frac{1}{R}+1\right]\,.\end{aligned}$$

So

$$\int\!\!\int_{R^2-B_1^2} div(X)^+ \;\le\; 2\pi\,M \;<\; \infty\,.$$

Therefore, $X\in\mathcal{A}^1(2,1,\infty)$ and

$$\begin{aligned}\mathcal{I}(X) \;&=\; \lim_{r\to\infty}\int_{\|(x,y)\|=r} <X(x,y),\,\frac{1}{\|(x,y)\|}(x,y)>\,ds\\ &=\; \lim_{r\to\infty}\int_0^{2\pi}\frac{1}{r}\,F(r)\,r\,d\theta \;=\; 2\pi\,a_0\,.\end{aligned}$$

For example, if

$$F(r) \;=\; F_\mu(r) \;=\; \left(\mu-\frac{1}{r}\right)\,\left(\mu-\frac{2}{r}\right)\cdots\left(\mu-\frac{n}{r}\right)$$

and $X \;=\; X_\mu$, we have

$$\mathcal{I}(X_\mu) \;=\; 2\pi\,\mu^n\,.$$

Then if n is odd it follows that the family X_μ verifies the conditions of Theorem 1.1, and we have a Hopf bifurcation at $\mu=0$. Observe that, for $\mu>0$ small, we have n limit cycles close to infinity.

Example 3.4. *Let*

$$X(x,y) \;=\; \frac{1}{r^{2m+2}}\,(P(x,y),Q(x,y))$$

with

$$\begin{aligned} P(x,y) &= p_0(x,y)+p_1(x,y)+\cdots p_{2m}(x,y)+p_{2m+1}(x,y) \quad \textit{and} \\ Q(x,y) &= q_0(x,y)+q_1(x,y)+\cdots q_{2m}(x,y)+q_{2m+1}(x,y)\,, \end{aligned}$$

where $p_k(x,y)$ *and* $q_k(x,y)$ *are homogeneus polynomials of degree* k. *Assume also that*

$$(p_{2m+1},q_{2m+1})(x,y) \;=\; G_{2m}(x,y)\,(-y,x) \;+\; F_{2m}(x,y)\,(x,y)$$

where $G_{2m}(x,y)$ *and* $F_{2m}(x,y)$ *are homogeneous polynomials of degree* $2m$ *which verify*

$$-y\,\frac{\partial G_{2m}}{\partial x} \;+\; x\,\frac{\partial G_{2m}}{\partial y} \;=\; 0 \quad \textit{and} \quad G_{2m}(x,y)^2 \;+\; F_{2m}(x,y)^2 > 0\,,$$

for all $(x,y) \neq (0,0)$.
We have

$$div(X) \;=\; \frac{1}{r^{2m+4}}\left[r^2\left(\frac{\partial P}{\partial x} \;+\; \frac{\partial Q}{\partial y}\right) \;-\; (2m+2)\,(x\,P \;+\; y\,Q)\right].$$

If we write

$$div(X) \;=\; \frac{1}{r^{2m+4}}\,[R_0+R_1+\cdots+R_{2m+2}]\;,$$

with R_k *a homogeneous polynomial of degree* k, *then*

$$R_{2m+2} \;=\; r^2\left[-y\,\frac{\partial G_{2m}}{\partial x} \;+\; x\,\frac{\partial G_{2m}}{\partial y} - 2m\,F_{2m} \;+\; x\,\frac{\partial F_{2m}}{\partial x} \;+\; y\,\frac{\partial F_{2m}}{\partial y}\right] \;=\; 0\,.$$

Therefore, $r^2\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) \;-\; (2m+2)\,(x\,P \;+\; y\,Q)$ *is a polynomial of degree at most* $2m+1$.
To prove that $div(X)$ *is Lebesgue integrable, it suffices to show*

$$\int\!\!\int_{\mathbb{R}^2-B_1^2} \frac{|x^{n-k}\,y^k|}{(\sqrt{x^2+y^2})^{2m+2}}\,dx\,dy \;<\; \infty\,,$$

for all $0 \le k \le n \le 2m-1$. *Since* $|x^{n-k}\,y^k| \;\le\; (\sqrt{x^2+y^2})^n$, *we have*

$$\begin{aligned} \int\!\!\int_{\mathbb{R}^2-B_1^2} \frac{|x^{n-k}\,y^k|}{(\sqrt{x^2+y^2})^{2m+2}}\,dx\,dy &\le \int\!\!\int_{\mathbb{R}^2-B_1^2} \frac{dx\,dy}{(\sqrt{x^2+y^2})^{2m+2-n}}\,dx\,dy \\ &= 2\pi\,\lim_{R\to\infty}\int_1^R \frac{r\,dr}{r^{2m+2-n}} \\ &= \frac{2\pi}{2m-n}\,, \end{aligned}$$

which implies that $div(X)$ *is Lebesgue integrable on* $\mathbb{R}^2 - B_1^2$.

