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Existence Results for a Impulsive Abstract Partial Differential Equation with State-Dependent Delay

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Resultados de Existência de Soluções para uma Equação Abstrata do Tipo Impulsiva com Retardamento Dependendo do Estado.

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Resumo

Neste artigo estudamos a existência de soluções fracas para uma classe de equações funcionais impulsivas com retardo dependendo do estado descritas na forma

$$\begin{aligned}
x'(t) &= Ax(t) + f(t, x_{\rho(t,x_t)}), \ \ t \in I = [0, a], \\
x_0 &= \varphi \in \mathcal{B}, \\
\triangle x(t_i) &= I_i(x_{t_i}), \ \ i = 1, \cdots, n,
\end{aligned}$$

onde A é o gerador infinitesimal de um C_0-semigrupo de operadores lineares limitados definido sobre um espaço de Banach X; $\mathcal{B}$ é um espaço de fase definido de maneira axiomática; $f : [0, a] \times \mathcal{B} \to X$, $\rho : \mathbb{R} \times \mathcal{B} \to \mathbb{R}$ são funções apropriadas; $0 < t_1 < .... < t_n < a$ são números pré-fixados e o símbolo $\triangle\xi(t)$ representa o impulso da função ξ em t, definido por $\triangle\xi(t) = \xi(t^+) - \xi(t^-)$.

Existence Results for a Impulsive Abstract Partial Differential Equation with State-Dependent Delay.

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Abstract

In this paper we establish the existence of mild solutions for a class of impulsive abstract partial functional differential equation with state-dependent delay.

1 Introduction

In this paper we establish the existence of mild solutions for an impulsive abstract functional differential equation with state-dependent delay described in the form

$$x'(t) = Ax(t) + f(t, x_{\rho(t,x_t)}), \quad t \in I = [0, a], \tag{1}$$
$$x_0 = \varphi \in \mathcal{B}, \tag{2}$$
$$\triangle x(t_i) = I_i(x_{t_i}), \quad i = 1, \cdots, n, \tag{3}$$

where A is the infinitesimal generator of a compact C_0-semigroup of bounded linear operators $(T(t))_{t\geq 0}$ defined on a Banach space X; the functions $x_s : (-\infty, 0] \to X$, $x_s(\theta) = x(s + \theta)$, belongs to some abstract phase space $\mathcal{B}$ described axiomatically; $f : I \times \mathcal{B} \to X$, $\rho : I \times \mathcal{B} \to (-\infty, a]$ are appropriate functions; $0 < t_1 < .... < t_n < a$ are pre-fixed numbers and the symbol $\triangle\xi(t)$ represents the jump of the function ξ at t, which is defined by $\triangle\xi(t) = \xi(t^+) - \xi(t^-)$.

The theory of impulsive differential equations has become an important area of investigation in recent years stimulated by their numerous applications to problems arising in mechanics, electrical engineering, medicine, biology, ecology, etc. Relative to ordinary impulsive differential equations, we cite among another works [15, 6, 12, 17, 22]. First order abstract partial differential equations with impulses are treated in Lakshmikantham [14], Liu [16] and Rogovchenko [20, 21].

The literature related to ordinary and partial functional differential equations with delay for which $\rho(t, \psi) = t$ is very extensive and we refer the reader to Hale & Lunel [8] and Wu [24] concerning this matter.

Functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received great attention in the last years, see for instance [2, 1, 3, 4, 5, 10, 11, 25, 23] and the references therein. The study of partial functional differential equations with state-dependent delay is a untreated topic and it is the motivation of our paper.

2 Preliminaries

Throughout this paper, $A : D(A) \subset X \to X$ is the infinitesimal generator of a compact semigroup of linear operators $(T(t))_{t\geq 0}$ defined on a Banach spaces X and $\widetilde{M}$ is a constant such that $\| T(t) \|\leq \widetilde{M}$ for every $t \in I = [0, a]$. Related semigroup theory, we suggest the Pazy book [19].

To consider the impulsive condition (3), it is convenient to introduce some additional concepts and notations. We say that a function $u : [\sigma, \tau] \to X$ is a normalized piecewise continuous function on $[\sigma, \tau]$ if u is piecewise continuous and left continuous on $(\sigma, \tau]$. We denote by $\mathcal{PC}([\sigma, \tau]; X)$ the space formed by the normalized piecewise continuous functions from $[\sigma, \tau]$ into X. In particular, we introduce the space $\mathcal{PC}$ formed by all functions $u : [0, a] \to X$ such that u is continuous at $t \neq t_i$, $u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exists, for all $i = 1, \cdots, n$. It is clear that $\mathcal{PC}$ endowed with the norm of the uniform convergence is a Banach space.

