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Lower Semicontinuity of Unstable Manifolds for Gradient Systems

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I. Introduction

Consider a C_0-semigroup $T(t)$ on a complete metric space X. In the study of the dynamics of $T(t)$, we are interested in the dependence of the long-term dynamics on a parameter. Understanding this allows us to sensibly balance the accuracy of our models against the ease of the mathematical analysis. It also alerts us to situations in which we must be concerned with possible inaccuracies in model parameters. The relevant information about the long-term dynamics is contained in the attractor. If we have a parameter dependent semigroup $T_\lambda(t)$, $0 \leq \lambda \leq \Lambda$, then we may have a sequence of attractors $\{\mathcal{A}_\lambda\}$. The first question we ask in the study of the long-term dynamics is "Are the attractors $\{\mathcal{A}_\lambda\}$ continuous with respect to λ?" In general, we expect to have upper semicontinuity of the attractors if $T_\lambda(t)$ satisfy some basic continuity conditions (see [5]), however, we need much stronger conditions to have lower semicontinuity. For example if the semigroup $T_0(t)$

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has a closed orbit, this orbit might be broken by the perturbations $T_\lambda(t)$ and the dimension of the attractor might decrease. One class of semigroups for which we do expect lower semicontinuity however, is the gradient systems. In a gradient system, we don't have any closed orbits. If a semigroup $T(t)$ is gradient, asymptotically compact and if each of the equilibria is hyperbolic and the set of equilibria is bounded, then the attractor is exactly the union of the unstable manifolds. In this case, we might expect that if the perturbations $T_\lambda(t)$ are continuous in λ, then the attractors will be lower semicontinuous. This is the question that we investigate here. In the next paragraph we define the technical terms used in this paper and then we go on to discuss previous results and our results about lower semicontinuity of the attractors.

A set $\mathcal{A}$ is an *attractor* for $T(t)$ if $\mathcal{A}$ is the maximal compact invariant set which attracts bounded sets under the flow. By *attracts* we mean that for any bounded set B, $dist(T(t)B, \mathcal{A}) \to 0$ as $t \to \infty$. We say that the family $\{\mathcal{A}_\lambda\}$ is *upper semicontinuous* in λ at $\lambda = 0$ if for every $\epsilon > 0$, there exists a $\bar{\lambda}$ such that $\mathcal{A}_\lambda \subset N_\epsilon(\mathcal{A}_0)$ for all $\lambda \leq \hat{\lambda}$ where N_ϵ is an ϵ-neighborhood. The sequence is *lower semicontinuous* in λ at $\lambda = 0$ if for every $\epsilon > 0$, there exists a $\bar{\lambda}$ such that $\mathcal{A}_0 \subset N_\epsilon(\mathcal{A}_\lambda)$ for all $\lambda \leq \hat{\lambda}$. If the sequence $\{\mathcal{A}_\lambda\}$ is both lower and upper semicontinuous, then it is continuous. A strongly continuous C^1-semigroup is a *gradient system* if each bounded orbit is precompact and there is a Lyapunov function $V : X \to I\!R$ in the sense of Hale [5]; that is $V(x)$ is bounded below; $V(x) \to \infty$ as $|x| \to \infty$; for each x, $V(T(t))x$ is non-increasing in t; and if $V(T(t)x) = V(x)$ for all t then x is an equilibrium point.

In 1989, Hale and Raugel [7] proved a result on lower semicontinuity of attractors for gradient systems. Basically, their result relies on four major hypotheses: the perturbed semigroups admit "nearby" attractors, the equilibria are continuous with respect to the parameter, the flow is continuous with respect to initial conditions and with respect to parameters for initial conditions in the attractor, and the local unstable manifolds are themselves lower semicontinuous. In this paper, we will discuss conditions under which the unstable manifolds are lower semicontinuous. We will show that if we assume certain uniformity conditions on the local unstable manifolds then continuity of the equilibria and continuity of the flow with respect to initial conditions for initial conditions in the attractor will together give us lower semicontinuity of the unstable manifolds. These results will make it easy for us to obtain lower semicontinuity of the attractor for an example from RFDE's (retarded functional differential equations) and an example from parabolic PDE's. In neither case are the perturbations C^1; in fact, the perturbation in the PDE example is singular.

One of the conditions on the local unstable manifolds that we will require is that the local unstable manifold attracts the flow, as long as the flow remains in a certain neighborhood of the equilibrium. This will be shown to be true under the same general conditions which give us existence of the unstable manifold. We will discuss this in Section IV. To look at the distance between the flow and the local unstable manifold, we will need to be able to decompose the solution into part in the unstable manifold and part in the stable manifold. To do this, we need a variation of constants formula in the phase space.

While this is easily obtained for parabolic PDE's, it is not so easy in the case of RFDE's (here we will restrict ourselves to the case of finite delay equations, so the phase space will be $C := C([-r, 0], I\!R^n)$). For example, the variation of constants formula given in [6] is not an equation in the phase space. For this reason, we will work instead in the dual of the sun-dual of C, à la Clément, et al. [3]. In this way we obtain a more practical variation of constants formula without too much trouble. Hale and Verduyn Lunel [8] do obtain a decomposition of the solution, but we prefer to work in $C^{\odot *}$ (pronounced "C sun star"). This is mostly a matter of taste.

In proving local attractivity of the unstable manifolds, it turns out to be practical for us also to prove existence of unstable manifolds for the nonlinear problem along with the facts that the nonlinear unstable manifold is homeomorphic, in a neighborhood, to the linear one and that the two are tangent at the origin, etc, even though this has been proved before (see [9] for the case of parabolic PDE's and [6] for the case of RFDE's). Our proof will be very general however and though it will be given specifically for RFDE's in the context of the sun-dual-dual formulation, it is trivially adapted to any C_0-semigroup in a Banach space. To adapt the proof to analytic semigroups generated by parabolic PDE's is also possible; one must work a little to account for the fact that the domain of the nonlinearity is in a fractional power space while its image is in the base space.

Next we will describe Hale and Raugel's result in more detail and then give a more technical version of our own result. We will then introduce the two problems to which we will apply our result. A discussion of the sun-dual-dual formulation for RFDE's will be postponed until Section III.

Let $T_\lambda(t)$ be a family of C^1-semigroups on a Banach space X for $0 \leq \lambda \leq \Lambda$. Suppose that the semigroup $T_0(t)$ is a gradient system, asymptotically compact with orbits of bounded sets bounded. Suppose also that each equilibrium of $T_0(t)$ is hyperbolic and that the set of equilibria, $E_0 = \{\phi_{10}, \phi_{20}, ..., \phi_{N0}\}$, is bounded. In this case $T_0(t)$ admits a global attractor $\mathcal{A}_0$ and $\mathcal{A}_0$ is exactly the union of the equilibria and their unstable manifolds. Suppose that the semigroups $T_\lambda(t)$ satisfy the following hypotheses.

(I1) For each $\lambda > 0$, there exists a neighbourhood U_0 of $\mathcal{A}_0$, which is independent of λ, such that $T_\lambda(t)$ has a local attractor $\mathcal{A}_\lambda$ which attracts U_0.
(I2) If E_λ is the set of equilibria for $T_\lambda(t)$, $\lambda \in [0, \Lambda]$, then there exists a neighborhood O of E_0 such that $O \cap E_\lambda = \{\phi_{1\lambda}, ..., \phi_{N\lambda}\}$ where each $\phi_{j\lambda}$ is hyperbolic and, for each j, $\phi_{j\lambda} \to \phi_{j0}$ as $\lambda \to 0$.
(I3) For each ϕ_j, the local unstable manifold at ϕ_j is lower semicontinuous as $\lambda \to 0$.
(I4) For any $\epsilon > 0$ and $t_0, \tau > 0$, there exists $\bar{\lambda} \leq \Lambda$ and $\delta > 0$ such that $\|T_\lambda(t)y - T_0(t)x\| < \epsilon$ for all $t \in [t_0, \tau]$, $\lambda \leq \bar{\lambda}$ and for all $x \in \mathcal{A}_0$, $y \in \mathcal{A}_\lambda$ with $\|x - y\| < \delta$.

Then, according to Hale and Raugel, the attractors $\mathcal{A}_\lambda$ are lower semicontinuous as $\lambda \to 0$.

For each $\phi_{j\lambda}$, let $S_{j\lambda}$ be the stable manifold for the linearized problem and $U_{j\lambda}$ be the unstable manifold for the linearized problem. Let $P_{j\lambda}$ be the associated spectral projection onto $U_{j\lambda}$ and $Q_{j\lambda}$ the spectral projection onto $S_{j\lambda}$. Let $W_{j\lambda}$ be the unstable manifold for the nonlinear problem. There is a neighborhood $N_{j\lambda}$ of $\phi_{j\lambda}$ and a homeomorphism $\sigma_{j\lambda}$ which maps $N_{j\lambda} \cap W_{j\lambda}$ to $U_{j\lambda}$. We will be interested in the distance between the flow and the unstable manifold. This distance is bounded above by the distance from the flow to the unstable manifold in the direction of the stable manifold, i.e. $\|Q_{j\lambda}T_\lambda(t)x_0 - \sigma_{j\lambda}(P_{j\lambda}T_{\lambda(t)}x_0)\|$. Under the same conditions required for existence of $W_{j\lambda}$ and $\sigma_{j\lambda}$, we will show that, for fixed λ, there are constants β and M such that

$$\|Q_{j\lambda}T_\lambda(t)x_0 - \sigma_{j\lambda}(P_{j\lambda}T_\lambda(t)x_0)\| \leq Me^{-\beta t}\|Q_{j\lambda}x_0 - \sigma_{j\lambda}(P_{j\lambda}x_0)\| \tag{1.1}$$

as long as $T_\lambda(t)x_0$ remains in a δ-ball of the origin.

In the case that (1.1) holds, we prove the following. Suppose the projections $P_{j\lambda}$ and $Q_{j\lambda}$ are uniformly bounded in λ and the maps $\sigma_{j\lambda}$ are Lipschitz with Lipshitz constants independent of λ. Also suppose that δ, the radius in which (1.1) holds, can be chosen independently of λ. Then, provided hypotheses (I2) and (I4) hold, (I3) holds. In fact, we can weaken hypothesis (I4) to continuity with respect to the parameter.

With this result in mind, consider the family of retarded functional differential equations

$$\dot{x}(t) = -\int_{-r}^{0} g(x(t+\theta))a_\lambda(\theta)d\theta \tag*{$(1.2)_\lambda$}$$

where a_λ is in $C^2([-r,0],I\!R)$. It can easily be shown that the solution semigroup for each problem $(1.2)_\lambda$ is C^1 (see [6]). If a_0 satisfies: $a(-r) = 0, a(\theta) \geq 0, \dot{a}(\theta) \geq 0, \ddot{a}(\theta) \geq 0$, for all $r \leq \theta \leq 0$, then we can exhibit a Lyapunov function for the problem $(1.2)_0$ (see [5]). In this way we can show that the limiting problem is a gradient system and exhibits a global attractor. In fact, the conditions given by Hale and Raugel for the limiting problem are satisfied. If a_λ fails at any point to have these properties then a Lyapunov function is not known. If a_λ, however, is close to a_0 in the L^1-norm, then we can use the results in [11] and [12] to show that for small λ, the problem $(1.2)_\lambda$ will still admit an attractor and the attractors are upper semicontinuous at $\lambda = 0$ provided $a_\lambda \to a_0$ in L^1. In fact, hypothesis (H4) holds for all initial conditions. Since the equilibria of $(1.2)_\lambda$ are given by the zeros of the function g, hypothesis (H2) also holds. Hence, we will be able to prove that the attractors $\mathcal{A}_\lambda$ are lower semicontinuous at $\lambda = 0$.

We will also consider reaction diffusion problems where the diffusion coefficient becomes large in a subregion which is interior to the physical domain. Suppose Ω is a bounded smooth domain in $I\!R^N$, $N \leq 3$ and suppose that the diffusion coefficient is very large in subregions $\Omega_{0,j}$ where each $\Omega_{0,j}$ is a smooth subdomain and $\bar{\Omega}_{0,i} \cap \bar{\Omega}_{0,j} = \emptyset$ for $i \neq j$. Let $\Omega_0 = \cup_{j=1}^m \Omega_{0,j}$ and $\Omega_1 = \Omega \setminus \bar{\Omega}_0$. Consider the parameter dependent problem

$$\begin{cases} u_t^\epsilon - \mathrm{Div}(a_\epsilon(x)\nabla u^\epsilon) + \lambda u^\epsilon + f(u^\epsilon) = 0, & \text{on } \Omega \\ \frac{\partial u^\epsilon}{\partial n_\epsilon} = 0, & \text{on } \Gamma \end{cases} \tag*{$(1.3)_\epsilon$}$$

where $\frac{\partial u^\epsilon}{\partial n_\epsilon} = a_\epsilon(x)\langle \nabla u, n\rangle$ and where, for each $0 \leq \epsilon \leq \epsilon_0$, a_ϵ is a regular bounded function in Ω satisfying $0 < m_0 \leq a_\epsilon(x)$ for every $x \in \Omega$ and $0 \leq \epsilon \leq \epsilon_0$. Suppose that $a_\epsilon(x) \to \infty$

uniformly on compact sets of Ω_0 and $a_\epsilon(x) \to a_0(x)$ uniformly on Ω_1. Then the problems $(1.3)_\epsilon$ will converge to some limiting problem (which we will refer to as problem $(1.3)_0$) for which u is constant on each of the subregions $\Omega_{0,j}$. Carvalho and Rodrigez-Bernal study this problem in [1] and prove that the problems $(1.3)_\epsilon$, $0 \le \epsilon \le \epsilon_0$, admit global attractors and the attractors are upper semicontinuous at $\epsilon = 0$. Condition (H4) is also satisfied for all initial conditions and condition (H2) is satisfied as well. Hence we can prove that, for this problem also, the attractors are lower semicontinuous.

