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Oscillation by Impulses for a Second Order Delay Differential Equation

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Resumo

Consideramos certa equação diferencial retardada de segunda ordem e provamos que suas soluções oscilam quando impomos controles de impulsos apropriados. Damos um exemplo.

Oscillation by impulses for a second order delay differential equation

L. P. Gimenes* and M. Federson †

Abstract

We consider a certain second order delay differential equation and prove that the all solutions oscillate when proper impulse controls are imposed. An example is given.

Keywords: Delay differential equations, oscillation, impulses.
2000 MSC: 34K45, 34K11.

1 Introduction

Recently, there has been an increasing interest on the oscillatory behavior of first order linear delay differential equation with impulses, but little has been done in the case of second order nonlinear delay differential equations with impulses. Some delay differential equations are nonoscillatory, but they may become oscillatory if some proper impulse controls are added to them. In the present paper we obtain sufficient conditions for the oscillation of a second order retarded differential equations with impulses.

In [2], Jiaowan Luo studies the oscillatory behavior of following second order quasilinear ODE with impulses

$$\begin{cases} (r(t)|x'(t)|^{\alpha-1}x'(t))' + f(t,x(t)) = 0, & t \geq t_0, \quad t \neq t_k \\ x(t_k^+) = I_k(x(t_k)), \quad x'(t_k{}^+) = J_k(x'(t_k)), & k = 1,2\ldots, \\ x(t_0^+) = x_0, \quad x'(t_0^+) = x_0', \end{cases} \tag{1}$$

where $0 \leq t_0 < t_1 < \ldots < t_k < \ldots$ with $\lim\limits_{k\to\infty} t_k = +\infty$.

In [3], Mingshu Peng and Weigao Ge prove an oscillation theorem for the second order delay differential with impulses equation

$$\begin{cases} (r(t)(x'(t))^\sigma)' + f(t,x(t),x(t-\tau)) = 0, & t \geq t_0, \quad t \neq t_k \\ x(t_k^+) = I_k(x(t_k)), \quad x'(t_k{}^+) = J_k(x'(t_k)), & k = 1,2\ldots, \\ x(t) = \phi(t), \quad t_0 - \tau \leq t \leq t_0, \end{cases} \tag{2}$$

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where $\tau > 0$, $0 \leq t_0 < t_1 < \ldots < t_k < \ldots$ with $\lim_{k\to\infty} t_k = +\infty$, $t_{k+1} - t_k > \tau$ and σ is any quotient of positive odd integers.

In this paper, we adapt the techniques applied [1] by Chen and Feng and in [3] to prove that the equation

$$\begin{cases} x''(t) + f(t, x(t), x'(t)) + g(t, x(t), x(t-\tau)) = 0, \quad t \geq t_0, \quad t \neq t_k \\ x(t_k) = I_k(x(t_k{}^-)), \quad x'(t_k) = J_k(x'(t_k{}^-)), \quad k = 1, 2\ldots, \\ x(t) = \phi(t), \quad t_0 - \tau \leq t \leq t_0, \end{cases} \tag{3}$$

oscillates, where $\tau > 0$, $0 \leq t_0 < t_1 < \ldots < t_k < \ldots$ with $\lim_{k\to\infty} t_k = +\infty$ and $t_{k+1} - t_k > \tau$.

Our paper is organized as follows. In Section 2, we present a lemma that plays an important role in the proof of the main result. In Section 3, we obtain the oscillatory behavior of (3) through impulse controls. An example is given in Section 4.

