UNIVERSIDADE DE SÃO PAULO

Instituto de Ciências Matemáticas e de Computação

ISSN 0103-2577

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Nº 233

NOTAS

Série Matemática

São Carlos – SP
Jul./2005

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[1]Apoio Parcial: PAPIIT IN110803-3 "Foliaciones Geométricas y Singularidades"
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Summary

We study here the four dimensional vector space of quasi-homogeneous polynomials in two variables of weights $(\frac{1}{3}, \frac{1}{3l})$, $l \geq 2$. This is a continuation of the work $(l = 2)$ done in [BKL]. We obtain similar models as before, varying if l is even or odd. We finish with the appropriate theorems of finite relative determination and a weak version of Stefan's Theorem.

Resumo

Este trabalho é a continuação de [BKL], no qual estuda-se os polinômios homogêneos de pesos $(\frac{1}{3}, \frac{1}{6})$ e grau 1. Aqui os pesos são $(\frac{1}{3}, \frac{1}{3l})$. Obtemos resultados semelhantes na mesma região fundamental, um toro, variando com a paridade de l. Os teoremas são análogos ao de determinação finita relativa a G, o grupo apropriado é sua relação com a determinação pela direita.

Introduction

In this work we study the quasi-homogeneous polynomials of weights $(\frac{1}{3}, \frac{1}{3l})$, $l \geq 2$ and degree 1.This is a continuation of [BKL], where we analyzed the case $(\frac{1}{3}, \frac{1}{6})$. In our case the models vary depending if l is even or odd. We obtain our fundamental region, a torus, as before. Here we also study our Lie group G depending in our models in T. We also prove the Theorems for finite relative determination and the equivalence to finite determination on the right. We finish with a weak version of Stefan's Theorem, since our version of Nakayma's Lemma doesn't apply directly.

We consider Q the set of quasi-homogeneous maps in two variables of weights $(\frac{1}{3}, \frac{1}{3l})$, $l \geq 2$ and degree 1. It is clear that they form a 4-dimensional vector space generated by: x^3, x^2y^l, xy^{2l} and y^{3l}. Now, consider G the group of maps g of the form $g(x,y) = (\alpha x + \beta y^l, \delta y)$ with $\alpha\delta \neq 0$. This group acts on Q by composition on the right. The diffeomorphisms in G can be seen as matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$$

with product

$$\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} * \begin{pmatrix} \alpha' & \beta' \\ 0 & \delta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' & \alpha\beta' + \beta\delta'^l \\ 0 & \delta\delta' \end{pmatrix}$$

We will be using G or our group triangular matrices.

We have two cases

1) If l is even, let T^+ the subgroup of triangular invertible matrices with term $(2,2)$ positive and G^+, the subgroup of G, with term $(2,2)$ positive too. Hence $h^+ : T^+ \to G^+$ defined by

$$h : \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \to \begin{pmatrix} \alpha & \beta \\ 0 & \delta^{1/l} \end{pmatrix}$$

is an isomorphism of groups.

2) If l is odd, h will be an isomorphism from T to G.

We have the following

Proposition 1. *Consider K the vector space of cubic in two variables under the action the group: T^+ if k is even or T if k is odd, and the vector space Q under the action of the group: G^+ if k is even or G if k is odd. Let ϕ be the natural isomorphism from K to Q and $h^+ : T^+ \to G^+$, $h : T \to G$ be defined by $h^+(\alpha, \beta, 0, \delta) = (\alpha, \beta, 0, \delta^{1/l})$ and $h(\alpha, \beta, 0, \delta) = (\alpha, \beta, 0, \delta^{1/l})$, respectively. Therefore we have that*

$\phi((ax^3 + bx^2y + cxy^2 + dy^3) \circ (\alpha x + \beta y, \delta y)) = \phi((ax^3 + bx^2y + cxy^2 + dy^3)) \cdot h(\alpha x + \beta y, \delta y)$

Proof. It is a simple calculation. ■

Classification of the quasi-homogeneous maps of weights $(\frac{1}{3}, \frac{1}{3l})$ and degree 1.

We start with $f(x,y) = ax^3 + bx^2y^l + cxy^{2l} + dy^{3l}$, hence $f(x,y) = \prod_{i=1}^{3}(a_ix + b_iy^l)$ with a_i, b_i complex numbers. For each case we will calculate its stabilizer.

First Case: $f(x,y) = (ax + by^l)^3$, a and b real numbers

1.1) If $a = 0$ hence $f(x,y) = b^3y^{3l}$ and we get

i) For l even, $f(x,y) = \pm(|b|^{1/l}y)^{3l}$, where we get $+$ if $b > 0$ and $-$ if $b < 0$.

ii) For l odd, $f(x,y) = (b^{1/l}y)^{3l}$.

Hence if l is even we get two orbits, namely $\pm y^{3l}$, and if l is odd we get one orbit, namely y^{3l}.

1.2) If $a \neq 0$, $f(x,y) = (ax + by^l)^3$ and we get one orbit namely x^3.

Stabilizers For 1.1): $\alpha \neq 0$ and $\delta^{3l} = 1$. For 1.2): $\alpha^3 = 1$, $\beta = 0$ and $\delta \neq 0$

Second Case: $f(x,y) = (ax + by^l)^2(cx + dy^l)$ such that $ad - bc \neq 0$.

2.1) If $a \neq 0$ and $c = 0$, hence $f(x,y) = (ax + by^l)^2dy^l$

i) For l even, $f(x,y) = \pm(ax + by^l)^2(|d|^{1/l}y)^l$ where we get $+$ if $d > 0$ and $-$ if $d < 0$.

ii) For l odd, $f(x,y) = (ax + by^l)^2(d^{1/l}y)^l$.

Hence, if l is even we get two orbits, namely, x^2y^l and $-x^2y^l$ and if l is odd we get one orbit, namely x^2y^l.

2.2) If $a \neq 0$ and $c \neq 0$, hence $f(x,y) = (ax+by^l)^2(cx+dy^l)$. For $x' = ax+by$, $y' = y$ we get that f is equivalent to $x'^2(\frac{c}{a}x' + \frac{ad-bc}{a}y'^2)$, with $x'' = \alpha x'$ and $y'' = \beta y'$ for appropriate α and β we have

i) For l even, we get two orbits, $x^3 + x^2y^l$ (if $\frac{ad-bc}{a} > 0$) and $x^3 - x^2y^l$ (if $\frac{ad-bc}{a} < 0$).

ii) For l odd, we get one orbit, $x^3 + x^2y^l$.

2.3) If $a = 0$ then $bc \neq 0$, hence $f(x,y) = (cx+dy^l)(by^l)^2 = (cx + dy^l)(|b|^{1/l}y)^2l$. Then we get one orbit, namely xy^{2l}.

Stabilizers For 2.1): $\alpha^2\delta^l = 1$, $\beta = 0$. For 2.2) $\alpha^3 = 1, \beta = 0$ and $\delta^l = \alpha$. We can see this stabilizer as

$$\{\begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}, \begin{pmatrix} \rho & 0 \\ 0 & \delta_0\mu \end{pmatrix}, \begin{pmatrix} \rho^2 & 0 \\ 0 & \delta_0^2\mu \end{pmatrix} \mid \rho^2+\rho+1 = 0, \mu^l = 1, \delta_0^l = \rho\}$$

It is an abelian group of order $3l$ and in fact cyclic isomorphic to $\mathbb{Z}_{3l}$. For 2.3) $\alpha\delta^l = 1, \beta = 0$.

Third Case: $f(x,y) = (ax+by^l)(cx+dy^l)(ex+gy^l)$ such that any two of the vectors in $\mathbb{R}^2$: $(a,b),(c,d),(e,f)$ are linearly independent. We can suppose $ac \neq 0$.

3.1) If $e = 0$, $f(x,y) = (ax+by^l)(cx+dy^l)(gy^l)$

i) For l even we get two orbits, $x(x+y^l)y^l$ and $-x(x+y^l)y^l$.

ii) For l odd, we get one orbit, $x(x+y^l)y^l$.

3.2) If $e \neq 0$,$f(x,y) = (ax+by^l)(cx+dy^l)(ex+gy^l)$

i) For l even we get two orbits, $\pm x(x+y^l)(x+\epsilon y^l)$, $\epsilon \neq 0$ and $\epsilon \neq 1$.

ii) For l odd, we get one orbit, $x(x+y^l)(x+\epsilon y^l)$, $\epsilon \neq 0$ and $\epsilon \neq 1$.

