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**Continuation of the Connection Matrix for
Singularly Perturbed Hyperbolic Equations**

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Continuation of the connection matrix for singularly perturbed hyperbolic equations

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Resumo

Sejam $\Omega \subset \mathbb{R}^N$, $N \leq 3$, um domínio limitado com fronteira suave, $\gamma \in L^2(\Omega)$ arbitrário e $\phi: \mathbb{R} \rightarrow \mathbb{R}$ uma função de classe C^1 satisfazendo uma condição de crescimento subcrítico. Para todo $\varepsilon \in]0, \infty[$ considere o semifluxo π_ε em $H_0^1(\Omega) \times L^2(\Omega)$ gerado pela equação da onda amortecida

$$\begin{aligned} \varepsilon \partial_{tt} u + \partial_t u &= \Delta u + \phi(u) + \gamma(x) & x \in \Omega, \quad t > 0 \\ u(x, t) &= 0 & x \in \partial\Omega, t > 0. \end{aligned}$$

Além disso, seja π' o semifluxo em $H_0^1(\Omega)$ gerado pela equação parabólica

$$\begin{aligned} \partial_t u &= \Delta u + \phi(u) + \gamma(x) & x \in \Omega, \quad t > 0 \\ u(x, t) &= 0 & x \in \partial\Omega, t > 0. \end{aligned}$$

Seja $\Gamma: H^2(\Omega) \rightarrow H_0^1(\Omega) \times L^2(\Omega)$ o mergulho $u \mapsto (u, \Delta u + \phi(u) + \gamma)$. Mostramos neste artigo que sempre que K' é um conjunto π' -invariante isolado e $(M'_p)_{p \in P}$ é uma decomposição de Morse parcialmente ordenada de K' , então os conjuntos $K = \Gamma(K')$ e $M_{p, \Gamma} = \Gamma(M'_p)$, $p \in P$, continuam, para $\varepsilon > 0$ pequeno, a um conjunto π_ε -invariante isolado K_ε , uma decomposição de Morse $(M_{p, \varepsilon})_{p \in P}$ de K_ε , relativamente a π_ε , tais que a trança índice homológica de $(\pi_\varepsilon, K_\varepsilon, (M_{p, \varepsilon})_{p \in P})$ é isomorfa à trança índice homológica de $(\pi', K', (M'_p)_{p \in P})$. Isto, em particular, implica um princípio de continuação para matrizes de conexão.

CONTINUATION OF THE CONNECTION MATRIX FOR SINGULARLY PERTURBED HYPERBOLIC EQUATIONS

MARIA C. CARBINATTO — KRZYSZTOF P. RYBAKOWSKI

ABSTRACT. Let $\Omega \subset \mathbb{R}^N$, $N \leq 3$, be a bounded domain with smooth boundary, $\gamma \in L^2(\Omega)$ be arbitrary and $\phi: \mathbb{R} \rightarrow \mathbb{R}$ be a C^1 -function satisfying a subcritical growth condition. For every $\varepsilon \in]0, \infty[$ consider the semiflow π_ε on $H_0^1(\Omega) \times L^2(\Omega)$ generated by the damped wave equation

$$\begin{aligned} \varepsilon \partial_{tt} u + \partial_t u &= \Delta u + \phi(u) + \gamma(x) & x \in \Omega, \quad t > 0 \\ u(x, t) &= 0 & x \in \partial\Omega, \quad t > 0. \end{aligned}$$

Moreover, let π' be the semiflow on $H_0^1(\Omega)$ generated by the parabolic equation

$$\begin{aligned} \partial_t u &= \Delta u + \phi(u) + \gamma(x) & x \in \Omega, \quad t > 0 \\ u(x, t) &= 0 & x \in \partial\Omega, \quad t > 0. \end{aligned}$$

Let $\Gamma: H^2(\Omega) \rightarrow H_0^1(\Omega) \times L^2(\Omega)$ be the imbedding $u \mapsto (u, \Delta u + \phi(u) + \gamma)$. We prove in this paper that whenever K' is a compact isolated π' -invariant set and $(M'_p)_{p \in P}$ is a partially ordered Morse decomposition of K' then the imbedded sets $K = \Gamma(K')$ and $M_{p, \Gamma} = \Gamma(M'_p)$, $p \in P$, continue, for $\varepsilon > 0$ small, to an isolated π_ε -invariant set K_ε a Morse decomposition $(M_{p, \varepsilon})_{p \in P}$ of K_ε , relative to π_ε , such that the homology index braid of $(\pi_\varepsilon, K_\varepsilon, (M_{p, \varepsilon})_{p \in P})$ is isomorphic to the homology index braid of $(\pi', K', (M'_p)_{p \in P})$. This, in particular, implies a connection matrix continuation principle.

1. INTRODUCTION

Let $N \in \{1, 2, 3\}$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, $\gamma \in L^2(\Omega)$ be arbitrary and $\phi: \mathbb{R} \rightarrow \mathbb{R}$ be a C^1 -function such that, for $N \geq 2$, there are constants $\bar{r}$ and $\bar{C} \in [0, \infty[$ with $|\phi'(u)| \leq \bar{C}(1 + |u|^{\bar{r}})$ for $u \in \mathbb{R}$. If $N = 3$ we also assume that $\bar{r} < 2$, i.e. that ϕ has subcritical growth.

For every $\varepsilon \in]0, \infty[$ consider the following damped wave equation

$$\begin{aligned} \varepsilon \partial_{tt} u + \partial_t u &= \Delta u + \phi(u) + \gamma(x) & x \in \Omega, \quad t > 0 \\ u(x, t) &= 0 & x \in \partial\Omega, \quad t > 0. \end{aligned} \tag{Hyp}_\varepsilon$$

It is well-known that equation (Hyp_ε) generates a local semiflow (actually, a local flow) π_ε on $H_0^1(\Omega) \times L^2(\Omega)$.

Setting, formally, $\varepsilon = 0$ in equation (Hyp_ε) we obtain the parabolic equation

$$\begin{aligned} \partial_t u &= \Delta u + \phi(u) + \gamma(x) & x \in \Omega, \quad t > 0 \\ u(x, t) &= 0 & x \in \partial\Omega, \quad t > 0. \end{aligned} \tag{Par}$$

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Again it is well-known that equation (Par) generates a local semiflow π' on $H_0^1(\Omega)$.

It is a natural question to ask whether, for $\varepsilon \rightarrow 0$, solutions of π_ε converge, in some sense, to solutions of π' , properly imbedded into $H_0^1(\Omega) \times L^2(\Omega)$. This question was considered in the context of attractors by Hale and Raugel [12] and in the context of Conley index theory by the second author in [17].

In fact, in [17] the following result was proved.

Theorem A. *Let K' be a compact (in $H_0^1(\Omega)$) isolated invariant set relative to π' . Then $K' \subset H^2(\Omega)$, and there is an $\tilde{\varepsilon}_0 > 0$ and for every $\varepsilon \in]0, \tilde{\varepsilon}_0]$ there is a compact isolated invariant set K_ε relative to π_ε such that the Conley index of $(\pi_\varepsilon, K_\varepsilon)$ is equal to the Conley index of (π', K') .*

Moreover, let $(M'_p)_{p \in P}$ be a $\prec$ -ordered Morse decomposition of K' relative to π' . Then $(M'_p)_{p \in P}$ continues to a $\prec$ -ordered Morse decomposition $(M_{p,\varepsilon})_{p \in P}$ of K_ε relative to π_ε , $\varepsilon \in]0, \tilde{\varepsilon}_0]$.

We refer to [17, Theorems 6.1 and 7.4] for the precise statement of a more general result.

The main goal of this paper is to show that a continuation result holds for homology index braids. More precisely, we shall prove

Theorem B. *Let K' be a compact isolated invariant set relative to π' and let $(M'_p)_{p \in P}$ be a $\prec$ -ordered Morse decomposition of K' relative to π' . Then there exists an $\varepsilon_0 \in]0, \infty[$ and for every $\varepsilon \in]0, \varepsilon_0]$ there exist a compact isolated invariant set K_ε relative to π_ε and a $\prec$ -ordered Morse decomposition $(M_{p,\varepsilon})_{p \in P}$ of K_ε relative to π_ε such that for all $\varepsilon \in]0, \varepsilon_0]$, the homology index braids of $(\pi', K', (M'_p)_{p \in P})$ and $(\pi_\varepsilon, K_\varepsilon, (M_{p,\varepsilon})_{p \in P})$ are isomorphic.*

We recall that if two homology index braids $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic, then, by [10, Proposition 1.5], $\mathcal{H}_1$ and $\mathcal{H}_2$ determine the same collection C -connection matrices. Thus Theorem B shows that, for small $\varepsilon > 0$, the Morse decomposition $(M_{p,\varepsilon})_{p \in P}$ of the invariant set K_ε relative to the damped wave equation (Hyp $_\varepsilon$) has the same set of C -connection matrices as the Morse decomposition $(M'_p)_{p \in P}$ of the invariant set K' relative to the reaction-diffusion equation (Par).

Theorem B will be proved by an application of some abstract continuation results for regularly and singularly perturbed semiflows, recently established in our papers [5] and [7]. Our approach to the proof of Theorem B is similar to the proof of Theorem A in [17]. As in [17], the first attempt would be to make a change of variables

$$\Phi: (u, v) \mapsto (u, w) := (u, v - \Delta u - \phi(u) - \gamma)$$

in (Hyp $_\varepsilon$), consider the corresponding conjugate semiflows $\tilde{\pi}_\varepsilon = \Phi^* \pi_\varepsilon$, $\varepsilon \in]0, \infty[$, and then apply the results of [7] to the family $\tilde{\pi}_\varepsilon$, $\varepsilon \in]0, \infty[$, where $\tilde{\pi}_0 = \pi'$. However, as it was pointed out in [17], there is an inherent difficulty in the present situation due to the fact that the transformation Φ is defined on the space $H^2(\Omega) \times L^2(\Omega)$, which is only a subset of the phase space $H_0^1(\Omega) \times L^2(\Omega)$ of the semiflows π_ε , so $\tilde{\pi}_\varepsilon$ is not well-defined for $\varepsilon \in]0, \infty[$. That is why we first study, in Section 3, a finite-dimensional perturbation problem (equations (3.1) and (3.2) below), to which a variable transformation like Φ is applicable. Results from [7] then yield a continuation result for homology index braids for the corresponding family of finite-dimensional semiflows (cf. Theorems 3.1 and 3.6 below).

Combining the compactness and smoothing results for parabolic equations and the boundedness and smoothing results for damped wave equations established

in [17] with the results from Section 3, we then obtain, in last section of the paper, our main result, Theorem 4.2, which implies Theorem B above.

2. PRELIMINARIES

The purpose of this section is to recall a few concepts from Conley index theory and some preliminary results needed later in this paper. We assume the reader's familiarity with the infinite-dimensional Conley index theory, as developed in [14, 15] and expounded in [16], and with the papers [9], [11], [5] and [6]. Moreover, we recall some results from the papers [1], [4], [7] and [17].

In this section, unless otherwise specified, Y is a metric space, π is a local semiflow on Y and all concepts are defined *relative to* π .

Suppose that Z is a subset of Y . By $\text{Inv}_\pi(Z)$ we denote the largest invariant subset of Z .