To simplify the notations, following we put $t_0 = 0$, $t_{n+1} = a$ and for $u \in \mathcal{PC}$ we denote by $\tilde{u}_i \in C([t_i, t_{i+1}]; X)$, $i = 0, 1, \cdots, n$, the function given by

$$\widetilde{u}_i(t) = \begin{cases} u(t), & \text{for } t \in (t_i, t_{i+1}], \\ u(t_i^+), & \text{for } t = t_i. \end{cases}$$

Moreover, for $B \subseteq \mathcal{PC}$ we denote by $\widetilde{B}_i$, $i = 0, 1, ...n$, the set $\widetilde{B}_i = \{\tilde{u}_i : u \in B\}$.

Lemma 1 *A set $B \subseteq \mathcal{PC}$ is relatively compact in $\mathcal{PC}$ if and only if each set $\widetilde{B}_i$ is relatively compact in $C([t_i, t_{i+1}]; X)$.*

In this work we will employ an axiomatic definition of the phase space $\mathcal{B}$ which is similar to that used in [13]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\| \cdot \|_{\mathcal{B}}$ and we will assume that $\mathcal{B}$ satisfies the following axioms:

(A) If $x : (-\infty, \sigma + b] \to X$, $b > 0$, is such that $x_\sigma \in \mathcal{B}$ and $x|_{[\sigma,\sigma+b]} \in \mathcal{PC}([\sigma, \sigma + b] : X)$, then for every $t \in [\sigma, \sigma + b)$ the following conditions hold:

(i) x_t is in $\mathcal{B}$,

(ii) $\| x(t) \|\leq H \| x_t \|_{\mathcal{B}}$,

(iii) $\| x_t \|_{\mathcal{B}}\leq K(t - \sigma) \sup\{\| x(s) \|: \sigma \leq s \leq t\} + M(t - \sigma) \| x_\sigma \|_{\mathcal{B}}$,

where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(B) The space $\mathcal{B}$ is complete.

Example: The phase space $\mathcal{PC}_h(X)$.

As usual, we said that $\varphi : (-\infty, 0] \to X$ is normalized piecewise continuous, if φ is left continuous and the restriction of φ to any interval $[-r, 0]$ is piecewise continuous.

Let $h : (-\infty, 0] \to [1, \infty)$, with $h(0) = 1$, be a continuous and nondecreasing function which satisfies the conditions (g-1), (g-2) in the terminology of [13]. Next we modify slightly the definition of the spaces C_h, C_h^0 in [13]. We denote by $\mathcal{PC}_h(X)$ the space formed by the normalized piecewise continuous functions φ such that $\frac{\varphi}{h}$ is bounded on $(-\infty, 0]$ and by $\mathcal{PC}_h^0(X)$ the subspace of $\mathcal{PC}_h(X)$ formed by the functions φ such that $\frac{\varphi(\theta)}{h(\theta)} \to 0$ as $\theta \to -\infty$. It is easy to see that $\mathcal{PC}_h(X)$ and $\mathcal{PC}_h^0(X)$ endowed with the norm $\| \varphi \|_{\mathcal{B}} := \sup_{\theta\in(-\infty,0]} \frac{\|\varphi(\theta)\|}{h(\theta)}$, are phase spaces in the sense considered in this work. Moreover, in these cases $K(s) \equiv 1$ for $s \geq 0$.

Example: The phase space $\mathcal{PC}_r \times L^2(h, X)$.

Let $h(\cdot) : (-\infty, -r] \to \mathbb{R}$ be a positive Lebesgue integrable function and $\mathcal{B} := \mathcal{PC}_r \times L^2(h; X)$, $r \geq 0$, be the space formed of all classes of functions $\varphi : (-\infty, 0] \to X$ such that $\varphi \mid_{[-r,0]} \in \mathcal{PC}([-r, 0], X)$, $\varphi(\cdot)$ is Lebesgue-measurable on $(-\infty, -r]$ and $h \mid \varphi \mid^p$ is Lebesgue integrable on $(-\infty, -r]$. The seminorm in $\| \cdot \|_{\mathcal{B}}$ is defined by

$$\| \varphi \|_{\mathcal{B}} := \sup_{\theta\in[-r,0]} \| \varphi(\theta) \| + \left(\int_{-\infty}^{-r} h(\theta) \| \varphi(\theta) \|^p \, d\theta \right)^{1/p}.$$

Assume that $h(\cdot)$ satisfies conditions (g-6), (g-7) in the terminology of [13]. Proceeding as in the proof of [13, Theorem 1.3.8] it follows that $\mathcal{B}$ is a phase space which verifies the axioms (**A**) and (**B**). Moreover, when $r = 0$ this space coincides with $C_0 \times L^2(h, X)$, $H = 1$; $M(t) = \gamma(-t)^{\frac{1}{2}}$ and $K(t) = 1 + \left(\int_{-t}^{0} h(\tau) d\tau \right)^{\frac{1}{2}}$, for $t \geq 0$ (see [13]).

