UNIVERSIDADE DE SÃO PAULO

Instituto de Ciências Matemáticas e de Computação

ISSN 0103-2577

A Generalization of the Euler Gamma Function

Mauro Spreafico

Nº 246

NOTAS

Série Matemática

São Carlos – SP

Out./2005

A GENERALIZATION OF THE EULER GAMMA FUNCTION

Mauro Spreafico
Departamento de Matemàtica
ICMC-USP São Carlos
Avenida Trabalhador Sãocarlenses, 400 Centro
Cx. Postal 668- CEP 13560-970 São Carlos, SP, Brazil
mauros@icmc.usp.br

Resumo Neste artigo é definida uma generalização da função gamma de Euler e são discutidas as suas propriedades principais.

A GENERALIZATION OF THE EULER GAMMA FUNCTION

Mauro Spreafico
ICMC, Universidade de São Paulo, CEP 13560-970 São Carols, Brazil

0.1. **Introduction.** Consider the regularized Weierstrass product

$$F_\beta(z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{n^\beta}\right)e^{\sum_{j=1}^{p}\frac{(-1)^j}{j}\frac{z^j}{n^{j\beta}}},$$

where β is real and positive, $p = \left[\frac{1}{\beta}\right]$, and $\gamma_\beta = -\log F_\beta(1)$. We define the function $\frac{1}{\Gamma_\beta(z)} = ze^{\gamma_\beta z}F_\beta(z)$. This is a an analytic function of z for all z up to simple poles for $z = 0, -1^\beta, -2^\beta, \ldots$, and $\Gamma_\beta(1) = 1$. The aim of this note is to announce the main properties of the function Γ_β, details and complete proofs will appear elsewhere. Since $\Gamma_1(z) = \Gamma(z)$, the function Γ_β is a natural generalization of the Euler function and we show that all the properties that characterize the Gamma function independently from the "linearity" of the terms in the defining sequence (see section 0.2) hold in a very natural and simple way for Γ_β. In particular, we provide a functional equation, the Taylor series and the asymptotic expansion for large z. Remarkably we obtain the latter by proving the existence of a full asymptotic expansion for the associated heat function (lemma 3). The last is of the type $\sum_{n=1}^{\infty} e^{-a_n t}$, for some sequence a_n, and it is in general very unlikely to provide a full expansion for these functions. Already the case of the two sphere is hard [4]. When $\beta > 1$, the Euler formula of the Gamma function generalizes to

$$F_\beta(z) = \lim_{m\to\infty}\frac{(1^\beta+z)(2^\beta+z)\ldots(m^\beta+z)}{1^\beta 2^\beta\ldots m^\beta},$$

that extends to $\beta < 1$ with the opportune regularising factors. This formula tells us that neither the Gamma's functional equation nor the integral representation are likely to generalize for Γ_β, since they depend on "linearity" of the general term.

Beside the abstract interest [6] [8], motivations for studying this type of functions arise from their appearance when dealing with the regularized determinant [5] [9] of real powers of the Laplace operator on the circle[1] [3]: $\det_\zeta(-\Delta^\beta_{S^1}+a)$ is given by $\Gamma_\beta(a)$, that generalizes the Lerch formula (lemma 3). In particular, the asymptotic of $\log\Gamma_\beta$ give the behavior for small and large mass term.

0.2. **Spectral functions.** Consider the sequence $S_{\beta,a} = \{a_n = n^\beta + a\}$, where $\beta \neq 1$ is fixed [2]. $S_{\beta,a}$ is a sequence with finite exponent of convergence $\frac{1}{\beta}$ whose only accumulation point is infinite. We associated to $S_{\beta,a}$ the following spectral functions. The zeta function $\zeta_\beta(s,a) = \sum_{n=1}^{\infty}(n^\beta+a)^{-s}$; the regularized Weierstrass product $F_\beta(z)$ that is an integral function of order $\frac{1}{\beta}$ and genus $p = \left[\frac{1}{\beta}\right]$; the heat function $f_\beta(t,a) = \sum_{n=0}^{\infty}e^{-(n^\beta+a)t}$; and the resolvent function defined by $R_\beta(\lambda,a) = \sum_{n=1}^{\infty}\frac{1}{\lambda-n^\beta-a} = -\zeta_\beta(1,a-\lambda)$, when $\beta > 1$ and where λ belongs to any