Z is called π -admissible if Z is closed and whenever $(x_n)_n$ and $(t_n)_n$ are such that $t_n \rightarrow \infty$ and $x_n \pi [0, t_n] \subset Z$ for all $n \in \mathbb{N}$, then the sequence $(x_n \pi t_n)_n$ has a convergent subsequence. We say that π *does not explode in* Z if whenever $x \in Y$ and $x\pi t \in Z$ as long as $x\pi t$ is defined, then $x\pi t$ is defined for all $t \in [0, \infty[$. Z is called *strongly* π -admissible if Z is π -admissible and π does not explode in Z .

Let N and Z be subsets of Y . The set Z is called *N -positively invariant* if whenever $x \in Z$, $t \geq 0$ are such that $x\pi [0, t] \subset N$, then $x\pi [0, t] \subset Z$.

Let N , Z_1 and Z_2 be subsets of Y . The set Z_2 is called an *exit ramp for* N *within* Z_1 if whenever $x \in Z_1$ and $x\pi t' \notin N$ for some $t' \in [0, \infty[$, then there exists a $t_0 \in [0, t']$ such that $x\pi [0, t_0] \subset N$ and $x\pi t_0 \in Z_2$.

If Z_1 and Z_2 are subsets of Y then Z_2 is called an *exit ramp for* Z_1 if Z_2 is an exit ramp for N within Z_1 , where $N = Z_1$.

Definition 2.1. ([11]) A pair (N_1, N_2) is called a *Franzosa-Mischaikow-index pair* (or *FM-index pair*) for S if:

- (1) N_1 and N_2 are closed subsets of Y with $N_2 \subset N_1$ and N_2 is N_1 -positively invariant;
- (2) N_2 is an exit ramp for N_1 ;
- (3) S is closed, $S \subset \text{Int}_Y(N_1 \setminus N_2)$ and S is the largest invariant set in $\text{Cl}_Y(N_1 \setminus N_2)$.

Given an isolated invariant set K having a strongly π -admissible isolating neighborhood we denote by $h(K) = h(\pi, K)$ the Conley-index of K and by $H(K) = H(\pi, K) = H(h(K))$ the homology Conley index, where H is the singular homology functor (with coefficients in some fixed R -module G , where R is a commutative ring).

For the rest of this paper let P be a fixed finite set and $\prec$ be a fixed strict partial order on P .

A set $I \subset P$ is called a $\prec$ -interval if whenever $i, j, k \in P$, $i, k \in I$ and $i \prec j \prec k$, then $j \in I$. By $\mathcal{I}(\prec)$ we denote the set of all $\prec$ -intervals in P . A set I is called a $\prec$ -attracting interval if whenever $i, j \in P$, $j \in I$ and $i \prec j$, then $i \in I$. By $\mathcal{A}(\prec)$ we denote the set of all $\prec$ -attracting intervals in P . Of course, $\mathcal{A}(\prec) \subset \mathcal{I}(\prec)$.

An *adjacent n -tuple of $\prec$ -intervals* is a sequence $(I_j)_{j=1}^n$ of pairwise disjoint $\prec$ -intervals whose union is a $\prec$ -interval and such that, whenever $j < k$, $p \in I_j$ and $p' \in I_k$, then $p' \not\prec p$ (i.e. $p \prec p'$ or else p and p' are not related by $\prec$). By $\mathcal{I}_n(\prec)$ we denote the set of all adjacent n -tuples of $\prec$ -intervals.

Recall the following definition.

Definition 2.2. (cf. [11]) Let π be a local semiflow on Y and S be a compact invariant set relative to π .

A family $(M_p)_{p \in P}$ of subsets of S is called a $\prec$ -ordered Morse decomposition of S (relative to π) if the following properties hold:

- (1) The sets M_p , $p \in P$, are closed, π -invariant and pairwise disjoint.
- (2) For every full solution σ of π lying in S either $\sigma(\mathbb{R}) \subset M_k$ for some $k \in P$ or else there are $k, l \in P$ with $k \prec l$, $\alpha(\sigma) \subset M_l$ and $\omega(\sigma) \subset M_k$.

Let $\mathcal{C}$ denote the set of all continuous functions from $\mathbb{R}$ to Y endowed with the metric introduced in [1]. Let $\mathcal{T}$ be a subset of $\mathcal{C}$. A set $S \subset Y$ is called $\mathcal{T}$ -invariant if $S = \text{Inv}_{\mathcal{T}}(S)$, i.e. if and only if for every $y \in S$ there is a $\sigma \in \mathcal{T}$ such that $\sigma(\mathbb{R}) \subset S$ and $y = \sigma(0)$. If π is a local semiflow on Y and $N \subset Y$ then we denote by $\mathcal{T}_{\pi, N}$ the set of all full solutions of π lying in N .

The previous concept can be generalized as follows:

Definition 2.3. (cf. [4]) Let $\mathcal{T}$ be a subset of $\mathcal{C}$. A family $(M_p)_{p \in P}$ of subsets of Y is called a $\prec$ -ordered $\mathcal{T}$ -Morse decomposition if the following properties hold:

- (1) The sets M_p , $p \in P$, are closed, $\mathcal{T}$ -invariant and pairwise disjoint.
- (2) For every $\sigma \in \mathcal{T}$ either $\sigma(\mathbb{R}) \subset M_k$ for some $k \in P$ or else there are $k, l \in P$ with $k \prec l$, $\alpha(\sigma) \subset M_l$ and $\omega(\sigma) \subset M_k$.

It is easily proved that, for π and S as in Definition 2.2, a family $(M_p)_{p \in P}$ of subsets of S is a $\prec$ -ordered Morse decomposition of S (relative to π) if and only if $(M_p)_{p \in P}$ is a $\prec$ -ordered $\mathcal{T}$ -Morse decomposition, where $\mathcal{T} := \mathcal{T}_{\pi, S}$.

If $A, B \subset Y$ then the $\mathcal{T}$ -connection set $\text{CS}_{\mathcal{T}}(A, B)$ from A to B is the set of all points $y \in Y$ for which there is a $\sigma \in \mathcal{T}$ with $\sigma(0) = y$, $\alpha(\sigma) \subset A$ and $\omega(\sigma) \subset B$. If π , S are as in Definition 2.2 and $\mathcal{T} := \mathcal{T}_{\pi, S}$, then we write

$$\text{CS}_{\pi, S}(A, B) := \text{CS}_{\mathcal{T}}(A, B).$$

Recall also the following definition.

Definition 2.4. (cf. [17]) We say that N is a $\mathcal{T}$ -isolating neighborhood (of a subset S of Y) if N is closed in Y and $\text{Inv}_{\mathcal{T}}(N) \subset \text{Int}_Y(N)$ (with $S = \text{Inv}_{\mathcal{T}}(N)$). If $S \subset Y$ and there exists a set $N \subset Y$ such that N is a $\mathcal{T}$ -isolating neighborhood of S then we call S a $\mathcal{T}$ -isolated invariant set.

For the rest of this paper, unless specified otherwise, let (X_0, d_0) be a metric space, ε_0 be a positive number and, for each $\varepsilon \in]0, \varepsilon_0]$, $(Y_\varepsilon, d_\varepsilon)$ be a metric space and $\theta_\varepsilon \in Y_\varepsilon$ be a distinguished point of Y_ε . The open ball in Y_ε of center in v and radius $\beta > 0$ is denoted by $B_\varepsilon(v, \beta)$.

For each $\varepsilon \in]0, \varepsilon_0]$ define the set $Z_\varepsilon := X_0 \times Y_\varepsilon$. Endow Z_ε with the metric

$$\Gamma_\varepsilon((u, v), (u', v')) := \max\{d_0(u, u'), d_\varepsilon(v, v')\}, \quad (u, v), (u', v') \in Z_\varepsilon.$$

Given a subset V of X_0 , $\beta > 0$ and $\varepsilon \in]0, \varepsilon_0]$ define the ‘inflated’ subset $[V]_{\varepsilon, \beta}$ of Z_ε as follows:

$$[V]_{\varepsilon, \beta} := \{(u, v) \in Z_\varepsilon \mid u \in V \text{ and } v \in \text{Cl}_\varepsilon B_\varepsilon(\theta_\varepsilon, \beta)\}.$$

Let π_0 be a local semiflow on X_0 and for every $\varepsilon \in]0, \varepsilon_0]$ let π_ε denote a local semiflow on Z_ε .

We say that the family $(\pi_\varepsilon)_{\varepsilon \in]0, \varepsilon_0]}$ of local semiflows *converges singularly* to the local semiflow π_0 if whenever $(\varepsilon_n)_n$ and $(t_n)_n$ are sequences of positive (resp. nonnegative) numbers such that $\varepsilon_n \rightarrow 0$, $t_n \rightarrow t_0$ as $n \rightarrow \infty$, for some $t_0 \in [0, \infty[$ and whenever $u_0 \in X_0$ and $w_n \in Z_{\varepsilon_n}$, $n \in \mathbb{N}$, are such that $\Gamma_{\varepsilon_n}(w_n, (u_0, \theta_{\varepsilon_n})) \rightarrow 0$ as $n \rightarrow \infty$ and $u_0 \pi_0 t_0$ is defined, then there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $w_n \pi_{\varepsilon_n} t_n$ is defined and $\Gamma_{\varepsilon_n}(w_n \pi_{\varepsilon_n} t_n, (u_0 \pi_0 t_0, \theta_{\varepsilon_n})) \rightarrow 0$ as $n \rightarrow \infty$.

Let η be a positive number and N be a closed subset of X_0 . We say that N is a *singularly strongly admissible set with respect to η and the family $(\pi_\varepsilon)_{\varepsilon \in]0, \varepsilon_0]}$* of local semiflows if the following conditions are satisfied:

- (1) N is a strongly π_0 -admissible set;
- (2) for each $\varepsilon \in]0, \varepsilon_0]$ the set $[N]_{\varepsilon, \eta}$ is strongly π_ε -admissible;
- (3) whenever $(\varepsilon_n)_n$ and $(t_n)_n$ are sequences of positive numbers such that $\varepsilon_n \rightarrow 0$, $t_n \rightarrow \infty$ as $n \rightarrow \infty$ and whenever $w_n \in Z_{\varepsilon_n}$, $n \in \mathbb{N}$, are such that $w_n \pi_{\varepsilon_n} [0, t_n] \subset [N]_{\varepsilon_n, \eta}$, $n \in \mathbb{N}$, then there exist a $u_0 \in N$ and a subsequence of the sequence $(w_n \pi_{\varepsilon_n} t_n)_n$ of endpoints, denoted again by $(w_n \pi_{\varepsilon_n} t_n)_n$, such that $\Gamma_{\varepsilon_n}(w_n \pi_{\varepsilon_n} t_n, (u_0, \theta_{\varepsilon_n})) \rightarrow 0$ as $n \rightarrow \infty$.

The following singular continuation result for Morse decompositions was established in [4].