Remark 1 *In retarded functional differential equations without impulses, the axioms of the abstract phase space $\mathcal{B}$ include the continuity of the function $t \to x_t$, see [13, 9] for details. Devoted to the impulsive effect, this property is unverified in impulsive delay system and for this reason have been eliminated in our abstract description of $\mathcal{B}$.*

Remark 2 *Let $\varphi \in \mathcal{B}$ and $t \leq 0$. The notation φ_t represents the function defined by $\varphi_t(\theta) = \varphi(t + \theta)$. Consequently, if the function $x(\cdot)$ in axiom **A** is such that $x_0 = \varphi$, then $x_t = \varphi_t$. We observe that φ_t is well defined for $t < 0$ since the domain of φ is $(-\infty, 0]$. We also note that in general $\varphi_t \notin \mathcal{B}$, consider for example the characteristic function $\mathcal{X}_{[-r,0]}$, $r > 0$, in the space $\mathcal{PC}_\mathbf{r} \times \mathbf{L^p(g; X)}$.*

Additional terminologies and notations are usual in functional analysis. In particular, for Banach spaces $(Z, \| \cdot \|_Z)$, $(W, \| \cdot \|_W)$, the notation $\mathcal{L}(Z, W)$ stands for the Banach space of bounded linear operators from Z into W. Moreover, $B_r(x, Z)$ denotes the closed ball with center at x and radius $r > 0$ in Z. For a bounded function $\xi : J \subset \mathbb{R} \to [0, \infty)$ and $t \in J$ we employ the notation ξ_t for $\xi_t = \sup\{\xi(s) : s \in J, s \leq t\}$.

The paper has four sections. In section 3 we establish the existence of mild solutions for system (1)-(3). Section 4 is reserved for examples.

To conclude this section, we recall the following well-known result for convenience.

Theorem 1 *[7, Theorem 6.5.4] Let D be a closed convex subset of a Banach space Z and assume that $0 \in D$. Let $G : D \to D$ be a completely continuous map. Then, either the map G has a fixed point in D or $\{z \in D : z = \lambda G(z),\ 0 < \lambda < 1\}$ is unbounded.*

3 Existence Results

In this section we establish the existence of mild solutions for the impulsive abstract Cauchy problem (1)-(3). To prove our results we always assume that $\rho : I \times \mathcal{B} \to (-\infty, a]$ is continuous and that φ and f verifies the next conditions.

$\mathbf{H}_\varphi$ Let $\mathcal{R}(\rho^-) = \{\rho(s,\psi) : (s,\psi) \in I \times \mathcal{B},\ \rho(s,\psi) \leq 0\}$. The function $t \to \varphi_t$ is well defined from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $J^\varphi : \mathcal{R}(\rho^-) \to \mathbb{R}$ such that $\| \varphi_t \|_{\mathcal{B}} \leq J^\varphi(t) \| \varphi \|_{\mathcal{B}}$ for every $t \in \mathcal{R}(\rho)$.

$\mathbf{H}_1$ The function $f : I \times \mathcal{B} \to X$ satisfies the following conditions:

(i) Let $x : (-\infty, a] \to X$ be such that $x_0 = \varphi$ and $x|_I \in \mathcal{PC}$. The function $t \to f(t, x_t)$ is continuous on $\mathcal{R}(\rho^-) \cup [0,a]$ and the function $t \to f(t, x_{\rho(t,x_t)})$ is strongly mensurable on $\mathcal{R}(\rho^-) \cup [0,a]$.

(ii) For each $t \in I$, the function $f(t, \cdot) : \mathcal{B} \to X$ is continuous.

(iii) There exists an integrable function $m : I \to [0, \infty)$ and a continuous nondecreasing function $W : [0,\infty) \to (0,\infty)$ such that

$$\| f(t,\psi) \| \leq m(t) W(\| \psi \|_{\mathcal{B}}), \qquad (t,\psi) \in I \times \mathcal{B}.$$

Remark 3 *We point here that condition $\mathbf{H}_\varphi$ is frequently verified by functions continuous and bounded. In fact, assume that $\mathcal{C}$ is a phase space which verifies axioms $\mathbf{A}, \mathbf{B}, \mathbf{C_2}$ in the nomenclature of [13]. In this case, there exists a positive constant L such that $\| \psi \|_{\mathcal{C}} \leq \mathrm{L} \sup_{\theta \leq 0} \| \psi(\theta) \|$ for every $\varphi \in \mathcal{C}$ continuous and bounded, see [13, Proposition 7.1.1] for details. Consequently,*

$$\| \psi_t \|_{\mathcal{C}} \leq L \frac{\sup_{\theta \leq 0} \| \psi(\theta) \|}{\| \psi \|_{\mathcal{C}}} \| \psi \|_{\mathcal{C}}, \qquad t \leq 0, \tag{4}$$

for every continuous and bounded function $\varphi \in \mathcal{C} \setminus \{0\}$.

