Instituto de Ciências Matemáticas de São Carlos

ISSN - 0103-2577

Two-weight norm inequalities for maximal functions on homogeneous spaces and boundary estimates

SÉRGIO LUÍS ZANI

Nº 23

NOTAS DO ICMSC
Série Matemática

São Carlos
Out. / 1994

TWO-WEIGHT NORM INEQUALITIES FOR MAXIMAL FUNCTIONS ON HOMOGENEOUS SPACES AND BOUNDARY ESTIMATES

SÉRGIO LUÍS ZANI

ABSTRACT. Let D be an open subset of a homogeneous space (X, d, μ). Consider the maximal function

$$M_\varphi f(x) = \sup \frac{1}{\varphi(B)} \int_{B \cap \partial D} |f| d\nu, \quad x \text{ in } D,$$

where the supremum is taken over all balls centered at suitable points of the boundary of D with radius larger than $t(x) = d(x, \partial D)$ and φ is a nonnegative set function defined for all Borel sets of X satisfying the (quasi-)monotonicity and doubling properties.

We give a necessary and sufficient condition on the weights w and v for the weighted norm inequality

$$\left(\int_D [M_\varphi(f)]^q w d\mu\right)^{1/q} \leq c \left(\int_{\partial D} |f|^p v d\nu\right)^{1/p} \tag{0.1}$$

to hold when $1 < p < q < \infty$ and $\sigma = v^{1-p'}$ is a doubling weight and give a sufficient condition for 0.1, when $1 < p \leq q < \infty$ without assuming that σ is a doubling weight but with an extra assumption on φ. Another characterization for 0.1 when $1 < p \leq q < \infty$ and D is of the form $Y \times (0, +\infty)$, where Y is a homogeneous space with group structure is also provided.

1. INTRODUCTION

We consider a homogeneous space (X, d, μ) where $d : X \times X \to [0, \infty)$ satisfies:

(1) $d(x, y) = 0$ if and only if $x = y$.
(2) $d(x, y) = d(y, x)$ for any x, y in X.
(3) There exists $A_o \geq 1$ such that

$$d(x, y) \leq A_o(d(x, z) + d(z, y)) \quad \text{for all } x, y, z \text{ in } X. \tag{1.1}$$

μ is a measure on the Borel subsets of X satisfying the doubling property: there exists a positive constant A_1 such that

$$\mu(B(x, 2r)) \leq A_1 \mu(B(x, r)) \quad \text{for all } x \text{ in } X \text{ and } r > 0, \tag{1.2}$$

where $B(x,r) = \{y \in X; d(x,y) < r\}$ is the ball of radius r with center x. If $B = B(x,r)$ and $\lambda > 0$, we will write λB for $B(x, \lambda r)$.

As mentioned in [C], Macias and Segovia (Cf. [MS]) have proved that given a quasi-metric d on X there exists a quasi-metric d' on X satisfying:

(1) $C^{-1}d'(x,y) \leq d(x,y) \leq Cd'(x,y)$ for all x,y in X, where C is a positive constant independent of x and y.

(2) The balls with respect to d', i.e., $B'(x,r) = \{y \in X; d'(y,x) < r\}, x \in X, r > 0$ are open.

Since only the order of magnitude of $d(x,y)$ will be relevant for us, we may assume that the balls defined by d are open. We will also assume that all annuli $A(x,r,R) = B(x,R) \setminus B(x,r)$ in X are nonempty for all $0 < r < R$. We immediately see that there exists $c > 0$ such that the diameter of $B(x,r)$

$$\delta(B(x,r)) = \sup\{d(y,z); y,z \in B(x,r)\} \geq cr$$

for all x in X and $r > 0$. In fáct, since $A = A(x, \frac{1}{2}r, r)$ is nonempty, there exists y in A. Thus, $r > d(x,y) \geq \frac{1}{2}r$. Therefore, $\delta(B(x,r)) \geq d(x,y) \geq \frac{1}{2}r$.

Now, for any z,y in $B(x,r)$, $d(z,y) \leq A_o(d(z,x) + d(x,y)) \leq 2A_o r$. Thus,

$$\frac{1}{2}r \leq \delta(B(x,r)) \leq 2A_o r,$$

and we say that the diameter of $B(x,r)$ is equivalent to r.

Let D be an open subset of X. For each x in X let $t(x) = d(x, \partial D)$ and select $a(x)$ in ∂D such that $d(x,a(x)) \leq \frac{3}{2}t(x)$. Let us denote by $\mathfrak{B}$ the set of all balls $B = B(z,r)$ with z in ∂D and $r > 0$.

Suppose φ is a nonnegative set function which is defined for all Borel sets of X satisfying the following conditions:

(1)(*quasi-monotonicity*) There exists $c_o > 0$ such that $\varphi(B') \leq c_o\varphi(B)$ whenever $B' \subset B$, $B', B \in \mathfrak{B}$.

(2) (*doubling*) There exists $c_o > 0$ such that $\varphi(B(z,2r)) \leq c_o\varphi(B(z,r))$ for all z in ∂D and $r > 0$.

Observe that (1) and (2) imply that for all $\gamma > 0$ there exists $c(\gamma) > 0$ such that

(3) $\varphi(B(z,\gamma r)) \leq c(\gamma)\varphi(B(z,r))$ for all z in ∂D and $r > 0$.

For φ, $t(x)$ and $a(x)$ as defined above define

$$M_{\varphi,a}f(x) = \sup_{r > t(x)} \frac{1}{\varphi(B(a(x),r))} \int_{\check{B}} |f| d\nu \tag{1.3}$$

where $\check{B}(a(x),r) = B(a(x),r) \cap \partial D$, $B(a(x),r) = \{y \in X; d(y,a(x)) < r\}$ and ν is a doubling measure on ∂D.

Note that if $b(x)$ is in ∂D and satisfies $d(x, b(x)) \leq \frac{3}{2}t(x)$ then it follows from the properties of φ that there exists $C > 0$ such that $C^{-1}M_{\varphi,b}f \leq M_{\varphi,a}f \leq CM_{\varphi,b}f$, for all f. For this reason we will write $M_\varphi f$ instead of $M_{\varphi,a}f$.

Observe that when $X = Y \times \mathbf{R}$, (Y, ν, ρ) is a homogeneous space, $D = Y \times (0, \infty) = \hat{Y}_+$ and $\varphi(E) = |E \cap \partial\hat{Y}_+|_\nu^{1-\gamma}$, where $|\cdot|_\nu$ denotes the $\nu-$measure and $0 \leq \gamma < 1$,

$$(1.4)\, M_\varphi f(x,t) = M_\gamma f(x,t) = \sup_{r>t} \frac{1}{|B(x,r)|_\nu^{1-\gamma}} \int_{B(x,r)} |f(y)|\, d\nu(y), \quad (x,t) \in \hat{Y}_+,$$

where $B(x,r) = \{y \in Y; \rho(x,y) < r\}$ and $r > 0$, is the fractional maximal function. In [WZ], R. Wheeden and S. Zhao showed that if $1 < p < q < \infty$, $w(x,t)$ and $v(y)$ are weights in $\hat{Y}_+$ and Y, respectively, and if $\sigma(y) = v(y)^{1-p'}$, $p' = p/(p-1)$ satisfies the doubling condition and μ is a Borel measure in $\hat{Y}_+$, then the the weighted norm inequality

$$(1.5)' \qquad \left(\int_{\hat{Y}_+} M_\gamma f(x,t)^q w(x,t) d\mu(x,t)\right)^{1/q} \leq c \left(\int_Y |f(y)|^p v(y) d\nu(y)\right)^{1/p}$$

holds if and only if there exists a positive constant C such that

$$(1.6) \qquad \left(\int_B d\nu\right)^{1-\gamma} \left(\int_{\hat{B}} w d\mu\right)^{1/q} \left(\int_B \sigma d\nu\right)^{1/p'} \leq C$$

for all balls $B \subset Y$, where $\hat{B} = B \times [0, r(B))$ and $r(B)$ is the radius of B. The first result we have for M_φ is

Theorem 1.1. *Suppose $1 < p < q < \infty$. Let w and v be weights defined on D and ∂D, respectively. Suppose that $\sigma d\nu = v^{1-p'}d\nu$ is a doubling measure. Then the weighted norm inequality*

$$(1.7) \qquad \left(\int_D |M_\varphi f|^q w d\mu\right)^{1/q} \leq c \left(\int_{\partial D} |f|^p v d\nu\right)^{1/p}$$

holds for all f, with c independent of f, if and only if

$$(1.8) \qquad \varphi(B)^{-1} \left(\int_{\hat{B}} w d\mu\right)^{1/q} \left(\int_{\check{B}} \sigma d\nu\right)^{1/p'} \leq C$$

for all balls $B = B(z,r)$, z in ∂D, $r > 0$, with C independent of B, where $\hat{B} = B \cap D$, $\check{B} = B \cap \partial D$.

In the next theorem we present a condition such that the weighted norm inequality

$$(1.9) \qquad \left(\int_D [M_\varphi f]^q w d\mu\right)^{1/q} \leq c \left(\int_{\partial D} |f|^p v d\nu\right)^{1/p},$$

holds for all f, where $1 < p \leq q < \infty$, without imposing the doubling condition on σ but with an additional condition on φ. The theorem is stated in terms of dyadic cubes

centered at points of the boundary of a fixed open subset D of a homogeneous space. Basically, a dyadic cube is a Borel set that contains and is contained in concentric balls that are centered at points of ∂D and whose diameters are comparable to the diameter of such a set. For a similar result for off-centered maximal operators, see [W2].

Theorem 1.2. *Suppose $1 < p \leq q < \infty$ and let w and v be weights defined on D and ∂D, respectively. Let $\sigma = v^{1-p'}$. Suppose that there exists $C > 0$ and $0 < \varepsilon \leq 1$ such that*

$$\left(\frac{|Q'|_\nu}{|Q|_\nu}\right)^\varepsilon \leq c\frac{\varphi(Q'^*)}{\varphi(Q^*)},$$

for all dyadic cubes Q' and Q such that $Q' \subset Q$, where Q^* is the exterior ball associated with Q. Suppose also that there exists $r > 1$ such that

$$\varphi(Q^*)|\check{Q}|_\nu^{1/p'}|3A_0^2\hat{Q}^*|_w^{1/q}\left(\frac{1}{|\check{Q}|_\nu}\int_{\check{Q}}\sigma^r d\nu\right)^{1/rp'} \leq C \tag{1.10}$$

for all dyadic cubes Q centered at points of ∂D, where Q^* is the exterior ball associated with Q, $\check{Q} = Q \cap \partial D$ and $3A_0^2\hat{Q}^* = (3A_0^2Q^*) \cap D$. Then, there exists $c > 0$ such that

$$\left(\int_D |M_\varphi f|^q w d\mu\right)^{1/q} \leq c\left(\int_{\partial D} |f|^p v d\nu\right)^{1/p} \tag{1.11}$$

for all f.

The next result is a characterization of weighted norm inequalities of type (p, q) with $1 < p \leq q < \infty$, when D is of the form $X \times (0, \infty)$, where (X, d, ν) is a homogeneous space. We will suppose additional conditions on X, d and ν. Let (X, d, ν) be a homogeneous space and define $\hat{X} = X \times \mathbb{R}$ and $\hat{X}_+ = X \times (0, \infty)$. Assume that X admits a group structure (not necessarily commutative) such that for all $x, y, z \in X$ and all balls $B \subset X$ we have

$$d(x + z, y + z) = d(x, y) \tag{1.12}$$

$$d(0, x) = d(0, -x) \tag{1.13}$$

where 0 is the identity element of X,

$$\nu(-B + x) = \nu(-B) \tag{1.14}$$

where $-B = \{x; -x \in B\}$, and

$$\nu(B) = \nu(-B). \tag{1.15}$$

Let us consider the following maximal function

$$M_\varphi f(x, t) = \sup_{r>t} \frac{1}{\varphi(B(x, r))}\int_{B(x,r)} |f| d\nu, \quad (x, t) \in \hat{X}_+ \tag{1.16}$$

where $B(x,r) \subset X$ and φ satisfies the (quasi-)monotonicity and doubling properties and is defined for all Borel sets of X. The following theorem characterizes 1.9 with M_φ as in 1.16, and it is based on the main theorem of [S1].

Theorem 1.3. *Let (X,d,ν) be a homogeneous space, where X has a group structure and d and ν satisfy conditions 1.12 - 1.15. Let w and v be weights in $\hat{X}_+$ and X, respectively, and suppose that $1 < p \leq q < \infty$. Then*

$$\left(\int_{\hat{X}_+} [M_\varphi f]^q w d\mu\right)^{1/q} \leq C\left(\int_X |f|^p v d\nu\right)^{1/p} \tag{1.17}$$

for all f if and only if

$$\left(\int_{\hat{Q}+z} [M_\varphi(\sigma\chi_{Q+z})]^q w d\mu\right)^{1/q} \leq C\left(\int_{Q+z} \sigma d\nu\right)^{1/p} \tag{1.18}$$

for all dyadic cube $Q \subset X$ and z in X, where $\hat{Q} = Q \times l(Q)$ and $l(Q)$ is the edgelength of Q.