Theorem 2.5. [4, Corollaries 4.14 and 4.15] *Assume $(\pi_\varepsilon)_{\varepsilon \in]0, \varepsilon_0]}$ is a family of local semiflows that converges singularly to the local semiflow π_0 , $\beta \in]0, \infty[$ and $\tilde{N}$ is a singularly strongly admissible set with respect to β and $(\pi_\varepsilon)_{\varepsilon \in]0, \varepsilon_0]}$. Moreover, suppose that $S_0 := \text{Inv}_{\pi_0}(\tilde{N})$ and $(M_{p,0})_{p \in P}$ is a $\prec$ -ordered Morse decomposition of S_0 relative to π_0 . For each $p \in P$, let $V_p \subset \tilde{N}$ be closed in X_0 and such that $M_{p,0} = \text{Inv}_{\pi_0}(V_p) \subset \text{Int}_{X_0}(V_p)$.*

Let $\eta \in]0, \beta]$. For $\varepsilon \in]0, \varepsilon_0]$ and $p \in P$ set $S_\varepsilon := \text{Inv}_{\pi_\varepsilon}([\tilde{N}]_{\varepsilon, \eta})$ and $M_{p,\varepsilon} := \text{Inv}_{\pi_\varepsilon}([V_p]_{\varepsilon, \eta})$. Then there is an $\tilde{\varepsilon} \in]0, \varepsilon_0]$ such that for every $\varepsilon \in]0, \tilde{\varepsilon}]$ and $p \in P$, $S_\varepsilon \subset \text{Int}_{Z_\varepsilon}([\tilde{N}]_{\varepsilon, \eta})$, $M_{p,\varepsilon} \subset \text{Int}_{Z_\varepsilon}([V_p]_{\varepsilon, \eta})$, $p \in P$, and the family $(M_{p,\varepsilon})_{p \in P}$ is a $\prec$ -ordered Morse decomposition of S_ε relative to π_ε .

Suppose that S is a compact invariant set relative to π and let $(M_p)_{p \in P}$ be a $\prec$ -ordered Morse decomposition of S (relative to π). An *index filtration* for $(\pi, S, (M_p)_{p \in P})$ is a family $\mathcal{N} = (N(I))_{I \in \mathcal{A}(\prec)}$ of closed subsets of Y such that

- (1) for each $I \in \mathcal{A}(\prec)$, the pair $(N(I), N(\emptyset))$ is an FM-index pair in $M(I)$,
- (2) for each $I_1, I_2 \in \mathcal{A}(\prec)$, $N(I_1 \cap I_2) = N(I_1) \cap N(I_2)$ and $N(I_1 \cup I_2) = N(I_1) \cup N(I_2)$.

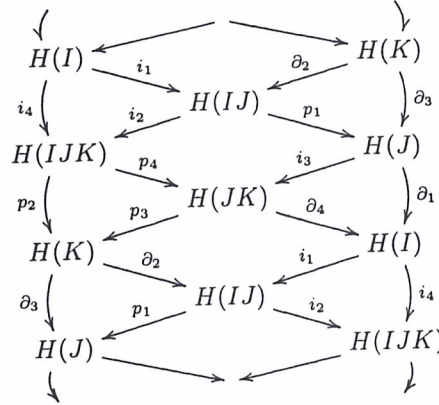
$\mathcal{N}$ is called *strongly π -admissible* if $N(P)$ is strongly π -admissible.

Let $\mathcal{N}$ be a strongly π -admissible index filtration for $(\pi, S, (M_p)_{p \in P})$. For $J \in \mathcal{I}(\prec)$ the set $M(J)$ is an isolated invariant set and we write $H(J) = H(\pi, J) := H(\pi, M(J))$. If $(I, J) \in \mathcal{I}_2(\prec)$, then $(M(I), M(J))$ is an attractor-repeller pair in $M(IJ)$, where $IJ := I \cup J$. Hence there is the corresponding homology index sequence

$$\longrightarrow H_q(I) \xrightarrow{i_{I,J}} H_q(IJ) \xrightarrow{p_{I,J}} H_q(J) \xrightarrow{\partial_{I,J}} H_{q-1}(I) \longrightarrow$$

of $(\pi, M(IJ), M(I), M(J))$. Using the filtration $\mathcal{N}$ one proves that for every triple $(I, J, K) \in \mathcal{I}_3(\prec)$ the following diagram, made up of the four homology index

sequences defined by (I, J, K) , commutes:



The collection of all the homology indices $H(\pi, M(J))$, $J \in \mathcal{I}(\prec)$, and all the maps $i_{I,J}$, $p_{I,J}$ and $\partial_{I,J}$, $(I, J) \in \mathcal{I}_2(\prec)$, is called the *homology index braid* of $(\pi, S, (M_p)_{p \in P})$ and is denoted by $\mathcal{H}(\pi, S, (M_p)_{p \in P})$.

For the rest of this section assume that, for $i = 1, 2$, π_i is a local semiflow on the metric space Y_i , S_i is an isolated invariant set and $(M_{p,i})_{p \in P}$ is a $\prec$ -ordered Morse decomposition of S_i , relative to π_i . Write $M_i(I) = M_{\pi_i, S_i}(I)$, $H_i(I) = H(\pi_i, M_i(I))$ and $\mathcal{H}_i := \mathcal{H}(\pi_i, S_i, (M_{p,i})_{p \in P})$, for $i = 1, 2$ and $I \in \mathcal{I}(\prec)$.

Suppose $\theta := (\theta(J))_{J \in \mathcal{I}(\prec)}$ is a family $\theta(J): H_1(J) \rightarrow H_2(J)$, $J \in \mathcal{I}(\prec)$, of maps such that, for all $(I, J) \in \mathcal{I}_2(\prec)$, the diagram

$$(2.1) \quad \begin{array}{ccccccc} \longrightarrow & H_{1,q}(I) & \xrightarrow{i_{I,J}} & H_{1,q}(IJ) & \xrightarrow{p_{I,J}} & H_{1,q}(J) & \xrightarrow{\partial_{I,J}} & H_{1,q-1}(I) & \longrightarrow \\ & \downarrow \theta(I) & & \downarrow \theta(IJ) & & \downarrow \theta(J) & & \downarrow \theta(I) & \\ \longrightarrow & H_{2,q}(I) & \xrightarrow{i_{I,J}} & H_{2,q}(IJ) & \xrightarrow{p_{I,J}} & H_{2,q}(J) & \xrightarrow{\partial_{I,J}} & H_{2,q-1}(I) & \longrightarrow \end{array}$$

commutes. Then we say that θ is a *morphism* from $\mathcal{H}_1$ to $\mathcal{H}_2$ and we write $\theta: \mathcal{H}_1 \rightarrow \mathcal{H}_2$. If each $\theta(J)$ is an isomorphism, then we say that θ is an *isomorphism* and that $\mathcal{H}_1$ and $\mathcal{H}_2$ are *isomorphic* homology index braids.

We may analogously define the concept of cohomology index braids and morphisms between them: just use cohomology instead of homology and reverse all the arrows.

Remark 2.6. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic (co)homology index braids, then, by [10, Proposition 1.5], $\mathcal{H}_1$ and $\mathcal{H}_2$ determine the same collection of C -connection matrices.

Proposition 2.7. Let X and X' be metric spaces, π be a local semiflow on X and π' be a local semiflow on X' . Let $\gamma: X \rightarrow X'$ be a homeomorphism which conjugates π with π' . Let S be an isolated π -invariant set which has a strongly π -admissible isolating neighborhood, and $(M_p)_{p \in P}$ be a Morse decomposition of S , relative to π . Set $S' = \gamma(S)$ and $M'_p = \gamma(M_p)$, $p \in P$.

Then S' is an isolated π' -invariant set which has a strongly π' -admissible isolating neighborhood, and $(M'_p)_{p \in P}$ is a Morse decomposition of S' , relative to π' .

Moreover, the (co)homology braid of $(\pi, S, (M_p)_{p \in P})$ is isomorphic to the (co)homology index braid of $(\pi', S', (M'_p)_{p \in P})$.

Proof. Intuitively, this result is clear. A rigorous proof follows by an application of [8, Theorem 3.2] in the case of homology and of [18, Theorem 4.2] in the case of cohomology. $\square$

We will require the following singular continuation principle for homology index braids and connection matrices established in [7].

Theorem 2.8. [7, Theorem 3.10] *Assume the hypotheses (and thus also the conclusions) of Theorem 2.5 and let $\tilde{\varepsilon} > 0$ be as in that theorem. Let $\tilde{\beta}_0 \in]0, \beta[$ be fixed. Suppose that there exists an $\beta_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$ and all $\eta \in]0, \beta_0]$ the set $\text{Cl}_{Y_\varepsilon}(B_\varepsilon(\theta_\varepsilon, \eta))$ is contractible. Then there exists an $\varepsilon_c \in]0, \tilde{\varepsilon}]$ such that the homology index braids of $(\pi_0, S_0, (M_p)_{p \in P})$ and $(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p \in P})$, $\varepsilon \in]0, \varepsilon_c]$, are isomorphic and so they determine the same collection of C -connection matrices.*

Theorem 2.8 refines the corresponding singular Conley index continuation principle established in [2].

Remark 2.9. Analogous definitions and results hold for Alexander-Spanier cohomology (with coefficients in G) with the obvious modifications (cf. [6]).

3. A FINITE DIMENSIONAL PERTURBATION PROBLEM

In this section we state and prove a continuation principle for homology index braids and connection matrices for a class of singularly perturbed ordinary differential equations defined on a finite dimensional space (cf. Theorems 3.1 and 3.6). Throughout this section, we assume reader's familiarity with the results from [17, Section 3].

Let $(E, |\cdot|)$ be a finite dimensional Banach space. Given a C^1 -map $g: E \rightarrow E$ and $\varepsilon > 0$ let $\pi_{\varepsilon,g}$ be the local (semi)flow generated by the following ordinary differential equation on $E \times E$:

$$(3.1) \quad \dot{u} = v, \quad \dot{v} = (1/\varepsilon)(-v + g(u)), \quad (u, v) \in E.$$

Furthermore, let π'_g be the local (semi)flow on E generated by the following ordinary differential equation on E :

$$(3.2) \quad \dot{u} = g(u), \quad u \in E.$$

One of the goals of this section is to prove the following result.

Theorem 3.1. *Let $N' \subset E$ be a compact isolating neighborhood relative to π'_g . Set $S' := \text{Inv}_{\pi'_g}(N')$ and let $(M'_p)_{p \in P}$ be a $\prec$ -ordered Morse decomposition for S' relative to π'_g . For each $p \in P$, let $V'_p \subset N'$ be closed in E such that $M'_p = \text{Inv}_{\pi'_g}(V'_p) \subset \text{Int}_E(V'_p)$. For every $\alpha > 0$, $\varepsilon > 0$ and $p \in P$, we set $S_{\varepsilon,\alpha} := \text{Inv}_{\pi_{\varepsilon,g}}(N'_\alpha)$ and $M_{p,\varepsilon,\alpha} := \text{Inv}_{\pi_{\varepsilon,g}}(V'_{p,\alpha})$, where*

$$N'_\alpha = N'_{\alpha,g} := \{ (u, v) \mid u \in N' \text{ and } |v - g(u)| \leq \alpha \},$$

$$V'_{p,\alpha} = V'_{p,\alpha,g} := \{ (u, v) \mid u \in V'_p \text{ and } |v - g(u)| \leq \alpha \}.$$

Then for every $\alpha \in]0, \infty[$, there exists an $\varepsilon_0 = \varepsilon_0(\alpha) \in]0, \infty[$ such that for every $\varepsilon \in]0, \varepsilon_0]$, the set N'_α is an isolating neighborhood of $S_{\varepsilon, \alpha}$ relative to $\pi_{\varepsilon, g}$, and for every $p \in P$, $V'_{p, \alpha}$ is an isolating neighborhood of $M_{p, \varepsilon, \alpha}$ relative to $\pi_{\varepsilon, g}$. Moreover, the family $(M_{p, \varepsilon, \alpha})_{p \in P}$ is a $\prec$ -ordered Morse decomposition for $S_{\varepsilon, \alpha}$ relative to $\pi_{\varepsilon, g}$ and the homology index braids of $(\pi_{\varepsilon, g}, S_{\varepsilon, \alpha}, (M_{p, \varepsilon, \alpha})_{p \in P})$ and $(\pi'_g, S', (M'_p)_{p \in P})$ are isomorphic and so they share the same collection of C -connection matrices.

