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Imagens iteradas e a Conjectura Jacobiana Planar

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Resumo

Provamos que as imagens iteradas de um par Jacobiano $f : \mathbb{C}^2 \to \mathbb{C}^2$ estabilizam; ou seja, para $k \in \mathbb{N}$ suficientemente grande, todos os conjuntos $f^k(\mathbb{C}^2)$ são iguais. Mais geralmente, seja X um subconjunto fechado algébrico de $\mathbb{C}^N$, e $f : X \to X$ uma aplicação polinomial aberta tal que $X - f(X)$ é um conjunto finito. Provamos que os conjuntos $f^k(X)$ estabilizam, e para qualquer subconjunto co-finito $\Omega \subseteq X$, com $f(\Omega) \subseteq \Omega$, os conjuntos $f^k(\Omega)$ estabilizam. Aplicamos estes resultados para obter uma nova caracterização da Conjectura Jacobiana Complexa bidimensional relacionada a questões de surjetividade.

ITERATED IMAGES AND THE PLANE JACOBIAN CONJECTURE

RONEN PERETZ, NGUYEN VAN CHAU, CARLOS GUTIERREZ, AND L. ANDREW CAMPBELL

ABSTRACT. We show that the iterated images of a Jacobian pair $f : \mathbb{C}^2 \to \mathbb{C}^2$ stabilize; that is, all the sets $f^k(\mathbb{C}^2)$ are equal for k sufficiently large. More generally, let X be a closed algebraic subset of $\mathbb{C}^N$, and $f : X \to X$ an open polynomial map with $X - f(X)$ a finite set. We show that the sets $f^k(X)$ stabilize, and for any cofinite subset $\Omega \subseteq X$ with $f(\Omega) \subseteq \Omega$, the sets $f^k(\Omega)$ stabilize. We apply these results to obtain a new characterization of the two dimensional complex Jacobian conjecture related to questions of surjectivity.

1. INTRODUCTION

The n-dimensional Jacobian conjecture for a field $\mathbf{k}$ of characteristic 0, which we denote as $JC(\mathbf{k}, n)$, asserts that any polynomial map $f : \mathbf{k}^n \to \mathbf{k}^n$, for which the Jacobian determinant $J(f)$ is a nonzero constant, is bijective and the inverse map is also polynomial ([BCW82, Ess00]). The full Jacobian conjecture aserts that $JC(\mathbf{k}, n)$ is true for all fields $\mathbf{k}$ of characteristic 0, and all integers $n > 0$; so far, no particular case $JC(\mathbf{k}, n)$ has been proved for any single $\mathbf{k}$ and any single $n > 1$. We consider only the complex case $\mathbf{k} = \mathbb{C}$ (it is known that for each $n > 0$, $JC(\mathbb{C}, n)$ implies $JC(\mathbf{k}, n)$ for any $\mathbf{k}$ of characteristic 0).

Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map with $J(f)$ a nonzero constant. Then f is an open map (even locally biholomorphic), $f(\mathbb{C}^n)$ is open and simply connected, and $\mathbb{C}^n - f(\mathbb{C}^n)$ is either empty or a closed algebraic subset of $\mathbb{C}^n$ of codimension at least 2 ([Cam73]). In the case $n = 2$ we call f a *Jacobian pair*. For a Jacobian pair f, the coimage $\mathbb{C}^2 - f(\mathbb{C}^2)$ is empty or at most 0-dimensional, hence finite in either case. We prove a general result about open polynomial endomorphisms with finite coimage, and then use that result to provide a recharacterization of $JC(\mathbb{C}, 2)$ as a conjecture about certain polynomial endomorphisms of cofinite subsets of the plane $\mathbb{C}^2$.

Let X be a closed complex algebraic subset of $\mathbb{C}^N$. We equip X with the strong topology that it inherits from the Euclidean topology on $\mathbb{C}^N$. Let $f : X \to X$ be a polynomial map. Let f^k denote the iterated map $f^k = f \circ f \circ \cdots \circ f$ (k composition factors). The sequence $X, f(X), f(f(X)), \ldots, f^k(X), \ldots$ of iterated images of X under f is a nested sequence of sets (each set contains the next or is equal to it).