The proofs of the preceding theorems can be found in Section 3.

2. Dyadic Balls, Dyadic Cubes and some Lemmas

In this section we will introduce the concept of dyadic balls and dyadic cubes in homogeneous spaces whose centers lie on the boundary of a fixed open subset. The construction of such sets will be based on the definitions of dyadic balls and dyadic cubes as presented in [SW2]. For a slightly different concept of dyadic cubes in homogeneous spaces, see [C]. We will also prove some lemmas that will be used in the next section.

Lemma 2.1. *Let (X,d,μ) be a separable homogeneous space, i.e., X has a countable dense subset. Let D be an open subset of X and ∂D be its boundary. Suppose $\{B_\alpha = B(z_\alpha, r_\alpha)\}_{\alpha\in A}$ is a family of balls with $z_\alpha \in \partial D$ and $B_\alpha \subset B(z,r)$ for some $z \in \partial D$ and $r > 0$.*

Then, there exists a countable subcollection $\{B_i\}_{i\in I}$ of these balls such that

1. *$B_i \cap B_j = \emptyset$, if $i \neq j$.*
2. *Given $\alpha \in A$ there exists $i \in I$ such that $B_\alpha \subset (A_o + 4A_o{}^2)B_i$.*
3. *$\mu\left(\bigcup_{\alpha\in A} B_\alpha\right) \leq c\sum_{i\in I}\mu(B_i)$ where c depends only on the positive constants A_o and A_1, as in 1.1 and 1.2, respectively.*

Proof

The proof is the same as in lemma 3.3 of [SW2] and it will be omitted.

As in [SW2], p. 843, we construct now a sequence of balls in X centered at points of ∂D.

Set $\lambda = A_o(2A_o+1)$, where A_o is as in 1.1. For each integer k select a sequence $\{z_j^k\}_j$ of points of ∂D such that the sequence of balls $\tilde{B}_j^k = B(z_j^k, \lambda^{k-1})$ is maximal with respect to the property that $\tilde{B}_i^k \cap \tilde{B}_j^k = \emptyset$ if $i \neq j$. Set $B_j^k = \lambda \tilde{B}_j^k$. Then,

(2.2) Given z in ∂D and an integer k there exists j such that $B(z, \lambda^{k-1}) \subset B_j^k$.

(2.3)
There exists $M > 0$ depending only on A_o and A_1, such that $\sum_j \chi_{B_j^k} \leq M$.

(2.4) $$\tilde{B}_i^k \cap \tilde{B}_j^k = \emptyset \text{ for } i \neq j,\ k \in \mathbb{Z}.$$

In fact, given $z \in \partial D$ and $k \in \mathbb{Z}$, by the maximality of $\{\tilde{B}_j^k\}$, there exists j such that $B(z, \lambda^{k-1}) \cap B(z_j^k, \lambda_m^k) \neq \emptyset$. Pick $z_o \in B(z, \lambda^{k-1}) \cap B(z_j^k, \lambda_m^k)$. Then, for $x \in B(z, \lambda^{k-1})$, we have

$$d(x, z_l^k) \leq A_o(d(x, z_o) + d(z_o, z_j^k)) \leq$$

$$\leq A_o(A_o(d(x,z) + d(z, z_o)) + \lambda^{k-1}) \leq A_o(2A_o\lambda^{k-1} + \lambda^{k-1}) =$$
$$= A_o(2A_o+1)\lambda^{k-1} = \lambda^k$$

and this proves 2.2. Now, suppose that $x \in B_j^k$, $1 \leq j \leq N$. If $y \in B(x, 2A_o\lambda^k)$ then

$$d(y, z_j^k) \leq A_o(d(y,x) + d(x, z_j^k)) \leq$$

$$\leq A_o(2A_o\lambda^k + \lambda^k) = A_o(2A_o+1)\lambda^k = \lambda^{k+1}.$$

Hence, $B(2A_o\lambda^k) \subset \lambda B_j^k = \lambda^2 \tilde{B}_j^k$. Thus,

$$\mu(B(2A_o\lambda^k)) \leq \mu(\lambda B_j^k) \leq c_\lambda \mu(\tilde{B}_j^k)$$

since μ is doubling. Also, if $w \in B_j^k$ then

$$d(w,x) \leq A_o(d(w, z_j^k) + d(z_j^k, x)) \leq A_o(\lambda^k + \lambda^k) = 2A_o\lambda^k,$$

that is, $B_j^k \subset B(x, 2A_o\lambda^k)$ for any j and, therefore, $\bigcup_j B_j^k \subset B(x, 2A_o\lambda^k)$. Since the B_j^k are pairwise disjoint in j, we obtain

$$N\mu(B(2A_o\lambda^k)) \leq c_\lambda \sum_{j=1}^{N} \mu(\tilde{B}_j^k) = c_\lambda \mu\left(\bigcup_{j=1}^{N} \tilde{B}_j^k\right) \leq c_\lambda \mu(B(x, 2A_o\lambda^k)).$$

Hence, $N \leq c_\lambda$ and this proves 2.3. Finally, 2.4 follows trivially from the definition of B_j^k. We refer to $\{B_j^k\}_{k,j}$ as the collection of ***dyadic balls***.

Lemma 2.5. *Suppose $\{B_\alpha\}_{\alpha\in A}$ is a family of dyadic balls as above. If $\{B_j\}_{j\in J}$ is a collection of maximal (with respect to inclusion) balls in $\{B_\alpha\}_{\alpha\in A}$, then the balls $\{\tilde{B}_j\}_{j\in J}$ are pairwise disjoint.*

Proof

We know that $\tilde{B}_i^k \cap \tilde{B}_j^k = \emptyset$ if $i \neq j$. Suppose there exists $x \in \tilde{B}_j^k \cap \tilde{B}_i^l$, where $l < k$. Then $B_i^l \subset B_j^k$. In fact, if $w \in B_i^l$ then

$$d(z_j^k, w) \le A_o[d(z_j^k, x) + A_o[d(x, z_i^l) + d(z_i^l, w)]] \le$$

$$\le A_o[\lambda^{k-1} + A_o(\lambda^{k-1} + \lambda^k)] \le A_o[\lambda^{k-1} + 2A_o\lambda^{k-1}] = \lambda^k.$$

The result follows now from the maximality of the balls. □

We now construct a grid of dyadic cubes centered at points of ∂D. This construction is analogous to the one that can be found in [SW2].

Set $\lambda = 8A_o^5$. For each $k \in \mathbb{Z}$ choose a sequence $\{z_j^k\}_{1\le j<n_k}$ of points on ∂D maximal with respect to the property that the balls $\{B(z_j^k, 3A_o^2\lambda^k)\}_{1\le j<n_k}$ are pairwise disjoint. Notice that since X is separable then the cardinality of $\{z_j^k\}$ is at most countable.

Observe that

$$\partial D = \bigcup_j B(z_j^k, 6A_o^3\lambda^k) \cap \partial D \tag{2.6}$$

for each $k \in \mathbb{Z}$. Indeed, if $z \in \partial D$ then by maximality, $B(z, 3A_o^2\lambda^k)$ intersects $B(z_j^k, 3A_o^2\lambda^k)$ for some j. Pick $y \in B(z, 3A_o^2\lambda^k) \cap B(z_j^k, 3A_o^2\lambda^k)$. We have

$$d(z, z_j^k) \le A_o(d(z,y) + d(y, z_j^k)) \le A_o(3A_o^2\lambda^k + 3A_o^2\lambda^k) = 6A_o^3\lambda^k$$

and, therefore, we get 2.6.

Fix $m \in \mathbb{Z}$ and define

$$E_1^m = B(z_1^m, 6A_o^3\lambda^m) - \left[\bigcup_{i\neq 1} B(z_i^m, \lambda^m)\right]$$

and for $j > 1$,

$$E_j^m = B(z_j^m, 6A_o^3\lambda^m) - \left[\bigcup_{i\neq j} B(z_i^m, \lambda^m)\right] - \left[\bigcup_{i<j} E_i^m\right].$$

Observe that $E_j^m \cap E_i^m = \emptyset$ if $i \neq j$.

Let $x \in E_j^m$. Then, $d(x, z_j^m) \le 6A_o^3\lambda^m < \lambda^{m+1}$, so that

$$E_j^m \subset B(z_j^m, \lambda^{m+1}).$$

Since the balls $\{B(z_j^m, 3A_o^2\lambda^m)\}$ are pairwise disjoint and $B(z_j^m, \lambda^m)$ is contained in $B(z_j^m, 3A_o^2\lambda^m)$ then the balls $\{B(z_j^m, \lambda^m)\}_j$ are also pairwise disjoint. Hence,

$$B(z_j^m, \lambda^m) \subset B(z_j^m, 3A_o^2\lambda^m) \setminus \bigcup_{i \neq j} B(z_i^m, \lambda^m).$$

It is clear from the definition of E_j^m that $B(z_j^m, \lambda^m)$ does not intersect $\bigcup_{i<j} E_i^m$. Thus, $B(z_j^m, \lambda^m) \subset E_j^m$.

Therefore,

$$B(z_j^m, \lambda^m) \subset E_j^m \subset B(z_j^m, \lambda^{m+1}). \tag{2.7}$$

We will show that

$$\partial D = \bigcup_j E_j^m \cap \partial D. \tag{2.8}$$

Let $z \in \partial D$. If $z \in \bigcup_j B(z_j^m, \lambda^m)$ then $z \in B(z_j^m, 3A_o^2\lambda^m)$ for some j and $z \notin B(z_i^m, \lambda^m)$ if $i \neq j$. Thus, since $B(z_j^m, \lambda^m)$ does not intersect $\bigcup_{i<j} E_i^m$, we conclude that $z \in E_j^m$. If $z \notin \bigcup_j B(z_j^m, \lambda^m)$ then by 2.6 there exists a positive integer j such that $z \in B(z_j^m, 6A_o^3\lambda^m)$. Let j_o be the least of such integers. Hence , $z \notin E_i^m$ for all $i < j_o$. Thus, $z \in E_{j_o}^m$ and this proves 2.8.

We now define $\{E_j^k\}_j$ for $k \geq m$. Suppose $\{E_j^l\}_{1 \leq j < n_l}$, has been defined for $l = m, m+1, \ldots, k$ for some $k \geq m$ so as to satisfy

$$B(z_j^l, \lambda^l) \cap \partial D \subset E_j^l \subset B(z_j^l, \lambda^{l+1}), \quad m \leq l \leq k, \quad 1 \leq j < n_l, \tag{2.9}$$

$$\partial D = \bigcup_j E_j^l \cap \partial D, \quad E_j^l \cap E_i^l = \emptyset \quad \text{if} \quad i \neq j, \tag{2.10}$$

and for $m \leq l_1 < l_2 \leq k$ and any i, j we have either

$$E_j^{l_2} \cap E_i^{l_1} = \emptyset \quad \text{or} \quad E_i^{l_1} \subset E_j^{l_2}. \tag{2.11}$$

For any $R > 0$ and $1 \leq j < n_{k+1}$, let

$$\tilde{B}(z_j^{k+1}, R) = \bigcup \{E_i^k; E_i^k \cap B(z_i^{k+1}, R) \neq \emptyset\}.$$

Define

$$E_1^{k+1} = \tilde{B}(z_1^{k+1}, 6A_o^3\lambda^{k+1}) - \left[\bigcup_{i \neq 1} \tilde{B}(z_i^{k+1}, \lambda^{k+1})\right]$$

and for $j > 1$,

$$E_j^{k+1} = \tilde{B}(z_j^{k+1}, 6A_o^3\lambda^{k+1}) - \left[\bigcup_{i \neq j} \tilde{B}(z_i^{k+1}, \lambda^{k+1})\right] - \left[\bigcup_{i<j} E_i^{k+1}\right].$$

Let us show that 2.9 holds with $1 \leq j < n_l$, $m \leq l \leq k+1$. Let $z \in \partial D \cap B(z_j^{k+1}, R)$. By 2.10 with $l = k$, there exists i_o such that $z \in E_{i_o}^k$. Thus, $z \in B(z_j^{k+1}, R) \cap E_{i_o}^k$. Therefore,

$$\tilde{B}(z_j^{k+1}, R) \supset \partial D \cap B(z_j^{k+1}, R). \tag{2.12}$$

Now, if $x \in B(z_j^{k+1}, R)$ then there exists $1 \leq i < n_{k+1}$ and $w \in X$ such that $x \in E_i^k$ and $w \in E_i^k \cap B(z_j^{k+1}, R)$. Thus, by 2.9 with $l = k$, we have

$$d(x, z_j^{k+1}) \leq A_o[d(x, z_i^k) + A_o[d(z_i^k, w) + d(w, z_j^{k+1})]] \leq A_o[\lambda^{k+1} + A_o[\lambda^{k+1} + R]].$$

Therefore, $\tilde{B}(z_j^{k+1}, R) \subset B(z_j^{k+1}, A_o[\lambda^{k+1} + A_o[\lambda^{k+1} + R]])$.