Stabilizers For 3.1):

$$\{\begin{pmatrix} \delta^l & 0 \\ 0 & \delta \end{pmatrix}, \begin{pmatrix} -\delta^l & -\delta^l \\ 0 & \delta \end{pmatrix} \mid \delta^{3l} = 1\}$$

This is an abelian group of order $6l$.

i) If l is even, this group is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_{3l}$.

ii) If l is odd, this group is isomorphic to the cyclic group $\mathbb{Z}_{6l}$.

For 3.2) We have the following cases:

i) If $\epsilon \notin \{-1, 2, \frac{1}{2}, \frac{1\pm\sqrt{3}i}{2}\}$ the stabilizer is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}, \begin{pmatrix} \rho & 0 \\ 0 & \delta\mu_0 \end{pmatrix}, \begin{pmatrix} \rho^2 & 0 \\ 0 & \delta\mu_0^2 \end{pmatrix} \mid \rho^2+\rho+1 = 0, \delta^l = 1, \mu_0^l = \rho \right\}$$

which is isomorphic to $\mathbb{Z}_{3l}$.

ii) If $\epsilon = -1$, the stabilizer is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}, \begin{pmatrix} \rho & 0 \\ 0 & \delta\mu_0 \end{pmatrix}, \begin{pmatrix} \rho^2 & 0 \\ 0 & \delta\mu_0^2 \end{pmatrix} \mid \rho^2+\rho+1 = 0, \delta^{2l} = 1, \mu_0^{2l} = \rho^2 \right\}$$

which is isomorphic to $\mathbb{Z}_{6l}$.

iii) $\epsilon = 2$, the stabilizer is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}, \begin{pmatrix} \rho & 0 \\ 0 & \delta\mu_0 \end{pmatrix}, \begin{pmatrix} \rho^2 & 0 \\ 0 & \delta^2\mu_0^2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & \widetilde{\delta} \end{pmatrix}, \begin{pmatrix} \rho & 2\rho \\ 0 & \widetilde{\delta}\mu_0 \end{pmatrix}, \begin{pmatrix} \rho^2 & 2\rho^2 \\ 0 & \widetilde{\delta}\mu_0^2 \end{pmatrix} \right\},$$

where $\rho^2 + \rho + 1 = 0, \delta^l = 1,\ \mu_0^l = \rho$ and $\widetilde{\delta}^l = -1$. This stabilizer is isomorphic to $\mathbb{Z}_{6l}$.

iv) $\epsilon = \frac{1}{2}$, the stabilizer is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}, \begin{pmatrix} \rho & 0 \\ 0 & \rho\mu_0 \end{pmatrix}, \begin{pmatrix} \rho^2 & 0 \\ 0 & \delta^2\mu_0^2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & \widetilde{\delta} \end{pmatrix}, \begin{pmatrix} \rho & \rho \\ 0 & \widetilde{\delta}\mu_0 \end{pmatrix}, \begin{pmatrix} \rho^2 & \rho^2 \\ 0 & \widetilde{\delta}^2\mu_0^2 \end{pmatrix} \right\},$$

where $\rho^2 + \rho + 1 = 0, \delta^l = 1, \mu_0^l = \rho$, and $\widetilde{\delta}^l = -1$ This is stabilizer is isomorphic to $\mathbb{Z}_{6l}$.

v) $\epsilon = \rho + 1$, the stabilizer is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}, \begin{pmatrix} \rho & 0 \\ 0 & \delta\mu_0 \end{pmatrix}, \begin{pmatrix} \rho^2 & 0 \\ 0 & \delta\mu_0^2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & \mu_0\delta \end{pmatrix}, \begin{pmatrix} 1 & -\rho^2 \\ 0 & \delta\mu_0^2 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} \rho & -1 \\ 0 & \delta \end{pmatrix}, \begin{pmatrix} \rho & \rho \\ 0 & \delta\mu_0^2 \end{pmatrix}, \begin{pmatrix} \rho^2 & \rho^2 \\ 0 & \delta \end{pmatrix}, \begin{pmatrix} \rho^2 & -\rho \\ 0 & \delta\mu_0 \end{pmatrix} \right\},$$

where $\rho^2 + \rho + 1 = 0, \delta^l = 1, \mu_0^l = \rho$. This is an abelian group of order $9l$ and in fact isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_{3l}$.

vi) $\epsilon = -\rho$, the stabilizer is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}, \begin{pmatrix} \rho & 0 \\ 0 & \rho\mu_0 \end{pmatrix}, \begin{pmatrix} \rho^2 & 0 \\ 0 & \delta\mu_0^2 \end{pmatrix}, \begin{pmatrix} 1 & -\rho \\ 0 & \mu_0\delta \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & \delta\mu_0^2 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} \rho & \rho \\ 0 & \delta \end{pmatrix}, \begin{pmatrix} \rho & -\rho^2 \\ 0 & \delta\mu_0^2 \end{pmatrix}, \begin{pmatrix} \rho^2 & -1 \\ 0 & \delta \end{pmatrix}, \begin{pmatrix} \rho^2 & \rho^2 \\ 0 & \delta\mu_0 \end{pmatrix} \right\},$$

where $\rho^2 + \rho + 1 = 0, \delta^l = 1, \mu_0^l = \rho$ As before, this abelian group is isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_{3l}$.

<u>Fourth Case</u>: $f(x, y) = (ax + by^l)(cx^2 + dxy^l + ey^{2l})$ with $d^2 - 4ce < 0$.

3.3) If $a = 0$, $f(x, y) = by^l(cx^2 + dxy^l + ey^{2l})$

i) For l even, we get two orbits, $y^l(x^2 + y^{2l})$ (if $c(bc - ad) > 0$) and $-y^l(x^2 + y^{2l})$ (if $c(bc - ad) < 0$).

ii) For l odd, we get one orbit, $y^l(x^2 + y^{2l})$.

3.4) $a \neq 0$, $f(x, y) = (ax + by^l)(cx^2 + dxy^l + ey^{2l})$

i) For l even, we have two orbits: $(x+\epsilon y^l)(x^2+y^l)$ and $-(x+\epsilon y^l)(x^2+y^l)$.

ii) For l odd, we have one orbit: $(x + \epsilon y^l)(x^2 + y^l)$.

Stabilizers For 3.3)

$$\left\{\begin{pmatrix} \pm 1 & 0 \\ 0 & \delta \end{pmatrix}, \begin{pmatrix} \pm\rho & 0 \\ 0 & \delta\mu_0 \end{pmatrix}, \begin{pmatrix} \pm\rho^2 & 0 \\ 0 & \delta\mu_0^2 \end{pmatrix} \mid \rho^2+\rho+1=0, \delta^l=1, \mu_0^l=\rho\right\}$$

It is an abelian group of order $6l$ and

i) If l is even, it is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_{3l}$.

ii) If l is odd, it is cyclic, of order $6l$.

For 3.4)

i) If $\epsilon \notin \{\pm\sqrt{3}, \pm 3i, 0\}$, it is isomorphic to $\mathbb{Z}_{3l}$.

ii) If $\epsilon \in \{\pm 3i, 0\}$, it is isomorphic to $\mathbb{Z}_{6l}$.

iii) If $\epsilon \in \{\pm\sqrt{3}\}$, it is isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_{3l}$.