The spaces $C_r \times L^p(g;X)$, $C_g^0(X)$ in [13] verify axioms $\mathbf{A}, \mathbf{B}, \mathbf{C_2}$, see [13, p.10] and [13, p.16] for details. Since, these spaces are continuously included in $\mathcal{PC}_r \times L^2(h, X)$ and $\mathcal{PC}_h(X)$ respectively, a inequality similar to (4) is also valid for bounded and continuous functions in $\mathcal{PC}_r \times L^2(h,X)$ and $\mathcal{PC}_h(X)$.

Remark 4 *In delay differential equations without impulses, the function f is usually assumed continuous. This turns out to be a poor choice of a condition for impulsive system since in general, $t \to x_t$ is discontinuous. This fact, is the justificative to consider condition $\mathbf{H_1}$-(i).*

In this work, we adopt the following concept of mild solution for (1)-(3).

Definition 1 *A function $x : (-\infty, a] \to X$ is called a mild solution of the abstract Cauchy problem (1)-(3) if $x_0 = \varphi$, $x_{\rho(s,x_s)} \in \mathcal{B}$ for every $s \in I$ and*

$$x(t) = T(t)\varphi(0) + \int_0^t T(t-s) f(s, x_{\rho(s,x_s)}) ds + \sum_{0<t_i<t} T(t-t_i) I_i(x_{t_i}), \quad t \in I.$$

The next Lemma is proved using the phase spaces axioms.

Lemma 2 *Let $x : (-\infty, a] \to X$ be a function such that $x_0 = \varphi$ and $x|_{[0,a]} \in \mathcal{PC}$. Then*

$$\| x_s \|_{\mathcal{B}} \leq (M_a + J_0^{\varphi}) \| \varphi \|_{\mathcal{B}} + K_a \| x \|_{\max\{0,s\}}, \quad s \in \mathcal{R}(\rho^-) \cup [0, a],$$

where $J_0^{\varphi} = \sup_{t \in \mathcal{R}(\rho^-)} J^{\varphi}(t)$.

Now, we prove our first existence result.

Theorem 2 *Assume that there are constants L_i, $i = 1, 2, \cdots, n$, such that*

$$\| I_i(\psi_1) - I_i(\psi_2) \| \quad \leq \quad L_i \| \psi_1 - \psi_2 \|_{\mathcal{B}}, \quad \psi_i \in \mathcal{B}, \, i = 1, 2, \cdots, n.$$

If

$$\widetilde{M} K_a \left[\liminf_{\xi \to \infty} \frac{W(\xi)}{\xi} \int_0^a m(s) ds + \sum_{i=1}^{n} L_i \right] < 1, \tag{5}$$

then there exists a mild solution of (1)-(3).

Proof: On the space $\mathcal{PC}$, we define the operator $\Gamma : \mathcal{PC} \to \mathcal{PC}$ by

$$\Gamma x(t) = T(t)\varphi(0) + \int_0^t T(t-s) f(s, \overline{x}_{\rho(s,\overline{x}_s)}) ds + \sum_{0 < t_i < t} T(t - t_i) I_i(\overline{x}_{t_i}), \quad t \in I,$$

where $\overline{x} : (-\infty, a] \to X$ is such that $\overline{x}_0 = \varphi$ and $\overline{x} = x$ on I. From axiom **A**, the strong continuity of $(T(t))_{t \geq 0}$ and our assumptions on φ and f, we infer that $\Gamma x(\cdot) \in \mathcal{PC}$.

Next, we prove that there exists $r > 0$ such that $\Gamma(B_r(0, \mathcal{PC})) \subset B_r(0, \mathcal{PC})$. If we assume this property is false, then for every $r > 0$ there exist $x^r \in B_r(0, \mathcal{PC})$ and $t^r \in I$ such that $r < \| \Gamma x^r(t^r) \|$. Then, by using Lemma 2 we find that