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In general, all we can say about the iterated images of X, is that they are constructible. But if f is an open map and $f(X)$ consists of all but finitely many points of X, we will show that the nested sequence $f^k(X)$ eventually stabilizes, with all $f^k(X)$ equal to the same subset of X for k large enough. In addition, we show that if $\Omega \subseteq X$ is a cofinite subset of X that is invariant under f (that is, $f(\Omega) \subseteq \Omega$), then the iterated images $f^k(\Omega)$ also stabilize. The second result reduces to the first when $\Omega = X$, but it is more general, because, if Ω is a proper cofinite subset of X, then Ω is usually not affine ([Har77]), hence cannot be embedded as a closed algebraic subset of $\mathbb{C}^N$ for any N. Combining both results into a single theorem yields

Theorem 1. *Let $f : X \to X$ be a polynomial map of a closed complex algebraic subset of $\mathbb{C}^N$ to itself. If f is an open map, $X - f(X)$ is finite, and Ω is a cofinite subset of X with $f(\Omega) \subseteq \Omega$, then there is a $K > 0$ such that $f^K(\Omega) = f^{K+1}(\Omega) = f^{K+2}(\Omega) = \dots$ (all $f^k(\Omega)$ are equal for $k \geq K$). The stable image $\bigcap_{k>0} f^k(\Omega)$ is an open cofinite subset of Ω.*

We will use the term *stability of iterated images* for the property that the iterated images are all eventually equal. The restriction of f to $S = \bigcap_{k>0} f^k(\Omega)$ is an open, surjective polynomial map of S to itself. In a slightly more general context, if V is a complex algebraic variety that can be embedded in $\mathbb{C}^N$ as a closed algebraic subset, if f is an open morphism (regular map) of a closed algebraic subset X of V to itself, and if $X - f(X)$ is finite, then stability of iterated images holds for f and any cofinite subset Ω of X satisfying $f(\Omega) \subseteq \Omega$. For an embedding of V induces an embedding of X, and f can be expressed as a polynomial map in the coordinates of $\mathbb{C}^N$.

Corollary 1. *Let $n > 1$. Then stability of iterated images holds for $f : \Omega \to \Omega$, if f is an étale regular endomorphism of a cofinite subset Ω of $\mathbb{C}^n$ and $f(\Omega)$ is a cofinite subset of Ω. The stable image of Ω is an open, simply connected, cofinite subset of Ω.*

Proof. $\mathbb{C}^n$ is simply connected; so is the complement in $\mathbb{C}^n$ of any closed complex analytic subset of codimension 2 or more ([Abh63]). If $f : \Omega \to \Omega$ is a regular map (a morphism of algebraic varieties), it is a rational map defined at all but finitely many points in $\mathbb{C}^n$, so it is polynomial. Let F be the unique extension of f to a polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$. The meaning of étale in this situation is that f is locally biholomorphic. As is known, this implies that the Jacobian determinant $J(F)$ is nonzero at every point of Ω. $J(F)$ is a polynomial, so it cannot vanish only on a finite set of points. Since Ω is cofinite, $J(F)$ must vanish nowhere. Thus F is open. Further $\mathbb{C}^n - F(\mathbb{C}^n)$ consists of, at most, some of the finitely many points of Ω omitted by f, and some of the images under F of the finitely many points of $\mathbb{C}^n$ not in Ω. So Theorem 1 applies. Since the stable image is cofinite in Ω, it is cofinite in $\mathbb{C}^n$, hence simply connected. □

The special case $\Omega = \mathbb{C}^2$ yields

Corollary 2. *The iterated images of a Jacobian pair stabilize. The stable image is an open, simply connected, cofinite subset of $\mathbb{C}^2$.*

It is well known that if f is a Jacobian pair, and f is injective, then f is an automorphism (bijective, with polynomial inverse). In fact, there is a general principle that an injective endomorphism of a complex algebraic variety is also surjective ([Ax69, CR91]), so bijectivity is easy. Moreover, f is an automorphism if its restriction $f|_\Omega$ to a dense open subset of C^2 is injective (the birational case of the Jacobian conjecture; this dates back to the origins of the Jacobian conjecture in [Kel39]). So $JC(\mathbb{C}, 2)$ holds if every Jacobian pair is injective.

In contrast, little is known about surjectivity of Jacobian pairs or general properties of surjective endomorphisms of complex algebraic varieties. However, applying Theorem 1 to Jacobian pairs, we show that

Theorem 2. $JC(\mathbb{C}, 2)$ *is equivalent to the following assertion*

- *If f is a Jacobian pair and Ω is a cofinite subset of $\mathbb{C}^2$, such that $f(\Omega) = \Omega$, then $f|_\Omega$ is injective.*

Remark . Note that the requirement $f(\Omega) = \Omega$ means that Ω is invariant under f and $f|_\Omega$ is surjective. In fact, using reasoning similar to that in Corollary 1 above, Theorem 2 can be restated as follows.