Remark 2. *We consider the case(3.2) and the quartic $y(x+y)x(x+\epsilon y)$ in the (y,x)-plane with slopes: $\lambda_1=\infty, \lambda_2=-1, \lambda_3=0$ and $\lambda_4=-\epsilon$. The cross ratios are repeated only for $\epsilon \in \{-1, 0, \frac{1}{2}, 1, 2, \frac{1\pm\sqrt{3}i}{2}\}$. Similarly for the quartic $y(x+iy)(x-iy)(x+\epsilon y)$ in the plane (y,x)-plane with slopes: $\lambda_1=\infty, \lambda_2=-i, \lambda_3=i$ and $\lambda_4=-\epsilon$. The cross ratios are repeated only for $\epsilon \in \{\pm i, \pm 3i, \pm\sqrt{3}i, 0\}$. Let $h:\mathbb{C}\to\mathbb{C}$ be defined by $h(z)=(1-2z)i$, hence $h(-1)=3i$, $h(2)=-3i$, $h(\frac{1}{2})=0$, $h(0)=i$, $h(1)=-i$, $h(\frac{1}{2}+\frac{\sqrt{3}}{2})=\sqrt{3}$ and $h(\frac{1}{2}-\frac{\sqrt{3}}{2})=-\sqrt{3}$. The diagram will be*

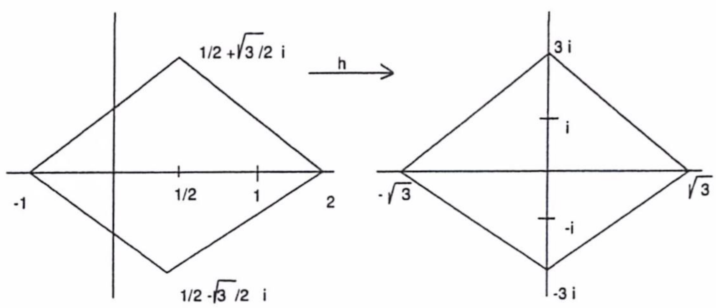

Figure 1:

We finish with the table

Type	Representant	Stabilizer	Codimension of the orbit
1.1)	$\pm y^{3l}$, l even y^{3l}, l odd	$\alpha \neq 0$, $\delta^{3l} = 1$	2
1.2)	x^3	$\alpha^3 = 1$, $\beta = 0$	1
2.1)	$\pm x^2 y^l$, l even $x^2 y^l$, l odd	$\alpha^2 \delta^l = 1$ $\beta = 0$	1
2.2)	$\pm(x^3 + x^2 y^l)$, l even $x^3 + x^2 y^l$, l odd	$\mathbb{Z}_{3l}$	0
2.3)	xy^{2l}	$\alpha \delta^l = 1$, $\beta = 0$	1
3.1)	$\pm x(x + y^l) y^l$, l even $x(x + y^l) y^l$, l odd	$\mathbb{Z}_2 \oplus \mathbb{Z}_{3l}$, l even $\mathbb{Z}_{6l}$, l odd	0
3.2)	$\pm x(x + y^l)(x + \epsilon y^l)$, l even $x(x + y^l)(x + \epsilon y^l)$, l odd	$\mathbb{Z}_{3l}$, $\epsilon \notin \{-1, 2, \frac{1}{2}, \frac{1 \pm \sqrt{3}i}{2}\}$ $\mathbb{Z}_{6l}$, $\epsilon \in \{-1, 2, \frac{1}{2}\}$ $\mathbb{Z}_3 \oplus \mathbb{Z}_{3l}$, $\epsilon \in \{\frac{1 \pm \sqrt{3}i}{2}\}$	0
3.3)	$\pm(x^2 + y^{2l}) y^l$, l even $(x^2 + y^{2l}) y^l$, l odd	$\mathbb{Z}_2 \oplus \mathbb{Z}_{3l}$, l even $\mathbb{Z}_{6l}$, l odd	0
3.4)	$\pm(x^3 + \epsilon x^2 y^l + x y^{2l} + \epsilon y^{3l})$	$\mathbb{Z}_{3l}$, $\epsilon \notin \{0, \pm 3i, \pm\sqrt{3}\}$ $\mathbb{Z}_{6l}$, $\epsilon \in \{0, \pm 3i\}$ $\mathbb{Z}_3 \oplus \mathbb{Z}_{3l}$, $\epsilon \in \{\pm\sqrt{3}\}$	0

Definition 3. *We say that a germ η is stable if there exist a neighborhood of η such that all the germs there are of the same kind.*

Theorem 4. *The quasi-homogeneous polynomials in two variables of weights $(\frac{1}{3}, \frac{1}{3l})$ satisfy*

i) If the stabilizer is infinite or isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_{3l}$, the germ is not finitely determined on the right.

ii) If the stabilizer is isomorphic to $\mathbb{Z}_{3l}$, and the germ stable, it is also finitely determined on the right.

iii) In all other cases, the germ is not finitely determined on the right.

Topology of the fibers of x^2y Consider the mapping $h : \mathbb{R}^4 \to \mathbb{R}^4$ given by $h(a, b, c, d) = (a^2c, a^2d + 2abc, b^2c + 2abd, b^2d)$. Remember that our codomain is the four-dimensional vector space Q with basis: $\{x^3, x^2y^l, xy^{2l}, y^{3l}\}$. If we denote our orbits by: $\mathcal{O}_{1,1}^{(\pm 1)^l}$, $\mathcal{O}_{1,2}$, $\mathcal{O}_{2,1}^{(\pm 1)^l}$, $\mathcal{O}_{2,2}^{(\pm 1)^l}$, $\mathcal{O}_{2,3}$, $\mathcal{O}_{3,1}^{(\pm 1)^l}$, $\mathcal{O}_{3,2,\epsilon}^{(\pm 1)^l}$, $\mathcal{O}_{3,3}^{(\pm 1)^l}$, $\mathcal{O}_{3,4,\epsilon}$

It is easy to see that the fibers $h^{-1}h(\alpha, \beta, \gamma, \delta)$ are

a) $\{a, \frac{\beta}{\alpha}a, \frac{\alpha^2\gamma}{a^2}, \frac{\alpha^2\delta}{a^2} | a \neq 0\}$

b) $\{0, b, \frac{\beta^2\gamma}{b^2}, \frac{\beta^2\delta}{b^2} | b \neq 0\}$

and the nearest points of such fibers to the origin are of the form

a) $(\pm\tau\alpha, \pm\tau\beta, \frac{\gamma}{\tau^2}, \frac{\delta}{\tau^2})$, where $\tau = (\frac{2(\gamma^2+\delta^2)}{\alpha^2+\beta^2})^{1/6}$

b) $(0, \pm\tau\beta, \frac{\gamma}{\tau^2}, \frac{\delta}{\tau^2})$, where $\tau = (\frac{2(\gamma^2+\delta^2)}{\beta^2})^{1/6}$

Such points are in the sphere S^3 if $27(\alpha^2 + \beta^2)^2(\gamma^2 + \delta^2) = 4$, therefore in polar coordinates we get:

$$27r^4s^2 = 4, \ \tau = \frac{2^{1/2}}{3^{1/2}r}$$

and our points are of the form

$$(\pm\frac{2^{1/2}}{3^{1/2}}\cos\theta, \pm\frac{2^{1/2}}{3^{1/2}}\sin\theta, \frac{1}{3^{1/2}}\cos\varphi, \frac{1}{3^{1/2}}\sin\varphi).$$

Hence we are in the torus $S^1_{\sqrt{2/3}} \times S^1_{\sqrt{1/3}}$

Remark 5. *1) The calculation done above includes both cases a) and b).*

2) Since $h^{-1}(\bar{0}) = \mathbb{R}^2 \times \{\bar{0}\} \cup \bar{0} \times \mathbb{R}^2$ we get $h^{-1}(\bar{0}) \cap (S^1_{\sqrt{2/3}} \times S^1_{\sqrt{1/3}}) = \emptyset$

3) Let $(\pm\tau\alpha, \pm\tau\beta, \frac{\gamma}{\tau^2}, \frac{\delta}{\tau^2}) \in S^1_{\sqrt{2/3}} \times S^1_{\sqrt{1/3}}$ hence $\tau^2 = (\frac{2}{3(\alpha^2+\beta^2)})$ and $\tau^4 = 3(\gamma^2+\delta^2)$, therefore $\tau^6 = 2(\frac{\gamma^2+\delta^2}{\alpha^2+\beta^2})$ and all the points in the torus $S^1_{\sqrt{2/3}} \times S^1_{\sqrt{1/3}}$ are of minimal length.

4) In the case $\gamma = \alpha$ and $\delta = \beta$ we get $\tau = 2^{1/6}$ and in the torus the points for this case are of the form

$$(\pm\frac{2^{1/2}}{3^{1/2}}\cos\theta, \pm\frac{2^{1/2}}{3^{1/2}}\sin\theta, \frac{1}{3^{1/2}}\cos\theta, \frac{1}{3^{1/2}}\sin\theta),$$

and their image is in the orbits $\mathcal{O}_{1,1}^{(\pm 1)^l}$ and $\mathcal{O}_{1,2}$.

In the case of our parametrization of the fiber: $(t, \frac{\beta}{\alpha}t, \frac{\alpha^2\gamma}{t^2}, \frac{\alpha^2\delta}{t^2})$, when we intersect with sphere S^3 we get the equation

$$t^6 - t^4(\frac{\alpha^2}{\alpha^2+\beta^2}) + \alpha^6(\frac{\gamma^2+\delta^2}{\alpha^2+\beta^2}) = 0.$$

If we substitute $z = t^2$ we get a cubic and its discriminant is zero if and only if $27(\alpha^2+\beta^2)^2(\gamma^2+\delta^2) = 4$.

