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TOPOLOGICAL $\mathcal{K}$-EQUIVALENCE OF ANALYTIC FUNCTION-GERMS

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ABSTRACT. We obtain some results on the topological $\mathcal{K}$-equivalence of function-germs $(\mathbb{R}^n, 0) \to \mathbb{R}$.

RESUMO. Neste artigo obtivemos alguns resultados sobre a $\mathcal{K}$-equivalência topológica dos germes de funções $(\mathbb{R}^n, 0) \to \mathbb{R}$.

1. Introduction

The $\mathcal{K}$-equivalence of singularities of differential maps was introduced by J. Mather and it was considered as an important step of investigation of $\mathcal{A}$-equivalence of these singularities. J. Montaldi [3] discovered a geometric nature of $\mathcal{K}$-equivalence, so-called contact of submanifolds. He proved that two map-germs $f, g\colon (\mathbb{R}^n, 0) \to \mathbb{R}^p$ are $\mathcal{K}$-equivalent if and only if their graphs have the "same contact" with $\mathbb{R}^n$.

The paper is devoted to topological $\mathcal{K}$-equivalence of function-germs $f\colon (\mathbb{R}^n, 0) \to \mathbb{R}$. We are going to introduce a complete invariant with respect to this equivalence relation. Our results can be considered as a confirmation of Nishimura's results. In [1], Benedetti and Shiota proved that the set of equivalence classes of germs of polynomial functions, with respect to topological $\mathcal{A}$-equivalence, is contable. This result was generalized by M. Coste [2] for functions definable in o-minimal structures. Clearly, these results are true for topological $\mathcal{K}$-equivalence, since topological $\mathcal{K}$-equivalence is weaker than topological $\mathcal{A}$-equivalence. Thus, the classification question can be asked. In the paper we resolve this classification question.

In Section 2, we present a complete invariant - so-called *tent function*. The theory is presented in the language of o-minimal structures . We use a definable version of the main lemma of Nishimura [4], which is the main technical tool of the result of the section. The Theorem 2.1 can be understood as a topological $\mathcal{K}$ version of the results of Montaldi.

Section 4 is devoted to a special case: analytic function-germs. For analytic function-germs the $\mathcal{K}$-invariant defined in Section 2 is very

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simple. It is a finite sequence of elements equal to 1 or to -1. We prove that all these invariants, in the case $n=2$, admit a polynomial realization. Finally, for $n=2$, we give a "normal form" for topological $\mathcal{K}$-finitely determined function-germs. Note that our invariants are created for all definable singularities and not only for generic singularities case as it is in the classical theory.

We would like to thank Maria Ruas and Feliciano Vitorio for very interesting discussions.

2. Tent functions

Let $M \subset \mathbb{R}^n$ be a convex $n-1$ dimensional polytope. Let $Z \subset M$ be a closed subset such that:

- Z is a union of some faces of M;
- Codimension of Z in M is not equal to zero.

Let $\{U_i\}$ be the family of connected components of $M - Z$. The collection $\{U_i\}$ is called a *$\mathcal{K}$-decomposition of M* and the set Z is called a *zero-locus of $\{U_i\}$*.

Let $\| \ \|_M$ be a norm in $\mathbb{R}^n$ such that the polytope M is realized as a unit sphere. Let

$$\tilde{Z} = \{x \in \mathbb{R}^n - \{0\} \ : \ \frac{x}{\|x\|_M} \in Z\} \text{ and } \tilde{U}_i = \{x \in \mathbb{R}^n - \{0\} \ : \ \frac{x}{\|x\|_M} \in U_i\}.$$

The collection $\{\tilde{U}_i\}$ is called *$\mathcal{K}$-decomposition of $\mathbb{R}^n$ associated to $\{U_i\}$* and the set $\tilde{Z}$ is called *zero-locus of $\{\tilde{U}_i\}$*.

Let M_1, M_2 be two polytopes, Z_1, Z_2 be two zero-loci and $\{U_i^1\}, \{U_j^2\}$ be two $\mathcal{K}$-decompositions. The $\mathcal{K}$-decompositions are called *combinatorially equivalent* if there exist triangulations of pairs $(M_1, Z_1), (M_2, Z_2)$ and a simplicial isomorphism between these triangulations.

A function $T_i \colon \mathbb{R}^n \to \mathbb{R}$ defined as follows:

$$T_i(x) = \begin{cases} dist(x, \partial\tilde{U}_i) & \text{se } x \in \tilde{U}_i, \\ 0 & \text{otherwise.} \end{cases}$$

is called an *elementary tent function*. The functions $\sum_i a_i T_i$, where a_i is equal to -1, 0 or 1, are called *tent functions associated to $\mathcal{K}$-decomposition $\{U_i\}$*.

