UNIVERSIDADE DE SÃO PAULO
Instituto de Ciências Matemáticas e de Computação

ISSN 0103-2577

Numerical Control of Equisingularity for Map Germs from $\mathbb{C}^n$ to $\mathbb{C}^3$, $n \geq 3$

V. H. Jorge Pérez
E. C. Rizziolli
M. J. Saia

Nº 261

NOTAS

Série Matemática

São Carlos – SP
Mai./2006

Numerical control of Equisingularity for map germs from $\mathbb{C}^n$ to $\mathbb{C}^3$, $n \geq 3$

V. H. Jorge Pérez,* E. C. Rizziolli and M. J. Saia†

Resumo

Neste artigo é dado o número mínimo de invariantes que garante a equisingularidade de Whitney em uma família a um parâmetro de germes de aplicações holomorfos de corank um $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^3, 0)$, com $n \geq 3$. A principal ferramenta utilizada na determinação destes invariantes é a relação entre os números de Lê que aparecem na hipersuperfície determinada pela imagem inversa por f do conjunto discriminante de f e as multiplicidades polares associadas aos tipos estáveis do germe.

Abstract

We give the minimal number of invariants that is sufficient for the Whitney equisingularity of a one parameter deformation of corank one finitely determined holomorphic map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^3, 0)$, with $n \geq 3$. We see from the work of Gaffney that the invariants needed to control the Whitney equisingularity are the 0-stable invariants and the polar multiplicities which appear in the stable types of a stable deformation of the germ. The key tool to give this minimal number is to use the relationship between the invariants in the stable types in the source and the target with the Lê numbers which appear in the inverse image by the germ f of the discriminant of f.

1 Introduction

Gaffney describes in [3] the following problem: "Given a 1-parameter family of map germs $F\colon (\mathbb{C}\times\mathbb{C}^n, (0,0)) \to (\mathbb{C} \times \mathbb{C}^p, (0))$, find analytic invariants whose constancy in the family implies the family is Whitney equisingular. "He shows that for the class of finitely determined map germs of discrete stable type, the Whitney equisingularity of such a family is guaranteed by the invariance of the zero stable types and the polar multiplicities associated to all stable types.

Since the number of invariants depends on the dimensions (n, p) and it can be very big according to n and p are big. Then a natural question is to find a minimal set of invariants that guarantee the Whitney equisingularity of the family.

In the case of map germs in $\mathcal{O}(2, 2)$ and $\mathcal{O}(2, 3)$, Gaffney reduced the number of invariants by finding relations among them. From the fact that they are upper semi-continuous, these relations allowed to reduce the number of invariants required in Gaffney's theorem. The method of Gaffney was applied in the cases

*The first named author is partially supported by CAPES - Grant 3131/04-3

†The third named author is partially supported by CNPq - Grant 300556/92-6.

of map germs in $\mathcal{O}(n,p)$, when $n = p$, [21], in $n < p$ [22], and in $(n,3)$ with $n \geq 3$, see [23]. However, we remark that this method does not allow to find the minimal number of invariants for these dimensions.

More recently, Gaffney and Vohra in [10] studied map germs from n-space to the plane, with $n \geq 3$. In this case, and also when $n \geq p$, there are some stable types which appear in the strata in the source which, possibly are with non isolated singularities. Then they showed that the Lê numbers, associated to the top dimensional stratum in the source, i. e., the inverse image of the discriminant control lower dimensional strata as well, and this leads to a considerable simplification, in terms of the numbers of invariants needed. In fact, they showed that this stratum carries all information necessary to determine if the family of map germs is Whitney equisingular.

In this paper we study map germs in $\mathcal{O}(n,3)$ with $n \geq 3$ under this point of view. Following this new method of Gaffney and Vohra, we also show that the Lê numbers of the inverse image of the discriminant control all information necessary to determine the Whitney equisingularity of the family.

2 Notation and preliminaries

We follow Gaffney in [3] and denote by $\mathcal{O}(n,p)$ the set of origin preserving germs of holomorphic mappings from $\mathbb{C}^n$ to $\mathbb{C}^p$, $\mathcal{O}_e(n,p)$ denotes the set of germs at the origin but not necessarily origin preserving.

For a germ $f \in \mathcal{O}_e(n,p)$, we denote the singular set of f by $S(f)$, it consists of all points where the rank of the derivative of f is less than $m = min(n,p)$. The ideal generated by the set of $m \times m$ minors of the derivative of f is denoted by $J(f)$.

The critical set $\Sigma(f)$ of f is the set of points $x \in \mathbb{C}^n$ such that $J(f)(x) = 0$. The discriminant $\Delta(f)$ of f is the image of $\Sigma(f)$ by f. When $n > p$, the sets $S(f)$ and $\Sigma(f)$ coincided.

We denote by $\mathcal{R}$ the group of local diffeomorphisms of the source $(\mathbb{C}^n, 0)$, and by $\mathcal{L}$ the group of local diffeomorphisms of the target $(\mathbb{C}^p, 0)$. The action of the product $\mathcal{L} \times \mathcal{R} =: \mathcal{A}$ leads to $\mathcal{A}$**−equivalence** of map-germs: $f, g \in \mathcal{O}(n,p)$ are $\mathcal{A}$-equivalent if they are equivalent by smooth coordinate changes of source and target. Similarly the action of the semi-direct product $\mathcal{K} := (\mathcal{R}.\mathcal{L})$ gives rise to $\mathcal{K}$**-equivalence** [also called **contact equivalence**] of maps germs. A given map-germ $f \in \mathcal{O}(n,p)$ is said to be $\mathbf{k} - \mathcal{G}$**−determined** ($\mathcal{G} = \mathcal{A}$ or $\mathcal{K}$) if whenever $g \in \mathcal{O}(n,p)$ and $j^k f(0) = j^k g(0)$, then g is $\mathcal{G}$-equivalent to f; f is **finitely** $\mathcal{G}$**-determined** if it is $k - \mathcal{G}-$ determined for some $k < \infty$. In order to lighten notation, and in accordance with standard practice, whenever we say "finitely determined" (without specifying whether $\mathcal{A}$ or $\mathcal{K}$), we shall mean "finitely $\mathcal{A}-$ determined".

Map germs which are finitely $\mathcal{K}-$ determined are called map germs of **finite singularity type**, [[27], p.201]. Using the symmetry between the theory of $\mathcal{A}$-equivalence and the theory of $\mathcal{K}$-equivalence, we know that, in particular if $f \in \mathcal{O}(n,p)$ is finitely $\mathcal{A}-$ determined, then f has finite singularity type. In this situation, the restriction $f|_{S(f)} : S(f) \to \Delta f$ is a **finite map** [any point in the target has at most a finite number of preimages], and hence $\dim S(f) = \dim \Delta f$.