---

[0] 2000 *Mathematics Subject Classification: 33E99.*

[1] In the standard case $F_2(z) = \frac{\operatorname{sh}\pi\sqrt{z}}{\pi\sqrt{z}}$.

[2] Notice that when $\beta = 1$, the general term of the homogeneous sequence $S_{1,0}$ is linear in the variable n: this property is in fact responsible of some of the main properties of the Gamma function, and this is why such properties do not hold for the general non linear case $\beta \neq 1$.

sector not containing the numbers a_n. For an arbitrary β we can only introduce derivatives $R^{(p)}_\beta(\lambda,a) = (-1)^p p! \sum_{n=1}^\infty \frac{1}{(\lambda-n^\beta-a)^{p+1}} = -p!\zeta_\beta(p+1,a-\lambda)$.

Using the Plana theorem as in [7], we provide a full asymptotic expansion for the heat function.

Lemma 1. *For $t\to 0^+$,* $\quad f_\beta(t,0) \sim \Gamma\left(1+\frac{1}{\beta}\right)t^{-\frac{1}{\beta}} + \frac{1}{2} + \sum_{m=1}^\infty \frac{(-1)^m}{m!}c_m t^m$,

$$c_{2m} = 2\sum_{j=0}^{m-1}(-1)^{j+1}\binom{2m}{2j+1}\cos^{2j+1}\frac{\pi}{2}\beta\sin^{2m-2j-1}\frac{\pi}{2}\beta\int_0^\infty \frac{y^{2m\beta}}{e^{2\pi y}-1}dy,$$

$$c_{2m+1} = 2\sum_{j=0}^{m}(-1)^{j+1}\binom{2m+1}{2j}\cos^{2j}\frac{\pi}{2}\beta\sin^{2m+1-2j}\frac{\pi}{2}\beta\int_0^\infty \frac{y^{(2m+1)\beta}}{e^{2\pi y}-1}dy.$$

0.3. **The zeta function.** The main properties of the generalized Hurwitz zeta function $\zeta_\beta(s,a)$ can be obtained by standard methods [2], but we present a simple proof of how the main zeta invariants are related to the Weierstrass product [9].

Lemma 2. *$\zeta_\beta(s,a)$ has a regular analytic continuation to the whole complex plane up to simple poles at $s=\frac{1}{\beta}-j$, for all $j=0,1,2,\dots$ when $\frac{1}{\beta}\notin\mathbb{N}$, but for $0\le j\le\frac{1}{\beta}-1$ when $\frac{1}{\beta}\in\mathbb{N}_{0,1}$.* $\mathrm{Res}_1\left(\zeta_\beta(s,a), s=\frac{1}{\beta}-j\right) = \frac{(-1)^j}{j!}\frac{\Gamma(1+\frac{1}{\beta})}{\Gamma(\frac{1}{\beta}-j)}a^j$.

Corollary 1. *The coefficients in the expansion of the heat function have the following alternative formula: $c_m = \zeta_\beta(-m,0) = \zeta(-\beta m)$.*