$$\begin{aligned}
r &< \| \Gamma x^r(t^r) \| \\
&\leq \widetilde{M} H \| \varphi \|_{\mathcal{B}} + \widetilde{M} \int_0^{t^r} m(s) W(\| \overline{x^r}_{\rho(s, \overline{x^r_s})} \|_{\mathcal{B}}) ds + \widetilde{M} \sum_{i=1}^{n} (L_i \| \overline{x}_{t_i} \|_{\mathcal{B}} + \| I_i(0) \|) \\
&\leq \widetilde{M} H \| \varphi \|_{\mathcal{B}} + \widetilde{M} W \left((M_a + J_0^{\varphi}) \| \varphi \|_{\mathcal{B}} + K_a \| x^r \|_a \right) \int_0^{t^r} m(s) ds \\
&\quad + \widetilde{M} \sum_{i=1}^{n} \left(L_i ((M_a + J_0^{\varphi}) \| \varphi \|_{\mathcal{B}} + K_a \| x^r \|_a) + \| I_i(0) \| \right),
\end{aligned}$$

and hence

$$1 \leq \widetilde{M} K_a \left[\liminf_{\xi \to \infty} \frac{W(\xi)}{\xi} \int_0^a m(s) ds + \sum_{i=1}^{n} L_i \right],$$

which is contrary to our assumption.

Let $r > 0$ be such that $\Gamma(B_r(0, \mathcal{PC})) \subset B_r(0, \mathcal{PC})$. Next, we will prove that Γ is condensing map on $B_r(0, \mathcal{PC})$. Consider the decomposition $\Gamma = \Gamma_1 + \Gamma_2$ where

$$
\begin{aligned}
\Gamma_1 x(t) &= T(t)\varphi(0) + \int_0^t T(t-s) f(s, \bar{x}_{\rho(s,\bar{x}_s)}) ds, \quad t \in I, \\
\Gamma_2 x(t) &= \sum_{0<t_i<t} T(t-t_i) I_i(x_{t_i}), \qquad t \in I.
\end{aligned}
$$

Step 1 The set $\Gamma(B_r(0, \mathcal{PC}))(t) = \{\Gamma x(t) : x \in B_r(0, \mathcal{PC})\}$ is relatively compact in X for every $t \in I$.

The case $t = 0$ is obvious. Let $0 < \varepsilon < t \leq a$. If $x \in B_r(0, \mathcal{PC})$, from Lemma 2 follows that

$$\| \bar{x}_{\rho(t,\bar{x}_t)} \|_{\mathcal{B}} \leq r^* := (M_a + J_0^{\varphi}) \| \varphi \|_{\mathcal{B}} + K_a(r + \| \varphi(0) \|)$$

and so that

$$\left\| \int_0^\tau T(\tau - s) f(s, \bar{x}_{\rho(s,\bar{x}_s)}) ds \right\| \leq r^{**} := \widetilde{M} W(r^*) \int_0^a m(s) ds, \qquad \tau \in I.$$

Consequently, for $x \in B_r(0, \mathcal{PC})$ we find that

$$
\begin{aligned}
\Gamma x(t) &= T(t)\varphi(0) + T(\varepsilon) \int_0^{t-\varepsilon} T(t - \varepsilon - s) f(s, \bar{x}_{\rho(s,\bar{x}_s)}) ds + \int_{t-\varepsilon}^t T(t-s) f(s, \bar{x}_{\rho(s,\bar{x}_s)}) ds \\
&\in \{T(t)\varphi(0)\} + T(\varepsilon) B_{r^{**}}(0, X) + C_\varepsilon,
\end{aligned}
$$

where $diam(C_\varepsilon) \leq 2\widetilde{M} W(r^*) \int_{t-\varepsilon}^t m(s) ds$, which proves that $\Gamma(B_r(0, \mathcal{PC}))(t)$ is relatively compact in X.

Step 2 The set of functions $\Gamma(B_r(0, \mathcal{PC}))$ is equicontinuous on I.

Let $0 < t < a$ and $\varepsilon > 0$. Since $(T(t))_{t \geq 0}$ is strongly continuous and $\Gamma(B_r(0, \mathcal{PC}))(t)$ relatively compact in X, we can choose $0 < \delta \leq a - t$ such that

$$\| T(h)x - x \| < \varepsilon, \qquad x \in \Gamma(B_r(0, \mathcal{PC}))(t), \ 0 < h < \delta.$$

Under these conditions, for $x \in B_r(0, \mathcal{PC})$ and $0 < h < \delta$ we get

$$
\begin{aligned}
\| \Gamma x(t+h) - \Gamma x(t) \| &\leq \| T(t)(T(h) - I)\varphi(0) \| + \| (T(h) - I)\Gamma x(t) \| \\
&\quad + \widetilde{M} \int_t^{t+h} m(s) W(r^*) ds \\
&\leq \widetilde{M} \| (T(h) - I)\varphi(0) \| + \varepsilon + \widetilde{M} W(r^*) \int_t^{t+h} m(s) ds,
\end{aligned}
$$

where $r^* = (M_a + J_0^{\varphi}) \| \varphi \|_{\mathcal{B}} + K_a(r + \| \varphi(0) \|)$. It proves that $\Gamma(B_r(0, \mathcal{PC}))$ is right equicontinuous at $t \in (0, a)$. Similarly, we can prove the right equicontinuous at zero and the left equicontinuous at $t \in (0, a]$. Thus, $\Gamma(B_r(0, \mathcal{PC}))$ is equicontinuos on I.