The remainder of this paper is organized as follows. In Section II, we present and prove the main theorem about lower semicontinuity of the unstable manifolds. In section III, we discuss the formulation of RFDE's in the space $C^{\odot *}$ and we obtain the variation of constants formula. In Section IV, we prove the existence of the unstable manifold $W_{j\lambda}$ and the homeomorphism $\sigma_{j\lambda}$ and we prove that (1.1) holds. The inequality (1.1) is an assumption in the main theorem. In Section V, we discuss applications. In V.1, we discuss applications to RFDE's where the delay depends on a parameter and we discuss, in particular, the example (1.2). In Section V.2, we discuss applications to the reaction diffusion problem (1.3).

II. The Main Theorem

Let X be a Banach space and let $T_\lambda(t)$ be a family of C^1-semigroups on X for $0 \le \lambda \le \Lambda$. Fix an equilibrium ϕ_j and translate all the equilibria so that this equilibrium is at the origin for the unperturbed problem. We will still use the notation $\phi_{j\lambda}$ for the translated equilibria. Then we have the following (we will use the notation presented in the introduction for the stable and unstable manifolds and the spectral projections).

Theorem 2.1. *Suppose that the family of semigroups $T_\lambda(t)$ satisfies the hypotheses*

(J1) For any $\epsilon > 0$ and $t_0, \tau > 0$, there exists $\bar\lambda \le \Lambda$ such that $||T_\lambda(t)x - T_0(t)x|| < \epsilon$ for all $t \in [t_0, \tau]$, $\lambda \le \bar\lambda$ and $x \in \mathcal{A}_0$.

(J2) $\phi_{j\lambda}$ is hyperbolic and for any $\epsilon > 0$ there is a $\bar\lambda \le \Lambda$ such that for all $\lambda \le \bar\lambda$, $||\phi_{j\lambda} - \phi_{j0}|| < \epsilon$.

Recall that $N_{j\lambda}$ is a neighborhood such that $W_{j\lambda} \cap N_{j\lambda}$ is homeomorphic to $U_{j\lambda} \cap N_{j\lambda}$. Suppose the following are true for the projections and for the neighborhoods $N_{j\lambda}$.

(J3) There exist positive real numbers p and q such that $||P_{j\lambda}|| \le p$ and $||Q_{j\lambda}|| \le q$ for all $\lambda \le \Lambda$

(J4) The Lipshitz constants of $\sigma_{j\lambda}$, L_{σ_j}, are independent of λ.

(J5) There exists a δ_1, independent of λ, such that $B_{\delta_1}(\phi_{j\lambda}) \subset N_{j\lambda}$ for all $\lambda \le \Lambda$.

(J6) There exists a δ_2 and an M such that for all $\lambda \le \Lambda$ the following estimate holds as long as $T_\lambda(t)x_0 \in B_{\delta_2}(\phi_{j\lambda})$:

$$||Q_{j\lambda}T_\lambda(t)x_0 - \sigma_{j\lambda}(P_{j\lambda}T_{j\lambda}(t)x_0)|| \le Me^{-\beta t}||Q_{j\lambda}x_0 - \sigma_{j\lambda}(P_{j\lambda}x_0)||$$

where $\beta \ge 0$ can depend on λ.

Let $\delta = \min(\delta_1, \delta_2)$ and let $r = \max(1, p)$. Define $\rho = \frac{\delta}{2r}$. Then the manifolds $V_{j\lambda} := W_{j\lambda} \cap B_\rho(\phi_{j\lambda})$ are lower semicontinuous at $\lambda = 0$.

Proof: Define $V_\lambda := V_{j\lambda}$ and drop all of the other j subscripts as well. Notice that V_0 is compact. We will show that for any $\epsilon > 0$ and for any $y \in V_0$, there exists a $\bar{\lambda}$ such that we have $y \in N_\epsilon(V_\lambda)$ for all $\lambda \leq \bar{\lambda}$. Since V_0 is compact, we can choose the parameter $\bar{\lambda}$ independently of y, hence we will have that $V_0 \subset N_\epsilon(V_\lambda)$ for all $\lambda \leq \bar{\lambda}$.

Choose $\epsilon > 0$. Define $\bar{\epsilon} := \min(\frac{\epsilon}{K}, \frac{\rho}{2})$ where $K = 1 + Mq + 2MpL_\sigma + M$ and choose $\tau > 0$ so that $T_0(-\tau)V_0 \subset B_{\bar{\epsilon}}(\phi_0)$. By hypothesis (J1) there exists λ_1 such that $||T_\lambda(t)x - T_0(t)x|| < \bar{\epsilon}$ for all $t \in [0, \tau]$, $\lambda \leq \lambda_1$ and $x \in \mathcal{A}_0$ and by hypothesis (J2) there exists a λ_2 so that $\phi_\lambda \in B_{\bar{\epsilon}}(\phi_0)$ for all $\lambda \leq \lambda_2$. Choose $\bar{\lambda} = \min(\lambda_1, \lambda_2)$.

Pick any $y \in V_0$. Define $x := T_0(-\tau)y$. Then

$$d(y, V_\lambda) \leq d(y, T_\lambda(\tau)x) + d(T_\lambda(\tau)x, V_\lambda).$$

Remember that $||Q_\lambda T_\lambda(\tau)x - \sigma_\lambda(P_\lambda T_\lambda(t)x)||$ is an upper bound for the distance between the flow and the unstable manifold. Using this and hypotheses (J1), (J3), (J4) and (J6), we get

$$\begin{aligned}
d(y, V_\lambda) &\leq ||T_0(\tau)x - T_\lambda(\tau)x|| + ||Q_\lambda T_\lambda(\tau)x - \sigma_\lambda(P_\lambda T_\lambda(\tau)x)|| \\
&\leq ||T_0(\tau)x - T_\lambda(\tau)x|| + M||Q_\lambda x - \sigma_\lambda(P_\lambda x)|| \\
&\leq \bar{\epsilon} + M||Q_\lambda x|| + M||\sigma_\lambda(P_\lambda x) - \sigma_\lambda(P_\lambda \phi_\lambda) + \sigma_\lambda(P_\lambda \phi_\lambda)|| \\
&\leq \bar{\epsilon} + M||Q_\lambda x|| + ML_\sigma||P_\lambda x - P_\lambda \phi_\lambda|| + M||\sigma_\lambda(P_\lambda \phi_\lambda)|| \\
&= \bar{\epsilon} + M||Q_\lambda x|| + ML_\sigma||P_\lambda x - P_\lambda \phi_\lambda|| + M||\phi_\lambda|| \\
&\leq \bar{\epsilon} + Mq||x|| + MpL_\sigma||x - \phi_\lambda|| + M||\phi_\lambda||
\end{aligned}$$

Notice that $\sigma_\lambda(P_\lambda \phi_\lambda) = \phi_\lambda$ since ϕ_λ is in both U_λ and W_λ. Also when we use these estimates, we are assuming that $T_\lambda(\tau)x$ and $P_\lambda x$ are in $B_\delta(\phi_\lambda)$. We will verify this at the end of the proof.

Now, using hypothesis (J2) and the fact that $x \in B_{\bar{\epsilon}}(\phi_0)$, we get

$$\begin{aligned}
d(y, V_\lambda) &\leq \bar{\epsilon} + Mq\bar{\epsilon} + 2MpL_\sigma\bar{\epsilon} + M\bar{\epsilon} \\
&= (1 + Mq + 2MpL_\sigma + M)\bar{\epsilon} \\
&\leq \epsilon
\end{aligned}$$

It is easy to see that $x \in B_\rho(\phi_\lambda)$ since $x \in B_{\bar{\epsilon}}(\phi_0)$ and $\phi_\lambda \in B_{\bar{\epsilon}}(\phi_0)$. Since $||P_\lambda x|| \leq p||x||$ this shows that $P_\lambda x \in B_\delta(\phi_\lambda)$. To see that $T_\lambda(\tau)x \in B_\delta(\phi_\lambda)$, notice that $||T_\lambda(\tau)x - T_0(\tau)x|| < \frac{\delta}{4r}$. Then, since $T_0(\tau)x \in B_{\frac{\delta}{2r}}(\phi_0)$, $T_\lambda(\tau)x \in B_{\frac{3\delta}{4r}}(\phi_0)$. Since $\phi_\lambda \in B_{\bar{\epsilon}}(\phi_0)$,

$$\begin{aligned}
||T_\lambda(t)x - \phi_\lambda|| &\leq \frac{3\delta}{4r} + \bar{\epsilon} \\
&\leq \frac{3\delta}{4r} + \frac{\delta}{4r} \\
&\leq \delta
\end{aligned}$$

where the last estimate holds because we chose $r \geq 1$.

This estimate holds for all $\lambda \leq \bar{\lambda}$. We can obtain such an estimate for any $y \in V_0$ and since V_0 is compact, we can pick $\bar{\lambda}$ independently of y. Hence we have the result. ∎

Now we will state the theorem for lower semicontinuity of the attractors. We will have to combine the hypotheses (I1) through (I4) and (J1) through (J6).

Theorem 2.2. *Suppose the C^1-semigroups $T_0(t)$ are gradient, asymptotically compact and that orbits of bounded sets are bounded. Let $\mathcal{A}_0$ be the attractor for $T_0(t)$. Suppose also that each equilbrium of $T_0(t)$ is hyperbolic and the set of equilibria, $E = \{\phi_{10}, \phi_{20}, ...\phi_{N0}\}$, is bounded. Suppose that $T_\lambda(t)$, $\lambda \leq \Lambda$, is a family of C^1-semigroups which satisfies the following.*

(H1) For each $\lambda > 0$, there exists a neighbourhood U_0 of $\mathcal{A}_0$, which is independent of λ, such that $T_\lambda(t)$ has a local attractor $\mathcal{A}_\lambda$ which attracts U_0.

(H2) For any $\epsilon > 0$ and $t_0, \tau > 0$, there exists $\bar{\lambda} \leq \Lambda$ and $\delta > 0$ such that $||T_\lambda(t)y - T_0(t)x|| < \epsilon$ for all $t \in [t_0, \tau]$, $\lambda \leq \bar{\lambda}$ and for all $x \in \mathcal{A}_0$, $y \in \mathcal{A}_\lambda$ with $||x - y|| < \delta$.

(H3) Let E_λ be the set of equilibria of $T_\lambda(t)$. There exists a neighborhood O of E_0 such that $O \cap E_\lambda = \{\phi_{1\lambda}, ..., \phi_{N\lambda}\}$ where each $\phi_{j\lambda}$ is hyperbolic and, for each j, $\phi_{j\lambda} \to \phi_{j0}$ as $\lambda \to 0$.

(H4) For each j, there exist positive real numbers p_j and q_j such that $||P_{j\lambda}|| \leq p_j$ and $||Q_{j\lambda}|| \leq q_j$ for all $\lambda \leq \Lambda$.

(H5) The Lipshitz constants of $\sigma_{j\lambda}$, L_{σ_j}, are independent of λ.

(H6) For each j, there exists a $\delta_1(j)$, independent of λ, such that $B_{\delta_1}(\phi_{j\lambda}) \subset N_{j\lambda}$ for all $\lambda \leq \Lambda$.

(H7) For each j, there exists a $\delta_2(j)$ and an $M(j)$ such that for all $\lambda \leq \Lambda$ the following estimate holds as long as $T_\lambda(t)x_0 \in B_{\delta_2}(\phi_{j\lambda})$:

$$||P_{j\lambda}T_\lambda(t)x_0 - \sigma_{j\lambda}(Q_{j\lambda}T_{j\lambda}(t)x_0)|| \leq Me^{-\beta t}||P_{j\lambda}x_0 - \sigma_{j\lambda}(Q_{j\lambda}x_0)||$$

where $\beta \geq 0$ can depend on λ.

Then the family of attractors $\{\mathcal{A}_\lambda\}$ is lower semicontinuous at $\lambda = 0$.