Lemma 3. $\quad \zeta_\beta(0,a) = \begin{cases} -\frac{1}{2}, & \frac{1}{\beta}\notin\mathbb{N}, \\ -\frac{1}{2}+(-1)^{\frac{1}{\beta}}a^{\frac{1}{\beta}}, & \frac{1}{\beta}\in\mathbb{N}_{0,1}, \end{cases}$

$$\zeta'_\beta(0,a) = -\frac{\beta}{2}\log 2\pi + \sum_{j=1}^{p-1}\frac{(-1)^j}{j}\zeta(\beta j)a^j - \log F_\beta(a) +$$

$$+\begin{cases} \frac{(-1)^p}{p}\zeta(\beta p)a^p, & \frac{1}{\beta}\notin\mathbb{N}, p\neq 0 \\ (-1)^{\frac{1}{\beta}}\left[(\beta+1)\gamma+\psi\left(\frac{1}{\beta}\right)\right]a^{\frac{1}{\beta}}, & \frac{1}{\beta}\in\mathbb{N}_{0,1}. \end{cases}$$

Proof. When $|a|<1$, we expand the powers of the binomial in the definition of the zeta function getting $\zeta_\beta(s,a) = \sum_{j=0}^\infty \binom{-s}{j}\zeta_\beta(s+j,0)a^j$, where $\zeta_\beta(s,0)=\zeta(\beta s)$. If $\frac{1}{\beta}\notin\mathbb{N}$, we can evaluate both the series and its s-derivative at $s=0$. If $\frac{1}{\beta}\in\mathbb{N}_{0,1}$, troubles come from the p-th term, that can be treated using expansions at $s=0$: $\binom{-s}{j}\zeta_\beta(s+j,0) = \frac{(-1)^j}{j}\{R_1(j)+[R_0(j)+(\gamma+\psi(j))R_1(j)]s\}+O(s^2)$, where $R_k(j) = \mathrm{Res}_k(\zeta_\beta(s,0), s=j) = \begin{cases}\gamma & k=0, \\ \frac{1}{\beta} & k=1\end{cases}$. ☐

Using lemma 3 we generalize the factorization lemma of Choi and Quine [1]:

Corollary 2. $\quad \zeta_{2\beta}(0,a^2) = \frac{1}{2}[\zeta_\beta(0,ia)+\zeta_\beta(0,-ia)]$,

$$\zeta'_{2\beta}(0,a^2) = \zeta'_\beta(0,ia)+\zeta'_\beta(0,-ia) - \begin{cases} 2(-1)^{\frac{p}{2}}\sum_{j=1}^{\frac{p}{2}}\frac{1}{2j-1}a^p, & \text{for even } p=\frac{1}{\beta}\in\mathbb{N}_{0,1}, \\ 0, & \text{otherwise.}\end{cases}$$

0.4. **Properties of the Gamma function.** Our first results follow from the definition[3].

Proposition 1. $$\Gamma_\beta(iz)\Gamma_\beta(-iz) = e^{\gamma_{2\beta} z^2}\Gamma_{2\beta}(z^2).$$

Proposition 2. *When* $\beta > 1$, $\left(1 - \frac{x}{z}\right) e^{-\gamma_\beta x} \prod_{n=1}^{\infty} \left(1 - \frac{x}{n^\beta + z}\right) = \frac{\Gamma_\beta(z)}{\Gamma_\beta(z-x)}$.

As a consequence, it is possible to evaluate the general class of infinite products of the form $\Pi = \prod_{n=1}^{\infty} \frac{P(n^\beta)}{Q(n^\beta)}$, where $P(x)$ and $Q(x)$ are two polynomial of the same degree J, and Q has no roots of the form n^β. Straightforward calculations give

$$\Pi = \prod_{j=1}^{J} \frac{b_j \Gamma_\beta(-b_j)}{a_j \Gamma_\beta(-a_j)} e^{\gamma_\beta (b_j - a_j)},$$

where the a_j, b_j are the roots of the two polynomials P and Q respectively.