We say f is a **stable germ** if every "nearby" germ is $\mathcal{A}$-equivalent to f. A **stable type** is an $\mathcal{A}$-equivalence class of stable germs. A stable type $\mathcal{Q}$ appears in a versal unfolding of f, F if for any representative $F = (id, f_u(x))$ of F, there exists a point $(u,y) \in \mathbb{C}^s \times \mathbb{C}^p$ such that the germ $f_u : (\mathbb{C}^n, S) \to (\mathbb{C}^p, y)$ is a stable germ of type $\mathcal{Q}$, $S = f^{-1}(y) \cap \Sigma(f_u)$. The points (u,y) and (u,x) with $x \in S$ are called points of stable type $\mathcal{Q}$ in the target and in the source, respectively.

A finitely determined germ f has **discrete stable type** if there exists a versal unfolding of f in which only a finite number of stable types occur. In particular, if the numbers (n,p) are Mather's "nice dimensions" [which is true for $(n,3)$, $n \geq 3$, our focus here] or on the boundary thereof, then every finitely determined germ $f \in \mathcal{O}(n,p)$ has discrete stable type.

Let $F : (\mathbb{C} \times \mathbb{C}^n, (0,0)) \to (\mathbb{C} \times \mathbb{C}^p, (0,0))$, $F = (t, \overline{f}(t,x))$, be a 1-parameter unfolding of a finitely determined germ f, such that $\overline{f}(t,-)$ preserves the origin for all t. Let $T := \mathbb{C} \times \{0\}$. F is a *good unfolding* of f if there exist neighborhoods U, W of the origin in $\mathbb{C} \times \mathbb{C}^n$ and $\mathbb{C} \times \mathbb{C}^p$ respectively such that $F^{-1}(W) = U$, F maps $U \cap \Sigma(F) - T$ to $W - T$ and if $(t_0, y_0) \in W - T$, then the germ $f_{t_0} : (\mathbb{C}^n, S) \to (\mathbb{C}^p, y_0)$ is stable, where $S = F^{-1}(t_0, y_0) \cap \Sigma(F)$.

A good unfolding is *excellent* if all the 0-stable invariants are constant in the unfolding and f is of discrete type.

An unfolding F of f is *Whitney equisingular* along the parameter space T if there exists a regular stratification of the source and the target, with T a stratum of the source and the target and these stratifications are Whitney equisingular along T, i.e. satisfy the Whitney conditions **a** and **b** and Thom's A_F condition.

One of the questions of main interest is to show when an excellent unfolding $F : (\mathbb{C} \times \mathbb{C}^n, (0,0)) \to (\mathbb{C} \times \mathbb{C}^p, (0,0))$ of a finitely determined germ $f \in \mathcal{O}(n,p)$ is Whitney equisingular.

Using the polar invariants, i.e, the polar multiplicities of the polar varieties of the stable types (defined by Teissier in [25]) and Thom's isotopy lemmas, Gaffney showed the following.

Theorem 2.1. ([3], p. 207) *Suppose that $F : (\mathbb{C} \times \mathbb{C}^n, (0,0)) \to (\mathbb{C} \times \mathbb{C}^p, (0,0))$ is an excellent unfolding of a finitely determined germ $f \in \mathcal{O}(n,p)$. Also suppose that the polar invariants of all stable types defined in:*

1. *the discriminant $\Delta(f_t) = f_t(\Sigma(f_t))$,*
2. *the singular set $\Sigma(f_t)$ and also in the set*
3. *$X(f_t) = \overline{(f_t^{-1}(\Delta(f_t)) - \Sigma(f_t))}$,*

are constant at the origin for all t. Then the unfolding is Whitney equisingular.

Remark 2.2. The theorem remains valid if we replace the term "an excellent unfolding" in the hypothesis by "a 1-parameter unfolding which, when stratified by stable types and by the parameter axis T, has only the parameter axis T as 1-dimensional stratum at the origin" ([10]).

3 The stable types in $\mathcal{O}(n,3)$

According to the Theorem of Gaffney, the constancy of the polar invariants of all stable types defined in $\Delta(f_t)$, $\Sigma(f_t)$ and $X(f_t)$ is the condition for the Whitney equisingularity.

The first step in the strategy to minimize the number of invariants is to describe all stable types which appear in these sets, as $(n,3)$ is in the range of the nice dimensions of Mather, any finitely determined map germ $f \in \mathcal{O}(n,3)$ is of discrete type, hence the stratification has a finite number of strata.

We can deal with the entire class of cases $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^3, 0)$, for all $n \geq 3$, "at one stroke", is a consequence of the fact that in each case, the stable singularities which appear in the target are precisely the same type, namely double points curve, edge cuspidal, triple point, swallowtail, cuspidal edge crossings transversally with a plane and the regular part.

In general, for any pair of dimensions (n,p) the description of the stable types can be done in terms of subschemes of multiple points of a germ f, as we can see in [21] for the case $n = p$ or in [22] for the case $n < p$.

Here we use the Thom-Boardman stratification to describe the stable types which appear in the singular set $\Sigma(f)$, then we show the stable types in the discriminant $\Delta(f)$ of f and finally the stable types of $X(f) = \overline{(f^{-1}(\Delta(f)) - \Sigma(f))}$. For any Boardman symbol $i = (i_1, \dots, i_r)$, we denote by $\Sigma^i(f)$ the set of points in $\Sigma(f)$ of type i.

Stratification of source.

In the source there are two sets which are stratified, one of them is the singular set $\Sigma(f)$ whose stratification is done by the smooth parts of the following sets:

1. The 2-dimensional set $\Sigma^{n-2}(f) = \Sigma(f)$;
2. The 1-dimensional set $\Sigma^{n-2,1}(f)$;
3. The 1-dimensional set of double points, denoted $D_1^2(f|\Sigma(f))$.

The other set to be stratified in the source is $X(f) = \overline{f^{-1}(f(\Sigma(f))) - \Sigma(f)} \subset \mathbb{C}^n$, which has its interior in the regular set of f.

This stratification is obtained by the inverse image of f of the stable types in the target and also the inverse image of the multiple points. It is formed by the smooth parts of the following sets.

1. The (n-1)-dimensional set $X(f) = \overline{(f^{-1}(f(\Sigma(f)) - \Sigma(f))}$;
2. the (n-2)-dimensional set $X_1(f) = \overline{(f^{-1}(f(\Sigma^{n-2,1}(f))) - \Sigma(f) \cap \Sigma^{n-2,1}(f))}$;
3. the (n-2)-dimensional set $X_2(f) = \overline{(f^{-1}(f(D_1^2(f|\Sigma(f)))) - \Sigma(f) \cap D_1^2(f|\Sigma(f)))}$;
4. the $(n-3)$-dimensional set $f^{-1}(0)$.