2 Preliminaries

Consider the impulsive differential equation

$$\begin{cases} x''(t) + f(t, x(t), x'(t)) + g(t, x(t), x(t-\tau)) = 0, \quad t \geq t_0, \quad t \neq t_k \\ x(t_k) = I_k(x(t_k{}^-)), \quad x'(t_k) = J_k(x'(t_k{}^-)), \quad k = 1, 2\ldots, \end{cases} \tag{4}$$

satisfying the initial value condition

$$x(t) = \phi(t), \quad t_0 - \tau \leq t \leq t_0, \tag{5}$$

where $\phi : [t_0 - \tau, t_0] \to \mathbb{R}$ has at most a finite number of discontinuities of first kind and is right continuous at these points. We assume that

(H$_1$) $f : [t_0 - \tau, +\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and non-negative;

(H$_2$) $g : [t_0 - \tau, +\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $ug(t, u, v) > 0$ for all $uv > 0$ and $\dfrac{g(t, u, v)}{\varphi(v)} \geq p(t)$, for all $v \neq 0$, where $p(t)$ is continuous in $[t_0 - \tau, \infty)$, $p(t) \geq 0$, $x\varphi(x) > 0$, for all $x \neq 0$ and $\varphi'(x) \geq 0$;

(H$_3$) I_k, $J_k : \mathbb{R} \to \mathbb{R}$ are continuous, $I_k(0) = J_k(0) = 0$, $k \in \mathbb{N}$ and there exist positive numbers a_k, b_k, c_k and d_k such that

$$a_k \leq \frac{I_k(x)}{x} \leq b_k, \quad c_k \leq \frac{J_k(x)}{x} \leq d_k, \quad x \neq 0, \quad k = 1, 2, \ldots.$$

(H$_4$) $\displaystyle\lim_{t\to+\infty} \int_{t_0}^{t} \prod_{t_0 < t_k < s} \frac{c_k}{b_k}\, ds = +\infty$

Now we define a solution of the impulsive problem (4)-(5).

Definition 2.1 *A function $x(t) : [t_0 - \tau, +\infty) \to \mathbb{R}$ is a solution of problem* (4)-(5) *if*

(i) $x(t)$ and $x'(t)$ are continuous on $[t_0, +\infty) \setminus \{t_k; k \in \mathbb{N}\}$, admit lateral limits at t_k and they are right continuous at t_k, $k \in \mathbb{N}$;

(ii) $x(t)$ fulfills (4)-(5)*;*

(iii) $x(t_k)$ and $x'(t_k)$ fulfill (4)*, for each $k \in \mathbb{N}$.*

Now we define an oscillatory solution of the impulsive problem (4)-(5).

Definition 2.2 *A solution of* (4)-(5) *is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, it is called oscillatory.*

Now we present a lemma which is a version of Theorem 1.4.1 in [4] replacing the left continuity by the right continuity of $m(t)$ and $m'(t)$ at t_k, $k \in \mathbb{N}$.

Lemma 2.3 *Suppose*

(i) the sequence $\{t_k\}_{k\in\mathbb{N}}$ satisfies $0 \leq t_0 < t_1 < \ldots < t_k < \ldots$ with $\lim_{k\to\infty} t_k = +\infty$.

(ii) $m \in PC^1(\mathbb{R}_+, \mathbb{R})$ is right continuous at t_k, for $k = 1, 2, \ldots$.

(iii) for $k = 1, 2, \ldots$ and $t \geq t_0$, we have

$$\begin{aligned} m'(t) &\leq p(t)\,m(t) + q(t), \qquad t \neq t_k &(6)\\ m(t_k) &\leq d_k m(t_k^{-}) + b_k, &(7)\end{aligned}$$

where $p, q \in C(\mathbb{R}_+, \mathbb{R})$, d_k and b_k are real constants with $d_k \geq 0$. Then the following inequality holds

$$\begin{aligned} m(t) \leq\ & m(t_0) \prod_{t_0<t_k<t} d_k \exp\left(\int_{t_0}^{t} p(s)ds\right) + \int_{t_0}^{t} \prod_{s<t_k<t} d_k \exp\left(\int_{s}^{t} p(u)du\right) q(s)ds \\ & + \sum_{t_0<t_k<t} \prod_{t_k<t_j<t} d_j \exp\left(\int_{t_k}^{t} p(s)ds\right) b_k, \qquad t \geq t_0. \end{aligned} \tag{8}$$

Remark 2.4 *If the inequalities* (6) *and* (7) *are reversed, then the inequality* (8) *is also reversed.*

3 Main result

This section is devoted to prove that every solution of (4)-(5) is oscillatory under hypotheses (H_1) to (H_4).