The Taylor series for $\log \Gamma_\beta$ can be deduced from lemma 3 as well. Alternatively, we can use the regularized resolvent function

$$\tilde{R}_\beta(-z, 0) = \sum_{n=1}^{\infty} \left[\frac{1}{-z - n^\beta} + \sum_{j=0}^{p-1} \frac{(-z)^j}{n^{(j+1)\beta}} \right],$$

that will be essential in the proof of proposition 4. The above series is uniformly convergent on each bounded domain and so are its first p derivatives, so we can take derivative and with the appropriate normalization: $\frac{d^k}{dz^k} \log F_\beta(z) = (-1)^k \tilde{R}_\beta^{(k-1)}(-z, 0)$. Taking the limit $\tilde{R}_\beta^{(k)}(-z, 0) = 0$, for $0 \leq k \leq p-1$. This proves:

Proposition 3. *For small* z, $\log \Gamma_\beta(z) = -\log z - \gamma_\beta z + \sum_{j=p+1}^{\infty} \frac{(-1)^j}{j} \zeta(\beta j) z^j$.

Our last result is the asymptotic expansion for large argument. The proof is in two steps. First, we use the Mellin transform as in the proof of lemma 2 to get the p-resolvent function $R_\beta^{(p)}(\lambda, 0)$ as the Laplace transform of the heat function; thus the asymptotic expansion of $R_\beta^{(p)}(\lambda, 0)$ is given by substitution using the expansion of the heat function provided by lemma 1. Since integration of asymptotic expansions is allowed, we obtain from the above expansion the expansion for the logarithm of the Weierstrass product, up to a finite set of constants b_j. Here some care is necessary to deal with the possible $\frac{1}{z}$ term. We need to distinguish two cases depending whether $\frac{1}{\beta} \in \mathbb{N}_{0,1}$ or not[4]. The second step is to determinate the still unknown constants b_j. For that, we need to introduce the following analytic representation of the zeta function:

$$\zeta_\beta(s, 0) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_{c,\theta}} e^{-\lambda t} \tilde{R}_\beta(\lambda, 0) d\lambda dt,$$

where $\Lambda_{c,\theta} = \left\{ \lambda \in \mathbb{C} \mid |\arg(\lambda - c)| = \frac{\theta}{2} \right\}$, for some positive real $c < 1$ and $0 < \theta < \pi$, modifies to a contour of Hankel type. Integrating by parts first in λ and then in t, we get $\log F_\beta$ instead of the regularized resolvent and an extra s factor. The presence of the last determinates a zero of second order at $s = 0$, and allows to

[3]Proposition 1 can also be obtained from corollary 2.

[4]Notice that in the first case necessarily $\beta < 1$, and hence the second case contains the case $\beta > 1$, $p = 0$. Notice also that $\frac{1}{\beta}$ can never vanish, since β is assumed finite.

use standard techniques[5] (see for example [2]) and the know expansion for $\log F_\beta$ to compute residua at $s=0$. By comparing with the values of the Riemann zeta function and its derivative, we deduce the constants b_j. We get[6]:

Proposition 4. *For large z with $\arg(z)\neq 0$:*

$$\log\Gamma_\beta(z)\sim -\frac{1}{2}\log z+\frac{\beta}{2}\log 2\pi+\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\zeta(-\beta m)z^{-m}-\gamma_\beta z+$$

$$+\begin{cases}\sum_{j=1}^{p}\frac{(-1)^{j+1}}{j}\zeta(\beta j)z^j+\frac{(-1)^{p+1}\pi}{\sin\pi\left(\frac{1}{\beta}-p\right)}z^{\frac{1}{\beta}}, & \text{if } \frac{1}{\beta}\notin\mathbb{N}, p=\left[\frac{1}{\beta}\right]\neq 0,\\ -\frac{\pi}{\sin\frac{\pi}{\beta}}z^{\frac{1}{\beta}}, & \text{if } \frac{1}{\beta}\notin\mathbb{N}, p=\left[\frac{1}{\beta}\right]=0,\\ \sum_{j=1}^{p-1}\frac{(-1)^{j+1}}{j}\zeta(\beta j)z^j+(-1)^{p+1}z^p[\log z+\gamma\beta], & \text{if } \frac{1}{\beta}=p\in\mathbb{N}_{0,1}.\end{cases}$$