The proof of Theorem 3.1 will be based on Theorem 2.8. We proceed similarly as in the proof of [17, Theorem 3.1]. Let U be a bounded open neighborhood of N' and $\tilde{g}: E \rightarrow E$ be a C^1 -map such that $g|_U = \tilde{g}|_U$ and $\sup_{u \in E} (|\tilde{g}(u)| + |D\tilde{g}(u)|) < \infty$. The existence of $\tilde{g}$ follows since E is finite-dimensional.

Since the differential equations defining $\pi_{\varepsilon, g}$ and $\pi_{\varepsilon, \tilde{g}}$ coincide on the open neighborhood $U \times E$ of N'_α in $E \times E$ it follows that N'_α (resp. $V'_{p, \alpha}$) is an isolating neighborhood relative to $\pi_{\varepsilon, g}$ if and only if N'_α (resp. $V'_{p, \alpha}$) is an isolating neighborhood relative to $\pi_{\varepsilon, \tilde{g}}$ and

$$\begin{aligned} \text{Inv}_{\pi_{\varepsilon, g}}(N'_\alpha) &= S_{\varepsilon, \alpha} = \text{Inv}_{\pi_{\varepsilon, \tilde{g}}}(N'_\alpha), \\ \text{Inv}_{\pi_{\varepsilon, g}}(V'_{p, \alpha}) &= M_{p, \varepsilon, \alpha} = \text{Inv}_{\pi_{\varepsilon, \tilde{g}}}(V'_{p, \alpha}), \quad p \in P. \end{aligned}$$

Moreover, $(M_{p, \varepsilon, \alpha})_{p \in P}$ is a $\prec$ -ordered Morse decomposition for $S_{\varepsilon, \alpha}$ relative to $\pi_{\varepsilon, g}$ if and only if $(M_{p, \varepsilon, \alpha})_{p \in P}$ is a $\prec$ -ordered Morse decomposition for $S_{\varepsilon, \alpha}$ relative to $\pi_{\varepsilon, \tilde{g}}$.

Similarly, N' (resp. V'_p) is an isolating neighborhood relative to π'_g if and only if N' (resp. V'_p) is an isolating neighborhood relative to $\pi'_{\tilde{g}}$ and

$$\text{Inv}_{\pi'_g}(N') = S' = \text{Inv}_{\pi'_{\tilde{g}}}(N') \quad \text{and} \quad \text{Inv}_{\pi'_g}(V'_p) = M'_p = \text{Inv}_{\pi'_{\tilde{g}}}(V'_p), \quad p \in P.$$

Furthermore, $(M'_p)_{p \in P}$ is a $\prec$ -ordered Morse decomposition for S' relative to π'_g if and only if $(M'_p)_{p \in P}$ is a $\prec$ -ordered Morse decomposition for S' relative to $\pi'_{\tilde{g}}$. Thus the homology index braids of $(\pi_{\varepsilon, g}, S_{\varepsilon, \alpha}, (M_{p, \varepsilon, \alpha})_{p \in P})$ and $(\pi'_g, S', (M'_p)_{p \in P})$ are isomorphic if and only if the homology index braids of $(\pi_{\varepsilon, \tilde{g}}, S_{\varepsilon, \alpha}, (M_{p, \varepsilon, \alpha})_{p \in P})$ and $(\pi'_{\tilde{g}}, S', (M'_p)_{p \in P})$ are isomorphic.

Hence we may assume, without loss of generality, that

$$C := \sup_{u \in E} (|g(u)| + |Dg(u)|) < \infty.$$

In particular, g is globally Lipschitzian and so both $\pi_{\varepsilon, g}$, $\varepsilon > 0$, and π'_g are global semiflows. We write $\pi_\varepsilon := \pi_{\varepsilon, g}$, $\varepsilon > 0$, and $\pi' := \pi'_g$ for short.

Notice that the map $\Phi: E \times E \rightarrow E \times E$, $\Phi(u, v) = (u, w) := (u, v - g(u))$ is a C^1 -diffeomorphism with inverse Φ^{-1} given by $\Phi^{-1}(u, w) = (u, v) := (u, w + g(u))$. Let $\tilde{\pi}_\varepsilon$ be the conjugate of π_ε via Φ i.e. $(u, w)\tilde{\pi}_\varepsilon t := \Phi((\Phi^{-1}(u, w))\pi_\varepsilon t)$, $(u, w) \in E \times E$, $t \in [0, \infty[$. Note that $\tilde{\pi}_\varepsilon$ is the semiflow generated by the equation

$$\begin{aligned} \dot{u} &= w + g(u) \\ \dot{w} &= -(1/\varepsilon)w - Dg(u)(w + g(u)). \end{aligned} \tag{3.3}$$

Let B_α be the closed ball in E with radius α centered at zero. It follows that $\Phi(N'_\alpha) = N' \times B_\alpha$ (resp. $\Phi(V'_{p, \alpha}) = V'_p \times B_\alpha$, $p \in P$).

We have the following

Proposition 3.2. *For every $\alpha \in]0, \infty[$, there exists an $\varepsilon_0 = \varepsilon_0(\alpha) \in]0, \infty[$ such that for every $\varepsilon \in]0, \varepsilon_0]$ and $p \in P$, the set $N' \times B_\alpha$ (resp. $V'_p \times B_\alpha$) is an isolating neighborhood of $\tilde{S}_{\varepsilon, \alpha}$ (resp. $\tilde{M}_{p, \varepsilon, \alpha}$) relative to $\tilde{\pi}_\varepsilon$, where*

$$\tilde{S}_{\varepsilon, \alpha} = \text{Inv}_{\tilde{\pi}_\varepsilon}(N' \times B_\alpha) \text{ and } \tilde{M}_{p, \varepsilon, \alpha} = \text{Inv}_{\tilde{\pi}_\varepsilon}(V'_p \times B_\alpha), p \in P.$$

Moreover, the family $(\tilde{M}_{p, \varepsilon, \alpha})_{p \in P}$ is a $\prec$ -ordered Morse decomposition for $\tilde{S}_{\varepsilon, \alpha}$ relative to $\tilde{\pi}_\varepsilon$ and the homology index braid of $(\tilde{\pi}_\varepsilon, \tilde{S}_{\varepsilon, \alpha}, (\tilde{M}_{p, \varepsilon, \alpha})_{p \in P})$ is isomorphic to the homology index braid of $(\pi', S', (M'_p)_{p \in P})$.

Proof. Define $X_0 = Y_\varepsilon = E$, $\theta_\varepsilon = 0$ and $d_0(u, u') = d_\varepsilon(u, u') = |u - u'|$ for all $\varepsilon > 0$ and $u, u' \in E$. It follows that $N' \times B_\alpha = [N']_{\varepsilon, \alpha}$ for all $\varepsilon > 0$ and $\alpha > 0$. Notice that $\text{Cly}_\varepsilon(B_\varepsilon(\theta_\varepsilon, \alpha)) = B_\alpha$, for all $\varepsilon > 0$ and $\alpha > 0$ and so $\text{Cly}_\varepsilon(B_\varepsilon(\theta_\varepsilon, \alpha))$ is contractible. The proof of [17, Lemma 3.6] shows that the set N' is singularly strongly admissible with respect to α and the family $(\tilde{\pi}_\varepsilon)_{\varepsilon > 0}$ singularly converges to π' . Now an application of Theorems 2.5 and 2.8 completes the proof. $\square$

Proposition 3.3. *Let $\alpha \in]0, \infty[$ and let $\varepsilon_0 = \varepsilon_0(\alpha) \in]0, \infty[$ as in Proposition 3.2. Then for every $\varepsilon \in]0, \varepsilon_0]$ and $p \in P$, the set N'_α (resp. $V'_{p, \alpha}$) is an isolating neighborhood of $S_{\varepsilon, \alpha}$ (resp. $M_{p, \varepsilon, \alpha}$) relative to π_ε , the family $(M_{p, \varepsilon, \alpha})_{p \in P}$ is a $\prec$ -ordered Morse decomposition for $S_{\varepsilon, \alpha}$ relative to π_ε and the homology index braids $(\pi_\varepsilon, S_{\varepsilon, \alpha}, (M_{p, \varepsilon, \alpha})_{p \in P})$ and $(\tilde{\pi}_\varepsilon, \tilde{S}_{\varepsilon, \alpha}, (\tilde{M}_{p, \varepsilon, \alpha})_{p \in P})$ are isomorphic.*

Proof. This follows from Proposition 2.7. $\square$

Proof of Theorem 3.1. This follows from Propositions 3.3 and 3.2. $\square$

We shall now generalize Theorem 3.1 to comprise isolating neighborhoods which are more general than N'_α . As in [17], define $\mathcal{T}_g$ to be the set of all functions $z: \mathbb{R} \rightarrow E \times E$ such that there is a full bounded solution $u: \mathbb{R} \rightarrow E$ of π'_g so that $z(t) = (u(t), g(u(t)))$ for all $t \in \mathbb{R}$. Thus, defining the map $\Gamma_g: E \rightarrow E \times E$ by $\Gamma_g(\xi) = (\xi, g(\xi))$, $\xi \in E$, we see that $\mathcal{T}_g$ is the set of all functions $z: \mathbb{R} \rightarrow E \times E$ such that there is a full bounded solution $u: \mathbb{R} \rightarrow E$ of π'_g with $z = \Gamma_g \circ u$.

Proposition 3.4. *Set $\pi'_g := \pi'$ and $\Gamma := \Gamma_g$. Let K' be a compact isolated invariant set relative to π' . Then $\Gamma(K')$ is a $\mathcal{T}_g$ -isolated invariant set. Let N be a bounded $\mathcal{T}_g$ -isolating neighborhood of $\Gamma(K')$. Let $\beta \in]0, \infty[$ and B'_β be the closed β -neighborhood of K' in E . For each $\alpha \in]0, \infty[$, define*

$$B'_{\beta, \alpha} := \{ (u, v) \mid u \in B'_\beta, |v - g(u)| \leq \alpha \}.$$

Then there exist an $\alpha \in]0, \infty[$ and a $\beta \in]0, \infty[$ such that B'_β is an isolating neighborhood of K' relative to π' and $B'_{\beta, \alpha} \subset N$. Moreover, there exists an $\varepsilon_1 \in]0, \infty[$ such that for every $\varepsilon \in]0, \varepsilon_1]$, $\text{Inv}_{\pi_\varepsilon}(B'_{\beta, \alpha}) = \text{Inv}_{\pi_\varepsilon}(N)$.