In the sequel, let $x(t)$ be a solution of (4)-(5).

Lemma 3.1 *Suppose (H_1) to (H_4) are fulfilled and there exists $T \geq t_0$ such that $x(t) > 0$ for $t \geq T - \tau$. Then $x'(t_k) \geq 0$ and $x'(t) \geq 0$ for $t \in [t_k, t_{k+1})$, where $t_k \geq T$.*

Proof. Suppose $x(t) > 0$, for $t \geq T - \tau$. Then $x(t-\tau) > 0$, $t \geq T$. At first, we prove that $x'(t_k^-) \geq 0$, $t_k \geq T$. If otherwise, there exists some $t_j \geq T$ such that $x'(t_j^-) < 0$. From (H_3) and (4), we obtain

$$x'(t_j) = J_j(x'(t_j^-)) \leq c_j x'(t_j^-) < 0.$$

Let $x'(t_j) = -\alpha$, $\alpha > 0$. By (H_1) and (H_2), given $t \in [t_j, t_{j+1})$, we have

$$x''(t) \leq -g(t, x(t), x(t-\tau)) \leq -p(t)\varphi(x(t-\tau)) \leq 0.$$

Then $x'(t)$ is decreasing in $t \in [t_j, t_{j+1})$. Moreover,

$$\begin{aligned} x'(t_{j+1}^-) &\leq x'(t_j) = -\alpha < 0, \\ x'(t_{j+2}^-) &\leq x'(t_{j+1}) = J_{j+1}(x'(t_{j+1}^-)) \leq c_{j+1}x'(t_{j+1}^-) \leq c_{j+1}(-\alpha) < 0, \\ x'(t_{j+3}^-) &\leq x'(t_{j+2}) = J_{j+2}(x'(t_{j+2}^-)) \leq c_{j+2}x'(t_{j+2}^-) \leq c_{j+2}c_{j+1}(-\alpha) < 0, \end{aligned}$$

and by induction one can prove that

$$x'(t_{j+n}^-) \leq -\prod_{i=1}^{n-1} c_{j+i}\,\alpha < 0. \tag{9}$$

Hence $x'(t)$ is decreasing in $[t_j, +\infty)$.

We now consider the impulsive differential inequalities

$$\begin{aligned} x''(t) &\leq 0, \quad t > t_j, \quad t \neq t_k, \quad k = j+1, j+2, \ldots \\ x'(t_k) &\leq c_k\, x'(t_k^-), \quad k = j+1, j+2, \ldots. \end{aligned}$$

By Lemma 2.3 with $m(t) = x'(t)$, we have

$$m(t) \leq m(t_j^-) \prod_{t_j < t_k < t} c_k,$$

that is,

$$x'(t) \leq x'(t_j^-) \prod_{t_j < t_k < t} c_k. \tag{10}$$

Now considering (10) and knowing that $x(t_k) = I_k(x(t_k^-)) \leq b_k x(t_k^-)$, $k = j+1, j+2, \ldots$, by Lemma 2.3, we conclude that

$$x(t) \leq \prod_{t_j < t_k < t} b_k \left[x(t_j^-) + x'(t_j^-) \int_{t_j}^{t} \prod_{t_j < t_k < s} \frac{c_k}{b_k}\, ds \right]. \tag{11}$$

By (H_4) and taking j sufficiently large, we find $x(t) \leq 0$. But this is a contradiction, since $x(t) > 0$, for $t \geq T - \tau$. Therefore $x'(t_k^-) \geq 0$, $t_k \geq T$.