If we set $R = 6A_o^3\lambda^{k+1}$, we get

$$\begin{aligned}\tilde{B}(z_j^{k+1}, 6A_o^3\lambda^{k+1}) &\subset B(z_j^{k+1}, A_o(A_o + 6A_o^4 + 1)\lambda^{k+1}) \\ &\subset B(z_j^{k+1}, 8A_o^5\lambda^{k+1}) = B(z_j^{k+1}, \lambda^{k+2}),\end{aligned}$$

that is,

$$\tilde{B}(z_j^{k+1}, 6A_o^3\lambda^{k+1}) \subset B(z_j^{k+1}, \lambda^{k+2}). \tag{2.13}$$

If we set $R = \lambda^{k+1}$, we obtain

$$\tilde{B}(z_j^{k+1}, \lambda^{k+1}) \subset B(z_j^{k+1}, A_o(1 + 2A_o)\lambda^{k+1}) \subset B(z_j^{k+1}, 3A_o^2\lambda^{k+1}). \tag{2.14}$$

Now, if $i \neq j$ then $B(z_i^{k+1}, 3A_o^2\lambda^{k+1})$ and $B(z_j^{k+1}, 3A_o^2\lambda^{k+1})$ are disjoint and it follows from 2.12 with $R = \lambda^{k+1}$, 2.14 and the definition of E_j^{k+1} that

$$\partial D \cap B(z_j^{k+1}, \lambda^{k+1}) \subset E_j^{k+1}, \quad 1 \leq j < n_{k+1}. \tag{2.15}$$

It also follows from the definition of E_j^{k+1} and 2.13 that

$$E_j^{k+1} \subset \tilde{B}(z_j^{k+1}, 6A_o^3\lambda^{k+1}) \subset B(z_j^{k+1}, \lambda^{k+2}). \tag{2.16}$$

Therefore, 2.15 and 2.16 together with the induction hypothesis show that 2.9 holds for $1 \leq j < n_{k+1}$, $m \leq l \leq k+1$.

Let us verify 2.10. Observe that by 2.6 and 2.12 with $R = 6A_o^3\lambda^{k+1}$,

$$\partial D = \bigcup_j B(z_j^{k+1}, 6A_o^3\lambda^{k+1}) \cap \partial D = \bigcup_j \tilde{B}(z_j^{k+1}, 6A_o^3\lambda^{k+1}) \cap \partial D.$$

Using 2.14 and the fact that the $B(z_j^{k+1}, 3A_o^2\lambda^{k+1})$ are pairwise disjoint in j, we obtain

$$\tilde{B}(z_j^{k+1}, \lambda^{k+1}) \subset \tilde{B}(z_j^{k+1}, 6A_o^3\lambda^{k+1}) \setminus \bigcup_{i \neq j} \tilde{B}(z_i^{k+1}, \lambda^{k+1}).$$

Therefore,

$$\bigcup_j \tilde{B}(z_j^{k+1}, \lambda^{k+1}) \subset \bigcup_j E_j^{k+1}. \tag{2.17}$$

If $z \in \partial D$, select the first j_o such that $z \in \tilde{B}(z_{j_o}^{k+1}, 6A_o^3\lambda^{k+1})$. If $z \in \tilde{B}(z_j^{k+1}, \lambda^{k+1})$ for some j then by 2.17, $z \in \bigcup_j E_j^{k+1}$. If $z \notin \bigcup_j \tilde{B}(z_j^{k+1}, \lambda^{k+1})$ then $z \in E_{j_o}^{k+1}$, by the definition of E_i^{k+1}.

Finally, let us prove 2.11. Using induction, suppose $m \leq l_1 < l_2 \leq k+1$ and $E_{i_1}^{l_1} \cap E_{j_1}^{l_2} \neq \emptyset$. By definition, $\tilde{B}(z_j^{l_2}, R) = \bigcup_{j \in J} E_j^{l_2-1}$ for some index set J. Since the $E_j^{l_2-1}$ are pairwise disjoint in j, it follows that each $E_j^{l_2}$ is also a union of certain $E_i^{l_2-1}$, say, $E_j^{l_2} = \bigcup_{i \in I} E_i^{l_2-1}$. Hence, if $E_{i_1}^{l_1} \cap E_{j_1}^{l_2} \neq \emptyset$, i.e., if $E_{i_1}^{l_1} \cap \left(\bigcup_{i \in I} E_i^{l_2-1}\right) \neq \emptyset$, then $E_{i_1}^{l_1} \cap E_i^{l_2-1} \neq \emptyset$ for some $i \in I$. Now, if $l_1 < l_2 - 1$ then $E_i^{l_1} \subset E_i^{l_2-1}$, by induction. If $l_1 = l_2 - 1$ then $E_{i_1}^{l_1} = E_i^{l_2-1}$ since $E_{i_1}^{l_1} \cap E_i^{l_2-1} \neq \emptyset$. This completes the proof of 2.11 since $E_i^{l_2-1} \subset \bigcup_{i \in I} E_i^{l_2-1} = E_j^{l_2}$.

We can summarize these results on dyadic cubes in the following lemma:

Lemma 2.18. *Suppose (X, d) is a separable quasi-metric space and $D \subset X$ is an open subset. Then there exists $\lambda > 1$ depending only on A_o, such that for every $m \in \mathbb{Z}$, there are points $z_j^k \in \partial D$ and Borel sets E_j^k, $1 \leq j < n_k$, $k \geq m$, where $n_k \in \mathbb{N} \cup \{\infty\}$, such that*

$$(2.19) \qquad B(z_j^k, \lambda^k) \cap \partial D \subset E_j^k \subset B(z_j^k, \lambda^{k+1}), \quad 1 \leq j < n_k, \quad k \geq m.$$

$$(2.20) \quad \partial D = \bigcup_j E_j^k \cap \partial D, \quad \text{for all} \quad k \geq m, \quad \text{and} \quad E_j^k \cap E_i^k = \emptyset \quad \text{if} \quad i \neq j.$$

Given i, j, k, l with $m \leq k < l$, then either

$$(2.21) \qquad E_j^k \subset E_i^l \quad \text{or} \quad E_j^k \cap E_i^l = \emptyset.$$

We will denote $\mathcal{D}_m = \{E_j^k; k \geq m, 1 \leq j < n_k\}$. For $Q = E_j^k$ in $\mathcal{D}_m$ write $\bar{Q} = B(z_j^k, \lambda^k) \cap \partial D$ and $Q^* = B(z_j^k, \lambda^{k+1})$. We will refer to $\bar{Q}$ and Q^* as the inner and outer balls, respectively, associated with Q. The diameter of Q will be called the edgelength of Q and will be denoted by $l(Q)$. Note that since any annulus is nonempty, $\lambda^k \leq$ edgelength of $Q \leq \lambda^{k+1}$.

The next lemma, which is an application of the Marcinkiewicz interpolation theorem, is analogous to Lemma 3.15 of [SW2].

Lemma 2.22. *Let $D \subset X$ be an open subset and ∂D its boundary. Suppose $a(B)$ is a nonnegative set function, defined for all balls B in $\mathfrak{B}$, that satisfies*

$$(2.23) \qquad a(B(z, 2r)) \leq ca(B(z, r)) \quad \text{for all} \quad z \in \partial D \quad \text{and } r > 0$$

and

$$(2.24) \qquad \sum_{B \in \Omega} a(B) \leq ca(B_o)$$

whenever Ω is a collection of pairwise disjoint balls $B = B(z,r)$ in $\mathfrak{B}$ contained in a ball $B_o = B(z_o, r_o)$ of $\mathfrak{B}$.

In addition, assume that $u(z) \geq 0$ on ∂D, $\beta \geq 1$ and Γ is a countable collection of dyadic balls centered in ∂D such that

$$\int_{\check{B}} u d\nu \leq c a(B), \quad \text{for all} \quad B \in \Gamma, \tag{2.25}$$

where $\check{B} = B \cap \partial D$ and $d\nu$ is a doubling measure on ∂D, i.e.,

$$|B(z,2r) \cap \partial D|_\nu \leq c|B(z,r) \cap \partial D|_\nu \quad \text{for all} \quad z \in \partial D \quad \text{and} \quad r > 0.$$

Furthermore, assume that

$$\sum_{\substack{B \in \Gamma \\ B \subset B_o}} a(B)^\beta \leq c a(B_o)^\beta, \quad \text{for all} \quad B_o \in \Gamma. \tag{2.26}$$

Then

$$\left(\sum_{B \in \Gamma} a(B)^\beta \left(\frac{1}{a(B)} \int_{\check{B}} f u d\nu \right)^t \right)^{1/t} \leq c_{s,t} \left(\int_{\partial D} f^s u d\nu \right)^{1/s} \tag{2.27}$$

for all $f \geq 0$ on ∂D and $t = s\beta$, $1 < s < \infty$.

Proof

Consider the map

$$f \longmapsto \left\{ \frac{1}{a(B)} \int_{\check{B}} f u d\nu \right\}_{B \in \Gamma}.$$

For $f \in L^\infty(\partial D, u d\nu)$, we have

$$\left| \frac{1}{a(B)} \int_{\check{B}} f u d\nu \right| \leq ||f||_\infty \frac{1}{a(B)} \int_{\check{B}} u d\nu \leq c||f||_\infty \quad \text{by} \quad 2.25.$$

Now, suppose $f : \partial D \to [0,\infty)$ is bounded with support contained in $B_1 \cap \partial D$, where $B_1 = B(z_1, r_1)$, $z_1 \in \partial D$. Let Γ' be a finite subset of Γ. For $\lambda > 0$, let $\{Q_j\}_{j \in \mathcal{J}'}$ be the maximal dyadic balls B in Γ' such that

$$\frac{1}{a(B)} \int_{\check{B}} f u d\nu > \lambda. \tag{2.28}$$

Note that $B \cap B_1 \neq \emptyset$ for any ball B that satisfies 2.28 since the support of f is contained in $B_1 \cap \partial D$. Thus,

$$\sum_{\substack{B \in \Gamma' \\ \frac{1}{a(B)} \int_{\check{B}} f u d\nu > \lambda}} a(B)^\beta \leq \sum_{j \in \mathcal{J}'} \sum_{B \subset Q_j} a(B)^\beta$$

$$\leq c \sum_{j\in\mathcal{J}'} a(Q_j)^{\beta}, \text{ by 2.26} \tag{2.29}$$

$$\leq c \left(\sum_{j\in\mathcal{J}'} a(Q_j) \right)^{\beta}, \text{ since } \beta \geq 1 \tag{2.30}$$

$$\leq c \left(\sum_{j\in\mathcal{J}'} a(\tilde{Q}_j) \right)^{\beta}, \text{ by 2.23} \tag{2.31}$$

where $\tilde{Q}_j$ denotes the ball concentric with Q_j and with radius $\frac{r(Q_j)}{A_o(2A_o+1)}$. Recall that the $\tilde{Q}_j$ are pairwise disjoint (see Lemma 2.5).