III. Unstable and Stable Manifolds for RFDE's

In the next section, we will show that (H7) holds under the same hypotheses which are required for existence of the nonlinear unstable manifold and the homeomorphism. We will give the proof for the case of RFDE's since in that case, the set-up is more complicated. In this section, we discuss the local theory for RFDE's. In order to prepare the way for the local theory, we will first develop the global theory and then give a variation of constants formula for the nonlinear problem. As mentioned in the introduction, the usual variation of constants formula is not actually a formula in the phase space $C([-r, 0], I\!R^n)$. If we work instead in the dual of the sun-dual, à la Clément, et al., we can get a variation of constants formula in the phase space. This will allow us to decompose the solution into the part in U and the part in S. This, along with the fact that there is a homeomorphism from U to W, will allow us to get our hands on the distance between the flow and the unstable manifold.

To begin, we will present the linear theory. Then in subsection 1, we will describe the sun dual spaces and their corresponding semigroups and generators. In subsection 2, we will show how to formulate the linear and nonlinear problems in these new spaces and we will give the variation of constants formula that we will use in the remainder of the paper. Finally, in subsection 3, we will give the decomposition of a solution in C.

Consider the retarded functional differential equation

$$\dot{u}(t) = Lx_t \tag{3.1}$$

where $u_t(\theta) := u(t+\theta)$, $u_t \in C := C([-r,0], I\!R^n)$, and L is a linear bounded functional from C into $I\!R^n$. Let $T_L(t)$ denote the solution semigroup associated with (3.1); that is, if ϕ is an initial condition in C then the solution through ϕ is given by $u_t(\cdot;\phi) = T_L(t)\phi$. $T_L(t)$ is a C_0-semigroup. The infintesimal generator of $T_L(t)$ is

$$[A\phi](\theta) = \begin{cases} \frac{d\phi}{d\theta} & -r \leq \theta < 0 \\ L(\phi) & \theta = 0 \end{cases}$$

where $D(A) = \{\phi \in C : \dot{\phi} \in C \text{ and } \dot{\phi}(0) = L(\phi)\}$.

The spectrum of A is entirely point spectrum and $\nu \in \sigma(A)$ if and only if

$$\Delta(\nu) := \det\left(\nu I - L(e^{\nu \cdot})\right) = 0 \tag{3.2}$$

We refer to equation 3.2 as the characteristic equation. The roots of equation 3.2 have real part bounded above and have finite multiplicity. If no roots of the characteristic equation lie on the imaginary axis then we say that the equilibrium is hyperbolic. Suppose this is indeed the case. Let $\Gamma = \{\nu : \Delta(\nu) = 0 \text{ and } Re(\nu) > 0\}$. There are a finite number of eigenvalues in Γ, each with algebraic multiplicity $k(\nu)$. Define $U = \cup_{\nu \in \Gamma} \mathcal{N}(A - \nu I)^{k(\nu)}$ and S so that $C = S \oplus U$. U and S will be called the unstable and stable manifolds, respectively. Both U and S are invariant with respect to $T_L(t)$ and if P and Q are the respective spectral projections then there are positive contants K and γ such that

$$\begin{aligned} \|T(t)P\phi\| &\leq Ke^{\gamma t}\|P\phi\|, & t \geq 0 \\ \|T(t)Q\phi\| &\leq Ke^{\gamma t}\|Q\phi\|, & t \leq 0 \end{aligned}$$

We want to develop the local theory for the nonlinear equation

$$\dot{u}(t) = Lu_t + f(u_t) \tag{3.3}$$

where $f : C \to \mathbb{R}^n$, $f(0) = 0$ and $Df(0) = 0$ (if we are interested in a nontrivial equilibrium, we just make a change of variables to translate the equilibrium to the origin). We denote by $T(t)$ the solution semigroup associated with this equation.

1. The Spaces $C^{\odot\odot}$ and $C^{\odot *}$

Equation 3.1 cannot be written as an abstract equation in C. If we choose instead to work in the phase space $C^{\odot\odot} := \{(\alpha, \phi) : \phi \in C \text{ and } \phi(0) = \alpha\}$, then we can write the equation as an abstract equation in the phase space. Define $v_t := (u(t), u_t)$ and $\hat{L}v_t := (Lu_t, \dot{u}_t)$. Then

$$\frac{d}{dt}v_t = \hat{L}v_t \tag{3.4}$$

Notice that C is isometrically isomorphic to $C^{\odot\odot}$ with the norm

$$\|(a, \phi)\| = \sup\{|\alpha|, \|\phi\|_\infty\}.$$

Unfortunately, we still can't write equation 3.3 as an abstract equation in $C^{\odot\odot}$. We can however write it as an abstract equation in $C^{\odot *} := \mathbb{R}^n \times L^\infty([0, h], \mathbb{R}^n)$. To this end, define $\hat{f}(v_t) = (f(u_t), 0)$. Then

$$\frac{d}{dt}v_t = \hat{L}v_t + \hat{f}(v_t)$$

To develop a variation of constants formula from this abstract equation, we will have to develop a bit of the theory of the adjoint spaces and adjoint semigroups. This theory can be found in [10].

Suppose $T_L(t)$ is a C_0-semigroup of bounded linear operators on a Banach space X. Then $T_L^*(t)$ is a semigroup of bounded linear operators on X^*, however the semigroup may no longer be C_0. For this reason, we define the subspace

$$X^\odot := \{x^* \in X^* : \lim_{t \downarrow 0} \|T_L^*(t)x^* - x^*\| = 0\}$$

Of course, $X^\odot$ is invariant with respect to $T_L^*(t)$. If we denote by $T_L^\odot(t)$ the restriction of $T_L^*(t)$ to $X^\odot$, then $T_L^\odot(t)$ is a C_0-semigroup in $X^\odot$.

Similarly, we can consider $X^{\odot *}$ and $T_L^{\odot *}(t)$ and define

$$X^{\odot\odot} := \{x^{**} \in X^{**} : \lim_{t \downarrow 0} \|T_L^{\odot *}(t)x^{**} - x^{**}\| = 0\}.$$

Suppose $j(X)$ is the natural embedding of X into $X^{**} \subset X^{\odot *}$. Since $T^{\odot *}(t)j(x) = j(T(t)x)$, $j(X) \subset X^{\odot\odot}$. If $j(X) = X^{\odot\odot}$, we say X is $\odot$-reflexive with respect to $T_L(t)$.

If A is the generator of $T_L(t)$, then A^* is the generator of $T_L^*(t)$ in the weak* sense. That is

$$D(A^*) = \{x^* \in X^* : \lim_{t\downarrow 0} \frac{1}{t}\langle T_L^*(t)x^* - x^*, x\rangle \text{ exists for all } x \in X\}$$

and if $x^* \in D(A^*)$ then

$$\langle A^*x^*, x\rangle = \lim_{t\downarrow 0} \frac{1}{t}\langle T_L^*(t)x^* - x^*, x\rangle.$$

It can be shown that $\overline{D(A^*)} = X^{\odot}$. We define $A^{\odot}$ to be the part of A^* in $X^{\odot}$; that is

$$D(A^{\odot}) = \{x^{\odot} \in D(A^*) : A^*x^{\odot} \in X^{\odot}\}$$

and if $x^{\odot} \in D(A^{\odot})$ then $A^{\odot}x^{\odot} = A^*x^{\odot}$. Then $A^{\odot}$ is the generator of $T_L^{\odot}(t)$. Similarly, we can define $A^{\odot *}$ and $A^{\odot\odot}$. It is important to note that if we perturb the generator A by a bounded linear operator, this does not change the spaces $X^{\odot}$ and $X^{\odot\odot}$.

The spaces $C^{\odot\odot}$ and $C^{\odot *}$ above are obtained from C and $T_L(t)$ in exactly this way. The generator of $T_L(t)$ is the sum of the generator of the shift semigroup and a bounded operator which depends on L. The problem $\dot{x}(t) = 0$ has the shift semigroup as its solution operator. According to the above remark then, the sun-dual with respect to $T_L(t)$ is the same as the sun-dual with respect to $T_Z(t)$ where Z is the zero operator. We will denote by A_Z the generator of $T_Z(t)$.

$$D(A_Z) = \{\phi \in C : \dot{\phi}(0) = 0\}$$

If $\phi \in D(A_Z)$, then $A_Z\phi = \dot{\phi}$. Below we will list the various dual spaces of C with respect to $T_Z(t)$. More details can be found in [4] or [15].

We will take for the dual space of C, $C^* = NBV([0,h], I\!R^n)$; that is, the space of normalized bounded variation functions. We will write $\langle x^*, x\rangle$ for $x^*(x)$. Then if $f \in C^*$ and $\phi \in C$, we define

$$< f, \phi > = \int_0^h \phi(-\theta)df(\theta).$$

¿From this we can find A_Z^*;

$$D(A_Z^*) = \{f \in C^* : f(\theta) = f(0^+) + \int_0^\theta g(\sigma)d\sigma, g \in \text{ NBV }, g(h) = 0\}$$

and $A_Z^* f = g$ for all $f \in D(A_Z^*)$.

If $T_Z(t)$ is the shift semigroup then

$$(T_Z^*(t)f)(\theta) = \begin{cases} f(t+\theta) & \theta > 0 \\ 0 & \theta = 0 \end{cases}.$$

Notice that $T_Z^*(t)$ is not C_0. $T_Z^*(t)$ is C_0 however in

$$C^{\odot} = \{f \in \mathrm{NBV} : f(\theta) = f(0^+) + \int_0^\theta g(\sigma)d\sigma, g \in L^1, g(\sigma) = 0 \text{ for a. a. } \sigma \geq h\}$$

Elements in $C^{\odot}$ are completely specified by $f(0)^+ \in \mathbb{R}^n$ and $g \in L^1(\mathbb{R}^+, \mathbb{R}^n)$. In fact, $C^{\odot}$ is isometrically isomorphic to $\mathbb{R}^n \times L^1$ with the norm

$$\|(c, g)\| = |c| + \|g\|_1$$

and so we will think of elements in $C^{\odot}$ as pairs $(c, g) \in \mathbb{R}^n \times L^1$. In this notation,

$$T_Z^{\odot}(t)(c, g) = \left(c + \int_0^t g(\sigma)d\sigma, g(t + \cdot)\right)$$

where we think of g as being extended to (h, ∞) by 0. $A_Z^{\odot}$ is the part of A_Z^* in $C^{\odot}$ so

$$D(A_Z^{\odot}) = \{(c, g) : g \in \text{ AC } (\mathbb{R}^+, \mathbb{R}^n)\}$$

and $A_Z^{\odot}(c, g) = (g(0), \dot{g})$.

Since $C^{\odot} = \mathbb{R}^n \times L^1$, we will take $C^{\odot *}$ to be $\mathbb{R}^n \times L^\infty$. The norm in $C^{\odot *}$ will be given by

$$\|(\alpha, \phi)\| = \sup\{|\alpha|, \|\phi\|_\infty\}$$

and the pairing between elements of $C^{\odot *}$ and $C^{\odot}$ will be given by

$$\langle(\alpha, \phi), (c, g)\rangle = \alpha c + \int_0^h \phi(-\theta)g(\theta)d\theta.$$

Next we find $A_Z^{\odot *}$ and $T_Z^{\odot *}(t)$.

$$D(A_Z^{\odot *}) = \{(\alpha, \phi) : \|\phi\|_\infty \text{ is finite and } \phi \text{ is Lipschitz with } \phi(0) = \alpha\}$$

and $A_Z^{\odot *}(\alpha, \phi) = (0, \dot{\phi})$. The solution operator in $C^{\odot *}$ is $T_Z^{\odot *}(t)(\alpha, \phi) = (\alpha, \phi_t^\alpha)$ where

$$\phi_t^\alpha(\theta) = \begin{cases} \phi(t + \theta) & \theta \leq -t \\ \alpha & \theta > -t \end{cases}.$$

$T_Z^{\odot *}$ is a C_0-semigroup in

$$C^{\odot\odot} = \{(\alpha, \phi) : \phi \in C, \phi(0) = \alpha\}.$$

Since $C^{\odot\odot}$ is isometrically isomorphic to C, C is sun-reflexive with respect to $T_Z(t)$.