Let $\{\tilde{U}_i\}$ and $\{\tilde{V}_j\}$ be $\mathcal{K}$-decompositions of $\mathbb{R}^n$. Let α, β be two tent functions associated to $\tilde{U}_i$ and $\tilde{V}_j$ respectively. The functions α and β are called *combinatorially equivalent* if there exists a germ of simplicial isomorphism $h \colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that, for each $\tilde{U}_i$, there exist $\tilde{V}_j$ such that $h(\tilde{U}_i) = \tilde{V}_j$ and $sign[\alpha(x)] = sign[\beta(h(x))]$. Clearly, if two tent functions are combinatorially equivalent then *$\mathcal{K}-$ decompositions* of the corresponding polytopes are combinatorially equivalent.

Definition . Let A be an o-minimal structure on $\mathbb{R}$. Let $f, g\colon (\mathbb{R}^n, 0) \to \mathbb{R}$ be two germs of definable in A continuous functions. The germs are called *topologically $\mathcal{K}$-equivalent in A* if there exist germs of definable in A homeomorphisms $h\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $H\colon (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ such that $H(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\}$ and the following diagram is commutative:

$$\begin{array}{ccccc} (\mathbb{R}^n,0) & \xrightarrow{(Id,f)} & (\mathbb{R}^n\times\mathbb{R},0) & \xrightarrow{\pi} & (\mathbb{R}^n,0) \\ h\downarrow & & H\downarrow & & h\downarrow \\ (\mathbb{R}^n,0) & \xrightarrow{(Id,g)} & (\mathbb{R}^n\times\mathbb{R},0) & \xrightarrow{\pi} & (\mathbb{R}^n,0) \end{array}$$

where $Id\colon \mathbb{R}^n \to \mathbb{R}^n$ is the identity map and $\pi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is the canonical projection $\pi(x,t) = x,\ (x,t) \in \mathbb{R}^n \times \mathbb{R}$.

Theorem 2.1. *Let A be an o-minimal structure on $\mathbb{R}$.*

1. *Let $f\colon (\mathbb{R}^n, 0) \to \mathbb{R}$ be a germ of a definable in A continuous function. Then there exist a $\mathcal{K}$-decomposition of $\mathbb{R}^n$ and a tent function $\alpha\colon (\mathbb{R}^n, 0) \to \mathbb{R}$ such that f and α are topologically $\mathcal{K}$-equivalent in A.*
2. *Two tent functions are topologically $\mathcal{K}$-equivalent in A if and only if they are combinatorially equivalent.*

We use the following version of Nishimura's Lemma [4].

Lemma 2.2. *Let A be an o-minimal structure on $\mathbb{R}$. Let U be a neighbourhood of $0 \in \mathbb{R}^n$. Let f and g be two germs of definable in A continuous functions. If $\dfrac{f(x)}{g(x)} > 0$, for each $x \in U - f^{-1}(0)$, then f and g are topologically $\mathcal{K}$-equivalent in A.*

Remark 2.3. *In the original version , see [4], the lemma is proved for some continuous maps. But it is easy to see that the arguments are "definable in A" and the homeomorphism constructed by Nishimura are definable in A.*

Proof of Theorem 2.1. 1. Let $f\colon (\mathbb{R}^n, 0) \to \mathbb{R}$ be a germ of a definable in A continuous function. Let us consider a triangulation of the pair $(\mathbb{R}^n, f^{-1}(0))$. Let (M, Y) be a link of this triangulation at 0. Let cY be a cone over Y considered as a union of all simplexes of this triangulation, such that $\{0\}$ is a vertex and all other vertices belong to Y. By [5] there exists a germ of a definable in A homeomorphism $h\colon (\mathbb{R}^n, cY) \to (\mathbb{R}^n, f^{-1}(0))$. Let Z be a union of all edges of Y with codimension more or equal to zero. Let $\{U_i\}$ be a $\mathcal{K}$-decomposition corresponding to M and Z. Let $\tilde{f}(x) = f(h(x))$. Let $\alpha = \sum_i a_i T_i$ be a tent function with $a_i = sign[\tilde{f}|_{\bar{U}_i}]$. By Nishimura's Lemma, $\tilde{f}$ and α are topologically $\mathcal{K}$-equivalent in A.

