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**Mixed Multiplicities for Arbitrary Ideals and  
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# Mixed Multiplicities for Arbitrary Ideals and Generalized Buchsbaum-Rim Multiplicities

R. Callejas-Bedregal\* and V. H. Jorge Pérez†

## Abstract

We introduce first the notion of mixed multiplicities for arbitrary ideals in a local  $d$ -dimensional noetherian ring  $(A, \mathfrak{m})$  which, in some sense, generalizes the concept of mixed multiplicities for  $\mathfrak{m}$ -primary ideals. We also generalize Teissier's Product Formula for a set of arbitrary ideals. We also extend the notion of the Buchsbaum-Rim multiplicity (in short way we write BR-multiplicity) of a submodule of a free module to the case where the submodule no longer has finite colength. For a submodule  $M$  of  $A^p$  we introduce a sequence of multiplicities  $e_{BR}^k(M)$ ,  $k = 0, \dots, d + p - 1$  which in the ideal case coincides with the multiplicity sequence  $c_0(I, A), \dots, c_d(I, A)$  defined for an arbitrary ideal  $I$  of  $A$  by Achilles and Manaresi in [AM]. In case that  $M$  has finite colength in  $A^p$  and it is totally decomposable we prove that our BR-multiplicity sequence essentially falls into the standard BR-multiplicity of  $M$ .

## Resumo

Introducimos primeiro a noção de multiplicidades mixtas para ideais arbitrários em um anel noetheriano local  $(A, \mathfrak{m})$   $d$ -dimensional, no qual em algum sentido, generaliza o conceito de multiplicidade mixtas para ideais  $\mathfrak{m}$ -primários. Também generalizamos a fórmula do produto de Teissier para um conjunto arbitrários de ideais. Também extendemos a noção a multiplicidade de Buchsbaum-Rim de submódulos livres para o caso onde o submódulo não tem colongitude finita. Para um Sumódulo  $M$  de  $A^p$ , introducimos uma sequência de multiplicidades

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$e_{BR}^k(M)$ ,  $k = 0, \dots, d + p - 1$  o qual no caso de ideais é igual com a sequência de multiplicidades  $c_0(I, A), \dots, c_d(I, A)$  definida para um ideal arbitrário  $I$  de  $A$  por Achilles e Manaresi em [AM].

## 1 Introduction

The theory of mixed multiplicities of primary ideals was introduced by Risler and Teissier. The roots of this theory go back to a 1928 paper of Van der Waerden [W] where he developed the theory of Hilbert functions of bigraded algebras over a field. This was generalized by Bhattacharya in 1955 to bigraded algebras over artinian rings [B]. He applied this to obtain Hilbert-Samuel polynomials of two ideals one of which is  $\mathfrak{m}$ -primary. The work of Bhattacharya was further generalized by Risler and Teissier to several ideals in a remarkable paper [T]. They applied them to the study of Milnor numbers of isolated singularities of complex analytic hypersurfaces. Risler and Teissier also showed that the mixed multiplicities of two  $\mathfrak{m}$ -primary ideals are Samuel multiplicities of ideals generated by *generic elements*, those which are generic linear combinations of a generating system of the two ideals. This notion of generic elements due to Risler and Teissier was further generalized by Rees in 1984 who introduced the concept of *join reduction* of a set of  $\mathfrak{m}$ -primary ideals [R]. He showed that each mixed multiplicity of a set of  $\mathfrak{m}$ -primary ideals is a multiplicity of certain join reduction of them.

Mixed multiplicities have found numerous applications. In 1988 Katz and Verma found a formula for the multiplicity of Rees algebras and extended Rees algebras in terms of mixed multiplicities (cf. [KV], [H], [V].)