2. Formulation of (3.3) in $C^{\odot *}$ and the Variation of Constants Formula

The equation $\dot{x}(t) = 0$, with solution semigroup $T_Z(t)$, can be written in $C^{\odot *}$ as

$$\frac{d}{dt}v_t = A_Z^{\odot *}v_t.$$

Equation 3.4 can be written as a bounded perturbation of this equation

$$\frac{d}{dt}v_t = (A_Z^{\odot *} + B)v_t$$

where $Bv_t = (Lu_t, 0)$. The solution operator $T_L(t)$ can be written as a perturbation of $T_Z(t)$ as follows. Remember that the sun-dual space with respect to $T_L(t)$ is the same as that with respect to $T_Z(t)$. If $\phi \in C$ $(C^{\odot\odot})$ then

$$T_L(t)\phi = T_Z(t-s)\phi + \int_0^t T_Z^{\odot *}(t-\tau)B(T_L(\tau)\phi)d\tau.$$

$T_L(t)$ is a C_0-semigroup in C. Each of the elements in this equation: $T_L(t)\phi$, $T_Z(t-s)\phi$ and $\int_0^t T_Z^{\odot *}(t-\tau)B(T_L(\tau)\phi)d\tau$ are in C. However, this is an equation in $C^{\odot *}$. We must be careful about the interpretation of the integral in this equation. Suppose $f \in C^{\odot}$. Then

$$\langle \int_s^t T_Z^{\odot *}(t-\tau)B(T_L(\tau)\phi)d\tau, f\rangle = \int_s^t \langle B(T_L(\tau)\phi), T_Z^{\odot}(t-\tau)f\rangle d\tau.$$

Let $\hat{A} := A_Z^{\odot *} + B$ and let A be the generator of $T_L(t)$. Then A is the part of $\hat{A}$ in X; that is

$$D(A) = \{x \in D(\hat{A}) : \hat{A}x \in C\}$$

and if $x \in D(A)$ then $Ax = \hat{A}x$ (recall that $D(B) = C$ and $D(A_Z^{\odot *}) \subset C^{\odot\odot} = C$).

The nonlinear abstract equation corresponding to (3.3) is

$$\frac{d}{dt}v_t = \hat{A}v_t + \hat{f}(v_t) \tag{3.5}$$

This is an equation in the larger space $C^{\odot *}$, though v_t itself is in C. The solution operator for (3.5) can be given in terms of the variation of constants formula in $C^{\odot *}$. We now write $u(t)$ for v_t.

$$u(t) = T_L(t-s)u(s) + \int_s^t T_L^{\odot *}(t-\tau)\hat{f}(u(\tau))d\tau \tag{3.6}$$

Again, each of the elements in this equation, $u(t)$, $T_L(t-s)u(s)$ and $\int_s^t T_L^{\odot *}(t-\tau)\hat{f}(u(\tau))d\tau$ are in C. However, the equation itself is an equation in $C^{\odot *}$.

Since we will be considering projections of u onto the stable and unstable manifolds, we want to understand how bounded operators act on the integral term of (3.6). If $L^*: C^* \to C^*$ is a bounded linear operator and also $L^*: C^\odot \to C^\odot$ then

$$L \int_s^t T_L^{\odot *}(t-\tau)\hat{f}(x(\tau))d\tau = \int_s^t L^{\odot *} T_L^{\odot *}(t-\tau)\hat{f}(x(\tau))d\tau. \tag{3.7}$$

3. Decomposition of the Solution in C

Just as in the case of the RFDE (3.1), the spectrum of $\hat{A}$ is entirely point spectrum and $\nu \in \sigma(\hat{A})$ if and only if ν satisfies (3.2). Hence, we similarly obtain the decomposition of $C^{\odot *}$ into the spaces $U^{\odot *}$ and $S^{\odot *}$ and we obtain the associated projections $P^{\odot *}$ and $Q^{\odot *}$. The spectrum of A (the part of $\hat{A}$ in C) is exactly the same as the spectrum of $\hat{A}$. The generalized eigenspaces associated with A and $\hat{A}$ are also the same; in particular, if $\nu \in \sigma(A)$ then $\mathcal{N}(\nu I - A_Z^{\odot *} - B)^k = \mathcal{N}(\nu I - A)^k$ for any k. Hence we also obtain the decompostion of $C^{\odot\odot}$ (C) into the stable and unstable manifolds, $U^{\odot\odot}$ (U) and $S^{\odot\odot}$ (S) and we obtain the respective projections $P^{\odot\odot}$ (P) and $Q^{\odot\odot}$ (Q). It can be shown (see [4]) that $P^{\odot *}|_C = P$ and Range($P^{\odot *}$) = Range(P). Therefore also $Q^{\odot *}|_C = Q$.

The subspaces $U^{\odot *}$ and $S^{\odot *}$ are invariant with respect to $T_L^{\odot *}(t)$ and its generator $A^{\odot *}$. On $U^{\odot *}$, $T_L^{\odot *}(t) = T_L(t)$ and $T_L(t)$ is a group. In fact, we have the exponential dichotomy as before. That is, there are positive constants K, ρ and β such that

$$\begin{aligned} \|T_L(s)u\| &\leq Ke^{\rho s}\|u\| \qquad && s \leq 0, u \in U \\ \|T_L^{\odot *}u^{\odot *}\| &\leq Ke^{-\beta s}\|u\| \qquad && s \geq 0, u^{\odot *} \in S^{\odot *} \end{aligned} \tag{3.8}$$

Now we can decompose the expression 3.6 into an expression for the part of $u(t)$ in S and the part in U. In the decomposition, we can use (3.7) since P^* and Q^* are bounded. We also use the fact that $T_L^{\odot *}(t)|_C = T_L(t)$.

$$\begin{aligned} x(t) := Qu(t) = Q^{\odot *}u(t) &= Q^{\odot *}T_L(t-s)u(s) + Q^{\odot *}\int_s^t T^{\odot *}(t-\tau)\hat{f}(u(\tau))d\tau \\ &= QT_L(t-s)u(s) + \int_s^t Q^{\odot *}T_L^{\odot *}(t-\tau)\hat{f}(u(\tau))d\tau \\ &= T_-(t-s)x(s) + \int_s^t T_-^{\odot *}(t-\tau)Q^{\odot *}\hat{f}(u(\tau))d\tau \\ y(t) := Pu(t) = P^{\odot *}u(t) &= P^{\odot *}T_L(t-s)u(s) + P^{\odot *}\int_s^t T_L^{\odot *}(t-\tau)\hat{f}(u(\tau))d\tau \\ &= PT_L(t-s)u(s) + \int_s^t P^{\odot *}T_L^{\odot *}(t-\tau)\hat{f}(u(\tau))d\tau \\ &= T_+(t-s)y(s) + \int_s^t T_+^{\odot *}(t-\tau)P^{\odot *}\hat{f}(u(\tau))d\tau \end{aligned}$$

where $T_-(t) = T_L(t)Q$ and $T_+(t) = T_L(t)P$. We should point out that each term in the expressions for $x(t)$ and $y(t)$ is an element of C.

IV. Existence and Attractivity of the Unstable Manifold

Now we will show existence and attractivity of the unstable manifold for the nonlinear problem (3.3). This proof can be easily adapted to any C_0-semigroup in a Banach space (just remove the "sun-stars"). It is also not hard to adapt the proof to analytic semigroups generated by parabolic PDE's; we must work a little to account for the fact that the domain of the nonlinearity is in a fractional power space while its image is in the base space. Here, though, we present the proof for the case of RFDE's since in this case the set-up is most complicated.

For any ϵ, define

$$W_\epsilon = \{\phi \in C : \|P\phi\| \leq \epsilon \text{ and } \|T(t)\phi\| \text{ exists and is bounded for all } t \leq 0\}.$$

We will show that, for small enough ϵ, W_ϵ exists and is invariant under $T(t)$. W_ϵ is homeomorphic to U in the ball B_ϵ of the origin and W_ϵ exponentially attracts solutions for as long as they remain in B_ϵ. The size of ϵ depends on the Lipschitz constant of f near 0; ϵ must be small enough that the Lipshcitz constant of f in B_ϵ is small.

To prove the general result, we will consider a sequence of problems

$$\begin{aligned} x(t) &= T_-(t-s)x(s) + \int_s^t T_-^{\odot *}(t-\tau)g_n(x(\tau),y(\tau))d\tau \\ y(t) &= T_+(t-s)y(s) + \int_s^t T_+^{\odot *}(t-\tau)h_n(x(\tau),y(\tau))d\tau \end{aligned} \tag{4.1$_n$}$$

where for each n, the functions g_n and h_n are globally Lipschitz, but their Lipschitz constants go to 0 as $n \to \infty$. We will show that, for n large enough, there exists a global unstable manifold $W = \{\phi : T(t)\phi$ exists and is bounded as $t \to -\infty\}$ where W is invariant, globally homeomorphic to U and globally attracting.

Such a sequence can be constructed from the original problem as follows. Let $u(t)$ be the solution to (3.5) in C. Denote by $g(x(\tau),y(\tau))$ the projection $Q^{\odot *}\hat{f}(u(\tau))$ and by $h(x(\tau),y(\tau))$ the other projection $P^{\odot *}\hat{f}(u(\tau))$. Then $g(0,0)=0$, $h(0,0)=0$, $Dg(0,0)=0$, and $Dh(0,0)=0$ and

$$\begin{aligned} x(t) &= T_-(t-s)x(s) + \int_s^t T_-^{\odot *}(t-\tau)g(x(\tau),y(\tau))d\tau \\ y(t) &= T_+(t-s)y(s) + \int_s^t T_+^{\odot *}(t-\tau)h(x(\tau),y(\tau))d\tau \end{aligned}$$

Now define

$$g_n(x,y) = \begin{cases} g(x,y) & \|x+y\| \le \frac{1}{n} \\ g(\frac{x}{n\|x+y\|}, \frac{y}{n\|x+y\|}) & \|x+y\| > \frac{1}{n} \end{cases}$$

and

$$h_n(x,y) = \begin{cases} h(x,y) & \|x+y\| \le \frac{1}{n} \\ h(\frac{x}{n\|x+y\|}, \frac{y}{n\|x+y\|}) & \|x+y\| > \frac{1}{n} \end{cases}$$

The Lipschitz constants for g_n and h_n are just the Lipshitz constants for g and h when restricted to $B_{\frac{1}{n}}$. That is $||g_n(y_1, z_1) - g_n(y_2, z_2)|| \le L_{g_n}(||y_1 - y_2|| + ||z_1 - z_2||)$ and $||h_n(y_1, z_1) - h_n(y_2, z_2)|| \le L_{h_n}(||y_1 - y_2|| + ||z_1 - z_2||)$ and L_{g_n} and L_{h_n} go to zero as n goes to infinity.

Once we have proved the existence of an N such that $(4.1)_n$ admits a global unstable manifold for all $n \ge N$, the local result for the original problem is trivial if we choose $\epsilon = \frac{1}{N}$.

Remember that we eventually want to study the parameter dependent problem

$$\dot{x}(t) = L_\lambda x_t + f_\lambda(x_t) \tag{4.2$_\lambda$}$$

so we will need the results about W to be uniform in the parameter λ.

We will first prove these results for fixed L and f and then will point out the requirements for uniformity in λ.

Theorem 4.1. *Consider the family of problems $(4.1)_n$. Suppose that for each n, g_n and h_n are globally Lipschitz and that the Lipschitz constants L_{g_n} and L_{h_n} go to zero as n goes to infinity. Suppose there are numbers $\gamma^- < 0$ and $\gamma^+ > 0$ such that $\sigma(A) \cap [\gamma^-, \gamma^+] = \emptyset$. Then there exists an N such that for $n \ge N$, the problem $(4.1)_n$ admits a global unstable manifold $W = \{\phi \in C : \|T(t)\phi\|$ exists and is bounded $\forall\, t \le 0\}$. W is invariant and there exists a homeomorphism $\sigma : U \to W$ such that $W = \{(x,y) : x = \sigma(y), y \in U\}$.*

Proof: We will use M_{g_n} and M_{h_n} for the maximum values of g_n and h_n respectively. M_{g_n} and M_{h_n} will also go to zero as n goes to infinity.

Define the class of functions

$$\mathcal{C}_{(D,\delta)} = \{\sigma : U \to S : |||\sigma||| := \sup_{y \in U} \|\sigma(y)\| \le D \text{ and } \|\sigma(y) - \sigma(y')\| \le \delta \|y - y'\|\}$$

This will be the class which contains the homeomorphism we seek. Next, for fixed τ, $\eta \in U$, and $\sigma \in \mathcal{C}$, define

$$y(t) = T_+(t-\tau)\eta + \int_\tau^t T_+^{\odot *}(t-s) h_n(\sigma(y(s)), y(s)) ds$$

$$y(\tau) = \eta$$

for $t \leq \tau$. $y(t)$ is a solution of

$$\frac{d}{dt}y_t = \hat{A}y_t + h_n(\sigma(y_t), y_t), \qquad y_t \in U.$$

Since U is finite dimensional and $y(t)$ is obviously bounded for all t, $y(t)$ exists and is unique for all $t < 0$. So we can define the map $G(\sigma)(\eta) : U \to S$ as

$$G(\sigma)(\eta) = \int_{-\infty}^{\tau} T_{-}^{\odot *}(\tau - s) g_n(\sigma(y(s)), y(s)) ds.$$

Notice that, by (3.7), $PG(\sigma)(\eta) = 0$, so $G(\sigma)(\eta)$ does indeed have range in S. We will prove that G has a fixed point in $\mathcal{C}$.