Step 3 The map $\Gamma_1(\cdot)$ is continuous on $B_r(0, \mathcal{PC})$.

Let $(x^n)_{n\in\mathbb{N}}$ be a sequence in $B_r(0,\mathcal{PC})$ and $x \in B_r(0,\mathcal{PC})$ such that $x^n \to x$ in $\mathcal{PC}$. From axiom **A**, it is easy to see that $\overline{x^n}_s \to \overline{x}_s$ as $n \to \infty$ uniformly for $s \in (-\infty, a]$. From this fact, condition $\mathbf{H_1}$ and the inequality

$$\begin{aligned} \| f(s,\overline{x^n}_{\rho(s,\overline{x^n_s})}) - f(s,\overline{x}_{\rho(s,\overline{x}_s)}) \| &\leq \| f(s,\overline{x^n}_{\rho(s,\overline{x^n_s})}) - f(s,\overline{x}_{\rho(s,\overline{x^n_s})}) \|_{\mathcal{B}} \\ &\quad + \| f(s,\overline{x}_{\rho(s,\overline{x^n_s})}) - f(s,\overline{x}_{\rho(s,\overline{x}_s)}) \|_{\mathcal{B}}, \end{aligned}$$

we infer that $f(s,\overline{x^n}_{\rho(s,\overline{x^n_s})}) \to f(\overline{x}_{\rho(s,\overline{x}_s)})$, as $n\to\infty$, for every $s \in I$. Now, a standard application of Lebesgue Dominated Convergence Theorem proves that $\Gamma x^n \to \Gamma x$ in $\mathcal{PC}$. Thus, $\Gamma_2(\cdot)$ is continuous.

Step 4 The map $\Gamma_2(\cdot)$ is a contraction on $B_r(0,\mathcal{PC})$.

The assertion follows directly from (5) and the estimative

$$\| \Gamma_2 x - \Gamma_2 y \|_a \leq K_a\widetilde{M}\sum_{i=1}^{n} L_i \| x-y \|_a .$$

The previous remarks prove that Γ is a condensing operator from $B_r(0,\mathcal{PC})$ into $B_r(0,\mathcal{PC})$. Now, the existence of a mild solution is a consequence of [18, Theorem 4.3.2]. The proof is ended.

Theorem 3 *Assume that $\rho(t,\psi) \leq t$ for every $(t,\psi) \in I\times\mathcal{B}$, the maps I_i completely continuous and that there are constants c_i^j, $i=1,2,\cdots,n$, $j=1,2$ such that $\| I_i(\psi) \| \leq c_i^1 \| \psi \|_{\mathcal{B}} + c_i^2$, for every $\psi \in \mathcal{B}$. If $\mu = 1 - K_a\widetilde{M}\sum_{i=1}^n c_i^1 > 0$ and*

$$\frac{K_a\widetilde{M}}{\mu}\int_0^a m(s)ds < \int_C^\infty \frac{ds}{W(s)},$$

where $C = \frac{1}{\mu}\left[(K_a\widetilde{M}H + M_a + J_0^\varphi) \| \varphi \|_{\mathcal{B}} + K_a\widetilde{M}\sum_{i=1}^n c_i^2\right]$, then there exists a mild solution of (1)-(3).

Proof: Let $\Gamma, \Gamma_i, i=1,2$, be the maps defined in the proof of Theorem 2. In order to use Theorem 1, next we will make *a priori* estimates for the solutions of the integral equation $z = \lambda\Gamma z$, $\lambda\in(0,1)$. If $x^\lambda = \lambda\Gamma x^\lambda$, $\lambda \in (0,1)$, then from Lemma 2 we have that

