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$\mathcal{K}$-Bi-Lipschitz Equivalence of Real Function-Germs

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$\mathcal{K}$-BI-LIPSCHITZ EQUIVALENCE OF REAL FUNCTION-GERMS

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ABSTRACT. In this paper we prove that the set of equivalence classes of germs of real polynomials of degree less than or equal to k, with respect to the $\mathcal{K}$-bi-Lipschitz equivalence, is finite.

RESUMO. Neste trabalho provamos que o conjunto das classes de equivalência dos polinômios reais de grau menor ou igual a k com respeito a $\mathcal{K}$-bi-Lipschitz equivalência, é finito.

1. INTRODUCTION

Finiteness theorems of different kinds appear in modern development of Singularity Theory. When one considers a classification problem, it is important to know if the problem is "tame" or not. In other words, how difficult the problem is and if there is any hope to develop a complete classification. For the problem of topological classification of polynomial function-germs, a finiteness theorem was obtained by Fukuda ([3]). He proved that the number of equivalence classes of polynomial function-germs of degree less than or equal to some k, with respect to the topological equivalence, is finite.

Mostowski ([7]) and Parusinski ([8]) proved that the set of equivalence classes of semi-algebraic sets with a complexity bounded from below by some k is finite.

A finiteness result does not hold for polynomial function-germs with respect to bi-Lipschitz equivalence. Henry and Parusinski ([4]) showed that this problem is not tame - has some "moduli".

Here we consider the problem of $\mathcal{K}$-bi-Lipschitz classification of polynomial function-germs ($\mathcal{K}$-equivalence is the contact equivalence defined by Mather ([5])). We show that

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this problem is still tame. The main idea of the proof is the following. First, we consider Lipschitz functions "of the same contact". Namely, f and g are of the same contact if $\frac{f}{g}$ is positive and bounded away from zero and infinity. We show that two functions of the same contact are $\mathcal{K}$-bi-Lipschitz equivalent. The next step is related to the geometry of contact equivalence. Recall that two function-germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are C^∞-contact equivalent if there exists a C^∞-diffeomorphism in the product space $(\mathbb{R}^n \times \mathbb{R}, 0)$ which leaves $\mathbb{R}^n$ invariant and maps the graph(f) to the graph(g). This result was first proved by Mather ([5]) for map-germs in dimensions (n, p). Montaldi extended this notion introducing a purely geometrical definition of contact: two pairs of submanifolds of $\mathbb{R}^n$ have the same contact type if there is a diffeomorphism of $\mathbb{R}^n$ taking one pair to the other, and relating this with the $\mathcal{K}$-equivalence of convenient map-germs. This geometrical interpretation also happens for the topological version of the $\mathcal{K}$-equivalence (cf. [1]). In this paper, we give a definition of the Montaldi's construction for the bi-Lipschitz case and show that the existence of a bi-Lipschitz analogue to Montaldi's construction, for two germs f and g, implies that $f \circ h$ and g are of the same contact, for some bi-Lipschitz map-germ h. Finally, the finiteness result follows from the Montaldi's construction and Mostowski-Parusinski ([8]) theorem on Lipschitz stratifications.

2. Basic definitions and results

Definition 2.1. *Two function-germs $f, g : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ are called* $\mathcal{K}$-bi-Lipschitz equivalent *(or* contact bi-Lipschitz equivalent*) if there exist two germs of bi-Lipschitz homeomorphisms $h : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ and $H : (\mathbb{R}^n \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}^n \times \mathbb{R}, 0)$ such that $H(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\}$ and the following diagram is commutative:*

$$\begin{array}{ccccc}
(\mathbb{R}^n, 0) & \xrightarrow{(id, f)} & (\mathbb{R}^n \times \mathbb{R}, 0) & \xrightarrow{\pi_n} & (\mathbb{R}^n, 0) \\
h \downarrow & & H \downarrow & & h \downarrow \\
(\mathbb{R}^n, 0) & \xrightarrow{(id, g)} & (\mathbb{R}^n \times \mathbb{R}, 0) & \xrightarrow{\pi_n} & (\mathbb{R}^n, 0)
\end{array}$$

where $id : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the identity map and $\pi_n : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ is the canonical projection.

Function-germs f and g are called $\mathcal{C}$-bi-Lipschitz equivalent *if $h = id$.*

In other words, two function-germs f and g are $\mathcal{K}$-bi-Lipschitz equivalent if there exists a germ of a bi-Lipschitz map $H : (\mathbb{R}^n \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}^n \times \mathbb{R}, 0)$ such that $H(x, y)$ can be written in the form $H(x, y) = (h(x), \tilde{H}(x, y))$, where h is a bi-Lipschitz map-germ, $\tilde{H}(x, 0) = 0$ and H maps the germ of the graph (f) onto the graph (g).