Proof. This follows from [17, Lemma 3.8 and proof of Theorem 3.9]. $\square$

We also need the following auxiliary result:

Proposition 3.5. *Set $\pi'_g := \pi'$ and $\Gamma := \Gamma_g$. Let K' be a compact isolated invariant set relative to π' and $K := \Gamma(K')$. Let $\mathcal{T}$ be the set of all $(u, v) \in \mathcal{T}_g$ such that $(u(t), v(t)) \in K$ for all $t \in \mathbb{R}$. Moreover, let $(M'_p)_{p \in P}$ be a $\prec$ -ordered Morse*

decomposition of K' relative to π' and define $M_{p,\Gamma} := \Gamma(M'_p)$, $p \in P$. For every $I \in \mathcal{I}(\prec)$ set

$$M'(I) := \bigcup_{p,q \in I} \text{CS}_{\pi',K'}(M'_p, M'_q)$$

and $M_\Gamma(I) := \Gamma(M'(I))$. Then $(M_{p,\Gamma})_{p \in P}$ is a $\prec$ -ordered $\mathcal{T}$ -Morse decomposition and

$$(3.4) \quad M_\Gamma(I) = \bigcup_{p,q \in I} \text{CS}_\mathcal{T}(M_p, M_q), \quad I \in \mathcal{I}(\prec).$$

Moreover, the sets K and $M_{p,\Gamma}$, $p \in P$, and $M_\Gamma(I)$, $I \in \mathcal{I}(\prec)$, are $\mathcal{T}_g$ -isolated invariant sets.

Proof. Write $M_p = M_{p,\Gamma}$ and $M(I) = M_\Gamma(I)$ for short. We see either directly or using [4, Corollary 3.6] that M'_p , $p \in P$, and $M'(I)$, $I \in \mathcal{I}(\prec)$, are compact isolated invariant sets relative to π' . By [17, Lemma 3.8], the sets K , M_p , $p \in P$, and $M(I)$, $I \in \mathcal{I}(\prec)$, are $\mathcal{T}_g$ -isolated invariant sets. Moreover, the sets M_p are closed in $E \times E$ (being compact in $E \times E$) and pairwise disjoint (Γ being one-to-one). If $p \in P$ and $(\bar{u}, \bar{v}) \in M_p$ are arbitrary, then $\bar{u} \in M'_p$ so there is a full solution u of π' lying in M'_p with $\bar{u} = u(0)$. Thus u lies in K' and so $z := \Gamma \circ u$ lies in $M_p \subset K$ and is an element of $\mathcal{T}_g$. Hence $z \in \mathcal{T}$, z lies in M_p and $z(0) = (\bar{u}, \bar{v})$. It follows that M_p is $\mathcal{T}$ -invariant.

Let $(u, v) \in \mathcal{T}$ be arbitrary. Then u is a full solution of π' lying in K' . Thus either u lies in M'_k for some $k \in P$, which implies that (u, v) lies in M_k or else there are $k, l \in P$ with $k \prec l$, $\alpha(u) \subset M'_l$ and $\omega(u) \subset M'_k$. In the latter case it is clear from the continuity of Γ that $\alpha(u, v) \subset M_l$ and $\omega(u, v) \subset M_k$. We have proved that $(M_p)_{p \in P}$ is a $\prec$ -ordered $\mathcal{T}$ -Morse decomposition.

Now let $p, q \in P$ be arbitrary. We claim that

$$\Gamma(\text{CS}_{\pi',K'}(M'_p, M'_q)) = \text{CS}_\mathcal{T}(M_p, M_q).$$

If u is a full solution of π' lying in K' with $\alpha(u) \subset M'_p$ and $\omega(u) \subset M'_q$ then, clearly, $z := \Gamma \circ u \in \mathcal{T}_g$, z lies in K , $\alpha(z) \subset M_p$ and $\omega(z) \subset M_q$. Thus

$$\Gamma(\text{CS}_{\pi',K'}(M'_p, M'_q)) \subset \text{CS}_\mathcal{T}(M_p, M_q).$$

Conversely, if $z \in \mathcal{T}$ with $\alpha(z) \subset M_p$ and $\omega(z) \subset M_q$ then $z = \Gamma \circ u$, where u is a full solution of π' lying in K' . Clearly, $\alpha(u) \subset M'_p$ and $\omega(u) \subset M'_q$. Hence

$$\Gamma(\text{CS}_{\pi',K'}(M'_p, M'_q)) \supset \text{CS}_\mathcal{T}(M_p, M_q).$$

This proves our claim, which, in turn, immediately implies (3.4). $\square$

The generalization of Theorem 3.1 reads as follows:

Theorem 3.6. *Assume the hypotheses (and hence the conclusions) of Proposition 3.5. Let N be a bounded $\mathcal{T}_g$ -isolating neighborhood of K and $N_p \subset N$ be a $\mathcal{T}_g$ -isolating neighborhood of $M_{p,\Gamma}$, $p \in P$. For $\varepsilon \in]0, \infty[$, set $\pi_{\varepsilon,g} := \pi_\varepsilon$, $K_\varepsilon := \text{Inv}_{\pi_\varepsilon}(N)$, and $M_{p,\varepsilon,\Gamma} := \text{Inv}_{\pi_\varepsilon}(N_p)$, $p \in P$.*

Then there is an $\tilde{\varepsilon}_0 \in]0, \infty[$ such that for every $\varepsilon \in]0, \tilde{\varepsilon}_0]$, N is an isolating neighborhood of K_ε relative to π_ε , and for every $p \in P$, N_p is an isolating neighborhood of $M_{p,\varepsilon,\Gamma}$ relative to π_ε . Moreover, the family $(M_{p,\varepsilon,\Gamma})_{p \in P}$ is a $\prec$ -ordered Morse decomposition of K_ε and the homology index braids of $(\pi_\varepsilon, K_\varepsilon, (M_{p,\varepsilon,\Gamma})_{p \in P})$ and $(\pi', K', (M'_p)_{p \in P})$ are isomorphic and so they share the same C -connection matrices.

Proof. Let $\beta \in]0, \infty[$. Let B'_β be the closed β -neighborhood of K' in E . For $p \in P$, let $B'_{p,\beta}$ be the closed β -neighborhood of M'_p in E . For each $\alpha \in]0, \infty[$, define

$$\begin{aligned} B'_{\beta,\alpha} &:= \{ (u, v) \mid u \in B'_\beta, |v - g(u)| \leq \alpha \} \\ B'_{p,\beta,\alpha} &:= \{ (u, v) \mid u \in B'_{p,\beta}, |v - g(u)| \leq \alpha \}, p \in P. \end{aligned}$$

Proposition 3.4 implies that there exist an $\alpha \in]0, \infty[$ and a $\beta \in]0, \infty[$ such that B'_β is an isolating neighborhood of K' relative to π' and $B'_{p,\beta}$ is an isolating neighborhood of M'_p relative to π' , for $p \in P$. Moreover, $B'_{\beta,\alpha} \subset N$ and $B'_{p,\beta,\alpha} \subset N_p$, for $p \in P$. Furthermore, there exists an $\varepsilon_1 \in]0, \infty[$ such that for every $\varepsilon \in]0, \varepsilon_1]$, $\text{Inv}_{\pi_\varepsilon}(B'_{\beta,\alpha}) = \text{Inv}_{\pi_\varepsilon}(N)$ and $\text{Inv}_{\pi_\varepsilon}(B'_{p,\beta,\alpha}) = \text{Inv}_{\pi_\varepsilon}(N_p)$, $p \in P$.

Define $N' := B'_\beta$ and $V'_p := B'_{p,\beta}$, $p \in P$. Hence for every $\varepsilon \in]0, \varepsilon_1]$, we have $N'_\alpha = B'_{\beta,\alpha}$ and $V'_{p,\alpha} = B'_{p,\beta,\alpha}$, $p \in P$. Thus

$$(3.5) \quad K' = \text{Inv}_{\pi'}(N') \subset \text{Int}_E(N') \text{ and } M'_p = \text{Inv}_{\pi'}(V'_p) \subset \text{Int}_E(V'_p), p \in P.$$

Moreover, for all $\varepsilon \in]0, \varepsilon_1]$,

$$(3.6) \quad \begin{aligned} K_\varepsilon &= \text{Inv}_{\pi_\varepsilon}(N) = \text{Inv}_{\pi_\varepsilon}(B'_{\beta,\alpha}) = \text{Inv}_{\pi_\varepsilon}(N'_\alpha), \\ M_{p,\varepsilon,\Gamma} &= \text{Inv}_{\pi_\varepsilon}(N_p) = \text{Inv}_{\pi_\varepsilon}(B'_{p,\beta,\alpha}) = \text{Inv}_{\pi_\varepsilon}(V'_{p,\alpha}), p \in P. \end{aligned}$$

Now an application of Theorem 3.1 together with (3.5) and (3.6) implies that there exists an $\tilde{\varepsilon}_0 \in]0, \varepsilon_1]$ such that for all $\varepsilon \in]0, \tilde{\varepsilon}_0]$, the family $(M_{p,\varepsilon,\Gamma})_{p \in P}$ is a $\prec$ -ordered Morse decomposition of K_ε relative to π_ε and the homology index braids of $(\pi_\varepsilon, K_\varepsilon, (M_{p,\varepsilon,\Gamma})_{p \in P})$ and $(\pi', K', (M'_p)_{p \in P})$ are isomorphic. $\square$

Remark 3.7. The results of this section also hold for cohomology index braids with analogous proofs.

4. CONTINUATION OF HOMOLOGY INDEX BRAID FOR SINGULARLY PERTURBED HYPERBOLIC EQUATIONS

In this section we will state and prove the main result of this paper (cf. Theorem 4.2). After recalling some notations from [17] we state a continuation result for Morse decomposition proved in [17] (cf. Theorem 4.1). Then we state our continuation principle for homology index braids and connection matrices for singularly perturbed hyperbolic equations (cf. Theorem 4.2). We assume the reader's familiarity with the results from [17].

Let X be a real Hilbert space and $A: D(A) \subset X \rightarrow X$ be a positive selfadjoint operator with compact resolvent. Then A is sectorial on X and so it generates the corresponding family $(X^\alpha)_{\alpha \in [0, \infty[}$ of fractional power spaces (cf. [13]). Let $(\phi_\nu)_{\nu \in \mathbb{N}}$ be a complete X -orthonormal basis of X consisting of eigenfunctions of A . Let $P_n: X \rightarrow X$ be the orthogonal projection of X onto the subspace spanned by the

first n eigenfunctions. Moreover, set $Q_n := I - P_n$ where I is the identity map on X .

As in [12] and [17], we will assume the following Standing Hypothesis:

- (4.1) $N \in \{1, 2, 3\}$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and such that $X = L^2(\Omega)$ and X^1 is continuously included in $H^2(\Omega)$;
 $\gamma \in L^2(\Omega)$ and $\phi: \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 -function such that, for $N \geq 2$, there are constants $\bar{r}$ and $\bar{C} \in [0, \infty[$ with $|\phi'(\xi)| \leq \bar{C}(1 + |\xi|^{\bar{r}})$ for all $\xi \in \mathbb{R}$.
 If $N = 3$ then $\bar{r} < 2$.