We remark that the set $f^{-1}(0)$ is an ICIS, while the other, possibly are not.

In the target the stratification is done in the discriminant of f, $\Delta(f) = f(\Sigma(f))$. It is formed by the smooth parts of the following sets.

1. The discriminant $\Delta(f) = f(\Sigma(f))$, which is 2-dimensional;
2. The 1-dimensional set $f(\Sigma^{(n-2,1)}(f))$;
3. The image of the double points of f, which is 1-dimensional and denoted by $f(D_1^2(f|\Sigma(f)))$.

To a k-dimensional variety are associated $k+1$ polar invariants. Since $\Sigma(f)$ and $\Delta(f)$ are of dimension 2 and the dimension of $D_1^2(f|\Sigma(f))$, $\Sigma^{n-2,1}(f)$, $f(\Sigma^{n-2,1}(f))$ and $f(D_1^2(f|\Sigma(f)))$ is 1, there are 14 polar invariants defined on these sets. We also have $3n-2$ polar multiplicities of the sets $X(f)$, $X_1(f)$, $X_2(f)$ and $n-2$ polar multiplicities of the set $f^{-1}(0)$.

Therefore to apply Theorem 2.1 to germs in $\mathcal{O}(n,3)$ we need the constancy of $4n+10$ invariants. In the following sections we show how to obtain the minimal number of invariants which guarantee the Whitney equisingularity, for any finitely determined map germs.

4 The Lê cycles and relative polar multiplicities

We now recall some general facts pertaining to our study: Lê cycles and relative polar multiplicities [see [18] and [6]].

Definition 4.1. *Let $h : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ define an (n-1)-dimensional, reduced analytic space germ $X \subset \mathbb{C}^n$, i.e., $X = h^{-1}(0)$. We consider the Blowup B of $\mathbb{C}^n$ along the Jacobian ideal $J(h)$:*

$$b : B = Bl_{J(h)}\mathbb{C}^n \to \mathbb{C}^n,$$

with ***exceptional divisor*** $E := b^{-1}(\nu(J(h))) = b^{-1}(S(h))$.

We observe that the map b, called the ***struture map*** *of Blowup, is a proper morphism and B equals the closure in $\mathbb{C}^n \times \mathbb{P}^{n-1}$ of the graph of the following map,*

$$J : (\mathbb{C}^n - \nu(J(h))) \to \mathbb{P}^{n-1}$$
$$z \mapsto ((h_{z_1})(z) : \ldots : (h_{z_n})(z)).$$

The ***relative polar variety of h****, of dimension, where $k = 0, \ldots, n-2$, denoted $\Gamma^k(h)$, is defined by:*

$$\begin{cases} \Gamma^0(h) := \emptyset, & para\ k = 0 \\ \Gamma^k(h) := b(H_1 \cap \ldots \cap H_{n-k} \cap Bl_{J(h)}\mathbb{C}^n), & para\ k = 1, \ldots, n-1 \\ \Gamma^n(h) := \mathbb{C}^n, & para\ k = n \end{cases}.$$

where the H_j are generic hyperplane of dimension j in $\mathbb{P}^{n-1}$.

The kth ***relative polar multiplicity of h****, where $k = 0, \ldots, n-2$, denoted $m_k(h)$, is defined by:*

$$m_k(h) = mult_0(\Gamma^k(h)).$$

Definition 4.2. *The* ***Lê cycle of h*** *of dimension i, with $i = 0, \ldots, n-1$, is denoted $\Lambda^i(h)$, and is defined as the pushforward by struture map of cycle formed by intersecting the exceptional divisor E in the blowup B with $(n-i-1)$ generics hyperplanes, i.e.,*

$$\Lambda^i(h) := b_*([E \cap H_1 \cap \ldots \cap H_i]), for\ i = 0, \ldots, n-1.$$

Note that each component of $\Lambda^i(h)$ has dimension i.

The associated ***Lê number*** *of dimension i is $\lambda^i(h) \equiv \lambda_{n-i}(h) := mult_0\ \Lambda^i(h)$, where $i = 0, \ldots, n-1$. When the singular locus of h has dimension s, say, then the numbers Lê of larger dimension, namely $\lambda^{s+1}(h), \ldots, \lambda^n(h)$, are zero.*

There are severals computational rules involving the objects above, namely:

1. The notation $[E].H^n$ refers to the cycle $[E \cap H_1 \cap \ldots \cap H_n]$ obtained when E is intersected by n hyperplanes in $\mathbb{P}^{n-1}$ that are **general with respect to B**; [see section 2 of [7] for details]. By convention, $[E].H^0$ is just $[E]$.

2. For a given cycle $[S] := \sum_i c_i W_i \subseteq B$, we define $b_*([S]) := \sum_i c_i.b(W_i)$, if $\dim W_i = \dim(b(W_i))$; and 0 otherwise.

3. $\Lambda^i(h) := b_*([E.H_i])$, for $i = 0, \ldots, n-1$.

4. The multiplicity [at the origin] of a given cycle $[S] := \sum_i c_i W_i$ is $mult_0\ S := \sum_i c_i(mult_0(W_i))$.

5. $\lambda^i := mult_0(\Lambda^i)$, for $i = 0, \dots, n-1$.

In addition, there is an another invariant, which involves Lê numbers and the relative polar multiplicities, namely: **Euler characteristic**, [see [18], p.39 and [7], p.726].

Definition 4.3. *The* ***reduced Euler characteristic of the Milnor fiber*** *of the $h \mid H^k$ at 0, denoted $\chi^{(\mathbf{k})}$, is defined by*

$$\chi^{(k)} := m_{n-k}(h) + \lambda^{n-k}(h) - \lambda^{n-(k-1)}(h) + \lambda^{n-(k-2)}(h) + \ldots + \lambda^{n-s}(h),\ n-s \leq k \leq n$$

with $s = \dim V(J(h))$.

The ***Euler characteristic****, denoted just $\chi^{(*)}$, is the following sequence:*

$$\chi^{(*)} := (\chi^{(n-s)}, \dots, \chi^{(n)}).$$

The next Lemma describes how the Lê numbers and the relative polar multiplicities are related.

Lemma 4.4. ((3.4) and (6.1), [7]) *Let $\lambda^j(h|H^k)$ denote the j-dimensional Lê numbers of the restriction of $X(t)$ to a generic k-dimensional plane section H^k.*

The following relations hold between the Lê numbers $\lambda^j(h)$, the relative polar multiplicities $m_j(h)$, and the Lê numbers $\lambda^j(h|H^k)$, with $k = n-s, \dots, n$, where $s = \dim V(J(h))$.