2. Let α and β be two tent functions associated to different $\mathcal{K}$-decompositions of $\mathbb{R}^n$. Suppose that α and β are topologically $\mathcal{K}$-equivalent. Then, by the definition of topological $\mathcal{K}$-equivalence, a definable in A homeomorphism $H\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}$ maps the graph of α to the graph of β. Since H is a definable homeomorphism, it can be triangulated. A triangulation of H makes a combinatorial equivalence between α and β. □

Remark 2.4. *Theorem 2.1 can be interpreted as a topological version of results of Montaldi* [3]*. Namely, we can define a definable topological contact between submanifolds X and Y of $\mathbb{R}^n$ at $x_0 \in X \cap Y$ in the following way: a pair of submanifolds $X, Y \subset \mathbb{R}^n$ definable in A have the same topological contact at $x_0 \in X \cap Y$ as a pair $X', Y' \subset \mathbb{R}^n$ definable in A at $x_0' \in X' \cap Y'$ if there exists a germ of a definable in A homeomorphism $H\colon (\mathbb{R}^n, x_0) \to (\mathbb{R}^n, x_0')$ such that $H(X) = X'$ and $H(Y) = Y'$.*

Then Theorem 2.1 can be translated in the following form: a germ of a definable function $f\colon (\mathbb{R}^n, 0) \to \mathbb{R}$ is topologically $\mathcal{K}$-equivalent in A to a germ of a definable function $g\colon (\mathbb{R}^n, 0) \to \mathbb{R}$ if the pairs $(graph(f), \mathbb{R}^n)$ and $(graph(g), \mathbb{R}^n)$ have the same definable topological contact.

3. Polynomials of 2 variables and $\mathcal{K}$-invariants

A $\mathcal{K}$-decomposition of $\mathbb{R}^2$ can be described as a finite collection of rays with initial point $(0,0) \in \mathbb{R}^2$. The sets $\tilde{N}_i$ are sections between these rays. An equivalence class of tent functions by a combinatorial equivalence described in Section 2 is a function $\eta\colon \{U_i\} \to \{-1, 0, 1\}$.

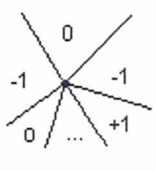

Figure 1

In other words, this invariant can be described as an equivalence class of finite collections of elements $-1, 0$ or 1 by cyclic permutations. An equivalence class η described above is called a *$\mathcal{K}$-invariant.* A $\mathcal{K}$-invariant is called *analytic* if

(1) the number of sectors is even;
(2) for all i, $\eta(i) \neq 0$.

Clearly, if a function-germ $f\colon (\mathbb{R}^n, 0) \to \mathbb{R}$ is analytic then the corresponding $\mathcal{K}-$invariant is analytic. We say that a $\mathcal{K}$-invariant η admits an *algebraic realization* if there exists a polynomial $f(x, y)$ of two variables such that the $\mathcal{K}$-invariant of the germ of this polynomial at $0 \in \mathbb{R}^2$ is equal to η.

Theorem 3.1. *Every analytic $\mathcal{K}$-invariant admits an algebraic realization.*

We need the following lemma.

Lemma 3.2. *Let $\gamma_1, \gamma_2 \colon [0, \epsilon) \to \mathbb{R}^2$ be two semialgebraic arcs, such that $\gamma_1(0) = \gamma_2(0) = 0$ and the tangent vectors at 0 to γ_1 and γ_2 are the same. Then there exists a polynomial $p(x, y)$ such that:*

(1) *the set $\{(x, y) \in \mathbb{R}^2 : p(x, y) = 0\}$ is a curve with two half-branches $\tilde{\gamma_1}$ and $\tilde{\gamma_2}$;*
(2) *the curve $\{(x, y) \in \mathbb{R}^2 : p(x, y) = 0\}$ have a singular point at $0 \in \mathbb{R}^2$;*
(3) *the unit tangent vectors to $\tilde{\gamma_1}$ and $\tilde{\gamma_2}$ at $0 \in \mathbb{R}^2$ are the same and equal to the unit tangent vector to γ_1 at $0 \in \mathbb{R}^2$;*
(4) *for small $t \neq 0$ we have $p(\gamma_1(t)) > 0$ and $p(\gamma_2(t)) > 0$;*
(5) *the set $\{(x, y) \in \mathbb{R}^2 : p(x, y) \leq 0\}$ is bounded by the curves $\tilde{\gamma_1}$ $\tilde{\gamma_2}$.*