Now, since $\mathcal{J}'$ is finite, $\bigcup_{j\in\mathcal{J}'} \tilde{Q}_j \subset B'$, for some ball B' centered at ∂D. Therefore, by Lemma 2.1, there exists a pairwise disjoint subcollection $\{Q_i\}_{i\in\mathcal{I}'}$ of the dyadic balls $\{Q_j\}_{j\in\mathcal{J}'}$ such that every Q_j is contained in some $Q_i^*, i \in \mathcal{I}'$, where Q_i^* is the ball concentric with Q_i and with radius $A_o(4A_o+1)r(Q_i)$. Thus,

$$\sum_{j\in\mathcal{J}'} a(\tilde{Q}_j) \leq \sum_{i\in\mathcal{I}'} \sum_{\substack{j\in\mathcal{J}' \\ Q_j\subset Q_i^*}} a(\tilde{Q}_j) \leq c \sum_{i\in\mathcal{I}'} a(Q_i^*), \quad \text{by} \quad 2.24 \quad \text{and Lemma 2.5.}$$

$$\leq c \sum_{i\in\mathcal{I}'} a(Q_i), \quad \text{by} \quad 2.23$$

$$\leq \frac{c}{\lambda} \sum_{i\in\mathcal{I}'} \int_{\check{Q}_i} fud\nu \leq \frac{c}{\lambda} \int_{\partial D} fud\nu,$$

since $\{Q_i\}_{i\in\mathcal{I}'}$ is a collection of pairwise disjoint balls. Thus, since Γ' was an arbitrary finite subset of Γ and the constant c above depends only on A_o and A_1, we conclude that the map

$$f \longmapsto \left\{ \frac{1}{a(B)} \int_{\check{B}} fud\nu \right\}_{B\in\Gamma}$$

is both of weak-type (∞,∞) and weak-type $(1,\beta)$, i.e., it takes $L^1(\partial D, ud\nu)$ to weak $l^{\beta}(\Gamma, a(B)^{\beta})$. Therefore, applying the Marcinkiewicz interpolation theorem we obtain 2.27. □

We will need the following:

Lemma 2.32. *Let $u(x) \geq 0$ on ∂D and for each $m \in \mathbf{Z}$, let $\{Q_i\}_i$ be a countable collection of dyadic cubes in $\mathcal{D}_m$. Suppose that there exist sequences $\{a_i\}_{i\in I}, \{b_i\}_{i\in I}$ of positive numbers and $\beta \geq 1$ satisfying*

$$\int_{\check{Q}_i} ud\nu \leq ca_i \tag{2.33}$$

$$\sum_{j;Q_j\subset Q_i} a_j^{\beta} \leq ca_i^{\beta}, \tag{2.34}$$

where $c > 0$ is independent of i and m. Then

$$\left(\sum_{i\in I} a_i^{\beta}\left(\frac{1}{a_i}\int_{\dot{Q}_i} fud\nu\right)^{q}\right)^{1/q} \leq c\left(\int_{\partial D} f^p ud\nu\right)^{1/p}$$

for all $f \geq 0$ on ∂D where $1 < p < \infty$, $q = \beta p$ and c is independent of m.

Proof

The map $f \mapsto \left\{\frac{1}{a_i}\int_{\dot{Q}_i} fud\nu\right\}$ takes $L_u^{\infty}(\partial D)$ into $l^{\infty}_{\{a_i^{\beta}\}}(I)$ by condition 2.33 and $L_u^1(\partial D)$ into weak $l^1_{\{a_i^{\beta}\}}(I)$ by condition 2.34, as we now show. Suppose f is bounded with compact support in ∂D. Let $t > 0$ and let $\{Q_j\}_{j\in J}$ be the maximal dyadic cubes from the collection $\{Q_i\}_{i\in I}$ such that $\frac{1}{a_i}\int_{\dot{Q}_i}|f|ud\nu > t$ (we may assume the collection $\{Q_i\}_{i\in I}$ is finite). Then

$$\sum_{i;\left|\frac{1}{a_i}\int_{\dot{Q}_i} fud\nu\right|>t} a_i^{\beta} \leq \sum_{j\in J}\sum_{i;Q_i\subset Q_j} a_i^{\beta} \tag{2.35}$$

$$\leq c\sum_{j\in J} a_j^{\beta} \quad \text{by 2.34} \tag{2.36}$$

$$\leq c\left(\sum_{j\in J} a_j\right)^{\beta} \quad \text{since} \quad \beta \geq 1 \tag{2.37}$$

$$\leq c\left(\frac{1}{t}\sum_j \int_{\dot{Q}_j}|f|ud\nu\right)^{\beta} \tag{2.38}$$

$$\leq c\left(\frac{1}{t}\int_{\partial D}|f|ud\nu\right)^{\beta} \tag{2.39}$$

since the maximal dyadic cubes are disjoint. The Marcinkiewicz interpolation theorem now completes the proof of Lemma 2.32. □

The next lemma can be found in [W1] and its proof is omitted.

Lemma 2.40 (Reverse Doubling). ***Suppose μ is a doubling measure on a homogeneous space X. Then μ satisfies the reverse doubling condition; that is, there exist $\alpha, \beta > 1$ such that $|B(x,\alpha r)|_\mu \geq \beta|B(x,r)|_\mu$ for all $x \in X$ and $r > 0$.***

We will need the concept of dyadic cubes in X. These sets are constructed as before with the only exception that the $\{z_j^k\}$ are chosen over all X instead of over the boundary of some open set. With this modification we can state the following version of Lemma 2.18 (cf. Lemma 3.21 of [SW2])

Lemma 2.41. *Suppose (X, d, ν) is a separable quasi-metric space. Let $\lambda = 8A_o^5$. Then for every $m \in \mathbb{Z}$, there are points z_j^k in X and Borel sets E_j^k, $1 \leq j < n_k$, $k \geq m$, where $n_k \in \mathbb{N} \cup \{\infty\}$, such that*

$$B(z_j^k, \lambda^k) \subset E_j^k \subset B(z_j^k, \lambda^{k+1}), \quad 1 \leq j < n_k, \quad k \geq m. \tag{2.42}$$

$$X = \bigcup_j E_j^k, \quad \text{for all} \quad k \geq m, \quad \text{and} \quad E_j^k \cap E_i^k = \emptyset \quad \text{if} \quad i \neq j. \tag{2.43}$$

Given i, j, k, l with $m \leq k < l$, then either

$$E_j^k \subset E_i^l \quad \text{or} \quad E_j^k \cap E_i^l = \emptyset.$$

We will denote $\mathcal{D}_m = \{E_j^k : k \geq m, 1 \leq j < n_k\}$. If $Q = E_j^k \in \mathcal{D}_m$, let $l(Q)$ be the edgelength of Q and define $\hat{Q} = Q \times (0, l(Q))$.

We will need the following lemma (cf. Lemma 3.20 of [SW1]).

Lemma 2.44. *Let $u(x) \geq 0$ on X. For each $m \in \mathbb{Z}$, suppose that $\{Q_i\}_i$ is a countable collection of dyadic cubes in $\mathcal{D}_m$, and for each $z \in X$, $\{a_i(z)\}_{i \in I}$ and $\{b_i(z)\}_{i \in I}$ are positive numbers satisfying*

$$\int_{Q_i + z} u d\mu \leq c a_i(z) \quad \text{for all} \quad i \in I \quad \text{and} \quad z \in X \tag{2.45}$$

$$\sum_{j; Q_j \subset Q_i} b_j(z) \leq c a_i(z) \quad \text{for all} \quad i \in I \quad \text{and} \quad z \in X, \tag{2.46}$$

with c independent of m. Then

$$\left(\sum_{i \in I} b_i(z) \left(\frac{1}{a_i(z)} \int_{Q_i + z} g u d\mu \right)^q \right)^{1/q} \leq c_q \left(\int_X g^q u d\mu \right)^{1/q}$$

for all $g \geq 0$ on X and $z \in X$, where $1 < q < \infty$ and c_q is independent of z, m and g.

Proof

The map $g \mapsto \left\{ \frac{1}{a_i(z)} \int_{Q_i + z} g u d\mu \right\}$ takes $L_u^\infty(X)$ into $l^\infty_{\{b_i(z)\}}(I)$ by condition 2.45 and $L_u^1(X)$ into weak $l^1_{\{b_i(z)\}}(I)$ by condition 2.46, as we now show. If g is bounded with compact support in X and $t > 0$, let $\{Q_j\}_{j \in J}$ be the maximal dyadic cubes from the collection $\{Q_i\}_{i \in I}$ such that $\frac{1}{a_i(z)} \int_{Q_i + z} |g| u d\mu > t$ (we may assume the collection

$\{Q_i\}_{i\in I}$ is finite). Then

$$\sum_{i;\left|\frac{1}{a_i(z)}\int_{Q_i+z} gud\mu\right|>t} b_i(z) \le \sum_{j\in J}\sum_{i;Q_i\subset Q_j} b_i(z) \tag{2.47}$$

$$\le c\sum_{j\in J} a_j(z) \quad \text{by 2.46} \tag{2.48}$$

$$\le \frac{c}{t}\sum_j \int_{Q_j+z} |g|ud\mu \tag{2.49}$$

$$\le \frac{c}{t}\int_X |g|ud\mu \tag{2.50}$$

since the maximal dyadic cubes are disjoint. The Marcinkiewicz interpolation theorem now completes the proof of Lemma 2.44. □

3. Proofs of the Main Theorems

In this section we will prove the theorems introduced in Section 1.

Proof of Theorem 1.1.

Let us prove that 1.7 implies 1.8. Given z in ∂D and $r > 0$, let $B = B(z, r)$ and set $f(y) = \sigma(y)\chi_{\check{B}}(y)$. Thus,

$$\left(\int_D M_\varphi^q(\sigma\chi_{\check{B}})wd\mu\right)^{1/q} \le c\left(\int_{\partial D}\sigma^p\chi_{\check{B}}vd\nu\right)^{1/p} \tag{3.1}$$

$$= c\left(\int_{\check{B}}\sigma d\nu\right)^{1/p} = c|\check{B}|_\sigma^{1/p}.$$

For $x \in X$ let $a(x)$ and $t(x)$ be as in the definition of M_φ. We claim that $B = B(z,r) \subset B(a(x), 3A_o^2[r + d(z,x)])$. Indeed, if $y \in B$ then

$$d(y,a(x)) \le A_o^2[d(y,z) + d(z,x) + d(x,a(x))]$$

$$\le A_o^2[r + d(z,x) + \frac{3}{2}d(z,x)] \le 3A_o^2[r + d(z,x)],$$

which proves our claim. Obviously $3A_o^2[r + d(z,x)] \ge t(x)$. Hence,

$$M_\varphi(\sigma\chi_{\check{B}})(x) \ge \frac{1}{\varphi(B(a(x), 3A_o^2[r + d(z,x)]))}\int_{\check{B}(a(x),3A_o^2[r+d(z,x)])}\sigma\chi_B d\nu \tag{3.2}$$

$$= \frac{|\check{B}|_\sigma}{\varphi(B(a(x), 3A_o^2[r + d(z,x)]))}$$

for any $x \in X$ and $r > 0$.

We claim that

$$B(a(x), 3A_o^2[r + d(z,x)]) \subset B(z, 18A_o^4 r)$$

for x in $B = B(z,r)$. In fact, for $y \in B(a(x), 3A_o^2[r + d(z,x)])$, we have

$$\begin{aligned} d(y,z) &\le A_o^2[d(y,a(x)) + d(a(x),x) + d(x,z)] \\ &\le A_o^2[3A_o^2[r + d(z,x)] + t(x) + d(x,z)] \\ &\le 9A_o^4[r + d(x,z)] \quad \text{since} \quad t(x) = d(x,\partial D) \le d(x,z) \\ &\le 18A_o^4 r. \end{aligned}$$

Thus, $\varphi(B(a(x), 3A_o^2[r + d(z,x)])) \le c\varphi(B(z,r))$. Therefore, it follows from 3.2 that

$$M_\varphi(\sigma\chi_{\check{B}})(x) \ge c\frac{|\check{B}|_\sigma}{\varphi(B(z,r))} \qquad \text{for all } x \text{ in } B. \tag{3.3}$$

Combining 3.1 and 3.3 we obtain

$$\varphi(B)^{-q}|\check{B}|_\sigma^q|\hat{B}|_w \le |\check{B}|_\sigma^{q/p}. \tag{3.4}$$

Thus, if $|\check{B}|_\sigma$ is neither 0 nor ∞, we obtain 1.8 from 3.4. If $|\check{B}|_\sigma = 0$ then 1.8 is obvious. Suppose $|\check{B}|_\sigma = \infty$. For any $\varepsilon > 0$ we have $[v(y) + \varepsilon]^{1-p'} \le \varepsilon^{1-p'} < \infty$ and, therefore, $|\check{B}|_{\sigma_\varepsilon} \le \varepsilon^{1-p'}|\check{B}|_\nu$ where $\sigma_\varepsilon(y) = [v(y) + \varepsilon]^{1-p'}$. Hence, since 1.7 implies 1.7 with v replaced by $v + \varepsilon$, we obtain 1.8 with v replaced by $v + \varepsilon$, i.e.,

$$\varphi(B)^{-1}|\hat{B}|_w^{1/q}|\check{B}|_{\sigma_\varepsilon}^{1/p'} \le C.$$

Letting ε tend to 0, we conclude that $w = 0$ a.e. (μ) since $|\check{B}|_\sigma = \infty$. Thus, 1.8 follows immediately.

Conversely, let us show that 1.8 implies 1.7. Let

$$m_\varphi^{dy} f(x) = \sup_{\substack{B \text{ dyadic ball} \\ x, a(x) \in B}} \frac{1}{\varphi(B)}\int_{\check{B}} |f| d\nu, \qquad x \in X$$

where a dyadic ball is a ball as defined just before Lemma 2.5. We claim that there exists $c > 0$ such that

$$M_\varphi f(x) \le c m_\varphi^{dy} f(x) \tag{3.5}$$

for all f and $x \in X$.