Risler and Teissier also proved the following Product Formula relating the Samuel multiplicity of a product of  $\mathfrak{m}$ -primary ideals  $I_1, \dots, I_q$ , in a  $d$ -dimensional local ring  $(A, \mathfrak{m})$ , with their mixed multiplicities

$$e(I_1 \cdots I_q, A) = \sum_{l_1 + \dots + l_q = d} \frac{d!}{l_1! \cdots l_q!} e_{l_1, \dots, l_q}(I_1 \mid \dots \mid I_q)$$

This formula plays a key role in Teissier work on the characterization of the Whitney conditions for families of isolated hypersurface singularities [T]. If we wish to generalize Teissier work for families of arbitrary hypersurface singularities we have to generalize the notion of Samuel multiplicities and the notion of mixed multiplicities for arbitrary ideals. The first target have been attained by Gaffney and Gassler in the analytic case by means of the



Segre numbers [GaG] and by Achilles and Manaresi for arbitrary ideals in a local noetherian ring [AM]. Both approaches agree in the analytic case [AR]. In this work we describe Achilles and Manaresi approach where they associate to every ideal  $I$  of a  $d$ -dimensional local noetherian ring  $(A, \mathfrak{m})$  a multiplicity sequence  $c_0(I, A), \dots, c_d(I, A)$  which when  $I$  is  $\mathfrak{m}$ -primary then  $c_0(I, A)$  coincides with the Hilbert-Samuel multiplicity  $e(I, A)$  of  $I$  and it is the only element of the sequence which is different from zero. In regard with the second target, inspired by the work of Achilles and Manaresi, we introduce a notion of mixed multiplicities for finitely many arbitrary ideals which, in some sense, generalizes the concept of mixed multiplicities for  $\mathfrak{m}$ -primary ideals (see Proposition 5.2, (2b)). This notion was first introduced in [CJ] for two arbitrary ideals. We also generalize Teissier's product formula for a set of arbitrary ideals (see Theorem 5.3). This is one of the important results we prove in this work.

On the other hand, the Buchsbaum-Rim multiplicity is a generalization of the Samuel multiplicity and is defined for submodules of free modules  $M \subset F$  such that  $F/M$  has finite length. These were first described by Buchsbaum and Rim in [BR]. The BR-multiplicity has been generalized, in the finite colength case, by Kirby [Ki], Kirby and Rees [KR1], Katz [K], Kleiman and Thorup [KT] and Simis, Ulrich and Vasconcelos [SUV]. For an extensive history of BR-multiplicity we refer to [KT].

In the last fifteen years the Buchsbaum-Rim multiplicity of a submodule of a free module has played an important role in the theory of equisingularity of families of complete intersections with isolated singularities (ICIS). The BR-multiplicity has been used in the context to control the  $A_f$ ,  $W_f$  and  $W$  conditions (cf. [Ga], [GaK], [GaM]).

The usefulness of the BR-multiplicity is restricted to families of ICIS, because it is only for these singularities that the submodules associated to the equisingularity conditions have finite colength, hence only for these types is the BR multiplicity well defined.

If we wish to generalize Gaffney's work for families of arbitrary complete intersection singularities we have to generalize the notion of BR-multiplicities for submodules  $M$  of a free module  $F$  of arbitrary colength. This have been done by Gaffney himself in [Ga2] by introducing a sequence of multiplicities  $e_i(M)$ ,  $0 \leq i \leq d$ . This sequence satisfies a Rees type theorem: Suppose that  $M \subset N \subset F$  are  $\mathcal{O}_{X,x}$  modules,  $F$  is free, and  $X^d$  is a complex analytic space which as a reduced space is equidimensional, and which is generically reduced. Suppose that  $e_i(M, x) = e_i(N, x)$ ,  $0 \leq i \leq d$ . Then  $N \subset \overline{M}$ . Also,