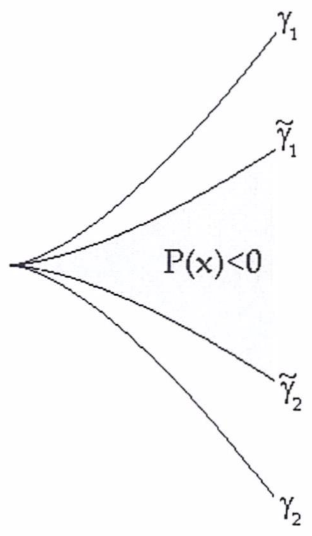

FIGURE 2

Proof. We can apply a blowing-up at $0 \in \mathbb{R}^2$ several times and obtain a picture where the inverse images of the curves γ_1 and γ_2 will have different tangent vectors. Let $\pi \colon X \to \mathbb{R}^2$ a composition of the blowing-up, where X is a 2-dimensional manifold. Let $\tilde{\gamma_i} = \pi^{-1}\gamma_i$ (for $i = 1, 2$) be the inverse images of these arcs. Since $\tilde{\gamma_1}$ and $\tilde{\gamma_2}$ have different tangent vectors at the $\tilde{\gamma_1}(0) = \tilde{\gamma_2}(0) = x_0 \in X$, there exists a real algebraic curve β with a cuspidal singularity such that β is contained in the area locally bounded by $\tilde{\gamma_1}$ and $\tilde{\gamma_2}$ and the singular point of β is equal to x_0 (see Figure 3).

Let $\tilde{p}(\tilde{x}) = 0$ be an algebraic equation of β such that $\tilde{p}(\tilde{x}) < 0$ in the cuspidal area. Applying the map π to the function $\tilde{p}$ we obtain a function p satisfying the statement of the lemma.

□

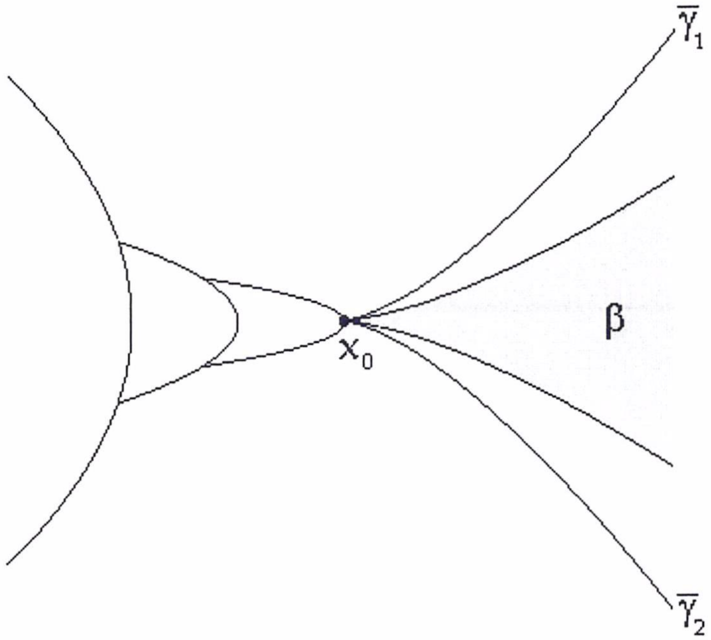

FIGURE 3

We say that an analytic $\mathcal{K}$-invariant η is *alternative* if, for each i, $\eta(i) \neq \eta(i+1)$.

Claim 1. *Let η be an analytic alternative $\mathcal{K}$-invariant. Then η admits the following algebraic realization:*

$$p(x, y) = (x + y)(2x + y) \cdots (sx + y).$$

Proof. The set $\{(x, y) \in \mathbb{R}^2 \ : \ p(x, y) = 0\}$ is the collection of the lines $l_i = \{(x, y) \in \mathbb{R}^2 \ : \ ix + y = 0\}$. Let $\tilde{U}_i, \tilde{U}_{i+1}$ be two sectors such that $\tilde{U}_i \cap \tilde{U}_{i+1}$ is a half-line of the line l_i. Observe that the function $ix + y$ have different signs in $int(\tilde{U}_i)$ and in $int(\tilde{U}_{i+1})$. So, the function p also have different signs in $int(\tilde{U}_i)$ and in $int(\tilde{U}_{i+1})$. Thus, the $\mathcal{K}$-invariant of p is alternative. □

An analytic $\mathcal{K}$-invariant η is called *double alternative* if, for each i, $\eta(i) \neq \eta(i+2)$. The corresponding sequence looks like

$$1, 1, -1, -1, 1, 1 \ldots$$

Claim 2. *An analytic double alternative $\mathcal{K}$-invariant admits the following algebraic realization:*

$$p(x,y) = (x+y)(2x+y)^2(3x+y)(4x+y)^2 \cdots (2sx+y)^2.$$

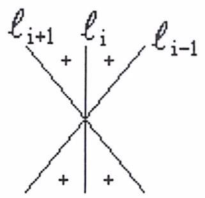

FIGURE 4

Proof. Let $\theta \in S^1$. Then, for all $r > 0$, we have $sign[p(r\theta)] = sign[p(\theta)]$. Suppose that θ moves along S^1. Then $p(\theta)$ does not change the sign when θ crosses a point $l_i \cap S^1$, for i even, and $p(\theta)$ change the sign when θ crosses a point $l_i \cap S^1$, for i odd, where $l_i \ : \ ix + y = 0$ (see figure 4). Thus, the $\mathcal{K}$-invariant of p is double alternative. □