Define the map $f: X^{1/2} \rightarrow X$ by

$$f(u) = \hat{\phi}(u) + \gamma, \quad u \in X^{1/2}.$$

We denote by π'_f the local semiflow on $X^{1/2}$ generated by the abstract parabolic equation (see [13])

$$\dot{u} = -Au + f(u), \quad u \in X^{1/2}.$$

For $\varepsilon \in]0, \infty[$ we also consider the local semiflow $\pi_{\varepsilon, f}$ on $X^{1/2} \times X$ generated by the damped hyperbolic equation

$$\dot{u} = v, \quad \dot{v} = (1/\varepsilon)(-v - Au + f(u)), \quad (u, v) \in X^{1/2} \times X.$$

Let $\Gamma_f: X^1 \rightarrow X^{1/2} \times X$ be the map defined by

$$\Gamma_f(u) := (u, -Au + f(u)), \quad u \in X^1.$$

Moreover, by $\mathcal{T}_f$ we denote the set of all maps $z: \mathbb{R} \rightarrow X^{1/2} \times X$ for which there is a full bounded solution u of π'_f such that $z(t) = \Gamma_f(u(t))$, $t \in \mathbb{R}$. In view of [17, Corollary 4.7], every full bounded solution of π'_f lies in X^1 , so the definition of $\mathcal{T}_f$ makes sense.

The following result has been proved in [17].

Theorem 4.1. ([17, Theorem 7.4]) *For $\varepsilon \in]0, \infty[$, set $\pi' := \pi'_f$, $\pi_\varepsilon := \pi_{\varepsilon, f}$ and $\Gamma := \Gamma_f$. Let K' be a compact isolated invariant set relative to π' and $K := \Gamma(K')$. Let $\mathcal{T}$ be the set of all $(u, v) \in \mathcal{T}_f$ such that $(u(t), v(t)) \in K$ for all $t \in \mathbb{R}$. Moreover, let $(M'_p)_{p \in P}$ a family of subsets of K' which is a Morse decomposition of K' relative to π' and let $M_p := \Gamma(M'_p)$, $p \in P$. For every $I \in \mathcal{I}(\prec)$ set*

$$M'(I) := \bigcup_{p, q \in I} \text{CS}_{\pi', K'}(M'_p, M'_q)$$

and $M(I) := \Gamma(M'(I))$. Then $(M_p)_{p \in P}$ is a $\prec$ -ordered $\mathcal{T}$ -Morse decomposition and

$$M(I) = \bigcup_{p, q \in I} \text{CS}_{\mathcal{T}}(M_p, M_q), \quad I \in \mathcal{I}(\prec).$$

Moreover, the sets K , M_p , $p \in P$ and $M(I)$, $I \in \mathcal{I}(\prec)$, are $\mathcal{T}_f$ -isolated invariant sets. Let N be a bounded $\mathcal{T}_f$ -isolating neighborhood of K , $V_p \subset N$ be a $\mathcal{T}_f$ -isolating

neighborhood of M_p , $p \in P$, and $V_I \subset N$ be a T_f -isolating neighborhood of $M(I)$, $I \in \mathcal{I}(\prec)$.

For $\varepsilon \in]0, \infty[$ set $K_\varepsilon := \text{Inv}_{\pi_\varepsilon}(N)$, $M_{p,\varepsilon} := \text{Inv}_{\pi_\varepsilon}(V_p)$, $p \in P$ and $M_\varepsilon(I) := \text{Inv}_{\pi_\varepsilon}(V_I)$, $I \in \mathcal{I}(\prec)$.

Then there is an $\tilde{\varepsilon} \in]0, \infty[$ such that for every $\varepsilon \in]0, \tilde{\varepsilon}]$, N (resp. V_p , resp. V_I) is an isolating neighborhood of K_ε (resp. $M_{p,\varepsilon}$, resp. $M_\varepsilon(I)$) relative to π_ε , for all $p \in P$ and all $I \in \mathcal{I}(\prec)$, the family $(M_{p,\varepsilon})_{p \in P}$ is a $\prec$ -ordered Morse decomposition of K_ε and

$$M_\varepsilon(I) = \bigcup_{p,q \in I} \text{CS}_{\pi_\varepsilon, K_\varepsilon}(M_{p,\varepsilon}, M_{q,\varepsilon}), \quad I \in \mathcal{I}(\prec).$$

We can now state our main result.

Theorem 4.2. Assume the hypotheses (and thus also the conclusions) of Theorem 4.1 and let $\tilde{\varepsilon} \in]0, \infty[$ be as in that theorem. Then there is an $\varepsilon_0 \in]0, \tilde{\varepsilon}]$ such that for every $\varepsilon \in]0, \varepsilon_0]$, the homology index braids of $(\pi_\varepsilon, K_\varepsilon, (M_{p,\varepsilon})_{p \in P})$ and $(\pi', K', (M'_p)_{p \in P})$ are isomorphic and so they share the same C -connection matrices.

Theorem 4.2 refines the corresponding Conley index continuation principle established in [17].

The rest of this section is devoted to the proof of our main theorem. The proof of Theorem 4.2 follows from a series of lemmas. For the rest of this section, for $n \in \mathbb{N}$ and $\theta \in [0, 1]$, let $f_{n,\theta}: X^{1/2} \rightarrow X$ be defined by

$$\begin{aligned} f_{n,\theta} &= (1 - \theta)f(u) + \theta P_n f(P_n u) \\ &= (1 - \theta)(\hat{\phi}(u) + \gamma) + \theta P_n(\hat{\phi}(P_n u) + \gamma), \quad u \in X^{1/2}. \end{aligned}$$

The following lemma has been proved in [17].

Lemma 4.3. ([17, Theorem 4.3 and Proposition 4.4]) Let $N' \subset X^{1/2}$ be bounded and closed. Furthermore, let $(n_m)_m$ be a sequence in $\mathbb{N}$ with $n_m \rightarrow \infty$ and $(\theta_m)_m$ be an arbitrary sequence in $[0, 1]$. For every $m \in \mathbb{N}$, let u_m be a full solution of $\pi'_{f_{n_m, \theta_m}}$ lying in N' . Then there is a sequence $(m_k)_k$ with $m_k \rightarrow \infty$ and there is a full solution u of π' lying in N' such that $u_{m_k}(t) \rightarrow u(t)$ in $X^{1/2}$, uniformly for t lying in compact subsets of $\mathbb{R}$.

Lemma 4.4. Let N' be bounded and closed in $X^{1/2}$ with $K' := \text{Inv}_{\pi'}(N') \subset \text{Int}_{X^{1/2}}(N')$. For each $p \in P$, let $V'_p \subset N'$ be closed in $X^{1/2}$ such that $M'_p = \text{Inv}_{\pi'}(V'_p) \subset \text{Int}_{X^{1/2}}(V'_p)$. For $n \in \mathbb{N}$, $\theta \in [0, 1]$ and $p \in P$ define $K'_{n,\theta} = \text{Inv}_{\pi'_{f_{n,\theta}}}(N')$ and $M'_{p,n,\theta} = \text{Inv}_{\pi'_{f_{n,\theta}}}(V'_p)$. Then there is an $n_0 \in \mathbb{N}$ so that whenever $n \geq n_0$ and $\theta \in [0, 1]$, then $K'_{n,\theta} \subset \text{Int}_{X^{1/2}}(N')$, $M'_{p,n,\theta} \subset \text{Int}_{X^{1/2}}(V'_p)$, $p \in P$, and the family $(M'_{p,n,\theta})_{p \in P}$ is a $\prec$ -ordered Morse decomposition of $K'_{n,\theta}$ relative to $\pi'_{f_{n,\theta}}$. Moreover, for all $n \geq n_0$ and all $\theta \in [0, 1]$, the homology index braid of $(\pi'_{f_{n,\theta}}, K'_{n,\theta}, (M'_{p,n,\theta})_{p \in P})$ is isomorphic to the homology index braid of $(\pi', K', (M'_p)_{p \in P})$.

Proof. This follows from [5, Corollaries 3.9 and 3.10]. $\square$

Given $n \in \mathbb{N}$ we consider the local semiflow π'_n generated on the finite dimensional space $Y_n := P_n(X^{1/2}) = P_n(X)$ by the ordinary differential equation

$$\dot{u} = -Au + P_n f(P_n u), \quad u \in Y_n. \quad (4.2)$$

The local semiflow π'_n is the n -Galerkin approximation of π' .

Moreover, let π''_n be the semiflow generated on $Z_n := Q_n(X^{1/2})$ by the evolution equation

$$\dot{u} = -Au, \quad u \in Z_n. \quad (4.3)$$

By [17, Proposition 4.2] and its proof, the space Y_n is positively invariant relative to the local semiflow $\pi'_{f_{n,1}}$ and every bounded $\pi'_{f_{n,1}}$ -invariant set is included in Y_n and is π'_n -invariant. Moreover, every π'_n -invariant set is $\pi'_{f_{n,1}}$ -invariant. Setting $K'_n := K'_{n,1}$ and $M'_{p,n} := M'_{p,n,1}$, $p \in P$, we thus see that, whenever $n \geq n_0$, where $n_0 \in \mathbb{N}$ is as in Lemma 4.4, the set K'_n is a compact π'_n -invariant set and the family $(M'_{p,n})_{p \in P}$ is a $\prec$ -ordered Morse decomposition of K'_n relative to π'_n . Moreover, the following continuation result has been proved in [5].

Lemma 4.5. *Let $n_0 \in \mathbb{N}$ be as in Lemma 4.4. Then for all $n \geq n_0$, the homology index braids of $(\pi'_{f_{n,1}}, K'_n, (M'_{p,n})_{p \in P})$ and $(\pi'_n, K'_n, (M'_{p,n})_{p \in P})$ are isomorphic.*

Now, Lemmas 4.4 and 4.5 imply the following result.