if  $M$  is of finite colength in  $F$ , then  $e_d(M)$  is the standard BR-multiplicity for modules of finite colength, and the others  $e_i$ 's are zero. Unfortunately, for ideals of non-finite colength, Gaffney's multiplicity sequence does not coincide with the Achilles-Manaresi multiplicity sequence, or with the Segre numbers of the ideal. Since Segre numbers are a good generalization of the Lê numbers introduced by Massey [M], which plays a key role in studying the Whitney condition for families of arbitrary hypersurface singularities, it would be important from the equisingularity sight to have a generalization of the BR-multiplicity for arbitrary submodules of  $F$  which coincides with the Segre numbers in case of ideals of arbitrary colength. In this work we extend the notion of the BR-multiplicity of a submodule of a free module to the case where the submodule no longer has finite colength. For a submodule  $M$  of  $A^p$  we introduce a sequence  $e_{BR}^k(M)$ ,  $k = 0, \dots, d+p-1$  which in the ideal case coincides with the multiplicity sequence  $c_0(I, A), \dots, c_d(I, A)$  defined for an arbitrary ideal  $I$  of  $A$  by Achilles and Manaresi [AM]. We prove also that if  $M = I_1 \oplus \dots \oplus I_p \subset A^p$  has finite colength then  $e_{BR}^0(M) = p!(e_{BR}(M))$  and  $e_{BR}^k(M) = 0$  for  $k = 1, \dots, d-1$ . Thus, for this kind of modules we have that our definition of BR-multiplicity sequence coincides (up to constant multiple) with the standard BR-multiplicity for modules of finite colength.

## 2 Hilbert functions of multigraded algebras

In this section we recall some well-known facts on Hilbert functions and Hilbert polynomials of multigraded algebras, which will play a central role along this work.

In the following, by a  $(q+1)$ -graded algebra we mean an algebra  $R = \bigoplus_{r, u_1, \dots, u_q}^\infty R_{r, u_1, \dots, u_q}$  such that

1.  $R_{r, u_1, \dots, u_q}$  are additive subgroups,
2.  $R_{r, u_1, \dots, u_q} \cdot R_{s, v_1, \dots, v_q} \subseteq R_{r+s, u_1+v_1, \dots, u_q+v_q}$  for all  $r, u_1, \dots, u_q, s, v_1, \dots, v_q$ ,
3.  $R$  is an  $R_{0, \dots, 0}$ -algebra finitely generated by elements of  $R_{0, \dots, 1, \dots, 0}$ , where 1 occurs only as the  $i$ -th component,  $i = 0, \dots, q$ .

Let  $R = \bigoplus_{r, u_1, \dots, u_q}^\infty R_{r, u_1, \dots, u_q}$  be a  $(q+1)$ -graded algebra of dimension  $d+q-1$  and assume that  $R_{0, \dots, 0}$  is an Artinian ring. The *Hilbert function* of  $R$  is defined to be

$$h_R(r, u_1, \dots, u_q) = \ell_{R_0, \dots, 0}(R_{r, u_1, \dots, u_q}).$$

For  $r, u_1, \dots, u_q$  sufficiently large, the function  $h_R(r, u_1, \dots, u_q)$  becomes a polynomial  $P_R(r, u_1, \dots, u_q)$ , of total degree  $d - 2$ , the *Hilbert polynomial* of  $R$ , which can be written in the form

$$P_R(r, u_1, \dots, u_q) = \sum_{\substack{k, l_1, \dots, l_q \geq 0 \\ k + l_1 + \dots + l_q \leq d - 2}} a_{k, l_1, \dots, l_q}(R) \binom{r}{k} \binom{u_1}{l_1} \cdots \binom{u_q}{l_q}$$

with  $a_{k, l_1, \dots, l_q}(R) \in \mathbb{Z}$  and  $a_{k, l_1, \dots, l_q}(R) \geq 0$  if  $k + l_1 + \dots + l_q = d - 2$  (see [W].)