(i) $\lambda^0(h|H^j) = \lambda^{n-j}(h) + m_{n-j}(h)$, *for* $j = n-s, \dots, n$;

(ii) $\lambda^{k-i}(h|H^k) = \lambda^{n-i}(h)$, *for* $i = n-s-1, \dots, k-1$ *and* $k = n-s, \dots, n$.

Definition 4.5. *The* ***conormal*** *of a given analytic space $Y \subseteq \mathbb{C}^n$, denoted $C(Y)$, is the closure in $\mathbb{C}^n \times \mathbb{P}^{n-1}$ of the pairs (y, H), where y is a smooth point of Y, and H is a tangent hyperplane to Y at such a point.*

Remark 4.6. Since the conormal $C(Y)$ encodes information about limiting tangent hyperplane to Y, it plays a role [as also does its relativized version] in stating equisingularity conditions, such as the Whitney conditions, or Thom's A_f condition; [see [6] and [8]].

5 Numeric Control of equisingularity in $\mathcal{O}(n,3),\ n \geq 3$

In this section we consider the following **setup**: Let $f \in \mathcal{O}(n,3),\ n \geq 3$ be a finitely determined map germ, and $F : (\mathbb{C} \times \mathbb{C}^n, (0,0)) \to (\mathbb{C} \times \mathbb{C}^3, (0,0))$ be a one-parameter unfolding of $f = f_0$, such that F is stratified by stable type, and by the parameter axis T. For each $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ in the family $\{f_t\}$, denote the equation that defines the discriminant Δf_t with reduced structure, by $g_t = 0$, where $g_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$. Let $\mathbb{X} = \{X(t)\}$ denotes the family of reduced hypersurfaces $X(t) := f_t^{-1}(\Delta f_t)$ in $\mathbb{C}^n$.

We show here that the inverse image $X(t)$ of the discriminant of f carries all information necessary to determine if the family of map germs is Whitney equisingular.

Lemma 5.1. *In the setup above:*

(i) $S(f_t)$*, the singular locus of* f_t*, is either empty or 2-dimensional;*

(ii) $S(f_t) \cap f_t^{-1}(0) = \{0\}$*, if* $S(f_t) \neq \emptyset$*;*

(iii) $f_t^{-1}(0)$ *is* $(n-3)$*-dimensional;*

(iv) g_t *is a submersion at each point of* $\Delta\backslash 0$*;*

(v) $S(f_t) \subseteq S(g_t \circ f_t) \subseteq S(f_t) \cup f_t^{-1}(0)$*;*

(vi) If $h_t := (g_t \circ f_t) : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$*, then* $X(t) = \mathcal{V}(h_t)$*, and* $\mathcal{I}(X(t)) = (h_t)$*.*

Proof **(i)** Assume $S(f_t) \neq \emptyset$, we use the fact that the singular locus, is nonempty, of any smooth stable map germ from $\mathbb{C}^n$ to $\mathbb{C}^3$, $n \geq 3$, is necessarily of dimension 2. We recall also the following result that relates finitely determined map germs and stable map germs, due to Mather: " If $f_t \in \mathcal{O}(n,p)$ is finitely determined, then there is an open neighborhood U of 0 in the source such that for each point z in $U\backslash\{0\}$, the germ of f_t at z is stable".

The desired result now follows, because at each point z different from 0 in the singular locus, the singular locus de f_t is locally 2-dimensional.

(ii) We know that $S(f_t)$, is nonempty, is 2-dimensional, and it is clear that both $S(f_t)$ and $f_t^{-1}(0)$ contain the point $\{0\}$ in the source. The desired result now holds because $f_t|_{S(f_t)}$ is a finite map.

(iii) Suppose $f_t := (\phi, \varphi, \psi)$; since $f_t^{-1}(0) = \mathcal{V}(\phi, \varphi, \psi) \subseteq \mathbb{C}^n$.

It follows that $\dim f_t^{-1}(0) = \dim \mathcal{V}(\phi, \varphi, \psi) \geq (n-3)$. We claim that $\dim \mathcal{V}(\phi, \varphi, \psi)$ cannot be n, $n-1$ or $n-2$. To see this, note that $\dim \mathcal{V}(\phi, \varphi, \psi)$ is either n, $n-1$ or $n-2$, then the kernel rank of f_t must be strictly greater than $n-3$ at points of $f_t^{-1}(0)$, implying that $f_t^{-1}(0) \subseteq S(f_t)$, and hence that $\dim S(f_t) > 2$, a contradiction of (i) above.

(iv) For all points $\overline{z} \in \Delta(f_t)\backslash 0$ sufficiently close to 0, we know $\Delta(f_t)$ is smooth at $\overline{z}$; since g_t defines $\Delta(f_t)$ with reduced structure, we conclude that $Dg_t(\overline{z}) \neq 0$.

(v) We first show that $S(f_t) \subseteq S(g_t \circ f_t)$. It clear that this inclusion holds if $S(f_t) = \emptyset$; consider the case of $S(f_t) \neq \emptyset$. Note that at any nonzero point in $S(f_t)$ close to 0, f_t is locally a fold, so it follows immediately that locally that inclusion $S(f_t) \subseteq S(g_t \circ f_t)$ holds. As regard the point $0 \in S(f_t)$, the connecteness of the curve $S(f_t)$, together with the inclusion just established for nonzero points in $S(f_t)$ shows that $0 \in S(g_t \circ f_t)$.

As regard the second inclusion $S(g_t \circ f_t) \subseteq S(f_t) \cup f_t^{-1}(0)$, consider a point $z \in S(g_t \circ f_t)\backslash S(f_t)$; it suffices to show that z belongs to $f_t^{-1}(0)$. At such point z, we know f_t is a submersion, and $D(g_t \circ f_t)(z) = 0$; it follows from the chain rule that we then must have $Dg_t(f_t(z)) = 0$. Since we also know that g_t is a submersion at each point of $\Delta f_t\backslash 0$, we conclude that $f_t(z) = 0$, i.e., $z \in f_t^{-1}(0)$.