It follows from (H_3) that $x'(t_k) \geq c_k\, x'(t_k^-) \geq 0$ for any $t_k \geq T$. Because $x'(t)$ is decreasing in $[t_k, t_{k+1})$, then $x'(t) \geq x'(t_k) \geq 0$, $t \in [t_k, t_{k+1})$, $t_k \geq T$ and the proof is complete. □

Remark 3.2 *When $x(t)$ is eventually negative, under hypotheses (H_1) to (H_4), one can prove in a similar way that $x'(t_k) \leq 0$ and $x'(t) \leq 0$ for $t \in [t_k, t_{k+1})$, where $t_k \geq T$.*

Theorem 3.3 *Suppose (H_1) to (H_4) are fulfilled and there exists a positive integer k_0 such that $a_k \geq 1$, for all $k \geq k_0$. If*

$$\sum_{k=0}^{+\infty} \int_{t_k}^{t_{k+1}} \prod_{t_0<t_k<u} \frac{1}{d_k} p(u)\, du = +\infty \tag{12}$$

then all solutions of (4)-(5) oscillate.

Proof. We suppose, without loss of generality, that $k_0 = 1$. Let $x(t)$ be a nonoscillatory solution of (4)-(5). We can assume that $x(t) > 0$, $t \geq t_0$. By Lemma 3.1, $x'(t) \geq 0$ and $x'(t_k) \geq 0$, $t \in [t_k, t_{k+1})$, where $t_k \geq t_0$.

By (H_3) and the fact that $a_k \geq 1$, $k = 1, 2, \ldots$, we obtain

$$x(t_0) < x(t_1^-) \leq x(t_1) \leq x(t_2^-) \leq \ldots.$$

It follows that $x(t)$ is decreasing in $[t_0, +\infty)$.

Now let

$$m(t) = \frac{x'(t)}{\varphi(x(t-\tau))} \tag{13}$$

Then $m(t_k) \geq 0$ and $m(t) \geq 0$, $t \geq t_0$. By (H_1) and equation (4), we have

$$\begin{aligned} m'(t) &= \frac{-f(t, x(t), x'(t)) - g(t, x(t), x(t-\tau))}{\varphi(x(t-\tau))} - \frac{x'(t)\varphi'(x(t-\tau))x'(t-\tau)}{\varphi^2(x(t-\tau))} \\ &\leq -p(t), \qquad t \geq t_0, \qquad t \neq t_k,\ t_k + \tau \end{aligned}$$

It follows from (H_3), equation (4), $a_k \geq 1$ and $\varphi'(x) \geq 0$ that

$$m(t_k) = \frac{x'(t_k)}{\varphi(x(t_k - \tau))} \leq \frac{d_k x'(t_k^-)}{\varphi(x(t_k^- - \tau))} = d_k m(t_k) \tag{14}$$

and

$$m(t_k + \tau) = \frac{x'(t_k + \tau)}{\varphi(x(t_k))} \leq \frac{x'(t_k^- + \tau)}{\varphi(a_k x(t_k^-))} \leq \frac{x'(t_k^- + \tau))}{\varphi(x(t_k^-))} = m(t_k^- + \tau) \tag{15}$$

Then using (14) and (15), by Lemma 2.3, we obtain

$$m(t) \leq m(s) \prod_{s<t_k<t} d_k - \int_s^t \prod_{u<t_k<t} d_k\, p(u)\, du, \qquad t_0 \leq s \leq t. \tag{16}$$

Let $s \to t_0$ and $t \to t_1^-$. It follows from (14) and (16) that

$$m(t_1) \leq d_1 m(t_1^-) \leq d_1 \left[m(t_0) - \int_{t_0}^{t_1} p(u)\, du\right] = d_1 m(t_0) - d_1 \int_{t_0}^{t_1} p(u)\, du.$$