We next show that $\Upsilon(r) \geq \frac{c}{r}$. For this, it suffices to prove

$$r^2 \, \|X(x,y)\|^2 \; = \; \frac{P^2(x,y)+Q^2(x,y)}{r^{4m+2}} \; \geq \frac{K}{2} > 0\,,$$

for all $\|(x,y)\| \; = \; r$ *sufficiently large, where*

$$K \; = \; \inf\{G_{2m}(\cos(\theta),\sin(\theta))^2 + F_{2m}(\cos(\theta),\sin(\theta))^2 \; : \; 0 \leq \theta \leq 2\pi\}\,.$$

The preceding inequality follows from

$$\lim_{r\to\infty} \frac{h_i(x,y)}{r^{4m+2}} \; = \; 0\,, \quad \text{for} \quad 0 \leq i < 4m+2\,, \quad \text{and}$$

$$\frac{h_{4m+2}(x,y)}{r^{4m+2}} \; = \; G_{2m}(\cos(\theta),\sin(\theta))^2 + F_{2m}(\cos(\theta),\sin(\theta))^2\,,$$

for $(x,y) \; = \; r(\cos(\theta),\sin(\theta))$. *Here* $h_i(x,y)$ *is the homogeneous part of degree* i *of the polynomial* $P^2(x,y)+Q^2(x,y)$.
Therefore, $X \in \mathcal{A}^1(2,1,\infty)$ *and*

$$\begin{aligned}
\mathcal{I}(X) \; &= \; \lim_{r\to\infty} \int_{\|(x,y)\|=r} < X(x,y), \frac{1}{\|(x,y)\|}(x,y) > ds \\
&= \; \lim_{r\to\infty} \frac{1}{r^{2m+1}} \int_0^{2\pi} [\cos(\theta)\, P(r\cos(\theta), r\sin(\theta)) + \\
&\qquad\qquad \sin(\theta)\, Q(r\cos(\theta), r\sin(\theta))]\; d\theta \\
&= \; \lim_{r\to\infty} \frac{1}{r^{2m+1}} \left(\sum_{k=0}^{2m} r^k \int_0^{2\pi} [\cos(\theta) p_k(\cos(\theta),\sin(\theta)) + \right. \\
&\qquad \left. \sin(\theta) q_k(\cos(\theta),\sin(\theta))]\; d\theta + \; r^{2m+1} \int_0^{2\pi} F_{2m}(\cos(\theta),\sin(\theta))\; d\theta \right) \\
&= \; \int_0^{2\pi} F_{2m}(\cos(\theta),\sin(\theta))\, d\theta\,.
\end{aligned}$$

Remark 3.5. *Many planar polynomial vector fields X that have a limit cycle at infinity are of the form $Y = (P,Q)$, with P and Q polynomials as in Example 3.4. For instance, in [4] and [11] the authors consider the case*

$$\begin{aligned}
P(x,y) \; &= \; \alpha x - y - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2 + (\alpha x - \lambda y)(x^2+y^2) \\
Q(x,y) \; &= \; x + \alpha y + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2 + (\lambda x + \alpha y)(x^2+y^2)\,.
\end{aligned}$$

They study the simultaneous generation of limit cycles from the origin, as well as from infinity. For $\lambda \neq 0$, a, and b, all of the possible configurations of these limit cycles are obtained, where a and b are certain positive multiples of the number $(\lambda_6 - \lambda_3)^2$. For λ as above, six is the greatest number of limit cycles which may be bifurcated: two (or four) from the origin, and four (or two) from infinity.

The corresponding vector field X is

$$X(x,y) \;=\; \frac{1}{(x^2+y^2)^2}\, Y(x,y)\,, \quad \textit{with} \quad F_2(x,y) \;=\; \alpha\,(x^2+y^2)\,.$$

Then

$$\mathcal{I}(X) \;=\; \int_0^{2\pi} F_2(\cos(\theta),\sin(\theta))\,d\theta \;=\; 2\,\pi\,\alpha\,.$$

Example 3.6. *For $k \in \mathbb{Z}$, consider the vector field $X_k = (P_k, Q_k)$ with*

$$P_k(x,y) = \Re[(x - y\,\imath)^k] \quad \textit{and} \quad Q_k(x,y) = \Im[(x - y\,\imath)^k]\,.$$

Then:

1) The Poincaré index of X_k at infinity is $k+2$.

2) The divergence of X_k is identically zero.

3) We have $||X_k(p)|| = ||p||^k$, which implies $\Upsilon(s) = \inf\{||X_k(p)|| : ||p|| = s\} = s^k$. Then, for $r > 0$, we have

$$\int_r^{\infty} \Upsilon(s)\,ds = \begin{cases} \infty\,, & \textit{for} \quad k \geq -1\,, \\ \frac{-r^{k+1}}{k+1}\,, & \textit{for} \quad k < -1\,. \end{cases}$$

Therefore, for all $k \geq -1$ and $r > 0$, the vector field $X_k \in \mathcal{A}^1(2,r,\infty)$ and verifies $\mathcal{I}(X_k) = 0$. Moreover, for $k \geq 0$, according to Theorem 1.2, any vector field $Y \in \mathcal{A}^1(2,r,\infty)$ with Poincaré index at ∞ equal to $k+2$ is equivalent to X_k in a neighborhood of ∞.
Finally, for $k \leq -2$, the vector field X_k has hyperbolic sectors at ∞, and therefore $X_k \notin \mathcal{A}^1(2,r,\infty)$.