$$\begin{aligned} \| x^\lambda(t) \| &\leq \widetilde{M}H \| \varphi \|_{\mathcal{B}} + \widetilde{M}\int_0^t m(s)W\left((M_a + J_0^\varphi) \| \varphi \|_{\mathcal{B}} + K_a \| x^\lambda \|_{\max\{0,\rho(s,\overline{x_s^\lambda})\}}\right) ds \\ &\quad + \widetilde{M}\sum_{0<t_i\leq t} c_i^1\left((M_a + J_0^\varphi) \| \varphi \|_{\mathcal{B}} + K_a \| x^\lambda \|_t\right) + \widetilde{M}\sum_{i=1}^n c_i^2 \\ &\leq \widetilde{M}H \| \varphi \|_{\mathcal{B}} + \widetilde{M}\int_0^t m(s)W\left((M_a + J_0^\varphi) \| \varphi \|_{\mathcal{B}} + K_a \| x^\lambda \|_s\right) ds \\ &\quad + \widetilde{M}\sum_{0<t_i\leq t} c_i^1\left((M_a + J_0^\varphi) \| \varphi \|_{\mathcal{B}} + K_a \| x^\lambda \|_t\right) + \widetilde{M}\sum_{i=1}^n c_i^2, \end{aligned}$$

since $\rho(s,\overline{x_s^\lambda}) \leq s$ for all $s\in I$. If $\xi^\lambda(t) = (M_a + J_0^\varphi) \| \varphi \|_{\mathcal{B}} + K_a \| x^\lambda \|_t$, we obtain that

$$\begin{aligned} \xi^\lambda(t) &\leq (K_a\widetilde{M}H + M_a + J_0^\varphi) \| \varphi \|_{\mathcal{B}} + K_a\widetilde{M}\sum_{i=1}^n c_i^2 + K_a\widetilde{M}\int_0^t m(s)W(\xi^\lambda(s))ds \\ &\quad + K_a\widetilde{M}\sum_{i=1}^n c_i^1\xi^\lambda(t), \end{aligned}$$

and so that

$$\xi^{\lambda}(t) \le \frac{1}{\mu}\left[(K_a\widetilde{M}H + M_a + J_0^{\varphi}) \parallel \varphi \parallel_{\mathcal{B}} + K_a\widetilde{M}\sum_{i=1}^{n} c_i^2\right] + \frac{K_a\widetilde{M}}{\mu}\int_0^t m(s)W(\xi^{\lambda}(s))ds$$

Denoting by $\beta_\lambda(t)$ the right hand side of the last inequality, follows that

$$\beta_\lambda'(t) \le \frac{K_a\widetilde{M}}{\mu} m(t)W(\beta_\lambda(t))$$

and hence

$$\int_{\beta_\lambda(0)=C}^{\beta_\lambda(t)} \frac{ds}{W(s)} \le \frac{K_a\widetilde{M}}{\mu}\int_0^a m(s)ds < \int_C^{\infty}\frac{ds}{W(s)},$$

which implies that the set of functions $\{\beta_\lambda(\cdot) : \lambda \in (0,1)\}$ is bounded in $C(I:\mathbb{R})$. Thus, $\{x^{\lambda}(\cdot) : \lambda \in (0,1)\}$ is bounded in $\mathcal{PC}$.

From the proof of Theorem 2, we know that Γ_1 is completely continuous. Next we prove that Γ_2 is also completely continuous. The continuity of Γ_2 can be prove using the phase space axioms. From the definition of Γ_2, for $r > 0$, $t \in [t_i, t_{i+1}] \cap (0, a]$ and $u \in B_r = B_r(0, \mathcal{PC})$ we find that

$$\widetilde{\Gamma_2 u}(t) \in \begin{cases} \sum\limits_{0<t_j<t} T(t-t_j)B_{r*}(0;X), & t \neq t_i,\ t \neq t_{i+1}, \\ \sum\limits_{j=0}^{i} T(t_{i+1}-t_j)B_{r*}(0;X), & t = t_{i+1}, \\ \sum\limits_{j=1}^{i-1} T(t_i - t_j)B_{r*}(0;X) + I_i(B_{r*}(0;X)), & t = t_i, \end{cases}$$

where $r^* := (M_a + J_0^{\varphi}) \parallel \varphi \parallel_{\mathcal{B}} + K_a(r + \parallel \varphi(0) \parallel)$, which proves that $[\widetilde{\Gamma_2(B_r)}]_i(t)$ is relatively compact in X for every $t \in [t_i, t_{i+1}]$. Moreover, using the compactness of the operators I_i and the strong continuity of $(T(t))_{t\ge 0}$, we can prove that $[\widetilde{\Gamma_2(B_r)}]_i$ is equicontinuous at t for every $t \in [t_i, t_{i+1}]$. Now, from Lemma 1 we conclude that Γ_2 is completely continuous.

The previous remarks shows that Γ is completely continuous. The existence of a mild solution for (1)-(3) is now a consequence of Theorem 1.