First, to see that $G(\sigma)(\eta)$ is bounded in U, we use the estimates (3.8).

$$||G(\sigma)(\eta)|| \leq \int_{-\infty}^{\tau} Ke^{-\beta}(\tau - s) M_{g_n} ds \leq KM_{g_n} \int_{-\infty}^{\tau} e^{-\beta(\tau - s)} ds = \frac{KM_{g_n}}{\beta}$$

We can pick n_1 large enough so that M_{g_n} is small enough so that $||G(\sigma)(\eta)|| \leq D$ for all $n \geq n_1$. Next we want to show that $G(\sigma)(\eta)$ is Lipschitz in η and that G is a contraction on $\mathcal{C}$. We will show these simultaneously. Denote by $y'(t)$, $y(t)$ with σ and η replaced by σ' and η'. We will first need an estimate of $||y(t) - y'(t)||$.

$$\begin{aligned}
&||y(t) - y'(t)|| \\
&\quad \leq ||T_+(t-\tau)\eta - T_+(t-\tau)\eta'|| \\
&\qquad + ||\int_{\tau}^{t} T_{+}^{\odot *}(t-s)\big(h_n(\sigma(y(s)), y(s)) - h_n(\sigma'(y'(s)), y'(s))\big) ds|| \\
&\quad \leq Ke^{\rho(t-\tau)}||\eta - \eta'|| + K\int_{t}^{\tau} e^{\rho(t-s)}||h_n(\sigma(y(s)), y(s)) - h_n(\sigma'(y'(s)), y'(s))|| ds \\
&\quad \leq Ke^{\rho(t-\tau)}||\eta - \eta'|| + K\int_{t}^{\tau} e^{\rho(t-s)} L_{h_n}\big(||\sigma(y(s)) - \sigma'(y'(s))|| + ||y(s) - y'(s)||\big) \\
&\quad \leq Ke^{\rho(t-\tau)}||\eta - \eta'|| + KL_{h_n}\int_{t}^{\tau} e^{\rho(t-s)}\big(|||\sigma - \sigma'||| + (1+\delta)||y(s) - y'(s)||\big) ds \\
&\quad \leq Ke^{\rho(t-\tau)}||\eta - \eta'|| + KL_{h_n}\int_{t}^{\tau} e^{\rho(t-s)}(1+\delta)||y(s) - y'(s)|| ds \\
&\qquad + \frac{KL_{h_n}}{\rho}|||\sigma - \sigma'|||\left(1 - e^{\rho(t-\tau)}\right)
\end{aligned}$$

Then we can use a Gronwall type estimate (see Lemma 4.4 at the end of this section) to get

$$||y(t) - y'(t)|| \leq K\left(||\eta - \eta'|| + \frac{L_{h_n}}{\rho}|||\sigma - \sigma'|||\right) e^{-(\rho - KL_{h_n}(1+\delta))(\tau - t)}$$

Now we compute $\|G(\sigma)(\eta) - G(\sigma')(\eta')\|$.

$$
\begin{aligned}
\|G(\sigma)(\eta) - G(\sigma')(\eta')\| &= \| \int_{-\infty}^{\tau} T_{-}^{\odot *}(\tau - s)\big(g_n(\sigma'(y'(s)), y'(s)) - g_n(\sigma(y(s)), y(s))\big) ds\| \\
&\leq K L_{g_n} \int_{-\infty}^{\tau} e^{-\beta(\tau-s)}\big(\|\sigma'(y'(s)) - \sigma(y(s))\| + \|y'(s) - y(s)\|\big) ds \\
&\leq K L_{g_n} \int_{-\infty}^{\tau} e^{-\beta(\tau-s)}\big((1+\delta)\|y'(s) - y(s)\| + |||\sigma - \sigma'|||\big) ds \\
&\leq I_1 \|\eta - \eta'\| + I_2 |||\sigma - \sigma'|||
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= K^2 L_{g_n}(1+\delta) \int_{-\infty}^{\tau} e^{-\beta(\tau-s)} e^{-(\rho - K L_{h_n}(1+\delta))(\tau - s)} ds \\
&= \frac{K^2 L_{g_n}(1+\delta)}{\beta + \rho - K L_{h_n}(1+\delta)} \\
I_2 &= K L_{g_n} \int_{-\infty}^{\tau} e^{-\beta(\tau-s)} \left(1 + K \frac{L_{g_n}}{\rho}(1+\delta) e^{-(\rho - K L_{h_n}(1+\delta))(\tau-s)}\right) ds \\
&= K L_{g_n} \int_{-\infty}^{\tau} e^{-\beta(\tau-s)} ds + K^2 L_{g_n} \frac{L_{g_n}}{\rho}(1+\delta) \int_{-\infty}^{\tau} e^{-\beta(\tau-s)} e^{-(\rho - K L_{h_n}(1+\delta))(\tau-s)} ds \\
&= K \frac{L_{g_n}}{\beta} + \frac{K^2 L_{g_n}^2 (1+\delta)}{\rho(\beta + \rho - K L_{h_n}(1+\delta))}
\end{aligned}
$$

Now we can find an n_2 so that I_1 and I_2 are small enough so that, for $n \geq n_2$,

$$
\|G(\sigma)(\eta) - G(\sigma')(\eta')\| \leq \delta \|\eta - \eta'\| + \theta |||\sigma - \sigma'|||
$$

where $\theta < 1$. Therefore, for $n \geq N := \max(n_1, n_2)$, G has a fixed point in $\mathcal{C}$. We will call this fixed point σ_n, so $\sigma_n(\eta) = G(\sigma_n)(\eta)$.

The homeomorphism σ_n gives us the unstable manifold. That is, we will define $W_n := \{(\sigma_n(y), y) : y \in U\}$. Now we need to prove that W_n is invariant and the solutions in W_n are exactly the ones that remain bounded as $t \to -\infty$.

Let ϕ be an initial condition in W_n. We can write ϕ as $(\sigma_n(P\phi), P\phi)$. A solution of $(4.1)_n$ through ϕ is

$$
u(t) = T_{+}(t) P\phi + \int_{-\infty}^{t} T_{-}^{\odot *}(t-s) g_n(x(s), y(s)) ds + \int_{0}^{t} T_{+}(t-s) h_n(x(s), y(s)) ds
$$

Notice that when $t = 0$ this just becomes

$$
\begin{aligned}
u(0) &= P\phi + \int_{-\infty}^{0} T_{-}^{\odot *}(t-s) g_n(x(s), y(s)) ds \\
&= P\phi + \sigma_n(P\phi) \\
&= \phi.
\end{aligned}
$$

Using the estimate (3.8) on U, we see clearly that this solution is bounded. In fact any bounded solution through ϕ can be written in this way. We just need to verify that $u(t) \in W_n$ for all t. The S component of $u(t)$ is

$$x(t) = Q \int_{-\infty}^{t} T_{-}^{\odot *}(t-s) g_n(x(s), y(s)) ds$$
$$= \int_{-\infty}^{t} T_{-}^{\odot *}(t-s) g_n(x(s), y(s)) ds = G(\sigma)(y(t)) = \sigma_n(y(t))$$

Hence $u(t) \in W_n$ and so W_n is invariant. ■

Now we want to show that W_n is exponentially attracting, at least for large enough n.

Theorem 4.2. *There exists an $\bar{N} \geq N$ such that for a solution $u(t)$ of $(4.1)_n$ with $n \geq \bar{N}$, we have*

$$\|Qu(t) - \sigma_n(Pu(t))\| \leq K e^{-\frac{\beta}{2}t} \|Qu(0) - \sigma_n(Pu(0))\|$$

where K and β are as above.

Proof: Let $(x(t), y(t))$ be a solution of $(4.1)_n$, $n \geq \bar{N}$. Define $\xi(t) = x(t) - \sigma_n(y(t))$. We want to obtain an estimate for $\xi(t)$. To do this, we will need the following estimates. Define

$$y^*(s,t) = T_+(s-t)y(t) + \int_t^s T_+^{\odot *}(s-\theta) h_n(\sigma_n(y^*(\theta,t)), y^*(\theta,t)) d\theta$$

Notice that $y^*(t,t) = y(t)$. Then we have the following.

$$\|y^*(s,t) - y(s)\| = \|T_+(s-t)y(t) + \int_t^s T_+^{\odot *}(s-\theta) h_n(\sigma_n(y^*(\theta,t)), y^*(\theta,t)) d\theta$$
$$- T_+(s-t)y(t) - \int_t^s T_+^{\odot *}(s-\theta) h_n(x(\theta), y(\theta)) d\theta\|$$
$$\leq \int_s^t K e^{\rho(s-\theta)} \|h_n(\sigma_n(y^*(\theta,t)), y^*(\theta,t)) - h_n(x(\theta), y(\theta))\| d\theta$$
$$\leq K L_{h_n} \int_s^t e^{\rho(s-\theta)} \big(\|\sigma_n(y^*(\theta,t)) - x(\theta)\| + \|y^*(\theta,t) - y(\theta)\|\big) d\theta$$
$$\leq K L_{h_n} \int_s^t e^{\rho(s-\theta)} \big(\|\sigma_n(y^*(\theta,t)) - \sigma_n(y(\theta))\| + \|\sigma_n(y(\theta)) - x(\theta)\|$$
$$+ \|y^*(\theta,t) - y(\theta)\|\big) d\theta$$
$$\leq K L_{h_n}(\delta+1) \int_s^t e^{\rho(s-\theta)} \|y^*(\theta,t) - y(\theta)\| + K L_{h_n} \int_s^t e^{\rho(s-\theta)} \|\xi(\theta)\| d\theta$$

So by Lemma 4.4, we have

$$\|y^*(s,t) - y(s)\| \leq K L_{h_n} \int_s^t e^{-(\rho - K L_{h_n}(1+\delta))(\theta - s)} \|\xi(\theta)\| d\theta$$

For $s \le t_0 \le t$, we also want to compute

$$\begin{aligned}
\|y^*(s,t) - y^*(s,t_0)\| &= \|T_+(s-t)y(t) + \int_t^s T_+(s-\theta)h_n(\sigma_n(y^*(\theta,t)), y^*(\theta,t))d\theta \\
&\quad - T_+(s-t_0)y(t) - \int_{t_0}^s T_+(s-\theta)h_n(\sigma_n(y^*(\theta,t_0)), y^*(\theta,t_0))d\theta\| \\
&= \|T_+(s-t)y(t) + \int_t^{t_0} T_+(s-\theta)h_n(\sigma_n(y^*(\theta,t)), y^*(\theta,t))d\theta - T_+(s-t_0)y(t)\| \\
&\quad + \int_{t_0}^s \|T_+(s-\theta)h_n(\sigma_n(y^*(\theta,t)), y^*(\theta,t)) - h_n(\sigma_n(y^*(\theta,t_0)), y^*(\theta,t_0))\|d\theta \\
&\le Ke^{\rho(s-\theta)}\|T_+(t_0-t)y(t) + \int_t^{t_0} T_+(t_0-\theta)h_n(\sigma_n(y^*(\theta,t)), y^*(\theta,t))d\theta - y(t_0)\| \\
&\quad + K\int_{t_0}^s e^{\rho(s-t_0)}\|h_n(\sigma_n(y^*(\theta,t)), y^*(\theta,t)) - h_n(\sigma_n(y^*(\theta,t_0)), y^*(\theta,t_0))d\theta\| \\
&\le Ke^{\rho(s-t_0)}\|y^*(t_0,t) - y(t_0)\| + KL_{h_n}(\delta+1)\int_{t_0}^s e^{\rho(s-\theta)}\|y^*(\theta,t) - y^*(\theta,t_0)\|d\theta
\end{aligned}$$

So, again by Lemma 4.4,

$$\|y^*(s,t) - y^*(s,t_0)\| \le K^2 L_{h_n}\int_{t_0}^t e^{-(\rho - KL_{h_n}(1+\delta))(\theta-s)}\|\xi(\theta)\|d\theta.$$

Now we are finally ready to estimate $\|\xi(t)\|$. To do this we will estimate $z(t) := \|\xi(t) - T_-(t-t_0)\xi(t_0)\|$.

$$\begin{aligned}
z(t) &= \|x(t) - \sigma_n(y(t)) - T_-(t-t_0)(x(t_0) - \sigma_n(y(t_0)))\| \\
&= \|T_-(t-t_0)x(t_0) + \int_{t_0}^t T_-^{\odot *}(t-\tau)g_n(x(\tau), y(\tau))d\tau \\
&\quad - \sigma_n(y(t)) - T_-(t-t_0)x(t_0) + T_-(t-t_0)\sigma_n(y(t_0))\| \\
&= \|\int_{t_0}^t T_-^{\odot *}(t-\tau)g_n(x(\tau), y(\tau))d\tau - \int_{-\infty}^t T_-^{\odot *}(t-\tau)g_n(\sigma_n(y^*(\tau,t)), y^*(\tau,t))d\tau \\
&\quad + T_-(t-t_0)\int_{-\infty}^{t_0} T_-^{\odot *}(t_0-\tau)g_n(\sigma_n(y^*(\tau,t_0)), y^*(\tau,t_0))d\tau\| \\
&= \|\int_{t_0}^t T_-^{\odot *}(t-\tau)\left(g_n(x(\tau), y(\tau)) - g_n(\sigma_n(y^*(\tau,t)), y^*(\tau,t))\right)d\tau \\
&\quad + \int_{-\infty}^{t_0} T_-^{\odot *}(t-\tau)\left(g_n(\sigma_n(y^*(\tau,t)), y^*(\tau,t)) - g_n(\sigma_n(y^*(\tau,t_0)), y^*(\tau,t_0))\right)d\tau\|
\end{aligned}$$