(vi) As $X(t) \subset (\mathbb{C}^n, 0)$ is a hypersurface germ, the ideal $\mathcal{I}(X(t))$ is principal; say the principal ideal $(\tilde{h}_t)$ corresponds to $X(t)$, $\tilde{h}_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ non-constant; in particular, $(\tilde{h}_t) := \mathcal{I}(X(t))$, We also know that by definition, $\mathcal{V}(h(t)) = X(t)$, so the ideal containment $(h_t) \subseteq (\tilde{h}_t)$ is clear. We claim that strict containment is untenable; this give us the desired result. Now if ideal (h_t) is strictly contained in $(\tilde{h}_t)$, then there must exist a component X' of X such that the defining equation $\tilde{h}'_t$ of X' is related to $h'_t := (g'_t \circ f_t)$ by $h'_t = (\tilde{h}'_t)^m$ for some integer $m \geq 3$. In order to lighten notation, we may assume, without loss of generality, that X is irreducible, and hence we must have $(g_t \circ f_t) = (\tilde{h}_t^m)$. We now consider the singular set $S(g_t \circ f_t)$ of $h_t \equiv (g_t \circ f_t)$, namely $S(g_t \circ f_t) := \mathcal{V}(g_t \circ f_t, \frac{\partial(g_t \circ f_t)}{\partial z_1}, \ldots, \frac{\partial(g_t \circ f_t)}{\partial z_n})$ and derive a contradiction. Since we have $(g_t \circ f_t) = (\tilde{h}_t^m)$, it follows that $\tilde{h}_t$ divides each of the functions defining $S(g_t \circ f_t)$, implying that these functions belong to the ideal $(\tilde{h}_t)$ and thus must vanish on $X(t)$. But then $S(g_t \circ f_t)$ is *all* of $X(t)$, which contradicts the fact that $S(g_t \circ f_t)$ is at most $(n-3)$–dimensional, whereas $X(t)$ is $(n-1)$–dimensional, $n \geq 3$. □

Remark 5.2. The proof of part (v) above [and using] (vi)] shows that the inclusion $S(f_t) \subseteq S(g_t \circ f_t)$ is in fact an equality if the defining equation g_t of $\Delta(f_t)$ satisfies $Dg_t(0) = 0$; in this case $\dim S(h_t) = n-3$.

Remark 5.3. From [part (v) of Lemma 5.1], and Remark 5.2, we can see that precisely one of the two inclusions in the nesting $S(f_t) \subseteq S(g_t \circ f_t) \subseteq S(f_t) \cup f_t^{-1}(0)$ is an equality, for a given f. The inclusion on the left side is an equality iff f is a fold, and in this case the analysis is particularly simple, as singular locus de X, namely $S(h)$, is necessarily 2-dimensional irrespective of how large the integer n [of source space $\mathbb{C}^n$] is. In all others situations, the inclusion on the right side is an equality, and in the sequel, we shall assume that it is this equality $S(g_t \circ f_t) = S(f_t) \cup f_t^{-1}(0)$ that holds.

We are going to obtain the Whitney equisingularity of a family of map germs in $\mathcal{O}(n, 3)$, $n \geq 3$. For this purpose: let $F : (\mathbb{C} \times \mathbb{C}^n, (0, 0)) \to (\mathbb{C} \times \mathbb{C}^3, (0, 0))$ with $n \geq 3$ be a *good* 1-parameter unfolding of a finitely determined map germ $f \in \mathcal{O}(n, 3)$.

Theorem 5.4. ([9], p. 32) *The pair $(\Delta(F), T)$ is Whitney equisingular iff the sequence $(m_1(g_t), m_2(g_t), \chi^*_{g_t})$ is independent of t.*

Theorem 5.5. *The pair $(X(F), T)$ is Whitney equisingular iff the sequence $(m_1(h_t), \ldots, m_{n-3}(h_t), \chi^*_{h_t})$ is independent of t.*

Proof First we remark that if the closure of a stratum S of $V(X_0(F)) = X(F)$ is the image of a component of the exceptional divisors of the Blow up $B = Bl_{J(X_0(F))}\mathbb{C}^{n+1}$, from the fact that the condition A_F of Thom is generic and also from dimensional reasons, this component is the conormal space of $\bar{S}$. Then, applying the theorem 6.3 of [9], the Blow up of this component by the pullback of m_T has a exceptional divisor which is equidimensional over T. From the theorem V.1.2 of [25], the pair (S, T) satisfies the condition W_F, hence the Whitney equisingularity.

Therefore it is enough to show that each stable type is the image of a component of the exceptional divisor of the Blow up $B = Bl_{J(X_0(F))}\mathbb{C}^{n+1}$.

First we show this for the mono germs. Here it is sufficient to show that if $f_{\mathcal{Q}}$ represents a minimal stable type $\mathcal{Q}$ which appears with positive dimension in F, then the relative polar curve of $X(f_{\mathcal{Q}})$ at the origin is not empty, but this curve is empty if and only if the intersection $H_1 \cap \ldots \cap H_{n-1} \cap Bl_{J(h)}\mathbb{C}^n$ is empty if and only if the fiber over the origin is not a component.

Therefore we conclude that the origin is the image of a component of the exceptional divisor. Since $X(f_{\mathcal{Q}})$ is a cartesian product along the strata of $X(F)$ which are different from T, the stratum which represents the stable type $\mathcal{Q}$ is also the image of a component of the exceptional divisor.

We remember that if the multiplicity of the relative polar curve is not zero, then the polar curve is not empty. Massey showed in ([19], p. 365) that this multiplicity is the number of spheres in the homotopy type of the link of the singularity, independently of the dimension of the singular locus. Here we only need to show now that this number is greater that zero.

For this we apply the theorem 3.1 of Damon in [2], pp. 14, which tell us that for t and ϵ small enough, $(X_0(f_{\mathcal{Q}}))^{-1}(0) \cap B_\epsilon$ or $(X(f_{\mathcal{Q}}) \cap B_\epsilon)$ is independent of the stabilization $f_{\mathcal{Q}}$ and is homotopically equivalent to a bouquet of n real spheres, since $n > 3$, this is different from zero.

Therefore the multiplicity of the relative polar curve is not zero, and we finish the proof for mono germs.

Now we consider the stable types of the multi germs. Here we have that if two smooth sheets of $X(F)$ intersect, then this has codimension 0 in $\Sigma(X_0(F))$, so this stratum must be the image of a component of the exceptional divisor. Otherwise, we can assume that the set $X(f_t)$ is locally the union of two hypersurfaces X_1 and X_2 embedded in $\mathbb{C}^k \times \mathbb{C}^{n-k}$, with equations $g_1(z_1, \ldots, z_k) = 0$ and $g_2(z_{k+1}, \ldots, z_n) = 0$. There are two cases: one with $k = n-1$ and $g_2 = z_n$ and the other is with both g_1, g_2 defining singular hypersurfaces.