Similarly, knowing that $t_2 - t_1 > \tau$ and using (15) and the above inequality, we get

$$\begin{aligned}
m(t_2) \le d_2 m(t_2^-) &\le d_2 \left[m(t_1 + \tau) - \int_{t_1+\tau}^{t_2} p(u)\, du \right] \\
&\le d_2 \left[m(t_1^- + \tau) - \int_{t_1+\tau}^{t_2} p(u)\, du \right] \\
&\le d_2 \left[m(t_1) - \int_{t_1}^{t_2} p(u)\, du \right] \\
&\le d_2\, d_1 m(t_0) - d_2\, d_1 \int_{t_0}^{t_1} p(u)\, du - d_2 \int_{t_1}^{t_2} p(u)\, du
\end{aligned}$$

By induction, we obtain

$$\begin{aligned}
m(t_n) &\le d_1 d_2 \cdots d_n m(t_0) - d_1 d_2 \cdots d_n \int_{t_0}^{t_1} p(u)\, du - d_2 \cdots d_n \int_{t_1}^{t_2} p(u)\, du \\
&- \cdots - d_{n-1} d_n \int_{t_{n-2}}^{t_{n-1}} p(u)\, du - d_n \int_{t_{n-1}}^{t_n} p(u)\, du \\
&= \prod_{t_0 < t_k < t_{n+1}} d_k \left[m(t_0) - \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \prod_{t_0 < t_k < u} \frac{1}{d_k} p(u)\, du \right]
\end{aligned}$$

Then in view of (12) and $m(t_n) \ge 0$, we find a contradiction as $n \to +\infty$ and the proof is finished. □

With the next corollaries we indent to show that the inequality (12) is fulfilled. We use Theorem 3.3 to conclude the results.

Corollary 3.4 *Suppose* (H_1) *to* (H_4) *are fulfilled and there exists a positive integer* k_0 *such that* $a_k \ge 1$ *and* $d_k \le 1$*, for all* $k \ge k_0$*. If*

$$\int_{t_0}^{+\infty} p(u)\, du = +\infty$$

then all solutions of (4)-(5) *oscillate.*

Proof. Suppose, without loss of generality, that $k_0 = 1$. Since $\dfrac{1}{d_k} \ge 1$, we have

$$\sum_{k=0}^{+\infty} \int_{t_k}^{t_{k+1}} \prod_{t_0 < t_k < u} \frac{1}{d_k} p(u)\, du = \lim_{n \to +\infty} \left(\sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \prod_{t_0 < t_k < u} \frac{1}{d_k} p(u)\, du \right)$$

$$
\begin{aligned}
&= \lim_{n\to+\infty}\left(\int_{t_0}^{t_1} p(u)\,du + \int_{t_1}^{t_2}\frac{1}{d_1}p(u)\,du + \int_{t_2}^{t_3}\frac{1}{d_1d_2}p(u)\,du + \cdots\right.\\
&\quad\left. + \int_{t_{n-1}}^{t_n}\frac{1}{d_1d_2\cdots d_{n-1}}p(u)\,du + \int_{t_n}^{t_{n+1}}\frac{1}{d_1d_2\cdots d_n}p(u)\,du\right)\\
&= \lim_{n\to+\infty}\left(\int_{t_0}^{t_1} p(u)\,du + \int_{t_1}^{t_2} p(u)\,du + \int_{t_2}^{t_3} p(u)\,du + \cdots\right.\\
&\quad\left. + \int_{t_{n-1}}^{t_n} p(u)\,du + \int_{t_n}^{t_{n+1}} p(u)\,du\right)\\
&= \lim_{n\to+\infty}\left(\int_{t_0}^{t_{n+1}} p(u)du\right) = +\infty
\end{aligned}
$$

Then condition (12) is satisfied. Hence, by Theorem 3.3, all solutions of the impulsive system (4)-(5) oscillate. □

Corollary 3.5 *Suppose (H_1) to (H_4) are fulfilled and there exist a positive integer k_0 and a constant $\alpha > 0$ such that $a_k \geq 1$ and $\frac{1}{d_k} \geq t_{k+1}^{\alpha}$, for all $k \geq k_0$. If*

$$\int_{t_1}^{+\infty} t^{\alpha} p(t)\,dt = +\infty$$

then all solutions of (4)-(5) oscillate.