The two dimensional complex Jacobian conjectire is true if, and only if, every surjective étale endomorphism of a cofinite subset of $\mathbb{C}^2$ is injective.

2. A lemma on the dynamics of a map

We begin with the following elementary observations on the dynamics of a map of a set to itself. Neither topology nor algebraic structure are involved.

Let $f : X \to X$ be a map of a nonempty set to itself. Let $f^0 = \text{Id}$, and $f^{k+1} = f \circ f^k$ for $k \geq 0$. Let E^k denote the set $X - f^k(X)$. Then it is easy to show that

$$\emptyset \subseteq E = E^1 \subseteq E^2 \subseteq E^3 \subseteq \ldots. \qquad (*)$$

and $E^{k+1} \subseteq E^k \cup f(E^k)$. ¿From this it follows easily that if E is finite, so are all the $E^k, k > 1$. Also, if, for some K, $E^K = E^{K+j}$ for any $j > 0$, then all the E^k from K on are the same. If there is such a K, we call the sequence (*) stable. We denote the union of all the E^k by E^∞; so (*) is stable if, and only if, $E^K = E^\infty$ for some $K > 0$.

Lemma 1. *Suppose that E is a finite set and (*) is not stable. Then there exists a point $e \in E$, such that all the points $f^k(e)$ for $k > 0$ belong to E^∞ and are distinct. For any such e there is a positive integer $M(e)$, such that the equation $f(x) = f^k(e)$ has exactly one solution for $k \geq M(e)$, namely $x = f^{k-1}(e)$.*

Proof. For $a \in X$ define the backward orbit of a as follows. $O(a)$ is a directed graph, whose vertices are all points $x \in X$ that satisfy $f^i(x) = a$ for some $i \geq 0$, and in which an edge is directed from x to y exactly when $f(y) = x$. Then, if $a \in E^k$, it is clear that $O(a)$ is a tree of maximum path length $k - 1$. Furthermore, if $a \notin E^\infty$, then $O(a)$ may contain directed cycles (arising from periodic points of f) and, in any case, has no maximum path length.

Now assume that (*) is not stable. Then E is nonempty. We claim that there exists an $e \in E$ such that $f^k(e) \in E^\infty$ for all $k \geq 0$. Note that if $x \notin E^\infty$ then $f(x) \notin E^\infty$. So if there is no $e \in E$ for which $f^k(e) \in E^\infty$ for all $k \geq 0$, then for each $e \in E$ there is a finite first value $k(e)$ for which $f^{k(e)} \notin E^\infty$. Since E is finite, we can define $m = \max_{e \in E} k(e)$. If $k > m$ and $a \in E^{k+1} - E^k$ then there is a path of length k starting at $a \in O(a)$ and terminating at some $e \in E$. So $f^k(e) = a \in E^\infty$. That contradicts the definition of m. So if $k > m$, it follows that $E^{k+1} = E^k$. That contradicts the assumed instability. We conclude that some $e \in E$ satisfies $f^k(e) \in E^\infty$ for all $k \geq 0$. For the remainder of this proof, we fix such an e. Note that all the points $f^k(e)$, $k > 0$ are distinct (an equality between any two with different values of k would imply that they are not in E^∞).

Consider the directed graph, Γ, whose set of vertices is the smallest set that contains e and all the points $f^k(e)$, $k > 0$, and contains every $x \in X$ for which $f(x)$ is a vertex. Since $f(x) \in E^{k+1}$ implies $x \in E^k$, all the vertices of Γ are contained in the graphs $O(f^k(e))$. The directed edges of Γ go from x to y exactly when $f(y) = x$. Let A be the set of points $f^k(e)$, $k \geq 0$. We claim that the vertices of Γ are the disjoint union of A and a finite set B. For suppose that v is a vertex and $v \notin A$. Then v is $f^m(e')$ for some $e' \in E$ that is different from e, and some $m \geq 0$. Since $f^{m+j}(e') = f^k(e)$ for some $j, k > 0$, only finitely many of the points $f^k(e')$, $k \geq 0$ do not belong to A. Since there are only finitely many possible e', B is finite. For all $k > 0$, the equation $f(x) = f^k(e)$ has the solution $f^{k-1}(e)$, and that is the only solution in A. Any other solution is a vertex of Γ, hence in B. Since B is finite, this can occur for only finitely many $k > 0$ (note that inverses of distinct elements are distinct). Thus there exists an $M(e) \in \mathbb{N}$ such that $f(x) = f^k(e)$ has the unique solution $f^{k-1}(e)$ if $k \geq M(e)$. □

3. Points with a limited number of inverse images

In this section, let X and Y be closed complex algebraic subsets of $\mathbb{C}^N$, and $f : X \to Y$ a polynomial map. For any $n \geq 0$, we define a set $A(f, n)$ as follows.

$$A(f, n) = \{y \in Y | \#(f^{-1}(y)) \leq n\}$$

That is, $A(f, n)$ is the set of points of Y which have n or fewer inverse images under f. Note that $A(f, 0) = E = E^1$ in the notation of the previous section.