Given $B = B(a(x), r)$ with $r > t(x)$, let k in $\mathbb{Z}$ be such that $\lambda^{k-1} < \frac{3}{2}r \le \lambda^k$ (λ as in Lemma 2.5). Thus, there exists a dyadic ball B_j^{k+1} of radius λ^{k+1} such that $B \subset B(a(x), \frac{3}{2}r) \subset B(a(x), \lambda^k) \subset B_j^{k+1}$. Clearly, $\varphi(B) \le c_o\varphi(B_j^{k+1})$. Let us show that $\varphi(B_j^{k+1}) \le c\varphi(B)$. Indeed, if $y \in B_j^{k+1} = B(z_j^{k+1}, \lambda^{k+1})$ then

$$d(y, a(x)) \le A_o[d(y, z_j^{k+1}) + d(z_j^{k+1}, a(x))] \le 2A_o\lambda^{k+1}.$$

Thus, $B_j^{k+1} \subset 3A_o\lambda^2 B$, and, hence, $\varphi(B_j^{k+1}) \le c\varphi(B)$. Therefore,

$$\frac{1}{\varphi(B)}\int_{\check{B}} |f|\, d\nu \le c\frac{1}{\varphi(B_j^{k+1})}\int_{\check{B}_j^{k+1}} |f|\, d\nu. \tag{3.6}$$

Clearly x and $a(x)$ lie in B_j^{k+1}. Hence, our claim follows from 3.6.

Due to 3.5 it is enough to prove 1.7 with M_φ replaced by m_φ^{dy}. We may assume $f \geq 0$. Let $D_k = \{x \in D; m_\varphi^{dy}(f\sigma)(x) > 2^k\}$, $k \in \mathbb{Z}$. Thus,

$$\int_D [m_\varphi^{dy}(f\sigma)]^q w\,d\mu = \sum_k \int_{D_k \setminus D_{k+1}} [m_\varphi^{dy}(f\sigma)]^q w\,d\mu \approx \sum_k 2^{kq} \int_{D_k \setminus D_{k+1}} w\,d\mu.$$

It follows from the definition of m_φ^{dy} that if $x \in D_k$ then there exists a dyadic ball B with $x, a(x) \in B$ satisfying

$$\frac{1}{\varphi(B)} \int_{\check{B}} f\,d\nu > 2^k. \tag{3.7}$$

Let $\{B_{k,j}\}$ be the collection of maximal dyadic balls with respect to inclusion which satisfy 3.7. Observe that $D_k \subset \bigcup_j \hat{B}_{k,j}$. In fact, if $x \in D_k$ then there exists a dyadic ball B with $x, a(x) \in B$ satisfying 3.7. Hence, by the maximality of the $B_{k,j}$, there exists j such that $x \in B \subset B_{k,j}$.

Thus,

$$\sum_k 2^{kq} \int_{D_k \setminus D_{k+1}} w\,d\mu \leq \sum_{k,j} \left[\frac{1}{\varphi(B_{k,j})} \int_{\check{B}_{k,j}} f\sigma\,d\nu\right]^q \int_{\hat{B}_{k,j}} w\,d\mu$$

$$= \sum_{k,j} b_{k,j} \left[\frac{1}{a_{k,j}} \int_{\check{B}_{k,j}} f\sigma\,d\nu\right]^q$$

where

$$a_{k,j} = \int_{\check{B}_{k,j}} \sigma\,d\nu$$

and

$$b_{k,j} = \left[\frac{a_{k,j}}{\varphi(B_{k,j})}\right]^q \int_{\hat{B}_{k,j}} w\,d\mu.$$

From 1.8 it follows that

$$b_{k,j} = a_{k,j}^q \varphi(B_{k,j})^{-q} |\hat{B}_{k,j}|_w \leq c a_{k,j}^q |\check{B}_{k,j}|_\sigma^{-q/p'} = c a_{k,j}^{q/p}.$$

We claim that there exists $c > 0$ such that

$$\sum_{B_{k,j} \subset B_{s,t}} a_{k,j}^{q/p} \leq c a_{s,t}^{q/p} \quad \textbf{for all} \quad s,t.$$

Indeed,

$$\sum_{B_{k,j} \subset B_{s,t}} a_{k,j}^{q/p} = \sum_{B_{k,j} \subset B_{s,t}} |\check{B}_{k,j}|_\sigma^{q/p} \leq \sum_{l=0}^{\infty} \sum_{\substack{B \text{ dyadic ball } \subset B_{s,t} \\ r(B)/r(B_{s,t}) = \lambda^{-l}}} |\check{B}_{k,j}|_\sigma^{q/p-1} |\check{B}_{k,j}|_\sigma$$

where $\lambda = A_o(2A_o + 1)$

$$\leq c\sum_{l=0}^{\infty} \sum_{\substack{B \text{ dyadic ball } \subset B_{s,t} \\ r(B)/r(B_{s,t})=\lambda^{-l}}} \left(\delta^l |\check{B}_{s,t}|_\sigma\right)^{q/p-1} |\check{B}_{k,j}|_\sigma$$

for some $\delta < 1$ since $\sigma d\nu$ is doubling and, therefore, it satisfies the reverse doubling condition (see Lemma 2.40)

$$\leq c\sum_{l=0}^{\infty} \left(\delta^l |\check{B}_{s,t}|_\sigma\right)^{q/p-1} M|\check{B}_{s,t}|_\sigma$$

by 2.3

$$= cM|\check{B}_{s,t}|_\sigma^{q/p} \sum_{l=0}^{\infty} \delta^{l(q/p-1)} = c|\check{B}_{s,t}|_\sigma^{q/p} = ca_{s,t}^{q/p}$$

since $0 < \delta < 1$ and $q > p$, and this finishes the proof of our claim.

We now obtain 1.7 from Lemma 2.22 with $a(B_{k,j}) = a_{k,j}$, $\beta = q/p$, $t = q$, $s = p$, $u = \sigma$ and $\Gamma = \{B_{k,j}\}$ since $\sigma d\nu$ is a doubling measure. This concludes the proof of Theorem 1.1. □

We now proceed to prove Theorem 1.2.

Proof

Let $\mathfrak{B} = \{B = B(z,r); z \in \partial D, r > 0\}$. Given $x \in D$ and $r > t(x)$, let $B = B(a(x), r)$ and $B_o = B(a(x), \frac{3}{2}r)$. Then, clearly $B \subset B_o$ and $x \in B_o$ since $d(x, a(x)) \leq \frac{3}{2}t(x) < \frac{3}{2}r$. Thus, since φ is doubling we have

$$\frac{1}{\varphi(B)} \int_{\check{B}} |f| d\nu \leq \frac{c}{\varphi(B_o)} \int_{\check{B}_o} |f| d\nu$$

$$\leq c \sup_{\substack{B \in \mathfrak{B} \\ x, a(x) \in B}} \int_{\check{B}} |f| d\nu = c\mathfrak{m}_\varphi f(x), \quad \text{say, and hence,}$$

$$M_\varphi f \leq c\mathfrak{m}_\varphi f.$$

For each $m \in \mathbb{Z}$, define

$$\mathfrak{m}_{\varphi,m} f(x) = \sup_{\substack{B \in \mathfrak{B} \\ x, a(x) \in B,\ r(B) > \lambda^m}} \int_{\check{B}} |f| d\nu,$$

where $\lambda = 8A_o^5$ as in Lemma 2.18.

It will be convenient to majorize $\mathfrak{m}_{\varphi,m} f$ by a suitable dyadic operator defined in terms of the dyadic cubes $Q \in \mathcal{D}_m$ that were introduced in Lemma 2.18. Given

$B = B(z,r) \in \mathfrak{B}, r > \lambda^m$, select $k > m$ such that $\lambda^k \le r < \lambda^{k+1}$. Suppose $B \cap Q \neq \emptyset, Q = E_j^k \in \mathcal{D}_m$ and $B = B(z,r)$ is as above. Then, if $y \in B$, we have

$$d(y, z_j^k) \le A_o^2[d(y,z) + d(z,x) + d(x, z_j^k)]$$

where $x \in B \cap Q$

$$\le A_o^2[\lambda^{k+1} + \lambda^{k+1} + \lambda^{k+1}]$$

since $Q \subset B(z_j^k, \lambda^{k+1})$ (see Lemma 2.18)

$$= 3A_o^2\lambda^{k+1}.$$

Thus,

$$B(z,r) \subset B(z_j^k, 3A_o^2\lambda^{k+1}). \tag{3.8}$$

Also, if $u \in Q$, we have

$$d(u,z) \le A_o^2[d(u,z_j^k) + d(z_j^k, x) + d(x,z)] \le A_o^2[\lambda^{k+1} + \lambda^{k+1} + \lambda^{k+1}] = 3A_o^2\lambda^{k+1}.$$

Thus,

$$Q = E_j^k \subset B(z, 3A_o^2\lambda^{k+1}). \tag{3.9}$$

Now, if $E_{j\alpha}^k \cap B \neq \emptyset, \alpha = 1, \ldots, N$, we obtain

$$N|\check{B}(z,r)|_\nu \le \sum_{\alpha=1}^{N} |\check{B}(z_{j\alpha}^k, 3A_o^2\lambda^{k+1})|_\nu \quad \text{by 3.8} \tag{3.10}$$

$$\le c\sum_{\alpha=1}^{N} |\check{B}(z_{j\alpha}^k, \lambda^k)|_\nu \quad \text{since } \nu \text{ is doubling} \tag{3.11}$$

$$\le c\sum_{\alpha=1}^{N} |\check{E}_{j\alpha}^k|_\nu \quad \text{by Lemma 2.18} \tag{3.12}$$

$$= c\left|\bigcup_{\alpha=1}^{N} \check{E}_{j\alpha}^k\right|_\nu \quad \text{since the cubes are disjoint} \tag{3.13}$$

$$\le c|\check{B}(z, 3A_o^2\lambda^{k+1})|_\nu \quad \text{by 3.9} \tag{3.14}$$

$$\le c|\check{B}(z,r)|_\nu \quad \text{since } \nu \text{ is doubling.} \tag{3.15}$$

Therefore, $N \le c$. Since $\partial D = \bigcup_j \check{E}_j^k$, for all $k \ge m$, by Lemma 2.18, any ball $B \in \mathfrak{B}$ with $r(B) > \lambda^m$ is contained in a finite union of dyadic cubes $\{E_{j\alpha}^k\}_{1\le\alpha\le N}$, say, and the number of such dyadic cubes is bounded by a constant that depends only on the doubling constant of ν. Thus, for any $B \in \mathfrak{B}$ with $r(B) > \lambda^m$ and $\lambda^k \le r < \lambda^{k+1}$, we have $\check{B} \subset \bigcup_{\alpha=1}^{N} \check{E}_{j\alpha}^k$ with $B \cap E_{j\alpha}^k \neq \emptyset$ for some $N \le c$. Hence,

$$\frac{1}{\varphi(B)}\int_{\check{B}} |f|\sigma d\nu \le \frac{1}{\varphi(B)}\int_{\bigcup_{\alpha=1}^{N} \check{E}_{j\alpha}^k} |f|\sigma d\nu \tag{3.16}$$

$$= \sum_{\alpha=1}^{N} \frac{1}{\varphi(B)} \int_{\dot{E}^k_{j\alpha}} |f|\sigma d\nu \leq \frac{N}{\varphi(B)} \int_{\dot{E}^k_{j\alpha_o}} |f|\sigma d\nu$$

for some $1 \leq \alpha_o \leq N$

$$\leq \frac{c}{\varphi(B)} \int_{\dot{E}^k_{j\alpha_o}} |f|\sigma d\nu.$$

It follows from 3.9 with j replaced by j_{α_o} that if B and k are as above then $E^k_{\alpha_o} \subset B(z, 3A_o^2\lambda^{k+1})$ and, therefore, $\varphi(E^{k\,*}_{\alpha_o}) \leq c\varphi(B(z,r)) = c\varphi(B)$, since φ is doubling, where $E^{k\,*}_{\alpha_o}$ is the outer ball associated with $E^k_{\alpha_o}$. Hence, it follows from 3.16 that

$$\frac{1}{\varphi(B)} \int_{\check{B}} |f|\sigma d\nu \leq \frac{c}{\varphi(E^{k\,*}_{\alpha_o})} \int_{E^k_{\alpha_o}} |f|\sigma d\nu. \tag{3.17}$$

Since $B \cap E^k_{\alpha_o} \neq \emptyset$, it follows from 3.8 that

$$B \subset B(z^k_j, 3A_o^2\lambda^{k+1}) = 3A_o^2 B(z^k_j, \lambda^{k+1}) = 3A_o^2 E^{k\,*}_{j\alpha}.$$