Proof. Suppose, without loss of generality, that $k_0 = 1$ and $t_1 \geq 1$. Since $\frac{1}{d_k} \geq t_{k+1}^{\alpha}$, for all $k \geq k_0$, we obtain

$$1 \leq t_1 < \ldots < t_k < t_{k+1} < \ldots$$

and

$$
\begin{aligned}
\frac{1}{d_1} &\geq t_2^{\alpha},\\
\frac{1}{d_1}\frac{1}{d_2} &\geq t_2^{\alpha}\,t_3^{\alpha} \geq t_3^{\alpha},\ldots,\\
\frac{1}{d_1}\frac{1}{d_2}\cdots\frac{1}{d_n} &\geq t_2^{\alpha}\,t_3^{\alpha}\cdots t_{n+1}^{\alpha} \geq t_{n+1}^{\alpha},\ldots.
\end{aligned}
$$

Thus

$$\sum_{k=0}^{+\infty}\int_{t_k}^{t_{k+1}}\prod_{t_0<t_k<u}\frac{1}{d_k}p(u)\,du = \lim_{n\to+\infty}\left(\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}}\prod_{t_0<t_k<u}\frac{1}{d_k}p(u)\,du\right)$$

$$
\begin{aligned}
&= \lim_{n\to+\infty}\left(\int_{t_0}^{t_1} p(u)\,du + \int_{t_1}^{t_2}\frac{1}{d_1}p(u)\,du + \int_{t_2}^{t_3}\frac{1}{d_1d_2}p(u)\,du + \cdots\right.\\
&\quad \left. + \int_{t_{n-1}}^{t_n}\frac{1}{d_1d_2\cdots d_{n-1}}p(u)\,du + \int_{t_n}^{t_{n+1}}\frac{1}{d_1d_2\cdots d_n}p(u)\,du\right)\\
&= \lim_{n\to+\infty}\left(\int_{t_0}^{t_1} p(u)\,du + \int_{t_1}^{t_2} t_2^\alpha\, p(u)\,du + \int_{t_2}^{t_3} t_3^\alpha\, p(u)\,du + \cdots\right.\\
&\quad \left. + \int_{t_{n-1}}^{t_n} t_n^\alpha\, p(u)\,du + \int_{t_n}^{t_{n+1}} t_{n+1}^\alpha\, p(u)\,du\right)\\
&\geq \lim_{n\to+\infty}\left(\int_{t_1}^{t_2} u^\alpha\, p(u)\,du + \int_{t_2}^{t_3} u^\alpha\, p(u)\,du + \cdots\right.\\
&\quad \left. + \int_{t_{n-1}}^{t_n} u^\alpha\, p(u)\,du + \int_{t_n}^{t_{n+1}} u^\alpha\, p(u)\,du\right)\\
&= \lim_{n\to+\infty}\left(\int_{t_1}^{t_{n+1}} u^\alpha\, p(u)du\right) = \int_{t_1}^{+\infty} u^\alpha\, p(u)\,du = +\infty
\end{aligned}
$$

Hence condition (12) is satisfied and Theorem 3.3 implies all solutions of (4)-(5) oscillate. □

Theorem 3.6 *Suppose (H_1) to (H_4) are fulfilled and $\varphi(ab) \geq \varphi(a)\varphi(b)$, for any $ab \neq 0$. If*

$$
\sum_{k=0}^{+\infty}\int_{t_k}^{t_{k+1}} \prod_{t_0<t_k<u}\frac{\varphi(a_k)}{d_k}\, p(u)\,du = +\infty \tag{17}
$$

then all solutions of (4)-(5) oscillate.