Recall that graph (f) is the set defined as follows:

$$\operatorname{graph}(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid y = f(x)\}.$$

Definition 2.2. *Two functions $f, g : \mathbb{R}^n \to \mathbb{R}$ are called* of the same contact at a point *$x_0 \in \mathbb{R}^n$ if there exist a neighbourhood U_{x_0} of x_0 in $\mathbb{R}^n$ and two positive numbers c_1 and c_2 such that, for all $x \in U_{x_0}$, we have*

$$c_1 f(x) \leq g(x) \leq c_2 f(x).$$

We use the notation: $f \approx g$.

Remark 2.3. *It is clear that if two function-germs f and g are of the same contact then the germs of the zero-sets are equal.*

The main results of the paper are the following:

Theorem 2.4. *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be two germs of Lipschitz functions. Then, f and g are $\mathcal{C}$-bi-Lipschitz equivalent if and only if one of the two conditions is true:*

i) $f \approx g$,

ii) $f \approx -g$.

Theorem 2.5. *Let $\mathcal{P}_k(\mathbb{R}^n)$ be the set of all polynomials of n variables with degree less than or equal to k. Then the set of equivalence classes of the germs at 0 of the polynomials in $\mathcal{P}_k(\mathbb{R}^n)$ with respect to the $\mathcal{K}$-bi-Lipschitz equivalence is finite.*

3. Functions of the same contact

Proof of Theorem 2.4:

Suppose that the germs of Lipschitz functions f and g are $\mathcal{C}$-bi-Lipschitz equivalent. Let $H : (\mathbb{R}^n \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}^n \times \mathbb{R}, 0)$ be the germ of a bi-Lipschitz homeomorphism satisfying the conditions of Definition 2.1. Let V_+ be the subset of $\mathbb{R}^n \times \mathbb{R}$ of the points (x,y) where $y>0$, and V_- be the subset of $\mathbb{R}^n \times \mathbb{R}$ where $y<0$. Clearly, we have one of the following options:

1) $H(V_+) = V_+$ and $H(V_-) = V_-$, or
2) $H(V_+) = V_-$ and $H(V_-) = V_+$.

Let us consider the first option. In this case, the functions f and g have the same sign on each connected component of the set $f(x) \neq 0$. Moreover,

$$|g(x)| = \| (x,0)-(x,g(x)) \| = \| H(x,0)-H(x,f(x)) \| \leq c_2 \ \| (x,0)-(x,f(x)) \| = c_2|f(x)|,$$

where c_2 is a positive real number. Using the same argument we can show

$$c_1|f(x)| \leq |g(x)|, \quad c_1 > 0.$$

Hence, $f \approx g$.

Let us consider the second option. Let $\xi : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ be a map-germ defined as follows:

$$\xi(x,y) = (x,-y).$$

Applying the same arguments to a map $\xi \circ H$, we will conclude that $f \approx -g$.

Reciprocally, suppose that $f \approx g$. Let us construct a map-germ

$$H : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$$

in the following may:

$$H(x,y) = \begin{cases} (x,0) & \text{se } y=0, \\ (x, \dfrac{g(x)}{f(x)}y) & \text{se } 0 \leq |y| \leq |f(x)|, \\ (x, y-f(x)+g(x)) & \text{se } 0 \leq |f(x)| \leq |y|, \\ (x,y) & \text{otherwise.} \end{cases} \tag{1}$$

The map $H(x,y) = (x, \tilde{H}(x,y))$ above defined is bi-Lipschitz. In fact, H is injective because, for any fixed $x^\star$, we can show that $\tilde{H}(x^\star, y)$ is a continuous and monotone function. Moreover, H is Lipschitz if $0 \leq |f(x)| \leq |y|$. Let us show that H is Lipschitz if

$0 \leq |y| \leq |f(x)|$. It is enough to show that all the partial derivatives $\dfrac{\partial \tilde{H}}{\partial x_i}$ are bounded in this domain, for all $i = 1, \ldots, n$. We have,

$$\frac{\partial \tilde{H}}{\partial x_i} = \frac{(\frac{\partial g}{\partial x_i} f(x) - \frac{\partial f}{\partial x_i} g(x))\, y}{(f(x))^2} = \frac{\partial g}{\partial x_i} \frac{y}{f(x)} - \frac{\partial f}{\partial x_i} \frac{g(x)}{f(x)} \frac{y}{f(x)}.$$

Since $|y| \leq |f(x)|$, then $\dfrac{y}{f(x)}$ is bounded. The expression $\dfrac{g(x)}{f(x)}$ is bounded since $f \approx g$. Moreover, $\dfrac{\partial g}{\partial x_i}$ and $\dfrac{\partial f}{\partial x_i}$ are bounded because f and g are Lipschitz functions.

Since H^{-1} can be constructed in the same form as (1), we conclude that H^{-1} is also Lipschitz and, thus, H is a bi-Lipschitz map. □

4. Montaldi's construction

Definition 4.1. *Two germs of Lipschitz functions are called* $\mathcal{K}$-$\mathcal{M}$-bi-Lipschitz equivalent *(or* contact equivalent in the sense of Montaldi*) if there exists a germ of a bi-Lipschitz map* $M : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ *such that* $M(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\}$ *and* $M(\text{graph}\,(f)) = \text{graph}\,(g)$. *A map* M *is called a* Montaldi map.