To conclude, notice that the constant γ_β is a natural generalization of the Euler constant $\gamma=\gamma_1$. This can be seen by writing explicitly the defining series or observing that $\gamma_\beta=\sum_{k=p+1}^{\infty}\frac{(-1)^k}{k}\zeta(\beta k)$. From the above results we also get

$$-\gamma_\beta=\lim_{z\to 0}\frac{d}{dz}\log z\Gamma_\beta(z)=\lim_{z\to 0}\left(\frac{d}{dz}\log\Gamma_\beta(z)+\frac{1}{z}\right),$$

consistently with the classical case, and independently from the functional equation. On the other side, for arbitrary β, it is no longer true that the constant γ_β coincides with the non singular part of the associated zeta function $\zeta_\beta(s,0)$ at $s=1$.

References

[1] J. Choi and J.R. Quine, *Zeta regularized products and functional determinants on spheres*, Rocky Mount. Jour. Math. 26 (1996) 719-729;

[2] P.B. Gilkey, *Invariance theorems, the heat equation, and the Atiyah-Singer index theorem*, Studies in Adv. Math. CRC Press, 1995;

[3] C. Morpurgo, *Sharp inequalities for functional integrals and traces of conformally invariant operators*, Duke Math. Jour. 114 (2002)477-573;

[4] H.P. Mulholland, *An asymptotic expansion for* $\sum_{n=1}^{\infty}(2n+1)e^{(n+\frac{1}{2})^2t}$, Proc. Cambridge Phil. Soc. 24 (1928) 280-289;

[5] P. Sarnak, *Determinants of Laplacians*, Comm. Math. Phys. 110 (1987) 113-120;

[6] R. Shuster, *A generalized Barnes G-function*, Zeit. An. Anw. 11 (1992) 229-236;

[7] M. Spreafico, *Zeta function and regularized determinant on projective spaces*, Rocky Mount. J. Maths 33 (2003);

[8] I. Vardi, *Determinants of Laplacians and multiple Gamma functions*, SIAM J. Math. Anal. 19 (1988) 493-507;

[9] A. Voros, *Spectral functions, special functions and the Selberg zeta function*, Comm. Math. Phys. 110 (1987) 439-465.

[5] Some care is necessary to modify the contour.

[6] Notice that the last formula covers also the case $p=1$.

NOTAS DO ICMC

SÉRIE MATEMÁTICA

245/2005 SPREAFICO, M. – Zeta function and regularized determinant on a disc and on a cone.

244/2005 SPREAFICO, M. – Zeta invariants for sequences of spectral type, special functions and the lerch formula.

243/2005 SPREAFICO, M. – Zeta invariants for dirichlet series.

242/2005 CARVALHO, A.N.; CHOLEWA, J.W. - Well posedness, asymptotics and regularity of solutions to semilinear strongly damped wave equations in the Banach spaces $W_0^{1,p}(\Omega) \times L^p(\Omega)$.

241/2005 BAPTISTTELLI, P.H.; MANOEL, M.G. – Some results on reversible – equivariant vector fields.

240/2005 GUTIERREZ, C.; RABANAL, R. – Injectivity of differentiable maps $R^2 \to R^2$ at infinity.

239/2005 FEDERSON, M. ; SCHWABIK, S. – Generalized ODEs approach to impulsive retarded differential equations.

238/2005 BRUSCHI, S.M.; CARVALHO, A.N.; CHOLEWA, J.W.; DLOTKO, T. – Uniform exponential dichotomy and continuity of attractors for singularly perturbed damped wave equations.

237/2005 HERNÁNDEZ, E. M.; DIAGANA, T. – On pseudo almost periodic solutions to some neutral functional-differential equations.

236/2005 HERNÁNDEZ, E. M.; DIAGANA, T. – Almost automorphic solutions to some partial neutral functional differential equations.