Example 3.7. *For $k \in \mathbb{Z}$, consider the vector field $Z_k = (R_k, S_k)$ with*

$$R_k(x,y) = \Re[(x + y\,\imath)^k] \quad \textit{and} \quad S_k(x,y) = \Im[(x + y\,\imath)^k]\,.$$

Then:

1) The Poincaré index of Z_k at ∞ is $2-k$.

2) We have $||X_k(p)|| = ||p||^k$, which implies $\Upsilon(s) = \inf\{||X_k(p)|| : ||p|| = s\} = s^k$. Then, for $r > 0$, we have

$$\int_r^{\infty} \Upsilon(s)\,ds = \begin{cases} \infty\,, & \textit{for} \quad k \geq -1\,, \\ \frac{-r^{k+1}}{k+1}\,, & \textit{for} \quad k < -1\,. \end{cases}$$

Therefore, for all $r \geq 1$ and $k < -1$, the vector field $Z_k \notin \mathcal{A}^1(2,r,\infty)$. Now for $k \geq 2$, we also have that $Z_k \notin \mathcal{A}^1(2,r,\infty)$ because Z_k has hyperbolic sectors at ∞. For the remaining values of k we have that

$$div(Z_{-1})(x,y) \;=\; \frac{2(y^2-x^2)}{(x^2+y^2)^2}\,, \quad div(Z_0)(x,y) \;=\; 0\,, \quad and \quad div(Z_1)(x,y) \;=\; 2$$

are Lebesgue integrable on $\mathbb{R}^2 - B_r^2$, for all $r > 0$. Therefore, $Z_{-1}, Z_0, Z_1 \in \mathcal{A}^1(2,r,\infty)$, and

$$\mathcal{I}(Z_{-1}) \;=\; \mathcal{I}(Z_0) \;=\; 0 \quad and \quad \mathcal{I}(Z_1) \;=\; \infty\,.$$

Remark 3.8. *Let $k \in \mathbb{Z}$. With the notation of Examples 3.6 and 3.7 we have the following two relationships:*

a) If $X_k = (P_k, Q_k)$, then $Z_k = (P_k, -Q_k)$.

b) If $(x,y) \neq (0,0)$, then $Z_{-k}(x,y) = \frac{1}{(x^2+y^2)^k} X_k(x,y)$.

References

[1] J. Billeke, E. Sáez and R. Vásquez, *About the generalized Hopf bifurcations at infinity for planar cubic systems*, Revista Colombiana de Matemticas **29:2** (1995), 89–93.

[2] P. Diamond, D. Rachinskii, and M. Yumagulov, *Stability of large cycles in a nonsmooth problem with Hopf bifurcation at infinity*, Nonlinear Analysis **42** (2000), 1017–1031.

[3] J. Glover, *Hopf bifurcation at infinity*, Nonlinear Analysis, Theory, Methods and Applications **13:12** (1989), 1393–1398.

[4] V. Guiñez, E. Sáez and I. Szántó, *Simultaneous Hopf bifurcations at the origin and infinity for cubic systems*, "Dynamical Systems", Pitman Research Notes in Math., Series 285 (1993), 40–51.

[5] C. Gutierrez and A. Sarmiento, *Injectivity of C^1 maps $\mathbb{R}^2 \to \mathbb{R}^2$ at infinity and planar vector fields*, Astérisque **287** (2003), 89–102.

[6] C. Gutierrez and J. Sotomayor, *Stable configurations of lines of principal curvature*, Astérisque **98–99** (1982), 195–215.

[7] C. Gutierrez and M. Teixeira, *Asymptotic stability at infinity of planar vector fields*, Bol. Soc. Bras. Mat. **26:1** (1995), 57–66.

[8] P. Hartman, *Ordinary Differential Equations*, John Wiley and Sons, Inc., 1973.

[9] M. Krasnosel'skii, N. Kuznetsov, and M. Yumagulov, *Localization and Construction of Cycles for Hopf Bifurcation at Infinity*, Doklady Mathematics **52:2** (1995), 223–226.

[10] M. Krasnosel'skii, N. Kuznetsov, and M. Yumagulov, *Conditions of cycle stability for the Hopf bifurcation at infinity*, Automation and Remote Control **58:1** (1997), 43–48.

[11] N. G. Lloyd, T. R. Blows, and M. C. Kalenge, *Some cubic systems with several limit cycles*, Nonlinearity **1:4** (1998), 653–669.

[12] H. Xiangian, *Hopf bifurcation at infinity with discontinous nonlinearities*, J. Austral. Math. Sec. B **33:2** (1991), 133–148.

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