Proof. Let $x(t)$ be a nonoscillatory solution of (4)-(5). We can assume that $x(t) > 0$, $t \geq t_0$. By Lemma 3.1, $x'(t) \geq 0$ and $x'(t_k) \geq 0$, $t \in [t_k, t_{k+1})$, where $t_k \geq t_0$. Now let $m(t)$ be defined by (13). Then $m(t_k) \geq 0$ and $m(t) \geq 0$, $t \geq t_0$. By (H_1) and equation (4), we have

$$
m'(t) \;\leq\; -p(t), \qquad t \geq t_0, \qquad t \neq t_k, \quad t_k + \tau.
$$

It follows from (H_3), equation (4), $\varphi(ab) \geq \varphi(a)\varphi(b)$ and $\varphi'(x) \geq 0$ that

$$
m(t_k) \;=\; \frac{x'(t_k)}{\varphi(x(t_k-\tau))} \leq \frac{d_k x'(t_k^-)}{\varphi(x(t_k^- - \tau))} = d_k m(t_k) \tag{18}
$$

and

$$
\begin{aligned}
m(t_k+\tau) \;&=\; \frac{x'(t_k+\tau)}{\varphi(x(t_k))} \leq \frac{x'(t_k^-+\tau)}{\varphi(a_k x(t_k^-))}\\
&\leq \frac{x'(t_k^-+\tau))}{\varphi(a_k)\varphi(x(t_k^-))} = \frac{1}{\varphi(a_k)} m(t_k^-+\tau)
\end{aligned} \tag{19}
$$

Then using (18) and (19), by Lemma 2.3, we obtain

$$m(t) \leq m(s) \prod_{s<t_k<t} d_k - \int_s^t \prod_{u<t_k<t} d_k\, p(u)\, du, \qquad t_0 \leq s \leq t. \tag{20}$$

Let $s \to t_0$ and $t \to t_1^-$. It follows from (19) and (20) that

$$m(t_1) \leq d_1 m(t_1^-) \leq d_1 \left[m(t_0) - \int_{t_0}^{t_1} p(u)\, du \right] = d_1 m(t_0) - d_1 \int_{t_0}^{t_1} p(u)\, du.$$

Similarly, knowing that $t_2 - t_1 > \tau$ and using (19) and the above inequality, we get

$$\begin{aligned} m(t_2) \leq d_2 m(t_2^-) &\leq d_2 \left[m(t_1 + \tau) - \int_{t_1+\tau}^{t_2} p(u)\, du \right] \\ &\leq d_2 \left[\frac{1}{\varphi(a_1)} m(t_1^- + \tau) - \int_{t_1+\tau}^{t_2} p(u)\, du \right] \\ &\leq d_2 \left[\frac{1}{\varphi(a_1)} m(t_1) - \int_{t_1}^{t_2} p(u)\, du \right] \\ &\leq \frac{d_2\, d_1}{\varphi(a_1)} m(t_0) - \frac{d_2\, d_1}{\varphi(a_1)} \int_{t_0}^{t_1} p(u)\, du - d_2 \int_{t_1}^{t_2} p(u)\, du \end{aligned}$$

Then by induction, we obtain

$$\begin{aligned} m(t_n) &\leq \frac{d_1 d_2 \cdots d_n}{\varphi(a_1)\varphi(a_2)\cdots\varphi(a_{n-1})} \left[m(t_0) - \int_{t_0}^{t_1} p(u)\, du - \frac{\varphi(a_1)}{d_1} \int_{t_1}^{t_2} p(u)\, du \right. \\ &\quad \left. - \cdots - \frac{\varphi(a_1)\varphi(a_2)\cdots\varphi(a_{n-2})}{d_1 d_2 \cdots d_{n-2}} \int_{t_{n-2}}^{t_{n-1}} p(u)\, du - \frac{\varphi(a_1)\varphi(a_2)\cdots\varphi(a_{n-1})}{d_1 d_2 \cdots d_{n-1}} \int_{t_{n-1}}^{t_n} p(u)\, du \right] \\ &= \frac{d_1 d_2 \cdots d_n}{\varphi(a_1)\varphi(a_2)\cdots\varphi(a_{n-1})} \left[m(t_0) - \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \prod_{t_0<t_k<u} \frac{\varphi(a_k)}{d_k} p(u)\, du \right] \end{aligned}$$