If $\rho = \alpha^2+\beta^2$ and $1-\rho = \gamma^2+\delta^2$, we get $27\rho^2(1-\rho) = 4$ or $\rho^3-\rho^2+\frac{4}{27} = 0$. The roots are $\rho = \frac{2}{3}$(twice) and $\rho = -\frac{1}{3}$, therefore $\alpha^2+\beta^2 = \frac{2}{3}$ and $\gamma^2+\delta^2 = \frac{1}{3}$. Our sextic remains $t^6 - \frac{3\alpha^2}{2}t^4 + \frac{\alpha^6}{2} = 0$, with roots: α(twice), $-\alpha$(twice), $\frac{\alpha i}{\sqrt{2}}$ and $-\frac{\alpha i}{\sqrt{2}}$.

From our previous results we get the following

Theorem 6. *Let $h : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $h(a,b,c,d) = (a^2c, a^2d+2abc, b^2c+2abd, b^2d)$ then:*

i) The fibers of h are of the form

a) $\{(t, \frac{\beta}{\alpha}t, \frac{\alpha^2\gamma}{t^2}, \frac{\alpha^2\delta}{t^2}) \mid t \neq 0\}$, $\alpha \neq 0$

b) $(0, t, \frac{\beta^2\gamma}{t^2}, \frac{\beta^2\delta}{t^2}) \mid t \neq 0\}$, $\alpha = 0$

ii) In the sphere S^3, the set of points of minimal length of each fibre coincides with the set of points where the square of the distance as two repeated roots. This set is the torus ${S^1}_{\sqrt{2/3}} \times {S^1}_{\sqrt{1/3}}$.

■

To conclude, we have our map $h \mid: {S^1}_{\sqrt{2/3}} \times {S^1}_{\sqrt{2/3}} \to \mathbb{R}^4$ defined by

$$h(\theta, \varphi) = (\frac{2}{3}\sqrt{\frac{1}{3}}(\cos^2\theta\cos\varphi, \cos^2\theta\sin\varphi + 2\cos\theta\sin\theta\cos\varphi,$$

$$\sin^2\theta\cos\varphi + 2\cos\theta\sin\theta\sin\varphi, \sin^2\theta\sin\varphi))$$

It is clear that $h(\theta_1, \varphi_1) = h(\theta_2, \varphi_2)$ if only if $\theta_1 = \theta_2$ or $\theta_2 = \theta_1 + \pi$ and $\varphi_1 = \varphi_2$. Hence we have the following diagram

$$\begin{tikzcd} {S^1}_{\sqrt{\frac{2}{3}}} \times {S^1}_{\sqrt{\frac{1}{3}}} \arrow[r, "h"] \arrow[d, "\pi \times Id"'] & \mathbb{R}^4 \\ \mathbb{RP}^1_{\sqrt{\frac{2}{3}}} \times {S^1}_{\sqrt{\frac{1}{3}}} \arrow[ur, "\widetilde{h}"'] & \end{tikzcd}$$

where $\widetilde{h}$ is the unique map defined by $(\frac{2}{3}\sqrt{\frac{1}{3}}(\cos^2\theta/2\cos\varphi, \cos^2\theta/2\sin\varphi + 2\cos\theta/2\sin\theta/2\cos\varphi, \sin^2\theta/2\cos\varphi + 2\cos\theta/2\sin\theta/2\sin\varphi, \sin^2\theta/2\sin\varphi))$.

We have the following

Theorem 7. *Let $\widetilde{h} : S^1 \times S^1 \to \mathbb{R}^4$ defined as above then*

i) The rank of $\widetilde{h}$ is one if and only if $\varphi = \theta/2$ or $\varphi = \theta/2 + \pi$

ii) In our singularities, the kernel is generated by the vector $v_i = (1, -1)$.

iii) The singularities are simple cusps.

Proof. The first two assertions is a simple calculation, and in fact in our fundamental region, our singular curve is

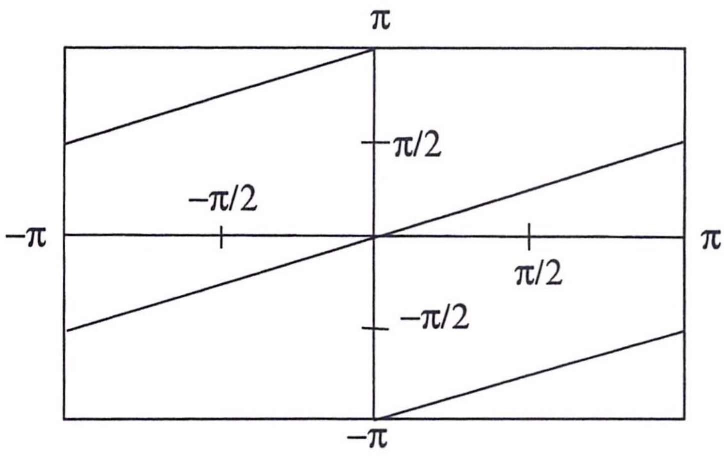

Figure 2: Fundamental region of the torus

For iii) since $v = (1, -1)$ is transversal to our curve we consider the map

$$g(t) = \widetilde{h}((t_0, t_0/2) + t(1, -1)) = \widetilde{h}(t_0 + t, t_0/2 - t)$$

Taking the 3-jet, we can write $g(t)$ as the following matrix times $(t^3, t^2, 0, 1)$

$$\begin{pmatrix} \frac{1}{4}(\sin\frac{t_0}{2}\cos^2\frac{t_0}{2} + \sin^3\frac{t_0}{2}) & -\frac{3}{4}(\sin^2\frac{t_0}{2}\cos\frac{t_0}{2} + \cos^3\frac{t_0}{2}) & & \cos^3\frac{t_0}{2} \\ -\frac{1}{4}(\cos^3\frac{t_0}{2} + \cos\frac{t_0}{2}\sin^2\frac{t_0}{2}) & -\frac{3}{4}(\sin\frac{t_0}{2}\cos^2\frac{t_0}{2} + \sin^3\frac{t_0}{2}) & e_i & 3\sin\frac{t_0}{2}\cos^2\frac{t_0}{2} \\ \frac{1}{4}(\sin^3\frac{t_0}{2} + \cos^2\frac{t_0}{2}\sin\frac{t_0}{2}) & -\frac{3}{4}(\sin^2\frac{t_0}{2}\cos\frac{t_0}{2} + \cos^3\frac{t_0}{2}) & & 3\sin^2\frac{t_0}{2}\cos\frac{t_0}{2} \\ -\frac{1}{4}(\sin\frac{t_0}{2}\cos^2\frac{t_0}{2} + \cos^3\frac{t_0}{2}) & -\frac{3}{4}(\sin\frac{t_0}{2}\cos^2\frac{t_0}{2} + \sin^3\frac{t_0}{2}) & & \sin^3\frac{t_0}{2} \end{pmatrix}$$

Where e_i has 1 in the i^{th}-coordinate and zero in the others. If $\cos\frac{t_0}{2} = 0$ use e_3, if $\sin\frac{t_0}{2} = 0$ use e_2. To conclude in all others cases one can use e_1 or

e_2, since the determinant will be

$$\sin^2\frac{t_0}{2}\cos\frac{t_0}{2} - 3\cos^3\frac{t_0}{2}\text{and}3\sin^2\frac{t_0}{2}\cos\frac{t_0}{2} - \cos^3\frac{t_0}{2}$$

respectively (module a map never zero).

We finish this section with a theorem that joins together the theory developed here.

Theorem 8. *Let $h : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be defined by $h(a,b,c,d) = (a^2c, a^2d + 2abc, b^2c + 2abd, b^2d)$, then h induces a differentiable homeomorphism $h : \mathbb{RP}^1 \times S^1 \rightarrow \mathbb{R}^4$. If we consider the orbits $\mathcal{O}_{1,1}^{(\pm 1)^l}$, $\mathcal{O}_{1,2}$, $\mathcal{O}_{2,1}^{(\pm 1)^l}$, $\mathcal{O}_{2,2}^{(\pm 1)^l}$, $\mathcal{O}_{3,1}^{(\pm 1)^l}$, hence the image of $\widetilde{h}$ is contained in these orbits and the preimage of $\mathcal{O}_{1,2}$ are simple cusps. Locally around a point of the orbit $\mathcal{O}_{1,1}$ we have the next picture*

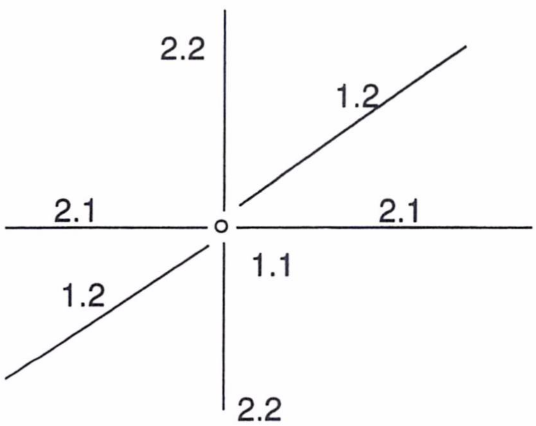

Figure 3: Local behavior around (1.1)

The complement are the orbits $\mathcal{O}_{2,2}^{(\pm 1)^l}$.