The following lemma expresses a well known result. For lack of a suitable refrence on an elementary level, we provide a simple proof.

Lemma 2. *If $f : X \to Y$ is an open polynomial map, then each set $A(f, n)$ is a closed algebraic subset of $\mathbb{C}^N$ (and of Y).*

Proof. If we fix n and allow the use of the coefficients of f and those of finite sets of polynomials defining X and Y, then there is a first order formula with free variables $z_1, \ldots, z_N$ that is true for precisely those $z = (z_1, \ldots, z_N)$ that belong to $A(f, n)$. So each $A(f, n)$ is constructible: it can be represented as a Boolean combination of finitely many closed complex algebraic subsets of $\mathbb{C}^N$. Furthermore, $A(f, n)$ is closed in $\mathbb{C}^N$. To see this, suppose $y \in Y$ does not belong to $A(f, n)$. Then there are $n + 1$ distinct points of X that map to y. Since $f : X \to Y$ is open, there is a neighborhood in Y of y consisting of points with at least $n + 1$ inverse images. Thus $A(f, n)$ is a closed subset of Y and hence of $\mathbb{C}^N$. But it is classic that a subset of $\mathbb{C}^N$ that is constructible and closed in the Euclidean topology of $\mathbb{C}^N$ is a closed algebraic subset of $\mathbb{C}^N$ (express the set as a disjoint union of finitely many sets of the form $C - D$, where C is closed, algebraic, and irreducible, and D is a proper closed algebraic subset of C; then observe that each C is the closure of $C - D$, hence the set is a finite union of the sets C). □

The same holds true if we assume only that $f : X \to Y$ is an open regular map (a morphism) of closed complex algebraic subsets of varieties that can be embedded in $\mathbb{C}^N$. For X and Y can then both be embedded in $\mathbb{C}^N$ for a large enough N, and f can be expressed as a polynomial map in the coordinates of $\mathbb{C}^N$.

4. The iterated image theorem

For many types of mathematical structures, there is a general principle that an endomorphism that is injective as a map of sets is also necessarily bijective. Obvious examples are finite sets and finite dimensional vector spaces. For complex algebraic sets we have

Fact 1. *Let $f : X \to X$ be a polynomial map of a closed algebraic subset of C^N to itself and suppose that f is injective. Then f is surjective.*

More general statements can be made ([Ax69, CR91]) but this will suffice.

Proof of Thorem 1. Assume that $f : X \to X$ is an open polynomial map of a closed algebraic subset of $\mathbb{C}^N$ to itself, that $X - f(X)$ is finite, and that $\Omega \subseteq X$ is a cofinite subset of X with $f(\Omega) \subseteq \Omega$.

For simplicity, we begin by considering the special case $\Omega = X$. Suppose that the sequence (*) is not stable. Then, by Lemma 1, there exist $e \in E$ and $M = M(e) > 0$, such that the points $f^k(e)$, $k \geq M$ are all distinct, they all belong to E^∞, and each has a single inverse image under f. Let $T = \{f^k(e)\}_{k>M}$. Then $T \subseteq A(f,1)$. Let $\bar{T}$ be the Zariski closure (in $\mathbb{C}^N$) of T; that is, the smallest closed algebraic subset of $\mathbb{C}^N$ containing T. Since $A(f,1)$ is closed algebraic by Lemma 2, $\bar{T} \subseteq A(f,1)$. Obviously $f(T) \subseteq T$ and so, by the Zariski continuity of f, we have $f(\bar{T}) \subseteq \bar{T}$. Thus the restriction of f to $\bar{T}$ is a polynomial map of $\bar{T}$ to itself. And it is injective because $\bar{T} \subseteq A(f,1)$. So it is surjective, and $f(\bar{T}) = \bar{T}$. From this it follows that each $t \in \bar{T}$ has an inverse image under f that is also in $\bar{T}$. Starting with $f^M(e) \in \bar{T}$, we can therefore construct an infinite sequence $\ldots, a_{-3}, a_{-2}, a_{-1}, f^M(e)$ in the backwards orbit $O(f^M(e))$, with $f(a_i) = a_{i+1}$, for $i < -1$. But that contradicts the fact that $f^M(e) \in E^\infty$. We conclude that the sequence (*) must be stable.