Theorem 4.2. *Two germs of Lipschitz functions* $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ *are* $\mathcal{K}$-$\mathcal{M}$-*bi-Lipschitz equivalent if and only if they are* $\mathcal{K}$-*bi-Lipschitz equivalent.*

Proof. It is clear that a $\mathcal{K}$-bi-Lipschitz equivalence implies a $\mathcal{K}$-$\mathcal{M}$-bi-Lipschitz equivalence. To prove the converse, let f and g be $\mathcal{K}$-$\mathcal{M}$-bi-Lipschitz equivalent, then

$$M(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\} \quad \text{and} \quad M(\text{graph}\,(f)) = \text{graph}\,(g).$$

Let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be defined by $h(x) = \pi_n(M(x, f(x)))$.

Claim 1. h is a bi-Lipschitz map-germ.

Proof of Claim 1.

Since g is a Lipschitz function, the projection $\pi_{n|_{\text{graph}\,(g)}}$ is a bi-Lipschitz map. By the same argument, a map $x \mapsto (x, f(x))$ is bi-Lipschitz. A map M is bi-Lipschitz by Definition 4.1 and Claim follows.

Claim 2. One of the following assertions is true:

i) $f \approx g \circ h$,

ii) $f \approx -(g \circ h)$.

Proof of Claim 2.

Since M is a bi-Lipschitz map, it follows that there exist two positive numbers c_1 and c_2, such that

$$c_1\,|f(x)| \ \leq \| \ M(x, f(x)) - M(x,0) \ \| \leq \ c_2|f(x)|.$$

By the above construction

$$\| \ M(x, f(x)) - M(x,0) \ \| = \| \ (h(x), g(h(x))) - M(x,0) \ \| \geq |g(h(x))|.$$

Therefore, $|g(h(x))| \leq c_2|f(x)|$.

Using the same procedure for the map M^{-1}, we obtain that

$$c_1|f(x)| \leq |g(h(x))|.$$

Since M is a homeomorphism and $M(\mathbb{R}^n \times \{0\}) = \mathbb{R}^n \times \{0\}$, the same argument as in Theorem 2.4 implies that, for all $x \in \mathbb{R}^n$,

$$\operatorname{sign} f(x) = \operatorname{sign} g(h(x)),$$

or, for all $x \in \mathbb{R}^n$,

$$\operatorname{sign} f(x) = -\operatorname{sign} g(h(x)).$$

Hence, Claim 2 is proved.

End of the proof of Theorem 4.2.

Using Claim 2 we obtain, by Theorem 2.4, that f and $g \circ h$ are $\mathcal{C}$-bi-Lipschitz equivalent. By Claim 1, f and g are $\mathcal{K}$-bi-Lipschitz equivalent. □

5. Finiteness Theorem

This section is devoted to a proof of a finiteness theorem (Theorem 2.5).

Let $\mathcal{P}_k(\mathbb{R}^n)$ be the set of all polynomials of n variables with degree less than or equal to k. Let $f \in \mathcal{P}_k(\mathbb{R}^n)$. Let X_f be the germ of the set $\mathbb{R}^n \times \{0\} \cup \operatorname{graph}(f)$. By Parusinski Theorem ([8]), there exists a finite semialgebraic stratification $\{\mathcal{S}_i\}_{i=1}^{p(k)}$ of $\mathcal{P}_k(\mathbb{R}^n)$ satisfying the following conditions:

i) Each $\mathcal{S}_i$ is a semialgebraic connected manifold.

ii) For each S_i and for each $f \in \mathcal{S}_i$, there exists an arcwise connected neighbourhood U_f such that, for any $\tilde{f} \in U_f$ and for every arc $\gamma(t)$ connecting f and $\tilde{f}$ (that is, $\gamma(0) = f$ and $\gamma(1) = \tilde{f}$), there exists a family of germs of bi-Lipschitz maps $\rho_t : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$, where $\rho_0 = id_{\mathbb{R}^n \times \mathbb{R}}$ and $\rho_t(X_f) = X_{\gamma(t)}$.

The condition ii) implies that, for small t, ρ_t is a Montaldi map.

Since $\mathcal{S}_i$ is a connected set, then, for any two polynomials $f_1, f_2 \in \mathcal{S}_i$, there exists a Montaldi map $M : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ such that $M(X_{f_1}) = X_{f_2}$.

Note, that any equivalence class of the $\mathcal{K}$-$\mathcal{M}$-bi-Lipschitz equivalence in $\mathcal{P}_k(\mathbb{R}^n)$ is a finite union of some strata of the stratification $\{S_i\}_{i=1}^{p(k)}$. By Theorem 4.2, the set of equivalence classes with respect to the $\mathcal{K}$-equivalence is also finite. $\square$

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