Now we apply the estimates (3.8) and use the fact that g is Lipschitz

$$
\begin{aligned}
z(t) \le KLg_n \int_{t_0}^{t} & e^{-\beta(t-\tau)}\big(\|x(\tau)-\sigma_n(y^*(\tau,t))\| + \|y(\tau)-y^*(\tau,t)\|\big)d\tau \\
&+ KL_{g_n}\int_{-\infty}^{t_0} e^{-\beta(t-\tau)}\big(\|\sigma_n(y^*(\tau,t))-\sigma_n(y^*(\tau,t_0))\| + \|y^*(\tau,t)-y^*(\tau,t_0)\|\big)d\tau \\
\le KL_{g_n}\int_{t_0}^{t} & e^{-\beta(t-\tau)}\|\xi(\tau)\|d\tau + KL_{g_n}(1+\delta)\int_{t_0}^{t} e^{-\beta(t-\tau)}\|y(\tau)-y^*(\tau,t)\|d\tau \\
&+ KL_{g_n}(1+\delta)\int_{-\infty}^{t_0} e^{-\beta(t-\tau)}\|y^*(\tau,t)-y^*(\tau,t_0)\|d\tau \\
= KL_{g_n}\int_{t_0}^{t} & e^{-\beta(t-\tau)}\|\xi(\tau)\|d\tau \\
&+ KL_{g_n}(1+\delta)\int_{t_0}^{t} e^{-\beta(t-\tau)}KL_{h_n}\int_{\tau}^{t} e^{-(\rho-KL_{h_n}(1+\delta))(\theta-\tau)}\|\xi(\theta)\|d\theta d\tau \\
&+ KL_{g_n}(1+\delta)\int_{t_0}^{t} e^{-\beta(t-\tau)}K^2L_{h_n}\int_{\tau}^{t} e^{-(\rho-KL_{h_n}(1+\delta))(\theta-\tau)}\|\xi(\theta)\|d\theta d\tau \\
\le KL_{g_n}\int_{t_0}^{t} & e^{-\beta(t-\tau)}\|\xi(\tau)\|d\tau \\
&+ K^3L_{g_n}L_{h_n}(1+\delta)\int_{t_0}^{t}\int_{t_0}^{\theta} e^{-\beta(t-\tau)}e^{-(\rho-KL_{h_n}(1+\delta))(\theta-\tau)}\|\xi(\theta)\|d\tau\, d\theta \\
&+ K^3L_{g_n}L_{h_n}(1+\delta)\int_{t_0}^{t}\int_{-\infty}^{t_0} e^{-\beta(t-\tau)}e^{-(\rho-KL_{h_n}(1+\delta))(\theta-\tau)}\|\xi(\theta)\|d\tau\, d\theta
\end{aligned}
$$

and so

$$
\begin{aligned}
z(t) \le KL_{g_n}\int_{t_0}^{t} & e^{-\beta(t-\tau)}\|\xi(\tau)\|d\tau \\
&+ K^3L_{g_n}L_{h_n}(1+\delta)\int_{t_0}^{t}\int_{-\infty}^{0} e^{-\beta(t-\tau)}e^{-(\rho-KL_{h_n}(1+\delta))(\theta-\tau)}\|\xi(\theta)\|d\tau\theta \\
= KL_{g_n}\int_{t_0}^{t} & e^{-\beta(t-\tau)}\|\xi(\tau)\|d\tau \\
&+ \frac{K^3L_{g_n}L_{h_n}(1+\delta)}{\beta+\rho-KL_{h_n}(1+\delta)}\int_{t_0}^{t} e^{-\beta t}e^{-(\rho-KL_{h_n}(1+\delta))\theta}\|\xi(\theta)\|e^{(\beta+\rho-KL_{h_n}(1+\delta))\theta}d\theta
\end{aligned}
$$

where the last estimate holds provided n is large enough that $\beta+\rho-KL_{h_n}(1+\delta)>0$. Now we have

$$
\begin{aligned}
z(t) &\le KL_{g_n}\int_{t_0}^{t} e^{-\beta(t-\tau)}\|\xi(\tau)\|d\tau + \frac{K^3L_{g_n}L_{h_n}(1+\delta)}{\beta+\rho-KL_{h_n}(1+\delta)}\int_{t_0}^{t}\|\xi(\theta\|e^{-\beta(t-\theta)}d\theta \\
&\le \bar{K}\int_{t_0}^{t} e^{-\beta(t-\theta)}\|\xi(\theta)\|d\theta
\end{aligned}
$$

where

$$\bar{K} = \frac{(K^3 - K^2)L_{g_n}L_{h_n}(1+\delta) + (\beta - \rho)KL_{g_n}}{\beta + \rho - KL_{h_n}(1+\delta)}$$

Now we can finally write this as an estimate for $\|\xi(t)\|$.

$$\|\xi(t)\| \leq \bar{K}\int_{t_0}^{t} e^{-\beta(t-\theta)}\|\xi(\theta)\|d\theta + \|T_-(t-\tau)\xi(t_0)\|$$

If we now apply Lemma 4.4, we get

$$\|\xi(t)\| \leq Ke^{(\bar{K}-\beta)(t-t_0)}\|\xi(t_0)\|.$$

To complete the proof, choose $\bar{N} > N$ so that $\bar{K} \leq \frac{\beta}{2}$ for all $n \geq \bar{N}$. ■

To get these results globally in the parameter λ, it is clear from the above estimates that all we need is to require the exponential dichotomy to be uniform in λ, the spectral gap to be maintained for all λ and the Lipschitz constants L_{g_n} and L_{h_n} to be uniformly small in λ, at least for n large. Recall that $g(x,y) = Q^{\odot *}\hat{f}(x+y) = Q^{\odot *}(f(x+y), 0)$ and $h(x,y) = P^{\odot *}\hat{f}(x+y) = P^{\odot *}(f(x+y), 0)$. Since $\|Q^{\odot *}\| = \|Q\|$ and $\|P^{\odot *}\| = \|P\|$, the Lipschitz constants of g and h being small in $B_{\frac{1}{n}}(0) \subset C^{\odot *}$ is equivalent to the Lipschitz constant of f being small in $B_{\frac{1}{n}}(0) \subset C$ (notice that $\|P\| = \|P*\| \geq \|P^{\odot}\| = \|P^{\odot *}\| \geq \|P^{\odot\odot}\| = \|P\|$). Hence we have the following hypotheses which replace hypotheses (H5) through (H7) in Theorem 2.2.

Theorem 4.3. *Suppose that the following are satisfied for the family of problems* $(4.2)_\lambda$.

(K1) There exists an $n \geq \bar{N}$ such that the Lipschitz constant of f_λ in the ball of radius $\frac{1}{n}$ of the origin in C can be chosen independently of λ.
(K2) There exists $\gamma^+ > 0$ and $\gamma^- < 0$ such that for every λ, $\sigma(A_\lambda) \cap [\gamma^-, \gamma^+] = \emptyset$.
(K3) There is a $K > 1$, $\beta > 0$ and $\rho > 0$, all of which are independent of λ, such that

$$\|T(s)x\| \leq Ke^{\rho s}\|x\| \qquad s \leq 0, x \in U$$
$$\|T^{\odot *}x^{\odot *}\| \leq Ke^{-\beta s}\|x\| \qquad s \geq 0, x^{\odot *} \in S^{\odot *}$$

Then, for fixed ϕ_λ, there exists a δ such that for all λ, $W_\lambda \cap B_\delta(\phi_\lambda)$ is homeomorphic to $U_\lambda \cap B_\delta(\phi_\lambda)$ and, as long as $u_\lambda(t)$ remains in $B_\delta(\phi_\lambda)$, then

$$\|Q_\lambda u(t) - \sigma_\lambda(P_\lambda u(t))\| \leq Ke^{-\frac{\beta}{2}t}\|Q_\lambda u(0) - \sigma_\lambda(P_\lambda u(0))\| \qquad (4.3)_\lambda$$

Also, the Lipshitz contants L_σ can be chosen uniformly in λ.

We conclude this section with a statement and proof of the Gronwall type estimate that was made use of in the above arguments.

Lemma 4.4. *Let a and b be real valued and continuous for $0 \le t \le \tau$ and suppose*

$$b(t) \le L \int_t^\tau b(\theta)d\theta + a(t), \qquad 0 \le t \le \tau. \tag{4.4}$$

Then

$$b(t) \le L \int_t^\tau a(\theta)e^{L(\theta - t)}d\theta + a(t). \tag{4.5}$$

Furthermore, if $a(t)$ is a decreasing function then

$$b(t) \le a(\tau)e^{L(\tau - t)} - \int_t^\tau \dot{a}(\theta)e^{L(\theta - t)}d\theta \le a(t)e^{L(\tau - t)} \tag{4.6}$$

Proof: Let

$$I(t) = L \int_t^\tau b(\theta)d\theta$$

then

$$\frac{d}{dt}(e^{Lt}I(t)) \ge -La(t)e^{Lt}.$$

Therefore

$$I(t) \le L \int_t^\tau a(\theta)e^{L(\theta - t)}d\theta.$$

¿From this inequality and (4.4) we obtain (4.5). Integrating (4.5) by parts, we get (4.6). ∎

V. Applications

To apply Theorem 2.2, we must know that the unperturbed semigroup $T_0(t)$ is a C^1-gradient system, asymptotically compact, and that orbits of bounded sets are bounded. It must also be true that each equilibrium is hyperbolic and that the set of equilibria are bounded. The perturbations $T_\lambda(t)$ must also be C^1-semigroups. Then we must verify hypotheses (H1) through (H4) of Theorem 2.2. Notice that to verify (H4), we only need to show that either $\|P_\lambda\|$ or $\|Q_\lambda\|$ is uniformly bounded, since if one is the other necessarily is. Finally we must verify hypotheses (K1) through (K3) of Theorem 4.3. If there is a t_0 such that the semigroups $\{T_\lambda(t)\}$ are compact for $t \ge t_0$, then condition (K3) in Theorem 4.3 is implied by condition (K2). So to verify the hypotheses of Theorem 4.3, we would just need to show that the gap persists under perturbation and that the nonlinearities have uniformly (in λ) small Lipschitz constants near the equilibria.

1. Retarded Functional Differential Equations

For finite delay equations in $C([-r,0],I\!R^n)$, the operators $T(t)$ are compact for $t \geq r$, so we need only verify conditions (K1) and (K2). For delay equations where only the delay depends on the parameter, it is not hard to satisfy condition (K2). Consider equations of the form

$$\dot{x}(t) = h\left(\int_{-r}^{0} g_1(x_t(\theta))d\eta_1(\theta), \ldots \int_{-r}^{0} g_k(x_t(\theta))d\eta_k(\theta)\right)$$

where the functions $\eta_1, \ldots \eta_k$ have bounded variation. Then it is shown in [13] that if there is a gap in the spectrum for the limiting problem, then it is maintained for small λ provided the measures $d\eta_j$ converge weakly.

Now we consider the example we presented in the introduction

$$\dot{x}(t) = -\int_{-r}^{0} g(x(t+\theta))a_\lambda(\theta)d\theta \qquad (5.1)_\lambda$$

where a_0 satisfies: $a(-r) = 0, a(\theta) \geq 0, \dot{a}(\theta) \geq 0, \ddot{a}(\theta) \geq 0$, for all $r \leq \theta \leq 0$. Suppose also that $g \in C_1$, the set of zeroes of g is bounded and finite, $a_\lambda \to a_0$ in L^1, and finally $\|a_\lambda\|_1 \leq M$ for all λ for some $M > 0$. If we define $x_t(\theta) = x(t+\theta)$ for all $\theta \in [-r,0]$, then we can think of the solution x_t as being an element of $C([-r,0],I\!R)$.

The equilibria are exactly the constant functions whose values are the zeros of g. In the notation, we will make no distinction between a zero of g and an equilibrium point, unless it is necessary. If x is an equilibrium with $g'(x) \neq 0$ then x is hyperbolic. We will assume that this is the case. Then the limiting problem satisfies all of the hypotheses of Theorem 2.2. Also, each equilibrium x_0 is stable if $g'(x_0) > 0$ and unstable with a one-dimensional unstable manifold if $g'(x_0) < 0$.

It was proved in [11,12] that for λ small enough the problem $(5.1)_\lambda$ admits an attractor $\mathcal{A}_\lambda$. Each attractor is "almost global" in the sense that for every bounded set $\Gamma \in C$ and for every $\epsilon > 0$ there exists a $\bar{\lambda}(\Gamma,\epsilon)$ such that for all $\lambda \leq \bar{\lambda}$, $\omega_\lambda(\Gamma) \subset N_\epsilon(\mathcal{A}_0)$. The family of attractors $\{\mathcal{A}_\lambda\}$ is upper semicontinuous as $\lambda \to 0$. In [11,12] it was also shown that, provided $a_\lambda \to a_0$ in L^1, then the flow is continuous with respect to the parameter. The continuity is uniform in bounded sets and in compact intervals of time. Also the flow is continuous with respect to initial conditions, uniformly in λ, uniformly in compact intervals of time, and uniformly in bounded sets of initial conditions. Hence (H1) and (H2) of Theorem 2.2 are satisfied.