Proof. As in a previous theorem we have the following picture

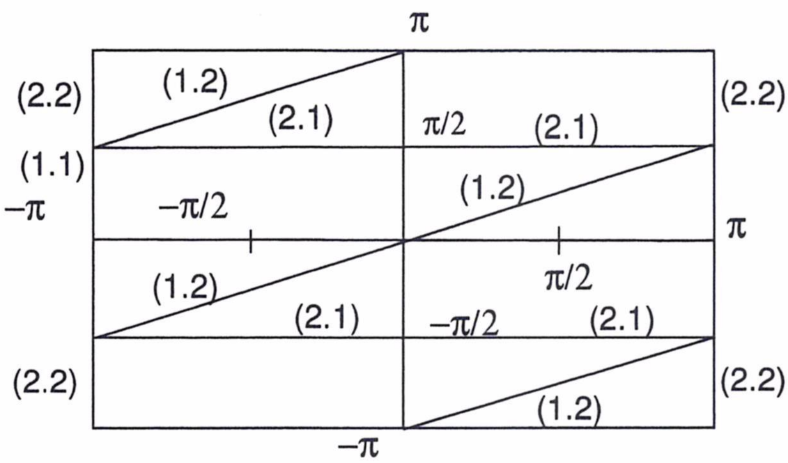

Figure 4: Region fundamental of the torus

and under the identification of the torus we get

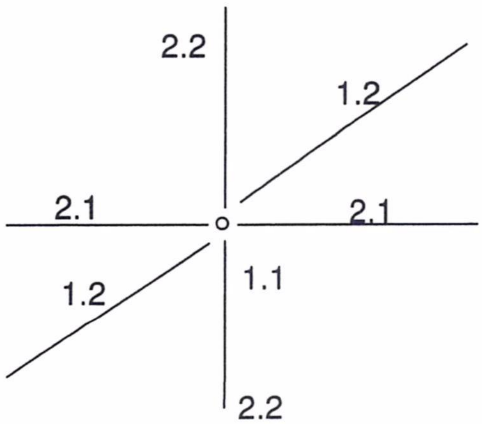

Figure 5: Local behavior around (1.1)

■

The Real Case

Consider $f_\epsilon(x,y) = x^3 + (\epsilon+1)x^2y^l + \epsilon xy^{2l}$ for $l \geq 2$. It was shown that for $l = 2$ f_ϵ is 6-determined on the right, and not less. Now for the general case, we want to show that

$$m(2)^{4l-1} \subseteq m(2)\langle 3x^2+2(\epsilon+1)xy^l+2l\epsilon y^{2l}+\frac{\partial\mu}{\partial x}, l(\epsilon+1)x^2y^{l-1}+2l\epsilon xy^{2l-1}+\frac{\partial\mu}{\partial y}\rangle \quad (1)$$

$\forall \mu \in m(2)^{4l-1}$.

From the equality

$$l(\epsilon+1)y^{l-1}(\frac{\partial(f_\epsilon+\mu)}{\partial x}) - 3(\frac{\partial(f_\epsilon+\mu)}{\partial y}) =$$

$$(\epsilon^2-\epsilon+1)xy^{2l-1} + \epsilon(\epsilon+1)ly^{3l-1} + l(\epsilon+1)y^{l-1}\frac{\partial\mu}{\partial x} - 3\frac{\partial\mu}{\partial y}$$

We get that the monomial $xy^{2l-1}y^{2l-1} = xy^{4l-2}$ is in our jacobian ideal, since $(3l-1)+(2l-1) \geq 4l$. Finally, if we consider

$$axy^{l-1}\frac{\partial f_\epsilon}{\partial x} + by^{2l-1}\frac{\partial f_\epsilon}{\partial x} + cx\frac{\partial f_\epsilon}{\partial y} + dy^l\frac{\partial f_\epsilon}{\partial y}$$

we get a system

$$\begin{cases} 3a + l(\epsilon+1)c = 0 \\ 2(\epsilon+1)a + 3b + 2l\epsilon c + l(\epsilon+1)d = 0 \\ \epsilon a + 2(\epsilon+1)lb + 2l\epsilon d = 0 \end{cases}$$

Which has solution for $b \neq 0$. Therefore y^{4l-1} is also in our jacobian ideal, and we get (1)

If f_ϵ is $(4l-3)$- determined on the right we get an equality of the form $y^{4l-2} = h_1(x,y)(3x^2+2(\epsilon+1)xy^l+\epsilon y^{2l}+\frac{\partial\mu}{\partial x}) + h_2(x,y)(l(\epsilon+1)x^2y^{2l-1} + 2l\epsilon xy^{2l-1}+\frac{\partial\mu}{\partial y})$ with $h_i \in m(2), i = 1,2$.

It is clear that $h_1(x,y) \in m(2)^l$ and if $h_1(x,y) = a_{1,l-1}xy^{l-1} + a_{0,l}y^l$, $h_2(x,y) = b_{1,0}x + b_{0,1}y$, we get

$$b_{1,0} = \frac{-3a_{1,l-1}}{l(\epsilon+1)} \text{ and } b_{0,1} = \frac{-3a_{0,l}}{l(\epsilon+1)}$$

It is clear that the factor of xy^{2l} is $a_{0,l}(\epsilon^2-\epsilon+1)$. Therefore $a_{0,l}=b_{0,1}=0$ and $h_1(x,y)$ has $(l+1)$ as smallest power of y.

We have as smallest power of y in the right hand side: $3l+1$. Now, considering the factors of $x^2y^{l+1},\cdots,x^2y^{2l-2}$, we get

$$3a_{0,l+1}+l(\epsilon+1)b_{0,2}=0,\cdots,3a_{0,2l-2}+l(\epsilon+1)b_{0,l-1}=0$$

equivalently

$$b_{0,2}=-\frac{3}{l(\epsilon+1)}a_{0,l+1},\cdots,b_{0,l-1}=-\frac{3}{l(\epsilon+1)}a_{0,2l-2} \tag{2}$$

The factors of $xy^{2l+1},\cdots,xy^{3l-2}$ are

$$2(\epsilon+1)a_{0,l+1}+2l\epsilon b_{0,2}=0,\cdots,2(\epsilon+1)a_{0,2l-2}+2l\epsilon b_{0,l-1}=0$$

equivalently

$$b_{0,2}=-\frac{\epsilon+1}{l\epsilon}a_{0,l+1},\cdots,b_{0,l-1}=-\frac{\epsilon+1}{l\epsilon}a_{0,2l-2} \tag{3}$$

From 2 and 3 we get

$$a_{0,l+1}=0,\cdots,a_{0,2l-2}=0$$

Therefore the minimum power of y that can appear in the right hand side is $(4l-1)$. Therefore f is $(4l-2)$ determined on the right and no less.

Definition 9. *Let A be an $\mathbb{R}$-algebra and B a vector subspace. If I is an ideal of A, we say that B is an I-module if for a natural number l, we have that*

$$I^lB\subseteq B$$

In this case, we say that B is a finitely generated I-module if there exist $v_1,\cdots,v_m$ in B such that for every vector $v\in B$, we have $v=a_1v_1+\cdots+a_mv_m$, where $a_i\in I^l, 1\leq i\leq m$.

We give a version of Nakayma's Lemma.