4 Example

In this section, $X = L^2([0,\pi])$ and $A : D(A) \subset X \to X$ is the operator $Af = f''$ with domain $D(A) := \{f \in X : f'' \in X,\ f(0) = f(\pi) = 0\}$. It is well known that A is the infinitesimal generator of a C_0-semigroup of bounded linear operators $(T(t))_{t\ge 0}$ on X. Moreover, A has discrete spectrum, the eigenvalues are $-n^2, n \in N$, with corresponding normalized eigenvectors $z_n(\xi) := \left(\frac{2}{\pi}\right)^{1/2}\sin(n\xi)$, the set $\{z_n : n \in N\}$ is an orthonormal basis of X and $T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} < x, z_n > z_n$ for every $x \in X$.

Consider the differential system

$$\begin{aligned}
\frac{\partial u(t,\xi)}{\partial t} &= \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + \int_{-\infty}^{0} \alpha(s)u(t+s-\sigma(u(t,0)),\xi)ds, \ t \in [0,a], \xi \in [0,\pi], && (6)\\
u(t,0) &= u(t,\pi) = 0, \qquad t \in [0,a] && (7)\\
u(\tau,\xi) &= \varphi(\tau,\xi), \qquad \tau \leq 0, \ \xi \in [0,\pi], && (8)\\
\Delta u(t_j,\xi) &= \int_{-\infty}^{t_j} \gamma_j(t_j - s)u(s,\xi)ds, \qquad j = 1,\cdots,n, && (9)
\end{aligned}$$

where $\varphi \in \mathcal{B} = \mathcal{PC}_0 \times L^2(h, X)$ and $0 < t_1 < .... < t_n < a$ are pre-fixed. To study this system we assume the next conditions.

(i) The functions $\alpha : \mathbb{R} \to \mathbb{R}$, $\sigma : \mathbb{R} \to \mathbb{R}^+$ are continuous, bounded and $L_f = \left(\int_{-\infty}^{0} \frac{\alpha^2(s)}{h(s)} ds\right)^{\frac{1}{2}} < \infty$.

(ii) The functions $\gamma_j : \mathbb{R} \to \mathbb{R}$ are continuous and $L_j := \left(\int_{-\infty}^{0} \frac{(\gamma_j(-s))^2}{h(s)} ds\right)^{\frac{1}{2}} < \infty$ for all $j = 1, 2, \cdots, n$.

By defining the functions $\rho(s,\psi) = -\sigma(\psi(0,0))$, $f(\psi)(\xi) = \int_{-\infty}^{0} \alpha(s)\psi(s,\xi)ds$ and $I_j(\psi)(\xi) = \int_{-\infty}^{0} \gamma_j(-s)\psi(s,\xi)ds$, $j = 1, 2, \cdots, n$, we can represent system (6)-(9) by the abstract impulsive Cauchy problem (1)-(3). Moreover, the maps f, I_j, $j = 1, \cdots, n$, are bounded linear operator, $\| f \|_{\mathcal{L}(\mathcal{B},X)} \leq L_f$ and $\| I_j \|_{\mathcal{L}(\mathcal{B},X)} \leq L_j$ for every $j = 1, \cdots, n$.

Proposition 1 *Let $\varphi \in \mathcal{B}$ and assume that the function $t \to \varphi_t$ continuous on $\mathcal{R}(\rho^-)$. If $\left(1 + \left(\int_{-a}^{0} h(\tau)d\tau\right)^{\frac{1}{2}}\right)(aL_f + \sum_{i=1}^{n} L_i) < 1$. Then there exits a mild solution of (6)-(8).*

Proof: In order to apply Theorem 2, it is remain to prove that condition $\mathbf{H_1}$-(i) is verified. Let $x : (-\infty, a] \to X$ such that $x_0 = \varphi$ and $x|_I \in \mathcal{PC}$. It's easy to see that $t \to f(t, x_{\rho(t,x_t)})$ can be approximate by simple functions. On the other hand, a straightforward estimation permit to show that

$$\begin{aligned}
\| f(x_{t+h}) - f(x_t) \| &\leq C_1 \| \varphi_h - \varphi \|_{\mathcal{B}} + C_2 \int_{-h}^{0} g(\theta) \| x(\theta + h) - \varphi(\theta) \|^2 d\theta \\
&\quad + C_3 \int_{0}^{t} \| x(\theta + h) - x(\theta) \|^2 d\theta, \qquad \text{for } t \geq 0,\\
\| f(x_{t+h}) - f(x_t) \| &\leq L_f \| \varphi_{t+h} - \varphi_t \|_{\mathcal{B}}, \qquad \text{for } t \leq 0,
\end{aligned}$$

where the constants C_i are independent of t. Since x is continuous $a.e$ on $[0, a]$ and $\varphi \in \mathcal{B}$, the last inequalities shows that $t \to f(x_t)$ is continuous on $\mathcal{R}(\rho^-) \cup [0, a]$. The proof is now completed.

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