Thus, if B is as above with $x \in \hat{B}$ and $a(x) \in \check{B}$ then 3.17 implies that

$$\frac{1}{\varphi(B)} \int_{\check{B}} |f|\sigma d\nu \leq c \sup_{\substack{Q \in \mathcal{D}_m; \\ x \in 3A_o^2\hat{Q}^*}} \frac{1}{\varphi(Q^*)} \int_{\check{Q}} |f|\sigma d\nu$$

$$= c\mathfrak{M}^{dy}_{\varphi,m}(f\sigma)(x), \qquad \text{say.}$$

Therefore, we have

$$\mathfrak{m}_{\varphi,m}(f\sigma)(x) \leq c\mathfrak{M}^{dy}_{\varphi,m}(f\sigma)(x), \quad x \in D. \tag{3.18}$$

For each $k \in \mathbb{Z}$ let $\Omega_{k,m} = \{x \in D; \mathfrak{M}^{dy}_{\varphi,m}(f\sigma)(x) > a^k\}$, where $a > 1$ will be chosen later. Thus, $x \in \Omega_{k,m}$ if and only if there exists $Q \in \mathcal{D}_m$ such that $x \in 3A_o^2\hat{Q}^*$ satisfying

$$\frac{1}{\varphi(Q^*)} \int_{\check{Q}} |f|\sigma d\nu > a^k. \tag{3.19}$$

Let $\{Q^k_j\}_j$ be the maximal (with respect to inclusion) dyadic cubes (in $\mathcal{D}_m$) which satisfy 3.19. We claim that $\Omega_{k,m} = \bigcup_j 3A_o^2\hat{Q}^{k\,*}_j$. Indeed, since any cube Q^k_j satisfies 3.19, then for any $x \in 3A_o^2\hat{Q}^{k\,*}_j$ we have

$$\mathfrak{M}^{dy}_{\varphi,m}(f\sigma)(x) \geq \frac{1}{\varphi(Q^k_j)} \int_{\check{Q}^k_j} |f|\sigma d\nu > a^k,$$

which implies that $x \in \Omega_{k,m}$. On the other hand, if $x \in \Omega_{k,m}$ then there exists $Q \in \mathcal{D}_m$ such that $x \in 3A_o^2\hat{Q}$ and 3.19 holds. Since the Q^k_j are maximal with respect to inclusion, there exists j_o such that $Q \subset Q^k_{j_o}$ and, therefore, $x \in 3A_o^2\hat{Q} \subset 3A_o^2\hat{Q}^{k*}_j$, that is, $x \in \bigcup_j 3A_o^2\hat{Q}^{k*}_j$, and this proves our claim.

Now

$$(3.20)\quad \left\|\mathfrak{M}^{dy}_{\varphi,m}(f\sigma)\right\|^q_{L^q_w(D)} = \int_D \left[\mathfrak{M}^{dy}_{\varphi,m}(f\sigma)\right]^q w d\mu \le a^q \sum_k a^{kq} \int_{\Omega_{k,m}\setminus\Omega_{k+1,m}} w d\mu$$

$$\le a^q \sum_{k,j} a^{kq} \int_{3A_o^2\hat{Q}^{k*}_j\setminus\Omega_{k+1,m}} w d\mu = a^q \sum_{k,j} a^{kq} |F^k_{j,m}|_w$$

where $F^k_{j,m} = 3A_o^2\hat{Q}^{k*}_j \setminus \Omega_{k+1,m}$

$$\le a^q \sum_{k,j} \left[\varphi(Q^{k*}_j)^{-1} \int_{\check{Q}^k_j} |f|\sigma d\nu\right]^q |F^k_{j,m}|_w$$

by 3.19 with $Q = Q^k_j$

$$= a^q \sum_{k,j} |F^k_{j,m}|_w \left[\varphi(Q^{k*}_j)^{-1}\mathcal{A}(Q^k_j)\right]^q \left[\frac{1}{\mathcal{A}(Q^k_j)} \int_{\check{Q}^k_j} |f|\sigma d\nu\right]^q$$

where $\mathcal{A}(Q) = |\check{Q}|_\nu^{1/r'} \left(\int_{\check{Q}} \sigma^r d\nu\right)^{1/r}$

$$\le a^q \sum_{k,j} |3A_o^2\hat{Q}^{k*}_j|_w \left[\varphi(Q^{k*}_j)^{-1}\mathcal{A}(Q^k_j)\right]^q \left[\frac{1}{\mathcal{A}(Q^k_j)} \int_{\check{Q}^k_j} |f|\sigma d\nu\right]^q$$

since $F^k_{j,m} \subset 3A_o^2\hat{Q}^{k*}_j$

$$= a^q \sum_{k,j} |3A_o^2\hat{Q}^{k*}_j|_w \varphi(Q^{k*}_j)^{-q} |\check{Q}^k_j|^q_\nu \left[|\check{Q}^k_j|^{-1}_\nu \mathcal{A}(Q^k_j)\right]^q \left[\frac{1}{\mathcal{A}(Q^k_j)} \int_{\check{Q}^k_j} |f|\sigma d\nu\right]^q.$$

We claim that

$$|3A_o^2\hat{Q}^{k*}_j|_w \varphi(Q^{k*}_j)^{-q} |\check{Q}^k_j|^q_\nu \left[|\check{Q}^k_j|^{-1}_\nu \mathcal{A}(Q^k_j)\right]^q \le c\mathcal{A}(Q^k_j)^{q/p} \quad \text{for all} \quad (k,j).$$

Indeed,

$$|3A_o^2\hat{Q}^{k*}_j|_w \varphi(Q^{k*}_j)^{-q} |\check{Q}^k_j|^q_\nu \left[|\check{Q}^k_j|^{-1}_\nu \mathcal{A}(Q^k_j)\right]^q$$

$$\le c|\check{Q}^k_j|^{q/p}_\nu \left(\frac{1}{|\check{Q}^k_j|_\nu} \int_{\check{Q}^k_j} \sigma^r d\nu\right)^{-q/rp'} \left[|\check{Q}^k_j|^{-1}_\nu \mathcal{A}(Q^k_j)\right]^q$$

by 1.10

$$= c|\check{Q}^k_j|^{-q/p'}_\nu \left(\frac{1}{|\check{Q}^k_j|_\nu} \int_{\check{Q}^k_j} \sigma^r d\nu\right)^{-q/rp'} \mathcal{A}(Q^k_j)^q$$

$$= c\left(\frac{|\check{Q}_j^k|_\nu^r}{|\check{Q}_j^k|_\nu}\int_{\check{Q}_j^k}\sigma^r d\nu\right)^{-q/rp'} \mathcal{A}(Q_j^k)^q$$

$$= c\left(|\check{Q}_j^k|_\nu^{\frac{r-1}{r}}\left(\int_{\check{Q}_j^k}\sigma^r d\nu\right)^{1/r}\right)^{-q/p'} \mathcal{A}(Q_j^k)^q$$

$$= c\mathcal{A}(Q_j^k)^{-q/p'}\mathcal{A}(Q_j^k)^q = c\mathcal{A}(Q_j^k)^{q/p},$$

which proves our claim. Thus, it follows from 3.20 that

$$\left\|\mathfrak{M}_{\varphi,m}^{dy}(f\sigma)\right\|_{L_u^q(D)}^q \leq c\sum_{k,j}\mathcal{A}(Q_j^k)^{q/p}\left[\frac{1}{\mathcal{A}(Q_j^k)}\int_{\check{Q}_j^k}|f|\sigma d\nu\right]^q. \tag{3.21}$$

We claim that

$$\sum_{k,j;\, Q_j^k\subset Q_i^l}\mathcal{A}(Q_j^k)^{q/p} \leq c\mathcal{A}(Q_i^l)^{q/p} \tag{3.22}$$

with $c > 0$ independent of l and i. First, let us show that $Q_j^k \subset Q_i^l$ implies $l \leq k$. If $Q_j^k \subsetneq Q_i^l$ then it follows from the maximality of the Q_j^k that

$$a^l < \varphi(Q_i^{l*})^{-1}\int_{\check{Q}_i^l}|f|\sigma d\nu \leq a^k, \tag{3.23}$$

which yields $l < k$, since $a > 1$. Now, suppose that $Q_j^k = Q_i^l$, and for any $Q \in \mathcal{D}_m$ let $Q^\star$ denote the smallest dyadic cube in $\mathcal{D}_m$ such that $Q^\star \supsetneq Q$. Thus, since ν is doubling, we have

$$a^l < \varphi(Q_i^{l*})^{-1}\int_{\check{Q}_i^l}|f|\sigma d\nu = \varphi(Q_j^{k*})^{-1}\int_{\check{Q}_j^k}|f|\sigma d\nu \tag{3.24}$$

$$\leq c\varphi(Q_j^{k**})^{-1}\int_{\check{Q}_j^{k*}}|f|\sigma d\nu \leq ca^k$$

since Q_j^k is maximal

$$\leq a^{k+1}$$

if we select $a \geq c$. Therefore, $l \leq k$.

Observe that the same argument employed in 3.24 can be used to obtain

$$a^k < \varphi(Q_j^k)^{-1}\int_{\check{Q}_j^k}|f|\sigma d\nu \leq a^{k+1}, \quad \text{for all } j,k. \tag{3.25}$$

Thus, since $q \geq p$, we have

$$(3.26) \qquad \sum_{k,j;\, Q_j^k \subset Q_i^l} \mathcal{A}(Q_j^k)^{q/p} \leq \left(\sum_{k,j;\, Q_j^k \subset Q_i^l} \mathcal{A}(Q_j^k) \right)^{q/p} = \left(\sum_{k=l}^{\infty} \sum_{\substack{j; \\ Q_j^k \subset Q_i^l}} \mathcal{A}(Q_j^k) \right)^{q/p}.$$

Now, for a fixed $k \geq l$, we have

$$(3.27) \qquad \sum_{\substack{j; \\ Q_j^k \subset Q_i^l}} \mathcal{A}(Q_j^k) = \sum_{\substack{j; \\ Q_j^k \subset Q_i^l}} |\check{Q}_j^k|_\nu^{1/r'} \left(\int_{\check{Q}_j^k} \sigma^r \, d\nu \right)^{1/r}$$

$$\leq \left(\sum_{\substack{j; \\ Q_j^k \subset Q_i^l}} |\check{Q}_j^k|_\nu \right)^{1/r'} \left(\sum_{\substack{j; \\ Q_j^k \subset Q_i^l}} \int_{\check{Q}_j^k} \sigma^r \, d\nu \right)^{1/r}$$

by Hölder's inequality

$$\leq \left(\sum_{\substack{j; \\ Q_j^k \subset Q_i^l}} |\check{Q}_j^k|_\nu \right)^{1/r'} \left(\int_{\check{Q}_i^l} \sigma^r \, d\nu \right)^{1/r}$$

since the Q_j^k are pairwise disjoint in j. Using the first inequality of 3.25 and the fact that $\left(\frac{|Q'|_\nu}{|Q|_\nu}\right)^\varepsilon \leq c \frac{\varphi(Q'^*)}{\varphi(Q^*)}$, we obtain

$$\sum_{\substack{j; \\ Q_j^k \subset Q_i^l}} |\check{Q}_j^k|_\nu \leq c \frac{|\check{Q}_i^l|_\nu}{\varphi(Q_i^{l*})^{1/\varepsilon}} \sum_{\substack{j; \\ Q_j^k \subset Q_i^l}} \left(a^{-k} \int_{\check{Q}_j^k} |f| \sigma \, d\nu \right)^{\frac{1}{\varepsilon}}$$

$$(3.28) \qquad \leq c \frac{|\check{Q}_i^l|_\nu}{\varphi(Q_i^{l*})^{1/\varepsilon}} \left(\sum_{\substack{j; \\ Q_j^k \subset Q_i^l}} a^{-k} \int_{\check{Q}_j^k} |f| \sigma \, d\nu \right)^{\frac{1}{\varepsilon}}$$

since $1/\varepsilon \geq 1$

$$\leq c \frac{|\check{Q}_i^l|_\nu}{\varphi(Q_i^{l*})^{1/\varepsilon}} \left(a^{-k} \int_{\check{Q}_i^l} |f| \sigma \, d\nu \right)^{\frac{1}{\varepsilon}}$$

since the Q_j^k are pairwise disjoint in j

$$\leq c\frac{|\check{Q}_i^l|_\nu}{\varphi(Q_i^{l*})^{1/\epsilon}}\left(a^{l+1-k}\varphi(Q_i^{l*})\right)^{\frac{1}{\epsilon}} = a^{\frac{l+1-k}{\epsilon}}|\check{Q}_i^l|_\nu$$

by the second inequality of 3.25 with $k = l$. Thus, combining 3.26, 3.27 and 3.28, we obtain

$$\sum_{k,j;\; Q_j^k\subset Q_i^l} \mathcal{A}(Q_j^k)^{q/p} \leq \left(\sum_{k=l}^{\infty}\left(a^{\frac{l+1-k}{1-\gamma}}|\check{Q}_i^l|_\nu\right)^{1/r'}\left(\int_{\check{Q}_i^l}\sigma^r d\nu\right)^{1/r}\right)^{q/p}$$

$$= \left(\sum_{k=l}^{\infty} a^{\frac{l+1-k}{(1-\gamma)r'}}|\check{Q}_i^l|_\nu^{1/r'}\left(\int_{\check{Q}_i^l}\sigma^r d\nu\right)^{1/r}\right)^{q/p}$$

$$= a^{\frac{q}{(1-\gamma)pr'}}\left(\sum_{k=0}^{\infty} a^{\frac{-k}{(1-\gamma)r'}}\mathcal{A}(Q_i^l)\right)^{q/p} = c\mathcal{A}(Q_i^l)^{q/p},$$

where $c = a^{\frac{q}{(1-\gamma)pr'}}\left(\sum_{k=0}^{\infty} a^{\frac{-k}{(1-\gamma)r'}}\right)^{q/p}$, and this yields 3.22.