Lemma 10. *Let A be an $\mathbb{R}$-algebra and B be a vector subspace. Suppose I is an ideal such that I^{k+1} is contained in the Jacobson radical of A and B be a finitely generated I-module(l=k). If $B \subseteq I^{k+1}B$, then $I^k B = \overline{0}$.*

Proof. Let $\{v_1, v_2, \cdots, v_m\}$ a minimal set of generators of the I-module B. Hence we get

$$v_1 = a_1 v_1 + a_2 v_2 + \cdots + a_m v_m$$

or

$$(1 - a_1)v_1 = a_2 v_2 + \cdots + a_m v_m$$

where $a_i \in I^{k+1}$, therefore $(1-a_1)$ is a unit and we conclude $I^k B = \{\overline{0}\}$. ■

Corollary 11. *let A be an $\mathbb{R}$-algebra, B and B' I-submodules of A. Suppose I is an ideal such that I^{k+1} is contained in the Jacobson radical of A. If $\frac{B+I^kB'}{I^kB'}$ is a finitely generated I-module ($l = k$) and $I^{k+1}B + I^kB' \supseteq I^kB$, then $I^kB' \supseteq I^kB$*

Proof. By hypothesis we have that

$$I^{k+1}(\frac{B + I^kB'}{I^kB'}) = \frac{I^{k+1}B + I^kB'}{I^kB'} \supseteq \frac{B + I^kB'}{I^kB'}$$

Hence by our previous lemma $I^k(\frac{B+I^kB'}{I^kB'}) = 0$ and therefore $I^kB' \supseteq I^kB$. ■

Let G be the group of germs of diffeomorphisms of the form

$$h(x, y) = (\alpha x + \beta y^l + h_1(x, y), \delta y + h_2(x, y))$$

where α, β and δ vary in the real numbers, $h_1 \in m(2)^{l+1}$ and $h_2 \in m(2)^2$.

Definition 12. *We say that a germ f is k-determined relative to G if given g a germ such that $j^k g(0) = j^k f(0)$, there exists a diffeomorphisms h in G such that $g = f \circ h$.*

We shall prove the following theorems

Theorem 13 (A). *A germ f is $(k+l)$-determined relative to G if*

$$m(2)^{k+1} \subseteq m(2)\{\frac{\partial f}{\partial x}(\alpha x + \beta y^l + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\} + m(2)^{k+2}$$

where α, β and δ vary over the real numbers, $h_1 \in m(2)^{l+1}$ and $h_2 \in m(2)^2$.

Theorem 14 (B). *If a germ f is k-determined relative to G, hence we get*

$$m(2)^{k+1} \subseteq \{\frac{\partial f}{\partial x}(\alpha x + \beta y^l + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\} + m(2)^{k+2}$$

where α, β and δ vary over the real numbers, $h_1 \in m(2)^{l+1}$ and $h_2 \in m(2)^2$.

Remark 15. *It is clear that the vector subspace $B' = \{\frac{\partial f}{\partial x}(\alpha x + \beta y^l + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\}$ and $B = m(2)^{k+1}$ are $I = m(2)$-modules for l, therefore our corollary implies that $m(2)^{k+l+1} \subseteq m(2)\{\frac{\partial f}{\partial x}(\alpha x + \beta y^l + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\}$*

From theorems A and B we get the following

Corollary 16. *The germ $f_\epsilon(x, y) = x^3 + (\epsilon + 1)x^2y^l + \epsilon xy^{2l}$ is $(5l-1)$-determined relative to G.*

Proof. We shall prove the inclusion $m(2)^{4l} \subseteq m(2)\{(3x^2 + 2(\epsilon+1)xy^l + \epsilon y^{2l})(\alpha x + \beta y^l + h_1) + (l(\epsilon+1)x^2y^{l-1} + 2l\epsilon xy^{2l-1})(\delta y + h_2)\} + m(2)^{4l+1}$ for $\epsilon \neq -1$

The inclusion is clear for monomials factorizing x^3 or x^2y^l. We have the equality $l(\epsilon+1)y^{l-1}(3x^2+2(\epsilon+1)xy^l+\epsilon y^{2l}) - (3(l(\epsilon+1)x^2y^{l-1}+2l\epsilon xy^{2l-1}) = 2l(\epsilon^2 - \epsilon + 1)xy^{2l-1} + l(\epsilon+1)y^{3l-1}$, hence we get $xy^{2l-1}y^{l+2} = xy^{3l+1}$ in our RHS. Finally, consider as before, with appropriate constants

$$axy^{l-1}\frac{\partial f_\epsilon}{\partial x} + by^{2l-1}\frac{\partial f_\epsilon}{\partial x} + cx\frac{\partial f_\epsilon}{\partial y} + dy^l\frac{\partial f_\epsilon}{\partial y},$$

which gives y^{4l-1} and therefore we get y^{4l}. Hence by Theorem A f_ϵ is $(5l-1)$-determined relative to G. For the case $\epsilon = -1$ the proof runs similar. ∎

Remark 17. *Consider the germ of the curve in G*

$$\gamma_{(x,y)}(t) = (x + t(\alpha x + \beta y^l + h_1), y + t(\delta y + h_2)) \tag{4}$$

with $(1+\alpha t)(1+\delta t) \neq 0$. *If* $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ *is a smooth function with* $\overline{0} \in U$, *then*

$$\frac{d}{dt}(f \circ \gamma_{(x,y)}(t))(0) = \frac{\partial f}{\partial x}(x,y)(\alpha x + \beta y^l + h_1) + \frac{\partial f}{\partial y}(x,y)(\delta y + h_2)$$

Proposition 18. *Let* $\pi_n : m(2) \to \frac{m(2)}{m(2)^{n+1}}$ *be the canonical projection,* $\mathcal{O}$ *be the orbit of* z*, the* k*-jet of* f*. Then we have the equality*

$$T_z(\pi_n(\mathcal{O}) = \pi(\{\frac{\partial f}{\partial x}(x,y)(\alpha x + \beta y^l + h_1) + \frac{\partial f}{\partial y}(x,y)(\delta y + h_2))\}$$

where α, β *and* δ *range over the real numbers,* $h_1 \in m(2)^{l+1}$ *and* $h_2 \in m(2)^2$.

We suppose the above proposition and prove Theorem B (14). Let z the k-jet of f. Since f is k-determined relative to G, we have that $z + m(2)^{k+1} \subseteq \mathcal{O}$, where $\mathcal{O}$ is the G-orbit of z. Then $\pi_{k+1}(z + m(2)^{k+1}) \subseteq \pi_{k+1}(\mathcal{O})$. Taking the tangent spaces at z, from the above proposition, we get

$$m(2)^{k+1} \subseteq \{\frac{\partial f}{\partial x}(\alpha x + \beta y^l + h_1) + \frac{\partial f}{\partial y}(\delta y + h_2)\} + m(2)^{k+2}$$

The previous proposition follows easily from the next theorem. Consider G a Lie Group acting smoothly on a manifold M. Denote by $x_0 G$ and $est(x_0)$, the orbit and the stabilizer of x_0 respectively and let $\pi : G \to G/est(x_0)$ the canonical projection.

Theorem 19. *Let* $c : (-\epsilon, \epsilon) \to x_0 G$ *a differentiable curve with* $c(0) = x_0$ *then there exists a differentiable curve* $\mathrm{C} : (-\delta, \delta) \to G$ *with* $\mathrm{C}(0) = Id$ *and* $c'(0) = \frac{d}{dt}(x_0 \mathrm{C}(t))(0)$.

Proof. We have the maps $\rho : G \to x_0 G$ and $\pi : G \to G/est(x_0)$ and the unique application $\overline{\rho} : G/est(x_0) \to x_0 G$ such that π is a submersion, $\overline{\rho}$ an one to one immersion and $\rho = \overline{\rho} \circ \pi$. Hence ρ has constant rank at each of its components. In particular there exists U and V open neighborhoods of the identity in G and x_0 in x_0G, charts $\phi : U \to U_1 \times V_1$, $\psi : V \to U_1 \times V_2$, such that $\psi \circ \rho \circ \phi^{-1}(x, y) = (x, 0)$. Consider the curve $C(t) = \phi \circ i \circ \psi \circ c(t)$, where $i : U_1 \times V_2 \to U_1 \times V_2$ is defined by $i(x, z) = (x, 0)$. It is clear that $\psi \circ \rho \circ C(t) = (\psi_1(c(t)), \overline{0})$. On the other side, $\psi_2(c(t)) = 0$ for $t \in (-\delta, \delta)$, δ small. Hence $(\psi_1, \psi_2)(c(t)) = (\psi_1, \psi_2) \circ \rho \circ C(t)$, therefore $c(t) = \rho \circ C(t)$ for $t \in (-\delta, \delta)$. ■

Proof of Theorem A (13). Let g be a germ such that $j^{k+l}g(0) = j^{k+l}f(0)$and define the homotopy $\phi(t, (x, y)) = tg(x, y) + (1 - t)f(x, y)$. Fix $t_0 \in [0, 1]$, we want a homotopy of germs of diffeomorphisms τ^t in G (for t near t_0) such that:

a) $\tau^{t_0} = Id$,

b) $\tau^t(\overline{0}) = \overline{0}$,

c) $\phi^t \circ \tau^t = \phi^{t_0}$.