Suppose first that $h_2 = z_n$. Then we can assume that ϕ_1 parametrizes a branch of the polar curve of h_1 and $h_{1_{z_1}} \circ \phi_1$ is zero for $1 \leq i \leq n-2$. As h_1 is in the integral closure of $J(h_1)$ we obtain that $(h_1/h_{1_{z_{n-1}}}) \circ \phi_1$ is an analytic germ ψ^n. Call $\varphi = (\phi_1, \psi_n)$ then if $f = z_n h_1 f_{z_i} \circ \varphi = 0$ for $1 \leq i \leq n-2$ and $(f_{z_n} - f_{z_{n-1}}) \circ \varphi = (h_1 - z_n h_{1_{z_{n-1}}}) \circ \varphi = 0$. Hence φ parametrizes the branch of the polar curve of f.

Now suppose that $f = h_1 h_2$ and assume that ϕ_1 and ϕ_2 are parametrizations of the branches of the polar curves h_1 and h_2, respectively. Then $\varphi = (\phi_1, \phi_2)$ parametrizes a polar surface of f and we obtain $f_{z_1} \circ \varphi = 0$ for $1 \leq i \leq k$ and $k+1 \leq i \leq n$. Therefore, for an appropriate choice of A and B, $(Af_{z_k} + Bf_{z_n}) \circ \varphi$ defines a curve of singularities which branch is parametrized by ψ, then $\varphi \circ \psi$ parametrizes a branch of the polar curve of f and we are done. □

Remark 5.6. Note that, in this setup initial, from the results 5.3 and 5.5 we have that the independence of the sequence $(m_1(h_t), \ldots, m_{n-3}(h_t), \chi^*_{h_t})$ also guarantee the Whitney equisingularity of other strata of f_t: $S(f_t)$ and $f_t^{-1}(0)$.

Hence, we obtain

Theorem 5.7. *Let $F : (\mathbb{C} \times \mathbb{C}^n, (0,0)) \to (\mathbb{C} \times \mathbb{C}^3, (0,0))$ with $n \geq 3$ be a good 1-parameter unfolding of a finitely determined map germ $f \in \mathcal{O}(n,3)$.*

The family $\{f_t\}$ is Whitney equisingular if, and only if, the sequences:

$$(m_1(g_t), m_2(g_t), \chi^*_{g_t}) \text{ and } (m_1(h_t), \ldots, m_{n-3}(h_t), \chi^*_{h_t})$$

are independent of t.

Proof Recall that an unfolding F of f is Whitney equisingular along the parameter space T if there exists a regular stratification of the source and the target, with T a stratum of the source and the target and these stratifications are Whitney equisingular along T, i.e. satisfy the Whitney conditions a and b and Thom's A_F condition. In this case, we consider four essential strata: the closure of the part smooth of $S(f_t)$, $f_t^{-1}(0)$, $X(t)$ (source) and $\Delta(f_t)$ (target). So that this claim follows immediately from the Theorems 5.4 and 5.5. □

Remark 5.8. *From the definition of the " Euler characteristic" 4.3 and [7], p.726 we can get the following equivalences:*

(i) $(m_1(h_t), \ldots, m_{n-3}(h_t), \chi^*_{h_t})$ *is independent of t iff* $(m_1(h_t), \ldots, m_{n-3}(h_t), \lambda^0(h_t), \ldots, \lambda^{n-3}(h_t))$ *is independent of t;*

(ii) $(m_1(g_t), m_2(g_t), \chi^*_{g_t})$ *is independent of t iff* $(m_1(g_t), m_2(g_t), \lambda^0(h_t), \lambda^1(h_t), \lambda^2(h_t))$ *is independent of t.*

Therefore, the theorem 5.7 can be rewritten.

Corollary 5.9. *Let $F : (\mathbb{C} \times \mathbb{C}^n, (0,0)) \to (\mathbb{C} \times \mathbb{C}^3, (0,0))$ with $n \geq 3$ be a good 1-parameter unfolding of a finitely determined map germ $f \in \mathcal{O}(n,3)$.*

The family $\{f_t\}$ is Whitney equisingular if, and only if, the sequences:

$$(m_1(h_t), \ldots, m_{n-3}(h_t), \lambda^0(h_t), \ldots, \lambda^{n-3}(h_t)) \text{ and } (m_1(g_t), m_2(g_t), \lambda^0(g_t), \lambda^1(g_t))$$

are independent of t.

Now we wish to minimize the number of the invariants listed above and still to control the Whitney equisingularity of the family $\{f_t\}$.

Lemma 5.10. *In the setup initial, if $\lambda^0(g_t), \lambda^1(g_t)$ are independent of t, then the map*

$$t \mapsto m_k(h_t),\ k = 0, \dots, n-3$$

is upper-semicontinuous.

Remark 5.11. To show that the above map $t \mapsto m_k(h_t)$, $k = 0, \dots, n-3$ is upper-semicontinuous we then must demonstrate that the fiber $\Gamma_z^{k+1}(H) \cap \{0\} \times \mathbb{C}^n$ of $\Gamma_z^{k+1}(H)$ over 0 specializes to $\Gamma^k(h_0)$, because if so, we can write $m_k(h_t) = e(\mathbf{m}_z, \mathcal{O}_{\Gamma_z^{k+1}(H)\cap\{t\}\times\mathbb{C}^n})$, [here $\mathbf{m}_z$ denotes the maximal ideal $(z_1, ..., z_n)$], and the desired result follows from the upper- semicontinuity of the classical ideal multiplicity (see [15]).

The problem is that there may well exist a component

$$C \subset S(h_0)$$

such that $\Gamma_z^{k+1}(H) \cap \{0\} \times \mathbb{C}^n = \Gamma^{k+1}(h_0) \cup C$, and the inclusion $\mathbb{C}^n \subset \Gamma^{k+1}(h_t) \subset \Gamma_z^{k+1}(H) \cap \{t\} \times \mathbb{C}^n$ is strict. More precisely, if there is no component of $\Gamma_z^{k+1}(H) \cap \{0\} \times \mathbb{C}^n$ in $S(h_0)$, then we have the desired specialization, i.e. $\Gamma^{k+1}(h_0) = \Gamma_z^{k+1}(H) \cap \{0\} \times \mathbb{C}^n$, and hence the upper-semicontinuity of the relative polar multiplicities $m_k(h_t)$, for $k = 0, ..., n-3$.

Proof Since $S(h_0) = S(f_0) \cup f_0^{-1}(0)$, it suffices by 5.11 above, to show that $k = 0, ..., n-3$,: **(i)** no component of $\Gamma_z^{k+1}(H) \cap \{0\} \times \mathbb{C}^n$ belong to $f_0^{-1}(0)$; and **(ii)** no component of $\Gamma_z^{k+1}(H) \cap \{0\} \times \mathbb{C}^n$ belong to $S(h_0)$.