Proposition 4.6. ([5, Theorem 3.11]) *Let $n_0 \in \mathbb{N}$ be as in Lemma 4.4. Then for all $n \geq n_0$, the homology index braid of $(\pi', K', (M'_p)_{p \in P})$ is isomorphic to the homology index braid of $(\pi'_n, K'_n, (M'_{p,n})_{p \in P})$.*

Lemma 4.7. *Let $(\varepsilon_m)_m$, $(n_m)_m$ and $(\theta_m)_m$ be sequences in $]0, \infty[$, $\mathbb{N}$ and $[0, 1]$, respectively. Suppose that $\varepsilon_m \rightarrow 0$ and $n_m \rightarrow \infty$. For each $m \in \mathbb{N}$, let (u_m, v_m) be a full solution of $\pi_{\varepsilon_m, f_{n_m, \theta_m}}$ lying in N . Then there is a subsequence of $((u_m, v_m))_m$, denoted again by $((u_m, v_m))_m$, and there is a full bounded solution u of π' such that $(u_m, v_m) \rightarrow (u, v)$ in $X^{1/2} \times X$, uniformly on compact subsets of $\mathbb{R}$. Here, $(u, v) = \Gamma \circ u$.*

Proof. This follows from [17, Theorem 5.13]. $\square$

Lemma 4.8. *For $\varepsilon \in]0, \infty[$, $n \in \mathbb{N}$, $\theta \in [0, 1]$, $p \in P$ and $I \in \mathcal{I}(\prec)$, define $K_{\varepsilon, n, \theta} = \text{Inv}_{\pi_{\varepsilon, f_{n, \theta}}}(N)$, $M_{p, \varepsilon, n, \theta} = \text{Inv}_{\pi_{\varepsilon, f_{n, \theta}}}(V_p)$ and $M_{\varepsilon, n, \theta}(I) = \text{Inv}_{\pi_{\varepsilon, f_{n, \theta}}}(V_I)$. Let $\tilde{\varepsilon} \in]0, \infty[$ be as in Theorem 4.1. Then there exist an $n' \in \mathbb{N}$ and an $\varepsilon' \in]0, \tilde{\varepsilon}]$ so that whenever $n \geq n'$, $\varepsilon \in]0, \varepsilon']$ and $\theta \in [0, 1]$, then $K_{\varepsilon, n, \theta} \subset \text{Int}_{X^{1/2} \times X}(N)$, $M_{\varepsilon, p, n, \theta} \subset \text{Int}_{X^{1/2} \times X}(V_p)$, $p \in P$, and the family $(M_{p, \varepsilon, n, \theta})_{p \in P}$ is a $\prec$ -ordered Morse decomposition of $K_{\varepsilon, n, \theta}$ relative to $\pi_{\varepsilon, f_{n, \theta}}$.*

Proof. It is well-known (cf. [3, Theorems 5.3 and 5.5]) that N is strongly $\pi_{\varepsilon, f_{n, \theta}}$ -admissible for all $\varepsilon \in]0, \infty[$, $n \in \mathbb{N}$ and $\theta \in [0, 1]$.

It follows from [17, Lemma 6.4] that there exist an $n'_0 \in \mathbb{N}$ and an $\varepsilon'_0 > 0$ such that for all $n \geq n'_0$, $\varepsilon \in]0, \varepsilon'_0]$ and $\theta \in [0, 1]$ the sets N , V_p , $p \in P$, and V_I , $I \in \mathcal{I}(\prec)$, are isolating neighborhoods relative to $\pi_{\varepsilon, f_{n, \theta}}$.

Notice that

$$(4.4) \quad W \subset N \text{ implies that } \text{Inv}_{\mathcal{T}_f}(W) = \text{Inv}_{\mathcal{T}}(W).$$

Suppose the conclusions of the lemma does not hold. Then for all $n' \in \mathbb{N}$ and all $\varepsilon' \in]0, \tilde{\varepsilon}]$ there exist an $n \geq n'$, an $\varepsilon \in]0, \varepsilon']$ and a $\theta \in [0, 1]$ such that the family $(M_{p, \varepsilon, n, \theta})_{p \in P}$ is not a $\prec$ -ordered Morse decomposition of $K_{\varepsilon, n, \theta}$ relative to $\pi_{\varepsilon, f_{n, \theta}}$. Hence there are sequences $(\varepsilon_m)_m$ in $]0, \infty[$, $(n_m)_m$ in $\mathbb{N}$ and $(\theta_m)_m$ in $[0, 1]$ such that $\varepsilon_m \rightarrow 0$, $n_m \rightarrow \infty$ as $m \rightarrow \infty$ and for all $m \in \mathbb{N}$, the family $(M_{p, \varepsilon_m, n_m, \theta_m})_{p \in P}$ is not a $\prec$ -ordered Morse decomposition of $K_{\varepsilon_m, n_m, \theta_m}$ relative to $\pi_{\varepsilon_m, f_{n_m, \theta_m}}$.

For each $m \in \mathbb{N}$, let $\mathcal{T}_{\varepsilon_m, n_m, \theta_m}$ denote the set of all full solution of $\pi_{\varepsilon_m, f_{n_m, \theta_m}}$ lying in N . It follows that

$$M_{p, \varepsilon_m, n_m, \theta_m} = \text{Inv}_{\mathcal{T}_{\varepsilon_m, n_m, \theta_m}}(V_p), \quad p \in P, \quad m \in \mathbb{N}$$

and

$$M_{\varepsilon_m, n_m, \theta_m}(I) = \text{Inv}_{\mathcal{T}_{\varepsilon_m, n_m, \theta_m}}(V_I), \quad I \in I(\prec), \quad m \in \mathbb{N}.$$

We claim that

$$(4.5) \quad \mathcal{T}_{\varepsilon_m, n_m, \theta_m} \rightarrow \mathcal{T} \text{ as } m \rightarrow \infty.$$

In fact, let $(m_k)_k$ be an arbitrary sequence in $\mathbb{N}$ with $m_k \rightarrow \infty$ and for every $k \in \mathbb{N}$ let (u_k, v_k) be a full solution of $\pi_{\varepsilon_{m_k}, f_{n_{m_k}, \theta_{m_k}}}$ lying in N . Thus

$$\sup_{k \in \mathbb{N}} \sup_{t \in \mathbb{R}} \|(u_k(t), v_k(t))\|_{X^{1/2} \times X} < \infty$$

so, by Lemma 4.7, there is a subsequence of $((u_k, v_k))_k$, denoted again by $((u_k, v_k))_k$, and there is a full bounded solution u of π' such that $(u_k, v_k) \rightarrow (u, v)$ in $X^{1/2} \times X$, uniformly on compact subsets of $\mathbb{R}$. Here, $(u, v) = \Gamma \circ u$. Since N is closed in $X^{1/2} \times X$, we see that (u, v) lies in N , so that (u, v) actually lies in K . Thus $(u, v) \in \mathcal{T}$ and the claim (4.5) is proved.

We also claim that

$$(4.6) \quad \mathcal{T} \text{ and } \mathcal{T}_{\varepsilon_m, n_m, \theta_m}, \quad m \in \mathbb{N}, \text{ are compact in } C(\mathbb{R} \rightarrow X^{1/2} \times X), \text{ translation and cut-and-glue invariant.}$$

Recall that

$$(4.7) \quad \mathcal{T}_{\varepsilon_m, n_m, \theta_m} = \mathcal{T}_{\pi_{\varepsilon_m, f_{n_m, \theta_m}}, N}, \text{ for all } m \in \mathbb{N}.$$

Now (4.7) and [4, Proposition 2.7] imply that for $m \in \mathbb{N}$, the set $\mathcal{T}_{\varepsilon_m, n_m, \theta_m}$ is compact in $C(\mathbb{R} \rightarrow X^{1/2} \times X)$, translation and cut-and-glue invariant.

In order to prove the compactness of $\mathcal{T}$, let $((u_\kappa, v_\kappa))_\kappa$ be an arbitrary sequence in $\mathcal{T}$. Thus, for every $\kappa \in \mathbb{N}$, u_κ is a full solution of π' lying in K' . Now an application of [17, Theorem 4.6] (with $f_\kappa \equiv f$) shows that there is a sequence $(\kappa_n)_n$ in $\mathbb{N}$ with $\kappa_n \rightarrow \infty$ and there is a full solution u of π' lying in K' such that $u_{\kappa_n} \rightarrow u$ in X^1 , uniformly on compact subsets of $\mathbb{R}$. Thus $(u_{\kappa_n}, v_{\kappa_n}) \rightarrow (u, v)$ in $X^{1/2} \times X$, uniformly on compact subsets of $\mathbb{R}$, where $(u, v) = \Gamma \circ u$. It follows that $(u, v) \in \mathcal{T}$ and so $\mathcal{T}$ is compact in $C(\mathbb{R} \rightarrow X^{1/2} \times X)$, as claimed. The translation and cut-and-glue invariance of $\mathcal{T}$ is obvious. This completes the proof claim (4.6).

An application of [4, Theorem 3.3] together with (4.4), (4.5) and (4.6) shows that there is an $m_0 \in \mathbb{N}$ such that for every $m \geq m_0$, the family $(M_{p, \varepsilon_m, n_m, \theta_m})_{p \in P}$ is a $\prec$ -ordered $\mathcal{T}_{\varepsilon_m, n_m, \theta_m}$ -Morse decomposition. This is a contradiction. The lemma is proved. $\square$

Lemma 4.9. *Let $n' \in \mathbb{N}$ and $\varepsilon' \in]0, \infty[$ be as in Lemma 4.8. Then for all $n \geq n'$, $\varepsilon \in]0, \varepsilon']$ and $\theta \in [0, 1]$, the homology index braid of $(\pi_{\varepsilon, f_{n, \theta}}, K_{\varepsilon, n, \theta}, (M_{p, \varepsilon, n, \theta})_{p \in P})$ is isomorphic to the homology index braid of $(\pi_{\varepsilon}, K_{\varepsilon}, (M_{p, \varepsilon})_{p \in P})$.*

Proof. Let $n \geq n'$ and $\varepsilon \in]0, \varepsilon']$ be fixed. Let $(\theta_k)_k$ be a sequence in $[0, 1]$ such that $\theta_k \rightarrow \theta_0$ in $[0, 1]$. Then $f_{n, \theta_k}(u) \rightarrow f_{n, \theta_0}(u)$ in X , locally uniformly in $u \in X^{1/2}$. This implies, by results in [16], that $\pi_{\varepsilon, f_{n, \theta_k}} \rightarrow \pi_{\varepsilon, f_{n, \theta_0}}$. Moreover, results in [3] imply that N is $(\pi_{f_{n, \theta_k}})_k$ -admissible. Now Lemma 4.8 and the abstract regular continuation principle [5, Theorem 3.7] complete the proof. $\square$

Given $\varepsilon \in]0, \infty[$ and $n \in \mathbb{N}$ we consider the local semiflow $\pi_{\varepsilon, n}$ generated on the finite dimensional space $X_n := P_n(X^{1/2}) \times P_n(X)$ by the ordinary differential equation

$$\dot{u} = v, \quad \dot{v} = (1/\varepsilon)(-v - Au + P_n f(P_n u)), \quad (u, v) \in X_n.$$

The local semiflow $\pi_{\varepsilon, n}$ is the n -Galerkin approximation of π_{ε} .