This assertion will be proved by a sequence of lemmas and using that $[0, 1]$ is compact and connected we finish the proof of Theorem A.

Lemma 20. *There exists a smooth germ* $\tau : (\mathbb{R} \times \mathbb{R}^2, (t_0, \overline{0})) \to \mathbb{R}^2$ *such that*

a) $\tau(t_0, (x, y) = (x, y)$,

b) $\tau(t, \overline{0}) = \overline{0}$,

c) $\phi(t, \tau(t, (x, y))) = \phi(t_0, \tau(t_0, (x, y)))$.

This lemma is equivalent to the following

Lemma 21. *There exist a smooth germ* $\tau : (\mathbb{R} \times \mathbb{R}^2, (t_0, \overline{0})) \to \mathbb{R}^2$ *such that*

a) $\tau(t_0, (x, y) = (x, y)$,

b) $\tau(t, \overline{0}) = \overline{0}$, c') $\frac{\partial\phi}{\partial x}(t,\tau(t,(x,y)))\frac{\partial\tau_1^t}{\partial t}(t,(x,y))+\frac{\partial\phi}{\partial y}(t,\tau(t,(x,y)))\frac{\partial\tau_2^t}{\partial t}(t,(x,y))+\frac{\partial\phi}{\partial t}(t,\tau(t,(x,y)))=0.$

We denote by A the algebra of germs of maps in $(t_0,\overline{0})$ and $\pi:\mathbb{R}^3\to\mathbb{R}^2$ the projection $\pi(t,(x,y))=(x,y)$. Consider the monomorphism $\pi^*:\epsilon_2\to A$

Lemma 22. *If f is a germ such that*

$$m(2)^{k+1}\subseteq\{\frac{\partial f}{\partial x}(\alpha x+\beta y^l+h_1)+\frac{\partial f}{\partial y}(\delta y+h_2)\}+m(2)^{k+2}$$

where α,β and δ range over the real number, $h_1\in m(2)^{l+2}$ and $h_2\in m(2)^2$, hence we also have

$$m(2)^{k+1}A\subseteq m(2)\{\frac{\partial\phi}{\partial x}(\alpha x+\beta y^l+h_1)+\frac{\partial\phi}{\partial y}(\delta y+h_2)\}+m(2)^{k+2}$$

and therefore $m(2)^{k+l+1}A\subseteq\{\frac{\partial\phi}{\partial x}(\alpha x+\beta y^l+h_1)+\frac{\partial\phi}{\partial y}\delta y+h_2)\}$

If this lemma is true, since $\frac{\partial\phi}{\partial t}\in g-f\in m(2)^{k+l+1}$

$$\frac{\partial\phi}{\partial t}=(g_1(x,y)(\alpha x+\beta y^l+h_1))\frac{\partial\phi}{\partial x}+(g_1(x,y)(\delta y+h_2))\frac{\partial\phi}{\partial y}\tag{5}$$

Define $\psi_1(x,y)=g_1(x,y)(\alpha x+\beta y^l+h_1)$, $\psi_2(x,y)=(g_1(x,y)(\delta y+h_2))$. Let us consider the differential equation

$$\frac{\partial\tau}{\partial t}(t,(x,y))=-(\psi_1,\psi_2)\circ\tau(t,(x,y))\tag{6}$$

with initial condition $\tau(0,(x,y))=(x,y)$. Since $\psi_1(\overline{0})=\psi_2(\overline{0})=0$, it is clear that $\tau(t,\overline{0})=\overline{0}$, and our equation 5 can be written as $\frac{\partial\phi}{\partial t}=\psi_1\frac{\partial\phi}{\partial x}+\psi_2\frac{\partial\phi}{\partial y}$, equivalently

$$\frac{\partial\phi}{\partial t}\circ\tau^t+\frac{\partial\tau_1^t}{\partial t}(\frac{\partial\phi}{\partial x}\circ\tau^t)+\frac{\partial\tau_2^t}{\partial t}(\frac{\partial\phi}{\partial y}\circ\tau^t)=0$$

■

Proof of the previous lemma: Consider the homotopy

$$\phi(t,(x,y))=(1-t)f(x,y)+tg(x,y),$$

where $j^{k+l}f(\overline{0}) = j^{k+l}g(\overline{0})$. We have that

$\frac{\partial\phi}{\partial x} = \frac{\partial f}{\partial x} + t\frac{\partial(g-t)}{\partial x}$ and $\frac{\partial\phi}{\partial y} = \frac{\partial f}{\partial y} + t\frac{\partial(g-f)}{\partial y}$

therefore

$$(\alpha x + \beta y^l + h_1)\frac{\partial f}{\partial x} \in \{(\alpha x + \beta y^l + h_1)\frac{\partial\phi}{\partial x})\} + A\,m(2)^{k+l+1}$$
$$(\delta y + h_2)\frac{\partial f}{\partial y} \in \{(\delta y + h_2)\frac{\partial\phi}{\partial y}\} + A\,m(2)^{k+l+1}.$$

Hence

$$m(2)^{k+1}A \subseteq m(2)\{(\alpha x + \beta y^l + h_1)\frac{\partial\phi}{\partial x} + (\delta y + h_2)\frac{\partial\phi}{\partial y}\} + m(2)^{k+l+1}$$
$$\subseteq m(2)\{(\alpha x + \beta y^l + h_1)\frac{\partial\phi}{\partial x} + (\delta y + h_2)\frac{\partial\phi}{\partial y}\} + m(2)^{k+2}$$

■

Remark 23. *Our differential equation 6 is the form*

$$\frac{\partial\tau}{\partial t}(t,(x,y)) = (-\psi_1, -\psi_2) \circ \tau(t,(x,y))$$

with $\psi_1, \psi_2 \in m(2)^{l+2}$, therefore $j^{l+1}\tau^t(x,y) = (x,y)$

The Complex Case

Consider $g_\epsilon(x,y) = x^3 + \epsilon x^2 y^l + xy^{2l} + \epsilon y^{3l}$. We want to find the minimum natural number k, such that for all $\mu \in m(2)^{k+1}$

$m(2)^{k+1} \subseteq m(2)\langle 3x^2 + 2\epsilon xy^l + y^{2l} + \frac{\partial\mu}{\partial x}, l\epsilon x^2 y^{l-1} + 2l\epsilon xy^{2l-1} + 3l\epsilon y^{3l-1} + \frac{\partial\mu}{\partial y}\rangle$

We shall see that $k+1 = 4l$.

It is clear for the monomials factorizing x^2. Now as in the real case we have

$$l\epsilon y^{l-1}(3x^2 + 2\epsilon xy^l + y^{2l} + \frac{\partial\mu}{\partial x}) - 3(l\epsilon x^2 y^{l-1} + 2l\epsilon xy^{2l-1} + 3l\epsilon y^{3l-1} + \frac{\partial\mu}{\partial y}) =$$
$$2l(\epsilon^2 - 3)xy^{2l-1} + (-8l\epsilon)y^{3l-1} + l\epsilon y^{l-1}\frac{\partial\mu}{\partial x} - 3\frac{\partial\mu}{\partial y}$$

a) If $\epsilon^2 \neq 3$, we get $xy^{2l-1}y^s$, $s \geq 2$ and $3l-1+s \geq k+2$,

b) If $\epsilon^2 = 3$, we get $(-8l\epsilon)y^{3l-1}y^s = (-8l\epsilon)y^{3l+s-1}$, for $s \geq 2$, and $(-8l\epsilon)y^{3l-1}xy^{s-1}$, $s \geq 2$.

c) As before (for $\epsilon^2 \neq 3$) we consider

$$axy^{l-1}\frac{\partial g_\epsilon}{\partial x} + by^{2l-1}\frac{\partial g_\epsilon}{\partial x} + cx\frac{\partial g_\epsilon}{\partial y} + dy^l\frac{\partial g_\epsilon}{\partial y} =$$