(i) Fix k and . Note first that at any point $z \in f^{-1}(0) - S(f_0)$, the map, f_0 is a submersion. Also, the hypothesis of $\lambda^0(g_t), \lambda^1(g_t)$ and $\lambda^2(g_t)$ are independent of t, by the generalization theorem of Lê- Saito ([9]), imply that Thom's conditions A_Q are satisfied by Q relative to the pair $(\mathbb{C} \times \mathbb{C}^3, T)$, so it follows that the map $H := Q \circ F$ satisfies the A_H condition for the pair $(\mathbb{C} \times \mathbb{C}^n, F^{-1}(T))$. This means that every limiting tangent plane to fibers of H contains the tangent space to $F^{-1}(T)$; in terms of the blowup $b : B = Bl_{J(H)}\mathbb{C} \times \mathbb{C}^n \to \mathbb{C} \times \mathbb{C}^n$, it follows, [since $\Gamma^{k+1}(H) := b(B \cap H_{n-k})$, (see p.47 [10]); that: there is a k-dimensional component of $\Gamma^{k+1}(H) \cap \{0\} \times \mathbb{C}^n$ in $f_0^{-1}(0)$ iff $H_{n-k} \cap E \cap b^{-1}(f_0^{-1}(0))$ has dimension k. But since at a generic point, the dimension of $b^{-1}(f_0^{-1}(0))$ is $n-1$, we get that $H_{n-k} \cap E \cap b^{-1}(f_0^{-1}(0))$ has dimension $(k-1)$, so we are done.

(ii) Note that at a generic point z of $S(f_0) - \mathcal{Q}(f_0) - \{0\}$ where $\mathcal{Q}(f_0)$ is a stable type 2-dimensional, f_0 is a fold, or f_0 is a cusp. Using the hypothesis of constancy of $\mu(g_t)$, we see that, for t small and by suitable choice of coordinates, the Jacobian $J(H)$ does not depend of t. Therefore, for each k, the relative polar variety $\Lambda^{k+1}(H)$ is independent of t and hence specializes as desired, so that $\Lambda^k(h_0) = \Lambda k + 1(H) \cap (0 \times \mathbb{C}^n)$.
□

Lemma 5.12. *In the setup initial. If $\lambda^0(h_t|H^3)$ is independent of t, then $\lambda^0(g_t), \lambda^1(g_t)$ are independent of t*

Proof Suppose that the hypothesis of constancy of the $\lambda^0(h_t|H^3)$ holds, but that the value of $\lambda^0(g_t), \lambda^1(g_t)$ change at 0, we derive a contradiction.

We may suppose that the singular sets of the maps g_t are of the form $\bigcup_t(S(g_t)) = T \cup C$, where C is a curve in $\mathbb{C} \times \mathbb{C}^3$, such that C is distinct from T, and that $T \cap C = (0, 0)$. Consider $F^{-1}(C) \cap (H^3 \times T)$; at each point $(t, c) \in F^{-1}(C)$, we know that $Dg_t(F(t, c)) = 0$. This imply that the maps $h_t|_{H^3}$ are singular along $F^{-1}(C)$, which in turn means that $\lambda^0(h_t|H^3)$ cannot be independent of t, a contradiction of the hypothesis.
□

Remark 5.13. *Consequently, the lemmas 5.10 and 5.12 above imply that if $\lambda^0(h_t|H^3)$ is independent of t, then the map*

$$t \mapsto m_k(h_t),\ k = 0, \dots, n-3$$

is upper-semicontinuous.

Proposition 5.14. *In the setup above, the $n-2$ zero-dimensional Lê- numbers*

$$\lambda^0(h_t|H^3),\ \lambda^0(h_t|H^4),\ \dots,\ \lambda^0(h_t|H^n)$$

are independent of t, iff all the Lê- numbers of all generic plane sections of $X(t)$, as well as the relative polar multiplicities $m_k(h_t)$ for $k = 1, \dots, n-3$, are independent of t.

Proof ($\Rightarrow$:) Recall the notation $\lambda^i(h_t) \equiv \lambda^i(h_t|H^n)$, and the fact that only the Lê numbers of dimension less than or equal to the dimension of the singular locus, can be non-zero. The hypothesis regarding the constancy of the Lê number $\lambda^0(h_t|H^3)$ implies, [see remark 5.13] that the relative polar multiplicities $m_k(h_t)$, for $k = 1, \dots, n-3$, are upper-semicontinuos.

Using this upper-semicontinuity property, and recalling that the Lê numbers are lexicographically upper-semicontinous (see cor. 4.5 [18]), it follows from formula of (i) of 4.4, that if the invariants $\lambda^0(h_t|H^j)$ are independent of t, then so are all the relevant Lê numbers of h_t, namely $\lambda^0(h_t), \dots, \lambda^{n-3}(h_t)$, as well the relative polar multiplicities $m_k(h_t)$, for $k = 1, \dots, n-3$. The remaining [possibly non-zero] Lê numbers are $\lambda^i(h_t|H^s)$, where $i = n-s-1, \dots, n-3$; $s = 3, \dots, n-1$ but by formulas (ii) of 4.4 the Lê number $\lambda^i(h_t|H^s)$ can be seen to be equal to the Lê number $\lambda^{n-s-i}(h_t|H^n) \equiv \lambda^{n-s-i}(h_t)$, and the latter is already known to be independent of t.

($\Leftarrow$:) This result is trivial. □

6 The main results

From the main Theorem of Gaffney, we need the constancy of $(4n+10)$ invariants to guarantee the Whitney equisingularity, for example, in the case $n = 4$ we need 26 invariants.

To minimize this number we apply all results shown here to obtain the following:

Theorem 6.1. *Let $F(t, x) = (t, f_t(x))$ be an good unfolding of a finitely determined map germ $f \in \mathcal{O}(n, 3)$, com $n \geq 4$.*

Then the family $\{f_t\}$ is Whitney equisingular along the parameter space T if and only if the following numbers are constant at the origin for all f_t:

$m_1(g_t)$, $m_2(g_t)$ and the $n-2$ zero-dimensional Lê- numbers $\lambda^0(h_t|L^3)$, $\lambda^0(h_t|L^4)$, ..., $\lambda^0(h_t|L^n)$.

Proof It follows immediately from results 5.9 and 5.14. □

Finally, we reduce the number of invariants from $\mathbf{4n+10}$ to $\mathbf{n}$.

Remark 6.2. *Remark that in the case $n = 3$ Jorge Pérez [20] reduces from 3 to 6, and we reduce from 6 to* 3.

References

[1] J. P. Brasselet and M. H. Schwartz, *Sur les classes de Chern d'un ensemble analytique complexe*, Asterisque **82-83**, 93–146, 1981.

[2] J. Damon, *Nonlinear sections of nonisolated complete intersections*, Kluwer Academic Publishers, 2001.