Now consider the case in which Ω is a proper subset of X, and suppose that the iterated images of Ω do not stabilize. Apply Lemma 1 to $f|_\Omega : \Omega \to \Omega$. Let $E^k_\Omega = \Omega - f^k(\Omega)$ and $E^\infty_\Omega = \bigcup_{k>0} E^k_\Omega$. We obtain an $e \in E^1_\Omega$, such that the points $f^k(e)$ for $k > 0$ are all distinct points of Ω and lie in E^∞_Ω. Also by Lemma 1, for large enough k, each $f^k(e)$ has exactly one inverse image under f that lies in Ω. If such a point has more than one inverse image under f in X, it must be the image of a point not in Ω. Since there are only finitely many of those, we see that for large enough k, say $k \geq M$, each $f^k(e)$ has exactly one inverse image under f, namely $f^{k-1}(e)$. Take such a point $f^k(e)$. $k \geq M$. It belongs to E^∞_Ω. Suppose that $f^k(e) \notin E^\infty$ (computed for $f : X \to X$). Then there is an infinite path in the backward orbit $O(f^k(e))$. All the points in that path are distinct (otherwise some points in the forward orbit would not be distinct). If we go back far enough the points cannot lie in Ω, since $f^k(e) \in E^\infty_\Omega$, and hence must be one of the finitely many points not in Ω. Since the path is infinite, this is a contradiction. Since we now have all $f^k(e) \in E^\infty \bigcap A(f,1)$ for $k \geq M$, we are in the same situation as in the case $\Omega = X$. Just take $T = \{f^k(e)\}_{k>M}$ and proceed as before. □

5. Iterated images of Jacobian pairs

Let f be a Jacobian pair, Ω a subset of $\mathbb{C}^2$ with finite complement, and suppose $f(\Omega) \subseteq \Omega$. Then Ω is an algebraic variety, $f|_\Omega : \Omega \to \Omega$ is open, and $f|_\Omega$ has finite coimage. In this situation, we have

Fact 2. *If $f|_\Omega$ is injective then so is f and f is an automorphism.*

For, since f is injective on a dense open subset of $\mathbb{C}^2$, it must have geometric degree 1, hence it is birational and therefore ([Kel39, Ess00]) an automorphism.

Little is known about surjectivity of Jacobian pairs or their $n > 2$ dimensional counterparts. In particular, whether Jacobian pairs are necessarily surjective is an open question of long standing ([Vit75]). It is also not known whether a surjective Jacobian pair must be an automorphism (but see [DM90] for the case in which $f : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map with integer coefficients and $f(\mathbb{Z}^n) = \mathbb{Z}^n$).

Theorem 2 relates the two dimensional complex Jacobian conjecture to surjective polynomial endomorphisms of cofinite subsets of the plane, and shows that $JC(\mathbb{C}, 2)$ is equivalent to the assumption that certain surjective endomorphisms are injective; thus seemingly turning on its head the usual principle that injectivity implies surjectivity.

Proof of Theorem 2. Let f be a Jacobian pair. If $JC(\mathbb{C}, 2)$ holds then f is an automorphism, which implies that its restriction $f|_\Omega$ to any subset whatsoever of the plane is injective. Conversely, let us now assume that $f|_\Omega$ is injective if Ω is any cofinite subset of $\mathbb{C}^2$ with $f(\Omega) = \Omega$. Theorem 1 applies to f, so let Ω be the stable image of f. By assumption, f is injective on Ω, hence by Fact 2, f is an automorphism. □

6. An example

For concreteness, we illustrate the application of Theorem 1 with a simple example. It is taken, with thanks, from [Ess00, pp. 292–294], where the properties we list below are stated. Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be given by

$$f(x, y) = (x - 2(xy + 1) - y(xy + 1)^2, -1 - y(xy + 1))$$

Then f is a flat map, hence open, and the image of $\mathbb{C}^2$ under f is $\Omega = \mathbb{C}^2 - (0, 0)$.

Since $f(\Omega) = \Omega$, we see that Ω is the stable image of $f : \mathbb{C}^2 \to \mathbb{C}^2$, as well as the stable image of $f|_\Omega : \Omega \to \Omega$. Note that $f|_\Omega$ is an open, surjective endomorphism of $\mathbb{C}^2 - (0, 0)$ that is not injective. It is, of course, not étale.

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