Since the zeroes of g do not change when λ does, hypothesis (H3) is trivially satisfied. Here we must show that (H4) and (K1) through (K3) hold. To verify hypotheses (K1) through (K3) we must satisfy the hypotheses of Theorem 4.3. According to the discussion above, we only need to verify the first hypothesis. To this end, consider the nonlinear part of g near an equilibrium x_0

$$s(z) = g(z) - g(x_0) - g'(x_0)(z - x_0).$$

Then define

$$f_\lambda(\phi) = -\int_{-r}^{0} s(\phi(\theta))a_\lambda(\theta)d\theta$$

We have the following Proposition

Proposition 5.1. *Let L_{f_λ} be the Lipschitz constant of f_λ restricted to the ball of radius $\frac{1}{n}$ of x_0 in C. Then $L_{f_\lambda} \to 0$ as $n \to \infty$ uniformly in λ.*

Proof:

$$\begin{aligned} f_\lambda(\phi) - f_\lambda(\psi) &= \int_{-r}^{0} (s(\phi(\theta)) - s(\psi(\theta)))a_\lambda(\theta)d\theta \\ &= \int_{-r}^{0} g(\phi(\theta)) - g(\psi(\theta)) - g'(x_0)(\phi(\theta) - \psi(\theta))a_\lambda(\theta)d\theta \\ &= \int_{-r}^{0} g'(c)(\phi(\theta) - \psi(\theta)) - g'(x_0)(\phi(\theta) - \psi(\theta))a_\lambda(\theta)d\theta \\ &= \int_{-r}^{0} (g'(c) - g'(x_0))(\phi(\theta) - \psi(\theta))a_\lambda(\theta)d\theta \end{aligned}$$

where c is some number in the interval $[x_0 - \frac{1}{n}, x_0 + \frac{1}{n}]$ which we obtain from the mean value theorem. Therefore

$$\|f_\lambda(\phi) - f_\lambda(\psi)\| \leq |g'(c) - g'(x_0)| \, \|a_\lambda\|_1 \, \|\phi - \psi\|.$$

Since we have assumed $\|a_\lambda\|_1 \leq M$ for all λ and $g \in C_1$, we have the result. ■

Hence hypotheses (K1) through (K3) hold.

It only remains to show that $\|P_\lambda\|$ is uniformly bounded. Of course, if $g'(x_0) > 0$, then $P_\lambda \equiv 0$ and since the eigenvalues will not cross the imaginary axis for small λ, we are done. For the case when $g'(x_0) < 0$ we will make use of the explicit decomposition of the phase space given in [6]. In this case, there is one positive eigenvalue, ν_λ, and the unstable manifold is the set of functions $\{ae^{\nu_\lambda \theta} : a \in I\!R\}$. For the eigenfunction ϕ_λ we will choose $a = 1$ so that $\|\phi_\lambda\| = 1$. Let ψ_λ be the corresponding eigenfunction for the adjoint equation

$$\dot{y}(t) = -\int_{-r}^{0} g(y(t-\theta))a_\lambda(\theta)d\theta.$$

Then $\psi_\lambda(s) = b_\lambda e^{-\nu_\lambda s}$ for $0 \leq s \leq r$. ¿From now on, we will write ν for ν_λ to simplify the notation. We will choose b_λ so that $(\psi_\lambda, \phi_\lambda)_\lambda = 1$ where

$$(\alpha, \beta)_\lambda = \alpha(0)\beta(0) - \int_{-r}^{0}\int_{0}^{\theta} \alpha(\xi - \theta)\beta(\xi)d\xi a_\lambda(\theta)d\theta.$$

Hence

$$b_\lambda = \frac{1}{1 - \int_{-r}^{0} \theta e^{\nu\theta} a_\lambda(\theta) d\theta}.$$

Then

$$P_\lambda x = \phi_\lambda (\psi_\lambda, x)_\lambda.$$

To get an estimate for $\|P_\lambda\|$, consider the following.

$$\begin{aligned}
\|P_\lambda x\| &= |(\psi_\lambda, x)_\lambda| \\
&= \left| x(0) b_\lambda + \int_{-r}^{0} b_\lambda e^{\nu\theta} \int_{\theta}^{0} e{-}\nu\xi x(\xi) d\xi a_\lambda(\theta) d\theta \right| \\
&\leq |x(0)|\, |b_\lambda| + r|b_\lambda| \|x\|_\infty \|a_\lambda\|_1 \\
&\leq \|x\|_\infty |b_\lambda| (1 + r\|a_\lambda\|_1) \\
&\leq \|x\|_\infty |b_\lambda| (1 + rM)
\end{aligned}$$

Therefore $\|P_\lambda\| \leq |b_\lambda|(1 + rM)$. Since $b_\lambda \leq 1$ for all λ, we have the result.

Proposition 5.2. *$\|P_\lambda\|$ is uniformly bounded in λ and so (H4) holds.*

Finally then we can conclude

Theorem 5.3. *For small λ, the problem $(5.1)_\lambda$ admits an attractor $\mathcal{A}_\lambda$ which is "almost global" and the family of attractors, $\{\mathcal{A}_\lambda\}$, is continuous at $\lambda = 0$.*

2. An Example from Parabolic PDE's

In this example we consider reaction diffusion problems for which the diffusion coefficient becomes large in a subregion which is interior to the physical domain of the differential equation. That situation can be found, for example, in composite materials, where the heat diffusion properties can change significantly from one part of the region to another; that is, heat may diffuse much faster in some subregions than in others. We intend to obtain continuity of attractors with respect to the attractor of a limiting problem which was obtained in [1]. In this direction, this paper represents the continuation of the analysis started in [1], where existence and upper semicontinuity of attractors with respect to the attractor of the limiting problem has been established. In what follows we briefly describe the problem following closely [1].

Let Ω be a bounded smooth domain in $I\!R^N$, $N \leq 3$, ϵ be a positive parameter, m be a positive integer, and $\Omega_0 = \cup_{i=1}^m \Omega_{0,i}$ be a subdomain of Ω where $\Omega_{0,i}$ is a smooth subdomain of Ω with $\overline{\Omega}_{0,i} \cap \overline{\Omega}_{0,j} = \emptyset$, for $i \neq j$. Let $\Gamma = \partial\Omega$, $\Gamma_{0,i} = \partial\Omega_{0,i}$ and $\Gamma_0 = \cup_{i=1}^m \Gamma_{0,i}$; these are, respectively, the boundary of Ω, $\Omega_{0,i}$ and Ω_0. Denote by Ω_1 the set $\Omega \backslash \overline{\Omega}_0$ and note that its boundary is given by $\partial\Omega_1 = \Gamma \cup \Gamma_0$.

Consider the family of parabolic equations

$$\begin{aligned} &u_t^\epsilon - \mathrm{Div}(a_\epsilon(x)\nabla u^\epsilon) + \lambda u^\epsilon + f(u^\epsilon) = 0, \quad \text{on} \quad \Omega \\ &\frac{\partial u^\epsilon}{\partial \vec{n}_\epsilon} = 0, \quad \text{on} \quad \Gamma \end{aligned} \tag{5.2}$$

where $\frac{\partial u}{\partial \vec{n}_\epsilon} = a_\epsilon(x)\langle \nabla u, \vec{n}\rangle$, a_ϵ, $0 \le \epsilon \le \epsilon_0$, is assumed to be a regular bounded function in Ω satisfying $0 < m_0 \le a_\epsilon(x)$ for every $x \in \Omega$ and $0 \le \epsilon \le \epsilon_0$. Assume that $a_\epsilon(x) \to \infty$ uniformly on compact subsets of Ω_0, as $\epsilon \to 0$, and $a_\epsilon(x) \to a_0(x)$ uniformly on Ω_1 as $\epsilon \to 0$.

Intuitively, we guess that for small values of ϵ, the solution u_ϵ of problem (5.2) should be approximately constant on Ω_0 as time increases. Therefore, suppose that u^ϵ converges to some function u, in some sense, and that u takes a time dependent spatially constant value on Ω_0 which we will call $u_{\Omega_0}(t)$.

If we formally take the limit in problem (5.2), inside Ω_1 we expect that

$$\begin{aligned} &u_t - \mathrm{Div}(a_0(x)\nabla u) + \lambda u + f(u) = 0, \quad \text{on} \quad \Omega_1 \\ &\frac{\partial u}{\partial \vec{n}_0} = 0, \quad \text{on} \quad \Gamma. \end{aligned}$$

where $\frac{\partial u}{\partial \vec{n}_0} = a_0(x)\langle \nabla u, \vec{n}\rangle$. The constant u_{Ω_0}, however, may not be arbitrary. Integrating the equation in Ω_0 and using the inward normal in the integration by parts, we obtain

$$\int_{\Omega_0} u_t^\epsilon + \int_{\Gamma_0} \frac{\partial u^\epsilon}{\partial \vec{n}_\epsilon} + \lambda \int_{\Omega_0} u^\epsilon + \int_{\Omega_0} f(u^\epsilon) = 0$$

and formally taking the limit and dividing by $|\Omega_0|$ we have

$$\dot{u}_{\Omega_0} + \frac{1}{|\Omega_0|}\int_{\Gamma_0} \frac{\partial u}{\partial \vec{n}_0} + \lambda u_{\Omega_0} + f(u_{\Omega_0}) = 0.$$

Thus, the limiting equation should be

$$\begin{aligned} &u_t - \mathrm{Div}(a_0(x)\nabla u) + \lambda u + f(u) = 0, \quad \text{on } \Omega_1 \\ &\frac{\partial u}{\partial \vec{n}_0} = 0, \quad \text{on } \Gamma \\ &u_{|\Omega_0} =: u_{\Omega_0} \quad \text{on } \Omega_0 \\ &\dot{u}_{\Omega_0} + \frac{1}{|\Omega_0|}\int_{\Gamma_0} \frac{\partial u}{\partial \vec{n}_0} + \lambda u_{\Omega_0} + f(u_{\Omega_0}) = 0. \end{aligned}$$

In fact, when $m \ge 1$ the limiting problem should be

$$\begin{aligned} &u_t - \mathrm{Div}(a_0(x)\nabla u) + \lambda u + f(u) = 0 \text{ on } \Omega_1 \\ &\frac{\partial u}{\partial \vec{n}_0} = 0 \text{ on } \Gamma \\ &u_{|\Omega_{0,i}} =: u_{\Omega_{0,i}} \text{ on } \Omega_{0,i}, \quad i = 1, \dots, m \\ &\dot{u}_{\Omega_{0,i}} + \frac{1}{|\Omega_{0,i}|}\int_{\Gamma_{0,i}} \frac{\partial u}{\partial \vec{n}_0} + \lambda u_{\Omega_{0,i}} + f(u_{\Omega_{0,i}}) = 0 \text{ on } \Omega_{0,i}, \quad i = 1, \dots, m. \end{aligned} \tag{5.3}$$

Assume that $f : I\!R \to I\!R$ is a C^1 function satisfying the following growth condition

$$\begin{aligned} &\lim_{|s|\to\infty} \frac{|f'(s)|}{e^{\eta\,|s|^2}} = 0, \forall\ \eta > 0,\ \text{ if } n = 2 \\ &|f'(s)| \le L(1 + |s|^2),\ \text{ if } n = 3 \end{aligned} \tag{5.4}$$

Observe that no growth assumptions are made for $n = 1$. Also assume that the following dissipativeness condition is satisfied

$$\liminf_{|u|\to\infty} \frac{f(u) + \lambda u}{u} > 0. \tag{5.5}$$

Under these conditions Carvalho and Rodriguez-Bernal prove in [1] that problems (5.2) and (5.3) define compact semigroups $\{S_\epsilon(t), t \ge 0\}$ and $\{S_0(t), t \ge 0\}$, respectively, on $H^1(\Omega)$ and $H^1_{\Omega_0}(\Omega)$, where

$$H^1_{\Omega_0}(\Omega) = \{u \in H^1(\Omega),\ u \text{ is constant on } \Omega_{0,i},\ i = 1, \ldots, m\}$$

Moreover, they also prove that these problems have global attractors $\mathcal{A}_\epsilon$ and $\mathcal{A}_0$ and that they are gradient systems in $H^1(\Omega)$ and $H^1_{\Omega_0}(\Omega)$ respectively, in the sense of Hale [].

Their main result is the following.

Theorem 5.4. *The global attractors, $\mathcal{A}_\epsilon$ and $\mathcal{A}_0$, are upper semicontinuous at $\epsilon = 0$.*

This theorem guarantees that hypothesis (H1) in Theorem 2.2 is satisfied. We will now prove lower semicontinuity of attractors for the same problem concluding that the family of attractors is continous at zero. This will be done by verifying that the remaining hypotheses of Theorem 2.2 are also satisfied.