Proof of Theorem 3.1. Observe that the number of sectors, for an analytic $\mathcal{K}$-invariant, is even, that is, $2r$, for some integer $r > 0$. We use induction on r. If $r = 1$ the statement is trivial. In fact, a sequence $(-1, 1)$ is realized for example by the function $p(x, y) = x$, a sequence $(1, 1)$ is realized by $p(x, y) = x^2$ and a sequence $(-1, -1)$ is realized by $p(x, y) = -x^2$. Let us suppose that all $\mathcal{K}$-invariants, for all $r \leq r_0$, are algebraically realized. Consider now a sequence with $2r_0 + 2$ elements. If the sequence is double alternative, then it is realized (cf Claim 2). Thus we can suppose that the sequence is not double alternative and, thus there exist three consecutive elements $\eta(i), \eta(i+1), \eta(i+2)$ such that $\eta(i) = \eta(i+2)$. Let us consider another $\mathcal{K}$-invariant obtained from the first one by taking these elements away and putting $\eta(i)$ instead of them:

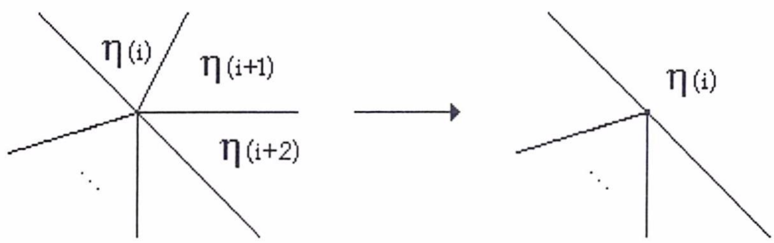

FIGURE 5

By the induction hypotheses, the new configuration is algebraically realized. Let $p_1(x, y)$ be a germ of a real polynomial realizing this new $\mathcal{K}$-invariant. Let γ_1 and γ_2 be the semialgebraic arcs bounding the area corresponding to $\eta(i)$. By Lemma 3.2, there exists a polynomial $p(x,y)$ satisfying the conditions of the lemma and positive outside the area bounded by the curve $\{(x,y) \in \mathbb{R}^2 \ : \ p(x,y) = 0\}$. Thus, the realization $\tilde{p}$ of our $\mathcal{K}$-invariant can be obtained as a product $\tilde{p} = p_1.p$ □

Corollary 3.3. *Let $f(x,y)$ be an analytic function at $0 \in \mathbb{R}^2$. Then there exists a polynomial $p(x,y)$ such that the germ of f at $0 \in \mathbb{R}^2$ is topologically $\mathcal{K}$-equivalent to the germ of p at $0 \in \mathbb{R}^2$.*

The following theorem gives a normal forms for all germs of topologically $\mathcal{K}$-finitely determined functions $(\mathbb{R}^2, 0) \to \mathbb{R}$.

Theorem 3.4. *Let $f(x,y)$ be a topologically $\mathcal{K}$-finitely determined germ of a definable in A function at $0 \in \mathbb{R}^2$. Then f is topologically $\mathcal{K}$-equivalent to a product of linear functions*

$$p(x,y) = (x+y)(2x+y)\cdots(sx+y).$$

Proof. Since f is topologically $\mathcal{K}$-finitely determined, we have $f^{-1}(0) \cap Sing(f) = \{0\}$ (cf [6]). Let γ be a half-branch of the curve $\{(x,y) \in \mathbb{R}^2 \ : \ f(x,y) = 0\}$. Since $\gamma - \{0\} \cap Sing(f) = \emptyset$, the function f changes a sign if one crosses γ. Thus, the $\mathcal{K}$-invariant of f is alternative. By the result of Claim 1, f is topologically $\mathcal{K}$-equivalent to a product of linear functions

$$p(x,y) = (x+y)(2x+y)\cdots(sx+y).$$

□

Corollary 3.5. *Two topologically $\mathcal{K}$-finitely determined germs are topologically $\mathcal{K}$-equivalent if and only if their zero-sets have the same number of half-branches.*

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