Thus, if we take $\beta = q/p$, I to be the set of indices k, j, $a_i = \mathcal{A}(Q_i)$ and $u = \sigma$, we obtain from 3.21 and Lemma 2.32 that

(3.29)

$$\left\|\mathfrak{M}_{\varphi,m}^{dy}(f\sigma)\right\|_{L_w^q(D)}^q \leq c\sum_{k,j}\mathcal{A}(Q_j^k)^{q/p}\left[\frac{1}{\mathcal{A}(Q_j^k)}\int_{\check{Q}_j^k}|f|\sigma d\nu\right]^q \leq c\left(\int_{\partial D} f^p u d\nu\right)^{q/p},$$

provided we verify 2.33 and 2.34. Note that 2.33 follows from the Hölder's inequality with $c = 1$. Condition 2.34 holds due to 3.22.

It follows from 3.18 that

$$\|\mathfrak{m}_{\varphi,m}(f\sigma)\|_{L_w^q(D)}^q \leq c\left(\int_{\partial D} f^p\sigma d\nu\right)^{q/p}, \tag{3.30}$$

where

$$\mathfrak{m}_{\varphi,m}(f\sigma)(x) = \sup_{\substack{B\in\mathfrak{B};\\ x,a(x)\in B, r(B)>\lambda^m}} \int_{\check{B}}|f|\sigma d\nu, \quad x\in D.$$

Therefore, letting m tend to $-\infty$ in 3.30, we obtain

$$\|\mathfrak{m}_{\varphi}(f\sigma)\|_{L_w^q(D)}^q \leq c\left(\int_{\partial D} f^p\sigma d\nu\right)^{q/p}. \tag{3.31}$$

Replacing f by $f\sigma^{-1}$ in 3.31 and using the fact that $M_\varphi f \leq c\mathfrak{m}_\varphi f$ (see the beginning of the proof of Theorem 1.2), we obtain 1.11. This concludes the proof of Theorem 1.2. □

In order to prove Theorem 1.3 it will be convenient to define the following dyadic maximal function: for $m \in \mathbb{Z}$ and $z \in X$ we define

$$\mathfrak{M}^{dy}_{\varphi,m,z} f(x,t) = \sup_{\substack{Q \in \mathcal{D}_m \\ (x,t) \in \hat{Q}+z}} \frac{1}{\varphi(Q+z)} \int_{Q+z} |f| d\nu, \quad (x,t) \in \hat{X}_+,$$

and for $k > m$, let

$$M^k_{\varphi,m} f(x,t) = \sup_{\lambda^k \geq r > \max\{\lambda^m, t\}} \frac{1}{\varphi(B(x,r))} \int_{B(x,r)} |f| d\nu.$$

We have

Lemma 3.32. *Let (X,d,μ) be a homogeneous space with group structure satisfying conditions 1.12 — 1.15. Suppose $1 \leq p < \infty$ and $w(x,t) \geq 0$ in $\hat{X}_+ = X \times (0,\infty)$. Then, there exists $C > 0$ such that*

$$\left(\int_{\hat{X}_+} (M_{\varphi,m} f(x,t))^p \, w(x,t) d\mu(x,t) \right)^{1/p}$$

$$\leq C \sup_{z \in X} \left(\int_{\hat{X}_+} \left(\mathfrak{M}^{dy}_{\varphi,m,z} f(x,t)\right)^p w(x,t) d\mu(x,t) \right)^{1/p}$$

for all $f \geq 0$ with C independent of m.

Proof

Fix $k > m$ and $(x,t) \in \hat{X}_+$. Suppose $B(x,r)$ is a ball in X satisfying

$$\frac{1}{\varphi(B(x,r))} \int_{B(x,r)} |f| d\nu > \frac{1}{2} M^k_{\varphi,m} f(x,t).$$

Now, select $m < l \leq k$ such that $\lambda^{l-1} < r \leq \lambda^l$. Let B_k be the ball in X of radius λ^k about the identity element of X. Define $\Omega = \{z \in B_{k+3}; \exists\, Q \in \mathcal{D}_m$ with $\lambda^k \leq l(Q) \leq \lambda^{l+1}$ and $B(x,r) \subset Q + z\}$. Thus, if $Q = E^{l+1}_i$ and $z \in \Omega$ then $Q + z \subset B(x, 2A_o\lambda^3 r)$ and, therefore, $\varphi(Q+z) \leq c\varphi(B(x,r))$. Hence,

$$\mathfrak{M}^{dy}_{\varphi,m,z} f(x,t) \geq \frac{1}{\varphi(Q+z)} \int_{Q+z} |f| d\nu$$

$$\geq \frac{c}{\varphi(B(x,r))} \int_{B(x,r)} |f| d\nu \geq c M^k_{\varphi,m} f(x,t).$$

Thus,

$$\int_{B_{k+3}} \mathfrak{M}^{dy}_{\varphi,m,z} f(x,t) d\nu(z) \geq c M^k_{\varphi,m} f(x,t) \nu(\Omega) \tag{3.33}$$

$$\geq c M^k_{\varphi,m} f(x,t) \nu(B_{k+3}), \tag{3.34}$$

provided we show $\nu(B_{k+3}) \leq c\nu(\Omega)$.

Let $\Gamma = \{j; E_j^{l+1} \cap B(x, \lambda^{k+2} \neq \emptyset\}$, where l is as in the definition of Ω. Now suppose z is in $-B(z_j^{l+1}, \lambda^l) + x$ where $-B = \{w \in X; -w \in B\}$. We show that $B(x,r) \subset E_j^{l+1} + z$. First note that

$$x - z \in B(z_j^{l+1}, \lambda^l) \subset B(z_j^{l+1}, \lambda^{l+1}) \subset E_j^{l+1},$$

which shows that $x \in E_j^{l+1} + z$. Now, let $y \in B(x,r)$. Since $x - z \in B(z_j^{l+1}, \lambda^l)$ and $d(y - z, x - z) = d(y, x) \leq \lambda^l$ by 1.12, we have

$$d(y - z, z_j^{l+1}) \leq A_o[d(y - z, x - z) + d(x - z, z_j^{l+1})] < A_o[\lambda^l + \lambda^l] < \lambda^{l+1}.$$

Thus, $y - z \in B(z_j^{l+1}, \lambda^{l+1}) \subset E_j^{l+1}$ and, therefore, $y \in E_j^{l+1} + z$.Now if $j \in \Gamma$ then $-B(z_j^{l+1}, \lambda^l) + x \subset B_{k+3}$. Indeed, if $w \in B(z_j^{l+1}, \lambda^l)$ and $u \in E_j^{l+1} \cap B(x, \lambda^{k+2})$ (which exists since $j \in \Gamma$), then

$$d(u, w) \leq A_o[d(u, z_j^{l+1}) + d(z_j^{l+1}, w)] < A_o[\lambda^{l+2} + \lambda^l] < 2A_o\lambda^{k+2},$$

which implies

$$d(x, w) \leq A_o[d(x, u) + d(u, w)] < A_o[\lambda^{k+2} + 2A_o\lambda^{k+2}] < 3A_o^2\lambda^{k+2},$$

which yields

$$d(0, w) \leq A_o[d(0, x) + d(x, w)] < A_o[\lambda^k + 3A_o^2\lambda^{k+2}] < 4A_o^3\lambda^{k+2},$$

since $x \in B_k$. Thus,

$$d(0, -w + x) \leq A_o[d(0, x) + d(x, -w + x)] < A_o[\lambda^k + d(0, w)]$$

since $x \in B_k$ and by 1.12 and 1.13

$$< A_o[\lambda^k + 4A_o^3\lambda^{k+2}] < 5A_o^4\lambda^{k+2} < \lambda^{k+3},$$

as desired. Now, since the sets $\{-B(z_j^{l+1}, \lambda^l) + x\}_{j\in\Gamma}$ are pairwise disjoint, we have

$$\begin{aligned}
&(3.35) \qquad & \nu(\Omega) &\geq \sum_{j\in\Gamma} \nu(-B(z_j^{l+1}, \lambda^l) + x) \quad \text{by 1.14 and 1.15}\\
&(3.36) & &\geq c\sum_{j\in\Gamma} \nu(E_j^{l+1}) \quad \text{since } \nu \text{ is doubling}\\
&(3.37) & &= c\nu\left(\bigcup_{j\in\Gamma} E_j^{l+1}\right) \geq c\nu(B(x, \lambda^{k+2})) \quad \text{by 2.43}\\
&(3.38) & &\geq c\nu(B(x, \lambda^{k+3})) \quad \text{since } \mu \text{ is doubling}\\
&(3.39) & &= c\nu(B(0, \lambda^{k+3}) + x) \quad \text{by 1.12}\\
&(3.40) & &= c\nu(B(0, \lambda^{k+3})) = c\nu(B_{k+3}) \quad \text{by 1.14.}
\end{aligned}$$

We are now able to finish the proof of this lemma. By 3.34 and Minkowski's inequality, we get

$$\left[\int_{B_k\times(0,r(B_k))}\left(M^k_{\varphi,m}f(x,t)\right)^p w(x,t)d\mu(x,t)\right]^{1/p}$$

$$\leq c\frac{1}{\mu(B_{k+3})}\int_{B_{k+3}}\left[\int_{\hat{X}_+}\left(\mathfrak{M}^{dy}_{\varphi,m,z}f(x,t)\right)^p w(x,t)d\mu(x,t)\right]^{1/p} d\nu(z)$$

$$\leq c\sup_{z\in B_{k+3}}\left[\int_{\hat{X}_+}\left(\mathfrak{M}^{dy}_{\varphi,m,z}f(x,t)\right)^p w(x,t)d\mu(x,t)\right]^{1/p},$$

and letting $k\to\infty$ finishes the proof of this lemma. □

Let us prove a version of Theorem 1.3 with $M_\varphi f$ replaced by $\mathfrak{M}^{dy}_{\varphi,m,z}f$. We have

Theorem 3.1. *Let (X,d,ν) be a homogeneous space, where X has a group structure and d and ν satisfy conditions 1.12 - 1.15. Let w and v be weights in $\hat{X}_+$ and X, respectively, and suppose that $1<p\leq q<\infty$. Then*

$$\left(\int_{\hat{X}_+}[\mathfrak{M}^{dy}_{\varphi,m,z}f]^q w d\mu\right)^{1/q}\leq C\left(\int_X |f|^p v d\nu\right)^{1/p} \tag{3.41}$$

for all f in $L^p_{vd\nu}$, m in $\mathbb{Z}$ and z in X if and only if

$$\left(\int_{\hat{Q}+z}[\mathfrak{M}^{dy}_{\varphi,m,z}(\sigma\chi_{Q+z})]^q w d\mu\right)^{1/q}\leq C\left(\int_{Q+z}\sigma d\nu\right)^{1/p} \tag{3.42}$$

for all dyadic cube Q in $\mathcal{D}_m$, m in $\mathbb{Z}$ and z in X, where $\hat{Q}=Q\times l(Q)$.

Proof

Let $f(x)=\sigma\chi_{Q+z}(x)$ where Q is a dyadic cube in in $\mathcal{D}_m$, m in $\mathbb{Z}$ and z in X. Now, if 3.41 holds we obtain

$$\left(\int_{\hat{Q}+z}[\mathfrak{M}_{\varphi,m,z}dy(\sigma\chi_{Q+z})]^q w d\mu\right)^{1/q}\leq\left(\int_{\hat{X}_+}[\mathfrak{M}_{\varphi,m,z}dy(\sigma\chi_{Q+z})]^q w d\mu\right)^{1/q}$$

$$\leq C\left(\int_X|\sigma\chi_{Q+z}|^p v d\nu\right)^{1/p}=\left(\int_{Q+z}\sigma d\nu\right)^{1/p},$$

which is 3.42.