[3] T. Gaffney, *Polar Multiplicities and Equisingularity of Map Germs*, Topology, **32**, No.1, pp.185-223, 1993.

[4] T. Gaffney, *Properties of Finitely Determined Germs*, Ph.D Thesis, Brandeis University, 1975.

[5] T. Gaffney, *Multiplicities and Equisingularities of I.C.I.S Germs*, Invent. Math. **123**, pp. 200-220, 1996.

[6] T. Gaffney, *Plane Sections, W_f and A_f, in Real and Complex Singularities*, edited by J. W. Bruce and F. Tari; Research Notes in Mathematics 412, Chapman & Hall, 2000.

[7] T. Gaffney and R. Gassler, *Segre Numbers and Hypersurfaces Singularities*, Algebraic Geometry **08**, pp. 695-736, 1999.

[8] T. Gaffney and S. L. Kleiman, *W_f and Integral Dependence, in real and Complex Singularities*, edited by J. W. Bruce and F. Tari; Research Notes in Mathematics 412, Chapman & Hall, 2000.

[9] T. Gaffney and D. Massey, *Trends in Equisingularity Theory in Singularity Theory*, edited by Bruce, J.W. and Mond. D., Cambridge University Press, 1999.

[10] T. Gaffney and R. Vohra, *A numerical characterization of equisingularity for map germs from n-space, $(n \geq 3)$, to the plane.* J. Dyn. Syst. Geom. Theor. **2**, pp. 43–55, 2004.

[11] G. Gonzalez-Sprinberg, *L'obstruction locale d'Euler et le théorème de Mac-Pherson*, Astérisque, **82-83**, pp. 7–33, 1979.

[12] V. V. Goryunov, *Semi-simplicial resolutions and homology of images and discriminants of mappings*, Proc. London Math. Soc. 3, **70**, pp.363-385, 1995.

[13] D. T. Lê, *Calculation of Milnor number of isolated singularity of complete intersection*, Funktsional'ny: Analizi Ego Prilozheniya, **8**, no.2, pp.45-49, 1974.

[14] D. T. Lê and B. Teissier, *Variétés polaires locales et classes de Chern de variétés singulères.* Annals of Math., **114**, pp. 457–491, 1981.

[15] J. Lipman, *Equimultiplicity, Reduction and Blowing Up*, Lecture Notes Pure Appl. Math. **68**, Marcel Dekker, New York, 1982.

[16] E. J. N. Looijenga, *Isolated singular points on complete intersections*, London Mathematical Soc., Lecture Note series, **77**, 1984.

[17] R. D. MacPherson, *Chern class for singular algebraic varieties*, Ann. of Math., **100**, pp. 423–432, 1974.

[18] D. Massey, *Le Cycles and Hypersurface Singularities*, Springer Lecture Notes in Mathematics, **1615**, 1995.

[19] D. Massey, *Numerical invariants of perverse sheaves*, Duke Math. Journal, **73**, ♯2 307-369, 1994.

[20] V.H. Jorge Pérez, Polar multiplicities and equisingularity of map germs from $\mathbb{C}^3$ to $\mathbb{C}^3$, Houston Journal of Mathematics. **29** n: 04, pp. 901–923, 2003.

[21] V. H. Jorge Pérez D. Levcovitz M. J. Saia, *Invariants, equisingularity and Euler obstruction of map germs from $\mathbb{C}^n$ to $\mathbb{C}^n$*, Notas do ICMC, n:**158** To appear in Journal fur die Reine und angewandte Mathematik, Crelle's Journal, 2002.

[22] V. H. J. Jorge Pérez and M. J. Saia. *Euler Obstruction, polar multiplicities and equisingularity of map germs in $O(n,p)$, $n<p$.* Notas do ICMC, no: **157**, 2002.

[23] V. H. J. Jorge Pérez, E. C. Rizziolli and M. J. Saia. *Whitney equisingularity, Euler obstruction and invariants of map germs from $\mathbb{C}^n$ to $\mathbb{C}^3$, $n>3$.* This paper is admited to the publication.

[24] B. Teissier, *Cycles Evanescents, Sections Planes et Conditions de Whitney*, II Proc. Symposia Pure Math., **40**, part 2, Amer. Math. Soc., pp.65-104, 1983.

[25] B. Teissier, *Variétés Polaires 2: Multiplicités Polaires, Sections Planes, et Conditions de Whitney*, Actes de la conference de géometrie algébrique ála Rábida, Sringer Lecture Notes, **961**, pp.314-491, 1981.

[26] C. T. C. Wall, *Finite Determinancy of Smooth Map Germs*, Bull. London Math. Soc. **13**, pp. 481-539, 1981.

[27] C. T. C. Wall, *Lectures on C^∞-Stability and Classifications*, Springer Lecture Notes in Mathematics, No.**192**, pp. 178-206, 1971.

First and Third named authors: Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos, SP, Brazil.
E-mails: vhjperes@icmc.usp.br, mjsaia@icmc.sc.usp.br

Second named author: Departamento de Matemática, Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista "Júlio Mesquita Filho" - Campus de Rio Claro, Caixa Postal 178, 13506-700 Rio Claro, SP, Brazil.
E-mail: eliris@igce.unesp.br

NOTAS DO ICMC

SÉRIE MATEMÁTICA

260/2006 HERNADES, M.E.; RODRIGUES, H.; RUAS, M.A.S – A_e – codimension of germs of analytic monomial curves.

259/2006 CARBINATTO, M.C.; RYBAKOWSKI, R.P. – Continuation of the connection matrix for singularly perturbed hyperbolic equations.

258/2006 RIZZIOLLI, E.C.; SAIA, M.J .– Polar multiplicities and Euler obstruction of the types in weighted homogeneus map germs from C^n to C^3, $n \geq 3$.

257/2006 ARRIETA, J.M.; CARVALHO, A.N.; LOZADA-CRUZ, G. – Dynamics in Dumbbell domains I. Continuity of the set of equilibria.

256/2006 SAIA, M.J.; SOARES JR., C.H. – Modified C□ - trivialization of real germs of functions and maps.

255/2006 GIMENES, L.P.; FEDERSON, M.; TÁBOAS, P. - Impulsive stability of systems of second order retarded differential equations.

254/2006 NABARRO, A.C.; RUAS, M.A.S. – Vector fields in R^2.with maximal index.

253/2005 GUTIERREZ, C.; PIRES, B.; RABANAL, R.. – Asymptotic stability at infinity for differentiable vector fields of the plane.

252/2006 CARBINATTO, M.C.; RYBAKOWSKI, R.P. – The suspension isomorphism for homology index.

251/2005 CARBONE, V.L; CARVALHO, A.N.; SILVA, K.S. – Continuity of attractors for parabolic problems with localized large diffusion.