But in view of (17) and $m(t_n) \geq 0$, we find a contradiction as $n \to +\infty$ and the proof is finished. □

Corollary 3.7 *Suppose (H_1) to (H_4) are fulfilled and there exist a positive integer k_0 and a constant $\alpha > 0$ such that $\dfrac{\varphi(a_k)}{d_k} \geq t_{k+1}^\alpha$, for all $k \geq k_0$. If*

$$\int_{t_1}^{+\infty} t^\alpha p(t)\, dt = +\infty$$

then all solutions of (4)-(5) oscillate.

The proof of Corollary 3.7 is omitted, since it can be deduced from Theorem 3.6 and it is similar to that of Corollary 3.5.

4 An example

Consider the impulsive delay differential equation

$$\begin{cases} x''(t) + x(t-\tau) + \arctan|x'(t)| = 0, & t \geq 0, \quad t \neq t_k \\ x(t_k) = \left(\dfrac{k+1}{k}\right) x(t_k^-), \quad x'(t_k) = x'(t_k^-), & k = 1, 2, \ldots, \\ x(t) = \phi(t), \quad -\tau \leq t \leq 0, \end{cases} \tag{21}$$

where $t_{k+1} - t_k > \tau$, $k = 1, 2, \ldots$ and $\phi : [-\tau, 0] \to \mathbb{R}$ is continuous.

Since $\varphi(v) = v$, $p(t) = 1$, $a_k = b_k = \dfrac{k+1}{k}$ and $c_k = d_k = 1$, $k = 1, 2, \ldots$, hypotheses (H_1) to (H_3) are satisfied. Notice that

$$\begin{aligned} \lim_{t\to+\infty} \int_{t_0}^{t} \prod_{t_0<t_k<s} \frac{c_k}{b_k}\, ds &= \int_{t_0}^{+\infty} \prod_{t_0<t_k<s} \frac{k}{k+1}\, ds \\ &= \int_{t_0}^{t_1} \prod_{t_0<t_k<s} \frac{k}{k+1}\, ds + \int_{t_1}^{t_2} \prod_{t_0<t_k<s} \frac{k}{k+1}\, ds + \int_{t_2}^{t_3} \prod_{t_0<t_k<s} \frac{k}{k+1}\, ds + \cdots \\ &= (t_1 - t_0) + \frac{1}{2}(t_2 - t_1) + \frac{1}{3}(t_3 - t_2) + \cdots \\ &= \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = +\infty. \end{aligned}$$

Thus (H_4) is also satisfied.

Let $k_0 = 1$. Then $a_k \geq 1$ and $d_k = 1$ for all $k \geq 1$. And since

$$\int_{t_0}^{+\infty} p(u)\, du = \int_{0}^{+\infty} du = +\infty,$$

it follows from Corollary 3.4 that all solutions $x(t)$ of (21) oscillate.

References

[1] Chen Yong-shao and Feng Wei-zhen, Oscillations of second order nonlinear ODE with impulses, *J. Math. Anal. Appl.*, **210** (1997), 150-169.

[2] Jiaowan Luo, Second-order quasilinear oscillation with impulses, *Computers and Math. Appl.*, **46** (2003), 279-291.

[3] Mingshu Peng and Weigao Ge, Oscillation criteria for second-order nonlinear differential equations with impulses, *Computers and Math. Appl.*, **39** (2000), 217-225.

[4] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of impulsive differential equations*, World Scientific, Singapore (1989) 32.

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