Moreover, let $\pi''_{\varepsilon, n}$ be the semiflow generated on $W_n := Q_n(X^{1/2}) \times Q_n(X)$ by the evolution equation

$$(4.8) \quad \dot{u} = v, \quad \dot{v} = (1/\varepsilon)(-v - Au), \quad (u, v) \in W_n.$$

By [17, Proposition 5.3 and its proof], the space X_n is positively invariant relative to the local semiflow $\pi_{\varepsilon, f_{n, 1}}$ and every bounded $\pi_{\varepsilon, f_{n, 1}}$ -invariant set is included in X_n and is $\pi_{\varepsilon, n}$ -invariant. Moreover, every $\pi_{\varepsilon, n}$ -invariant set is $\pi_{\varepsilon, f_{n, 1}}$ -invariant. Setting $K_{\varepsilon, n} := K_{\varepsilon, n, 1}$ and $M_{p, \varepsilon, n} := M_{p, \varepsilon, n, 1}$, $p \in P$, we thus see that, whenever $n \geq n'$, $\varepsilon \in]0, \varepsilon']$, where $n' \in \mathbb{N}$ and $\varepsilon' \in]0, \infty[$ are as in Lemma 4.8, then $K_{\varepsilon, n}$ is a compact $\pi_{\varepsilon, n}$ -invariant set and $(M_{p, \varepsilon, n})_{p \in P}$ is a $\prec$ -ordered Morse decomposition of $K_{\varepsilon, n}$, relative to $\pi_{\varepsilon, n}$. Furthermore,

Lemma 4.10. *Let $n' \in \mathbb{N}$ and $\varepsilon' \in]0, \infty[$ be as in Lemma 4.8. Then for $n \geq n'$ and $\varepsilon \in]0, \varepsilon']$, the homology index braid of $(\pi_{\varepsilon, n}, K_{\varepsilon, n}, (M_{p, \varepsilon, n})_{p \in P})$ is isomorphic to the homology index braid of $(\pi_{\varepsilon, f_{n, 1}}, K_{\varepsilon, n}, (M_{p, \varepsilon, n})_{p \in P})$.*

Proof. Let $\varepsilon \in]0, \varepsilon']$ and $n \geq n'$ be fixed arbitrary. Let $\mathcal{N}'_{\varepsilon, n} = (N'_{\varepsilon, n}(I))_{I \in \mathcal{A}(\prec)}$ be a strongly $\pi_{\varepsilon, n}$ -admissible index filtration for $(\pi_{\varepsilon, n}, K_{\varepsilon, n}, (M_{p, \varepsilon, n})_{p \in P})$. (Strong $\pi_{\varepsilon, n}$ -admissibility means, in this finite-dimensional case, simply that $N'_{\varepsilon, n}(P)$ is bounded in X_n .) Let $B = B_n$ be the unit ball in W_n . Since $|u \pi''_{\varepsilon, n} t|_{W_n} \leq e^{-\beta_n t} |u|_{W_n}$ for some $\beta_n \in]0, \infty[$ and all $u \in W_n$ and $t \in [0, \infty[$ it follows that, relative to $\pi''_{\varepsilon, n}$, B is an isolating block for $\{0\}$ with empty exit set, so in particular, B is positively invariant.

We define $N_{\varepsilon, n}(I) := N'_{\varepsilon, n}(I) + B \cong N'_{\varepsilon, n}(I) \times B$, $I \in \mathcal{A}(\prec)$. It is now a simple exercise to show that $\mathcal{N}_{\varepsilon, n} = (N_{\varepsilon, n}(I))_{I \in \mathcal{A}(\prec)}$ is a strongly $\pi_{\varepsilon, f_{n, 1}}$ -admissible index filtration for $(\pi_{\varepsilon, f_{n, 1}}, K_{\varepsilon, n}, (M_{p, \varepsilon, n})_{p \in P})$. We claim that

$$(4.9) \quad \text{there is an isomorphism } \theta_{\mathcal{N}'_{\varepsilon, n}, \mathcal{N}_{\varepsilon, n}} = (\theta_{N'_{\varepsilon, n}, N_{\varepsilon, n}}(J))_{J \in \mathcal{I}(\prec)} \text{ from the} \\ \text{homology index braid of } (\pi_{\varepsilon, n}, K_{\varepsilon, n}, (M_{p, \varepsilon, n})_{p \in P}) \text{ to the homology index} \\ \text{braid of } (\pi_{\varepsilon, f_{n, 1}}, K_{\varepsilon, n}, (M_{p, \varepsilon, n})_{p \in P}).$$

In fact, let $J \in \mathcal{I}(\prec)$ be arbitrary. Choose $I, K \in \mathcal{A}(\prec)$ with $(I, J) \in \mathcal{I}_2(\prec)$ and $K = IJ$. Let $\phi: N'_{\varepsilon, n}(K)/N'_{\varepsilon, n}(I) \rightarrow N_{\varepsilon, n}(K)/N_{\varepsilon, n}(I)$ be inclusion induced and

$\psi: N_{\varepsilon,n}(K)/N_{\varepsilon,n}(I) \rightarrow N'_{\varepsilon,n}(K)/N'_{\varepsilon,n}(I)$ be induced by the canonical projection $y + z \mapsto y$ of $X^{1/2} \times X = X_n \oplus W_n$ onto X_n . It follows that $\psi \circ \phi$ is the identity on $N'_{\varepsilon,n}(K)/N'_{\varepsilon,n}(I)$ while $\phi \circ \psi$ is homotopic to the identity on $N_{\varepsilon,n}(K)/N_{\varepsilon,n}(I)$ via the homotopy $N_{\varepsilon,n}(K)/N_{\varepsilon,n}(I) \times [0, 1] \rightarrow N_{\varepsilon,n}(K)/N_{\varepsilon,n}(I)$ induced by the homotopy $(y + z, \tau) \mapsto y + \tau z$ on $X^{1/2} \times X = X_n \oplus W_n$. The homotopy axiom for singular homology now implies that the map

$$\theta_{N'_{\varepsilon,n}, N_{\varepsilon,n}}(J): H(\pi_{\varepsilon,n}, M_{\varepsilon,n}(J)) \rightarrow H(\pi_{\varepsilon, f_{n,1}}, M_{\varepsilon,n}(J))$$

induced by ϕ is an isomorphism. Claim (4.9) is proved which in turn concludes the proof of the lemma. $\square$

Now, Lemmas 4.9 and 4.10 imply the following proposition.

Proposition 4.11. *Let $n' \in \mathbb{N}$ and $\varepsilon' \in]0, \infty[$ be as in Lemma 4.8. Then for $n \geq n'$ and $\varepsilon \in]0, \varepsilon']$, the homology index braid of $(\pi_{\varepsilon,n}, K_{\varepsilon,n}, (M_{p,\varepsilon,n})_{p \in P})$ is isomorphic to the homology index braid of $(\pi_{\varepsilon}, K_{\varepsilon}, (M_{p,\varepsilon})_{p \in P})$.*

Lemma 4.12. *There exists an $n_1 \in \mathbb{N}$ such that for every $n \geq n_1$, the sets N and V_p , $p \in P$, are $\mathcal{T}_{f_{n,1}}$ -isolating neighborhoods of $K_n := \Gamma_{f_{n,1}}(K'_n)$ and $M_{p,n} := \Gamma_{f_{n,1}}(M'_{p,n})$, $p \in P$, respectively, where N' and V'_p , $p \in P$, are as in Theorem 4.1.*

Proof. This follows from [17, Theorem 4.11]. $\square$

We can now give a

Proof of Theorem 4.2. Let $n_0 \in \mathbb{N}$ be as in Lemma 4.4, $n' \in \mathbb{N}$ be as in Lemma 4.8 and $n_1 \in \mathbb{N}$ be as in Lemma 4.12. Define $\tilde{n} := \max\{n_0, n', n_1\}$. Fix $n \geq \tilde{n}$ arbitrarily. Proposition 4.6 implies that

$$(4.10) \quad \text{the homology index braids of } (\pi'_n, K'_n, (M'_{p,n})_{p \in P}) \text{ and } (\pi', K', (M'_p)_{p \in P}) \text{ are isomorphic.}$$

Notice that

$$\begin{aligned} K'_n &= \text{Inv}_{\pi'_{f_{n,1}}}(N') = \text{Inv}_{\pi'_n}(N' \cap Y_n) \subset \text{Int}_{Y_n}(N' \cap Y_n), \\ M'_{p,n} &= \text{Inv}_{\pi'_{f_{n,1}}}(V'_p) = \text{Inv}_{\pi'_n}(V'_p \cap Y_n) \subset \text{Int}_{Y_n}(V'_p \cap Y_n), \quad p \in P. \end{aligned}$$

Moreover, with the notation introduced in Section 3, it follows from Lemma 4.12

$$\begin{aligned} \text{Inv}_{\mathcal{T}_{g_n}}(N \cap X_n) &= \text{Inv}_{\mathcal{T}_{f_{n,1}}}(N) = K_n \subset \text{Int}_{X^{1/2} \times X}(N), \\ \text{Inv}_{\mathcal{T}_{g_n}}(V_p \cap X_n) &= \text{Inv}_{\mathcal{T}_{f_{n,1}}}(V_p) = M_{p,n} \subset \text{Int}_{X^{1/2} \times X}(V_p), \quad p \in P, \end{aligned}$$

where $g_n: Y_n \rightarrow Y_n$ is defined by

$$g_n(u) = -Au + P_n(\hat{\phi}(u) + \gamma), \quad u \in Y_n.$$

Hence

$$\begin{aligned} \text{Inv}_{\mathcal{T}_{g_n}}(N \cap X_n) &\subset \text{Int}_{X_n}(N \cap X_n), \\ \text{Inv}_{\mathcal{T}_{g_n}}(V_p \cap X_n) &\subset \text{Int}_{X_n}(V_p \cap X_n), \quad p \in P. \end{aligned}$$

It follows from Theorem 3.6 that there exists an $\tilde{\varepsilon}_0 \in]0, \infty[$ such that for all $\varepsilon \in]0, \tilde{\varepsilon}_0]$, the family $(M_{p,\varepsilon,\Gamma_{g_n}})_{p \in P}$ is a $\prec$ -ordered Morse decomposition of $\tilde{K}_{\varepsilon,n}$ and

the homology index braids of $(\pi_{\varepsilon,n}, \tilde{K}_{\varepsilon,n}, (M_{p,\varepsilon,\Gamma_{g_n}})_{p \in P})$ and $(\pi'_n, K'_n, (M'_{p,n})_{p \in P})$ are isomorphic, where

$$\begin{aligned}\tilde{K}_{\varepsilon,n} &= \text{Inv}_{\pi_{\varepsilon,n}}(N \cap X_n) = \text{Inv}_{\pi_{\varepsilon,f_{n,1}}}(N) = K_{\varepsilon,n}, \\ M_{p,\varepsilon,\Gamma_{g_n}} &= \text{Inv}_{\pi_{\varepsilon,n}}(V_p \cap X_n) = \text{Inv}_{\pi_{\varepsilon,f_{n,1}}}(V_p) = M_{p,\varepsilon,n}, \quad p \in P.\end{aligned}$$

Hence

(4.11) the homology index braid of $(\pi_{\varepsilon,n}, K_{\varepsilon,n}, (M_{p,\varepsilon,n})_{p \in P})$ and the homology index braid of $(\pi'_n, K'_n, (M'_{p,n})_{p \in P})$ are isomorphic, for all $\varepsilon \in]0, \tilde{\varepsilon}_0]$.

Define $\varepsilon_0 := \min\{\varepsilon', \tilde{\varepsilon}_0\}$, where $\varepsilon' \in]0, \infty[$ is as in Lemma 4.8, and let $\varepsilon \in]0, \varepsilon_0]$. Proposition 4.11 implies that

(4.12) the homology index braid of $(\pi_{\varepsilon,n}, K_{\varepsilon,n}, (M_{p,\varepsilon,n})_{p \in P})$ and the homology index braid of $(\pi_{\varepsilon}, K_{\varepsilon}, (M_{p,\varepsilon})_{p \in P})$ are isomorphic.

Now (4.10), (4.11) and (4.12) imply the homology index braid of $(\pi_{\varepsilon}, K_{\varepsilon}, (M_{p,\varepsilon})_{p \in P})$ is isomorphic to the homology index braid of $(\pi', K', (M'_p)_{p \in P})$ for all $\varepsilon \in]0, \varepsilon_0]$ and the proof is complete. $\square$

Remark 4.13. Theorem 4.2 also holds for cohomology index braids with an analogous proof.

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