In [1], they actually prove a little more than upper semicontinuity of attractors; they also prove that

Lemma 5.5. *If $u^\epsilon(t)$, $t \in I\!R$ denotes a solution of (5.2) in $\mathcal{A}_\epsilon$; then $\{u^\epsilon(t),\ t \in I\!R\}_{\epsilon>0}$ is relatively compact in $H^1(\Omega)$. Furthermore,*

$$u^\epsilon \to u \text{ in } C(I\!R, H^1(\Omega))$$

uniformly in compact subsets of $I\!R$, where u is a solution of (5.3) in $\mathcal{A}_0$.

As an immediate consequence of the above result we have that the following corollary holds.

Corollary 5.6. *If $T_\epsilon(t)$ denotes the semigroup defined by (5.2) and $T_0(t)$ denotes the flow defined by (5.3). Then $\{T_\epsilon\}_{\epsilon\geq 0}$ is continous at zero in $\mathcal{A}_0$.*

This corollary proves that hypothesis (H2) of Theorem 2.2 is satisfied.

Denote by $\mathcal{E}_\epsilon$, $0 < \epsilon \leq \epsilon_0$, the set of equilibrium points for (5.2) and by $\mathcal{E}_0$ the set of equilibrium points for (5.3). Assume that all the equilibrium points for the problem (5.3) are hyperbolic. Therefore, the problem (5.3) has a finite number of equilibrium points and, since the solution semigroup is gradient, the global attractor $\mathcal{A}_0$ for (5.3) can be written as

$$\mathcal{A}_0 = \cup_{\phi\in\mathcal{E}_0} W^u(\phi).$$

In what follows we prove that there exists $\epsilon_1 > 0$ such that all the equilibrium points for (5.2) are hyperbolic. We start by proving the particular case of Lemma 5.5 concerning convergence of equilibrium points. Our first result gives us bounds for $\mathcal{E}_\epsilon$ uniformly in ϵ, $0 < \epsilon \leq \epsilon_0$.

Lemma 5.7. *Under the growth and dissipativeness conditions above, the following estimates hold*

$$\sup_{\epsilon}\sup_{\phi\in\mathcal{E}_\epsilon} \|\phi\|_{L^\infty(\Omega)} < \infty$$

and for every $\alpha \leq 1$

$$\sup_{\epsilon}\sup_{\phi\in\mathcal{E}_\epsilon} \|\phi\|_{X_\epsilon^\alpha} < \infty.$$

In particular $\cup_{\epsilon>0}\mathcal{E}_\epsilon$ is a bounded subset of $H^1(\Omega)$.

Proof: It has been proved in [2] that $\mathcal{A}_\epsilon$ is bounded in $L^\infty(\Omega)$ and that the bound does not depend on the diffusion coefficient but only on the nonlinearity. Thus we have that $\cup_{\epsilon>0}\mathcal{A}_\epsilon$ is bounded in $L^\infty(\Omega)$ uniformly in ϵ and the result follows. ∎

Using this result we can prove the following lemma.

Lemma 5.8. *The sets of equilibrium points $\mathcal{E}_\epsilon$, $0 \leq \epsilon \leq \epsilon_0$ are upper semicontinuous at $\epsilon = 0$.*

Proof: Assume that we have proved that the set $\{\phi_\epsilon\}_{\epsilon>0}$, where $\phi_\epsilon \in \mathcal{E}_\epsilon$, is relatively compact in $H^1(\Omega)$. Then, taking subsequences if necessary,

$$\phi_\epsilon \to \phi_0 \in H^1(\Omega).$$

It follows from the fact that $\int_\Omega a_\epsilon(x)|\nabla\phi|^2dx$ is bounded that ϕ_0 must be constant in Ω_0 and therefore, $\phi_0 \in H^1_{\Omega_0}(\Omega)$.

It follows from this that $f(\phi_\epsilon) \to f(\phi_0)$ and for any $\psi \in H^1_{\Omega_0}(\Omega)$ we have that

$$\int_\Omega a_\epsilon\nabla\phi_\epsilon\nabla\psi = \int_{\Omega_1} a_\epsilon\nabla\phi_\epsilon\nabla\psi \to \int_{\Omega_1} a_0\nabla\phi_0\nabla\psi$$

and we have that

$$\int_\Omega a_0 \nabla \phi_0 \nabla \psi + \lambda \int_\Omega \phi_0 \psi + \int_\Omega f(\phi_0)\psi = 0, \quad \forall \psi \in H^1_{\Omega_0}(\Omega)$$

or equivalently $\phi_0 \in \mathcal{E}_0$.

We now must prove that $\{\phi_\epsilon\}_{\epsilon>0}$ is indeed relatively compact in $H^1(\Omega)$. Note that $\{\phi_\epsilon\}_{\epsilon>0}$ is a bounded subset of $H^1(\Omega)$ and of $L^\infty(\Omega)$. Thus $\{f(\phi_\epsilon)\}_{\epsilon>0}$ is a bounded subset in $H^1(\Omega)$ and therefore, taking subsequences if necessary, $\{f(\phi_\epsilon)\}_{\epsilon>0}$ converges in $L^2(\Omega)$. Since, again taking subsequences if necessary, $\phi_\epsilon \to \phi_0 \in H^1_{\Omega_0}(\Omega)$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$, we have that $\phi_0 \in \mathcal{E}_0$ and

$$\int_\Omega a_\epsilon |\nabla \phi_\epsilon|^2 \to \int_\Omega f(\phi_0)\phi_0 = \int_\Omega a_0 |\nabla \phi_0|^2$$

or $\|\phi_\epsilon\|_{X_\epsilon^{\frac{1}{2}}} \to \|\phi_0\|_{X_0^{\frac{1}{2}}}$. Therefore, $\|\phi_\epsilon\|_{H^1(\Omega)} \to \|\phi_0\|_{H^1(\Omega)}$ and $\phi_\epsilon \to \phi_0$ strongly in $H^1(\Omega)$ and $\{\phi_\epsilon\}_{\epsilon>0}$ is relatively compact in $H^1(\Omega)$.

Suppose that there exists a $\delta > 0$ such that for any $\epsilon > 0$ there exists $\phi_\epsilon \in \mathcal{E}_\epsilon$ which is not in a $\delta-$neigborhood of $\mathcal{E}_0$. Then, $\{\phi_\epsilon, 0 < \epsilon < \epsilon_0\} \cap \mathcal{E}_0 = \emptyset$, which contradicts what we just proved. ■

This result implies that if $\mathcal{A}_\epsilon \ni u_0^\epsilon \to u_0 \in \mathcal{A}_0$ in $H^1(\Omega)$ then $u^\epsilon(t, u_0^\epsilon) \to u(t, u_0)$ uniformly in compact subsets of $I\!R$.

Next we prove that there exists $\epsilon_1 > 0$ such that all the equilibrium points for (5.2) are hyperbolic, if $0 < \epsilon < \epsilon_1$. Let $\{\phi_{\epsilon_n}\}$ be any sequence of equilibrium points such that $\phi_{\epsilon_n} \in \mathcal{E}_{\epsilon_n}$ and such that $\phi_{\epsilon_n} \to \phi_0 \in \mathcal{E}_0$ in $H^1(\Omega)$. It follows from the results in [14] that the eigenvalues μ_ϵ^i of the problem

$$\begin{aligned} &-\operatorname{Div}(a_\epsilon(x)\nabla u_\epsilon) + \lambda u_\epsilon + f'(\phi_\epsilon)u_\epsilon = \mu_\epsilon u_\epsilon \\ &\frac{\partial u}{\partial \vec{n}_\epsilon} = 0 \end{aligned}$$

converge to the eigenvalues μ_0^i of the problem

$$\begin{aligned} &-\operatorname{Div}(a_0(x)\nabla u) + \lambda u + f'(\phi_\epsilon)u = \mu\, u \\ &\frac{\partial u}{\partial \vec{n}_0} = 0 \\ &\frac{1}{|\Omega_0|}\int_{\Gamma_0} \frac{\partial u}{\partial \vec{n}_0} + \lambda u_{\Omega_0} + f'(\phi_{0|_{\Omega_0}})u_{\Omega_0} = \mu\, u_{\Omega_0}. \end{aligned}$$

Therefore, since ϕ_0 is a hyperbolic equilibrium for (5.3) we have that ϕ_ϵ is also a hyperbolic equilibrium for (5.2).

Since the set $\{\mathcal{E}_\epsilon, \epsilon_0 \geq \epsilon \geq 0\}$ is upper-semicontinuous at zero we have that, given $\delta > 0$, there exists $\epsilon_1 = \epsilon_1(\delta) > 0$ such that $\mathcal{E}_\epsilon$ is in a δ neighborhood of $\mathcal{E}_0$, for every $\epsilon < \epsilon_1$. This proves the following result

Lemma 5.9. *Assume that all the equilibrium points for (5.3) are hyperbolic. Then, all the equilibrium points for (5.2) are hyperbolic, $0 < \epsilon < \epsilon_1$.*

Remark. *Let $\phi_0 \in \mathcal{E}_0$; then, for δ small enough, there may be only one equilibrium point of (5.2) in a $\delta-$neighborhood of ϕ_0.*

We assume that given an equilibrium point ϕ_0 for the problem (5.3) and $\delta > 0$ there is an $\epsilon_2 > 0$ such that (5.2) has an equilibrium point in a δ neighborhood of ϕ_0 for $0 < \epsilon < \epsilon_2$. With this assumption and the above remark, then (H3) holds.

The hypothesis (H4) is trivially satisfied since the projections onto the linear unstable manifold have finite rank and the eigenfunctions converge in $H^1(\Omega)$. Since $T(t)$ is compact for all $t > 0$, we don't need to verify (K3), and (K2) is clearly satisfied since the eigenvalues converge. We have left then to verify (K1). That is, the Lipschitz constants of the nonlinearities in (5.2) are uniformly close to zero in a small neighborhood of the equilibrium.

Lemma 5.10. *Let $\{\phi_\epsilon, 0 < \epsilon < \epsilon_0\}$ be a sequence of equilibrium points of (5.2) that converge to an equilibrium point ϕ_0 of (5.3). The Lipschitz constant of the map $g_\epsilon^e : H^1(\Omega) \to L^2(\Omega)$ given by*

$$g_\epsilon^e(u)(x) = g_\epsilon(x, u(x))$$

where $g_\epsilon(x,s) = f(s) - f(\phi_\epsilon(x)) - f'(\phi_\epsilon(x))(s - \phi_\epsilon(x))$ in the ball of radius $\frac{1}{n}$ tends to zero as $n \to \infty$ uniformly with respect to ϵ.

Proof: Let u and v be points in $H^1(\Omega)$ in a ball of radius $\frac{1}{n}$ around ϕ_0

$$\begin{aligned}
&\|g_\epsilon^e(u) - g_\epsilon^e(v)\|_{L^2(\Omega)}^2 = \|f(u) - f(v) - f'(\phi_\epsilon)(u-v)\|_{L^2(\Omega)}^2 \\
&= \int_\Omega [f(u(x)) - f(v(x)) - f'(\phi_\epsilon(x))(u(x) - v(x))]^2 dx \\
&= \int_\Omega \int_0^1 [f'(u(x) - t(u(x) - v(x))) - f'(\phi_\epsilon(x))]^2 dt |u(x) - v(x)|^2 dx \\
&\le \left(\int_\Omega \left(\int_0^1 [f'(u(x) - t(u(x) - v(x))) - f'(\phi_\epsilon(x))]^2 dt \right)^{\frac{3}{2}} dx \right)^{\frac{2}{3}} \left(\int_\Omega |u(x) - v(x)|^6 dx \right)^{\frac{1}{3}} \\
&\le K \left(\int_\Omega [f'(u(x) - t(x)(u(x) - v(x))) - f'(\phi_\epsilon(x))]^3 dx \right)^{\frac{2}{3}} \|u - v\|_{H^1(\Omega)} \\
&\le L_\epsilon \|u - v\|_{H^1(\Omega)},
\end{aligned}$$

where,

$$L_\epsilon = K \left(\int_\Omega [f'(u(x) - t(x)(u(x) - v(x))) - f'(\phi_0(x)) + f'(\phi_0(x)) - f'(\phi_\epsilon(x))]^3 dx \right)^{\frac{2}{3}}.$$

Then, since $\phi_\epsilon \to \phi_0$ in $H^1(\Omega)$ as $\epsilon \to 0$, we have that L_ϵ is small in the ball of radius $\frac{1}{n}$ uniformly in ϵ and the result is proved. ■

The hypothesis (K2) and (K3) are easily seen to be satisfied from the convergence of eigenvalues. This verifies the hypothesis of Theorem 2.2 and the following result holds.

Theorem 5.11. *The family of attractors $\{\mathcal{A}_\epsilon\}_{\epsilon\geq 0}$ is continuous at zero.*

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