Conversely, assume that 3.42 holds. Define

$$\mathfrak{M}^{dy,R}_{\varphi,m,z}f(x,t)=\sup_{\substack{Q\in\mathcal{D}_m\\(x,t)\in\hat{Q}+z\\ \operatorname{diam}(Q^*)\leq R}}\frac{1}{\varphi(Q^*+z)}\int_{Q+z}|f|d\nu.$$

Let f in $L^p_{vd\nu}$ and for each k in $\mathbb{Z}$ and $R > 0$ define

$$\Omega_k^R = \{(x,t) \in \hat{X}_+; \mathfrak{M}^{dy,R}_{\varphi,m,z}(f\sigma)(x,t) > 2^k\}.$$

For a fixed m in $\mathbb{Z}$ let

$$\Omega_{k,m}^R = \{(x,t) \in \Omega_k^R; \exists Q \in \mathcal{D}_m, (x,t) \in \hat{Q} + z, Q^* + z \subset \Omega_k^R\}.$$

Let $\{Q_j^k\}_{j \in J_k^m}$ denote the dyadic cubes maximal among those dyadic cubes $Q \in \mathcal{D}_m$ with the property that Q^* is contained in Ω_k^R. Then,

$$\Omega_{k,m}^R = \bigcup_{j \in J_k^m} (\hat{Q}_j^k + z) \qquad \text{for } k \text{ in } \mathbb{Z}, \tag{3.43}$$

$$Q_i^k \cap Q_j^k = \emptyset \qquad \text{for } i \neq j, k \text{ in } \mathbb{Z}, \tag{3.44}$$

$$\frac{1}{\varphi(Q_j^{k^*} + z)} \int_{Q_j^k + z} |f| \sigma d\nu > 2^k \qquad \text{for } j \text{ in } J_k^m, k \text{ in } \mathbb{Z}. \tag{3.45}$$

We have

$$\varphi(Q_j^{k^*} + z) \leq 2^{-k} \int_{Q_j^k + z} |f| \sigma d\nu \qquad \text{by 3.45}$$

$$\leq 2^{-k} \left(\int_{Q_j^k + z} |f|^p \sigma d\nu \right)^{1/p} \left(\int_{Q_j^k + z} \sigma d\nu \right)^{1/p'} \qquad \text{by Hölder's inequality.}$$

Let $\tilde{Q}_j^k$ be such that $\tilde{Q}_j^k + z = (\hat{Q}_j^k + z) \setminus \Omega_{k+1,m}^R$. We have

$$\int_{\hat{X}_+} [\mathfrak{M}^{dy,R}_{\varphi,m,z}(f\sigma)]^q w d\mu \leq 2^q \sum_{k \in \mathbb{Z}} 2^{kq} \int_{\Omega_{k,m}^R \setminus \Omega_{k+1,m}^R} w d\mu$$

$$\leq 2^q \sum_{\substack{k \in \mathbb{Z} \\ j \in J_k^m}} |\tilde{Q}_j^k + z|_{wd\mu} \left(\frac{1}{\varphi(Q_j^{k^*} + z)} \int_{Q_j^k + z} |f| \sigma d\nu \right)^q \qquad \text{by 3.43 and 3.45}$$

$$= 2^q \sum_{\substack{k \in \mathbb{Z} \\ j \in J_k^m}} |\tilde{Q}_j^k + z|_{wd\mu} \left(\frac{1}{\varphi(Q_j^{k^*} + z)} \int_{Q_j^k + z} \sigma d\nu \right)^q \left(\frac{1}{|Q_j^k + z|_{\sigma d\nu}} \int_{Q_j^k + z} |f| \sigma d\nu \right)^q$$

$$\leq 2^q \sum_{\substack{k \in \mathbb{Z} \\ j \in J_k^m}} \left(\int_{\tilde{Q}_j^k + z} \left(\mathfrak{M}^{dy}_{\varphi,m,z} \left(\sigma \chi_{Q_j^k + z} \right) \right)^q w d\mu \right) \left(\frac{1}{|Q_j^k + z|_{\sigma d\nu}} \int_{Q_j^k + z} |f| \sigma d\nu \right)^q$$

$$= 2^q \sum_{\substack{k \in \mathbb{Z} \\ j \in J_k^m}} b_j^k(z) \left(\frac{1}{a_j^k(z)} \int_{Q_j^k + z} |f| \sigma d\nu \right)^q,$$

where

$$b_j^k(z) = \int_{\tilde{Q}_j^k+z} \left(\mathfrak{M}_{\varphi,m,z}^{dy}\left(\sigma\chi_{Q_j^k+z}\right)\right)^q w d\mu.$$

and

$$a_j^k(z) = |Q_j^k + z|_{\sigma d\nu}.$$

Let us show that

$$\sum_{\substack{k\in\mathbb{Z}, j\in J_k^m \\ Q_j^k\subset Q_o}} b_j^k(z) \leq c a_o(z)^{q/p}$$

for all Q_o in $\mathcal{D}_m$, z in X and m in $\mathbb{Z}$, where $a_o(z) = |Q_o + z|_{\sigma d\nu}$.

If Q_o is as above then

$$\sum_{\substack{k\in\mathbb{Z}, j\in J_k^m \\ Q_j^k\subset Q_o}} b_j^k(z) = \sum_{\substack{k\in\mathbb{Z}, j\in J_k^m \\ Q_j^k\subset Q_o}} \int_{\tilde{Q}_j^k+z} \left(\mathfrak{M}_{\varphi,m,z}^{dy}\left(\sigma\chi_{Q_j^k+z}\right)\right)^q w d\mu$$

$$\leq \int_{\hat{Q}_o+z} \left(\mathfrak{M}_{\varphi,m,z}^{dy}\left(\sigma\chi_{Q_o+z}\right)\right)^q w d\mu \quad \text{since the } \tilde{Q}_j^k\text{'s are disjoint in } k \text{ and } j$$

$$\leq c\left(\int_{Q_o+z} \sigma d\nu\right)^{q/p} \quad \text{by 3.42}$$

$$= c a_o(z)^{q/p}.$$

Using Lemma 2.44 we obtain

$$\left(\int_{\hat{X}_+} [\mathfrak{M}_{\varphi,m,z}^{dy,R} f]^q w d\mu\right)^{1/q} \leq C\left(\int_X |f|^p v d\nu\right)^{1/p}$$

and letting R tend to infinity we have 3.41. $\square$

Now we are able to finish the proof of Theorem 1.3.

***Proof of Theorem* 1.3**

If we set $f = \sigma\chi_{Q+z}$ where $Q \in \bigcup_m \mathcal{D}_m$ and z is in X in 1.17 we obtain 1.18.

Now suppose that 1.18 holds. It is enough to show 1.17 with M_φ replaced by $M_{\varphi,m}$ with c independent of m, since we obtain the desired result by letting m tend to $-\infty$. By Lemma 3.32 it is enough to show 1.17 with M_φ replaced by $\mathfrak{M}_{\varphi,m,z}^{dy}$, with c independent of m and z. By Theorem 3.1 it is enough to show that

$$\left(\int_{\hat{Q}+z} [\mathfrak{M}_{\varphi,m,z}^{dy}(\sigma\chi_{Q+z})]^q w d\mu\right)^{1/q} \leq C\left(\int_{Q+z} \sigma d\nu\right)^{1/p},$$

for all $Q \in \bigcup_m \mathcal{D}_m$ and z is in X.

Let Q_o be a dyadic cube in $\mathcal{D}_m$ and let $\bar{Q}_o$ and $\hat{Q}_o^*$ be the inner and outer balls, respectively, associated with Q_o. Let $z \in X$. If $x - z \in Q_o$ then

$$\frac{1}{\varphi(Q_o+z)}\int_{Q_o+z}\sigma d\nu \le \frac{c}{\varphi(\bar{Q}_o+z)}\int_{Q_o+z}\sigma d\nu$$

$$\le \frac{c}{\varphi(Q_o^*+z)}\int_{Q_o^*+z}\chi_{Q_o+z}\sigma d\nu \le cM_{\varphi,m}(\chi_{Q_o+z}\sigma)(x,t),$$

since $x - z \in Q_o$ and the radius of Q_o^* is larger that t.

Hence,

$$\mathfrak{M}^{dy}_{\varphi,m,z}(\sigma\chi_{Q_o+z})(x,t) \le cM_{\varphi,m}(\chi_{Q_o+z}\sigma)(x,t),$$

for all (x,t) in $\hat{Q}_o + z$ and. therefore,

$$\left(\int_{\hat{Q}_o+z}[\mathfrak{M}^{dy}_{\varphi,m,z}(\sigma\chi_{Q_o+z})]^q w d\mu\right)^{1/q}$$

$$\le c\left(\int_{\hat{Q}_o+z}[M_{\varphi,m}(\sigma\chi_{Q_o+z})]^q w d\mu\right)^{1/q} \le \left(\int_{Q_o+z}\sigma d\nu\right)^{1/p},$$

by 1.18. This finishes the proof of Theorem 1.3.

□

Remark. Theorem 1.1 and Theorem 1.2 in the special case when $\varphi(B) = |B \cap \partial D|_\nu^{1-\gamma}$, where ν is a doubling measure in ∂D, were first proved by the author in his Ph.D. thesis (see [Z]).

Acknowledgements. I wish to thank Professor Richard L. Wheeden for his constant support and suggestions while I was writing my Ph.D. thesis that led to this paper.

REFERENCES

[C] M. Christ, *A T(b) theorem with remarks on analytic capacity and the Cauchy integral,* Colloq. Math. **60/61** (1990), 601–628.

[GGK] I. Genebashvili, A. Gogatishvili, and V. Kokilashivili, *Criteria of general weak type inequalities for integral transforms with positive kernels,* Proc. Georgian Acad. Sci., Math. **1** (1992), 11–34.

[MS] R. Macias and C. Segovia, *Lipschitz functions on spaces of homogeneous type,* Adv. in Math. **33** (1979), 257–270.

[MW] B. Muckenhoupt and R. L. Wheeden, *Some weighted weak-type inequalities for the Hardy-Littlewood maximal function and the Hilbert transform,* Indiana Univ. Math. J. **26** (1977), 801–816.

[S] E. M Stein, *Singular integrals and differentiability properties of functions,* Princeton, Princeton Univ. Press, (1970). (1988), 533–545.

[S1] E. T. Sawyer, *A characterization of two weight norm inequalities for fractional and Poisson integrals,* Trans. Amer. Math. Soc. **308** (1988), 533–545.

[SW1] E. T. Sawyer and R. L. Wheeden, *Carleson conditions for the Poisson integral,* Indiana Univ. Math. J. **40** (1991), 639–676.

[SW2] ———, *Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces,* Amer. J. Math. **114** (1992), 813– 874.

[W1] R. L. Wheeden, *A characterization of some weighted norm inequalities for the fractional maximal function,* Studia Math.

[W2] ———, *Norm inequalities for off-centered maximal operators,* Pub. Mat.

[WZ] R. L. Wheeden and S. Zhao, *Weighted norm inequalities for operators of potential type and fractional maximal functions,* to appear.

[Z] S. L. Zani, *Norm inequalities and boundary estimates for a class of positive operators and fractional maximal functions on homogeneous spaces,* Ph.D. Thesis, Rutgers University, New Brunswick, N.J., U.S.A.

P.O. Box 668 - São Carlos - SP - 13.560-000 - Brazil
E-mail address: szani@icmsc.sc.usp.br

NOTAS DO ICMSC

SÉRIE MATEMÁTICA

022/94 LABOURIAU, I. S. and RUAS, M. A. S., *Singularities of equations of Hodgkin-Huxley type*

021/94 BRUCE, J.W. and MARAR, W.L., *Images and varieties*

020/94 LADEIRA, L.A.C., *Differentiability with respect to delays for a neutral differential-difference equation*

019/94 CARVALHO, A.N. and CUMINATO, J.A., *Reaction-diffusion problems in cell tissues*

018/94 CARVALHO, A.N. and RODRIGUEZ-BERNAL, A., *Global attractors for parabolic problems with nonlinear boundary conditions in fractional power spaces*

017/94 RODRIGUEZ-BERNAL, A., *Localized spatial homogenization and large diffusion*

016/94 GIONGO, M.A.P.A.P. and TABOAS, P.Z., *Effect of a vaccination on an epidemic model*

015/94 OLIVA, W.M., *Realidade Matemática: a controvérsia dos computadores, extraindo ordem do caos, medindo simetria e outros ensaios*

014/94 MARAR, W.L. and TARI, F., *On the geometry of simple germs of corank 1 maps from C^3 to C^3*

013/94 RODRIGUES, H.M. and RUAS FILHO, J.G., *Homoclinics and subharmonics of nonlinear two dimensional system. Uniform boundedness of generalized inverses*