$(3a+cl\epsilon)x^3y^{l-1} + (2a\epsilon + 3b + 2cl\epsilon + 2ld\epsilon)x^2y^{2l-1} + (a + 2b\epsilon + 3cl\epsilon + 2dl)xy^{3l-1} + (b + 3dl\epsilon)y^{4l-1} + axy^{l-1}\frac{\partial\mu}{\partial x} + by^{2l-1}\frac{\partial\mu}{\partial x} + cx\frac{\partial\mu}{\partial y} + dy^l\frac{\partial\mu}{\partial y}$

It is clear that we can vanish the first three summands with $b+3dl\epsilon \neq 0$. Therefore for $\epsilon^2 \neq 3$ we get

$m(2)^{4l-1} \subseteq m(2)\langle\frac{\partial(g_\epsilon+\mu)}{\partial x}, \frac{\partial(g_\epsilon+\mu)}{\partial x}\rangle + m(2)^{4l}$, $\forall\mu \in m(2)^{4l-1}$ and therefore g_ϵ is $(4l-2)$-determined on the right, and not $(4l-3)$-determined on the right, for if $y^{4l-1} = h_1(x,y)(3x^2 + 2\epsilon xy^l + y^{2l}) + \frac{\partial\mu}{\partial x} + h_2(x,y)(l\epsilon x^2y^{l-1} + 2lxy^{2l-1} + 3l\epsilon y^{3l-1} + \frac{\partial\mu}{\partial y})$

As in the real case we start with $h_1 \in m(2)^l$, $h_2 \in m(2)$ and in the notation used there

$$b_{1,0} = -\frac{3a_{1,l-1}}{l\epsilon} \text{ and } b_{0,l-1} = -\frac{3a_{0,2l-2}}{l\epsilon}$$

The factor of xy^{2l} is zero. Therefore $2\epsilon a_{0,l} + 2lb_{0,1} = 0$. Since $\epsilon^2 \neq 3$ we get $a_{0,l} = 0 = b_{0,1}$

We continue with the equations

$$b_{0,2} = -\frac{3}{l\epsilon}a_{0,l+1}, \cdots, b_{0,l-1} = -\frac{3}{l\epsilon}a_{0,2l-2} \tag{7}$$

Now, the factor of xy^{3l-2} is

$$2\epsilon a_{0,2l-2} + 2lb_{0,l-1} = 0$$

and $h_1(x,y)$ has $(l+1)$ as smallest power of y. Again, since $\epsilon^2 \neq 3$ we get $a_{0,2l-2} = 0$ and $b_{0,l-1} = 0$. Therefore the minimal power of y that may appear non-trivially is $4l-1$.

For $\epsilon^2 = 3$, ~~it can be~~ prove that we shall
$m(2)^{3l+1} \subseteq m(2)\langle \frac{\partial(g_\epsilon+\mu)}{\partial x}, \frac{\partial(g_\epsilon+\mu)}{\partial x}\rangle$, $\forall \mu \in m(2)^{3l+1}$
(It is clear from b) that xy^{3l} and y^{3l+1} don't have any problem.)

Now let $\mu = cy^{3l}$, c to be found, and suppose that

$$y^{3l} = h_1(x,y)(3x^2 + 2\epsilon xy^l + y^{2l}) + h_2(x,y)(l\epsilon x^2 y^{l-1} + 2lxy^{2l-1} + 3l(c+\epsilon)y^{3l-1}) \tag{8}$$

From previous remarks, $h_1 \in m(2)^l$, $h_2 \in m(2)$ and

$$b_{1,0} = -\frac{3}{l\epsilon}a_{1,l-1}, \cdots, b_{0,l} = -\frac{3}{l\epsilon}a_{0,l} \tag{9}$$

The coefficient of y^{3l} in the RHS of 8 is

$$a_{0,l} + 3l(c+\epsilon)b_{0,1} = a_{0,l}\left(1 - \frac{9(c+\epsilon)}{\epsilon}\right)$$

Choose c with $\frac{9(c+\epsilon)}{\epsilon} = 1$ and we get a contradiction. Therefore we resume the complex case in the following

Theorem 24. *Let $g_\epsilon(x,y) = x^3 + \epsilon x^2 y^l + xy^{2l} + \epsilon y^{3l}$, then we have*

i) If $\epsilon^2 \neq 3$, g_ϵ is exactly $(4l-2)$-determined on the right.

ii) If $\epsilon^2 = 3$, g_ϵ is exactly $3l$-determined on the right.

Remark 25. *For $l = 2$, both numbers coincide.*

We now see g_ϵ under the action of G.

Proposition 26. *The germ $g_\epsilon(x,y) = x^3 + \epsilon x^2 y^l + xy^{2l} + \epsilon y^{3l}$ is $(5l-1)$-determined relative to G if $\epsilon^2 \neq 3$ and $(4l)$-determined if $\epsilon^2 = 3$.*

Proof. We shall prove for $\epsilon^2 \neq 3$ that

$m(2)^{4l} \subseteq m(2)\{(3x^2+2\epsilon xy^l + y^{2l})(\alpha x + \beta y^l + h_1) + (l\epsilon x^2y^{l-1}+2lxy^{2l-1} + 3l\epsilon y^{3l-1})(\delta y+h_2)\}+m(2)^{4l+1}$ The inclusion is clear for monomials factorizing x^3 or x^2y^l. From the equality

$l\epsilon y^{l-1}(3x^2+2\epsilon xy^l+y^{2l})-3(l\epsilon x^2y^{l-1})+2lxy^{2l-1}+3l\epsilon y^{3l-1}) = 2l(\epsilon^2-3)xy^{2l}+ (-8l\epsilon)y^{3l-1}$

As before we get

a) If $\epsilon^2 \neq 3$, we get $xy^{2l-1}y^s$, $s \geq 2$ and $s+3l-1 \geq 4l \Leftrightarrow s > l+1$,

b) If $\epsilon^2 = 3$, we get xyy^{3l-1} and y^2y^{3l-1}.

c) In a very similar way we conclude that

i) $m(2)^{4l} \subseteq m(2)\langle\frac{\partial g_\epsilon}{\partial x}, \frac{\partial g_\epsilon}{\partial x}\rangle + m(2)^{4l+1}$, $\epsilon^2 \neq 3$

ii) $m(2)^{3l+1} \subseteq m(2)\langle\frac{\partial g_\epsilon}{\partial x}, \frac{\partial g_\epsilon}{\partial x}\rangle + m(2)^{3l+2}$, $\epsilon^2 = 3$

Therefore from Theorem A we get

i) If $\epsilon^2 \neq 3$, g_ϵ is $5l-1$-determined relative to G.

ii) If $\epsilon^2 = 3$ g_ϵ is $(4l)$-determined relative to G.

<u>Another Version of Theorem A</u>

Theorem 27. *Let f be a germ, suppose that for all μ germ in $m(2)^{k+1}$, we have that*

$$m(2)^{k+1} \subseteq \{\frac{\partial(f+\mu)}{\partial x}(\alpha x + \beta y^l + h_1) + \frac{\partial(f+\mu)}{\partial y}(\delta y + h_2)\}$$

where α, β and δ range in the real numbers, $h_1 \in m(2)^{l+1}$ and $h_2 \in m(2)^2$. Hence f is k-determined relative to G.

Proof. Consider a germ g with $j^kg(0) = j^kf(0)$ and $\phi^t = f + t(g-f)$. Therefore $\frac{\partial \phi}{\partial t} = g - f$ is an element in $m(2)^{k+1}$, so we can write

$$\frac{\partial \phi^t}{\partial t}(x,y) = \frac{\partial \phi^t}{\partial x}(\alpha x + \beta y^l + h_1) + \frac{\partial \phi^t}{\partial y}(\delta y + h_2) \tag{10}$$

If we denote by $\psi_1(x,y) = \alpha x + \beta y^l + h_1$ and $\psi_2(x,y) = \delta y + h_2$. Consider as before the differential equation

$$\frac{\partial \tau}{\partial t} = (-\psi_1, -\psi_2) \circ \tau(t,(x,y))$$

with initial condition $\tau(0,(x,y)) = (x,y)$. Since the germs ψ_1 and ψ_2 belong to $m(2)$ we get $\tau^t(t,\overline{0}) = \overline{0}$, for all t.

The equation 10 is equivalent to

$$\frac{\partial \phi^t}{\partial t}(t,\tau(t,(x,y))) + \frac{\partial \tau_1^t}{\partial t}(x,y)\frac{\partial \phi^t}{\partial x}(x,\tau(t,(x,y))) + \frac{\partial \tau_2^t}{\partial t}(x,y)\frac{\partial \phi^t}{\partial y}(x,\tau(t,(x,y))) = 0.$$

We finish as in our proof of Theorem A. ■

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