Let  $\mathbf{0}_s$  denotes  $(0, \dots, 0) \in \mathbb{N}^s$  and let

$$h_R^{(1, \mathbf{0}_q)}(r, u_1, \dots, u_q) := \sum_{i=0}^r h_R(i, u_1, \dots, u_q).$$

From this description it is clear that, for  $r, u_1, \dots, u_q$  sufficiently large, also  $h_R^{(1, \mathbf{0}_q)}$  becomes a polynomial with integer coefficients of degree at most  $d - 1$ , which can be written in the form

$$P_R^{(1, \mathbf{0}_q)}(r, u_1, \dots, u_q) = \sum_{\substack{k, l_1, \dots, l_q \geq 0 \\ k + l_1 + \dots + l_q \leq d - 1}} a_{k, l_1, \dots, l_q}^{(1, \mathbf{0}_q)}(R) \binom{r}{k} \binom{u_1}{l_1} \cdots \binom{u_q}{l_q}$$

with  $a_{k+1, l_1, \dots, l_q}^{(1, \mathbf{0}_q)}(R) = a_{k, l_1, \dots, l_q}(R)$  for  $k, l_1, \dots, l_q \geq 0$ ,  $k + l_1 + \dots + l_q \leq d - 2$ .

**Definition 2.1.** For the coefficients of the terms of highest degree in  $P_R^{(1, \mathbf{0}_q)}$  we introduce the symbols

$$c_k^{l_1, \dots, l_q}(R) := a_{k, l_1, \dots, l_q}^{(1, \mathbf{0}_q)}(R),$$

for  $k = 0, \dots, d - 1$  and  $l_1 + \dots + l_q = d - 1 - k$ , which are called the **Mixed Multiplicity sequence** of  $R$ .

Let  $R = \bigoplus_{r,u_1,\dots,u_q}^{\infty} R_{r,u_1,\dots,u_q}$  be a  $(q+1)$ -graded algebra of dimension  $d+q-1$  with  $R_{0,\dots,0}$  an Artinian ring. For each  $s \in \{1, \dots, q\}$  consider the  $(q-s+2)$ -graded algebra

$${}^s R := \bigoplus_{r,u,u_{s+1},\dots,u_q}^{\infty} \left( \bigoplus_{u_1+\dots+u_s=u} R_{r,u_1,\dots,u_q} \right)$$

The formula bellow was incorrectly stated by Herrmann et al in the proof of [H], Theorem (4.3) p. 329. Following their own notation, they wrote:

$$e({}^p S; k, k_{p+1}, \dots, k_r) = \sum_{k_1+\dots+k_p=k} e(S; k_1, \dots, k_r)$$

where  $k + k_{p+1} + \dots + k_r = d-1$ .

In fact, a careful reading of their proof shows that their induction argument and their previous  $p=2$  case of p. 331, l. 11-14, was wrongly applied. Following their own notation again, we observe that a correct version of their formula of p. 330, l. 2, must be written as:

$$e({}^p S; k, k_{p+1}, \dots, k_r) = \sum_{k_1+\dots+k_p=k-p+1} e(S; k_1, \dots, k_r) \quad (*)$$

where  $k + k_{p+1} + \dots + k_r = d+p-2$ .

In our context, the above formula  $(*)$  should be written as:

$$a_{k,l,l_{s+1},\dots,l_q}({}^s R) = \sum_{l_1+\dots+l_s=l-s+1} a_{k,l_1,\dots,l_q}(R)$$

for  $k + l + l_{s+1} + \dots + l_q = d+s-3$ .

Then, as a direct consequence of the above formula we obtain the following:

**Proposition 2.2.**

$$a_{k,l,l_{s+1},\dots,l_q}^{(1,0_{q-s+1})}({}^s R) = \sum_{l_1+\dots+l_s=l-s+1} a_{k,l_1,\dots,l_q}^{(1,0_q)}(R)$$

for  $k + l + l_{s+1} + \dots + l_q = d+s-2$ .



### 3 Achilles-Manaresi's multiplicity sequence

Let  $R = \bigoplus_{r,u}^{\infty} R_{r,u}$  be a  $d$ -dimensional bigraded algebra. Following the notation of the previous section, we consider the Hilbert sum

$$h_R^{(1,1)}(r, u) := \sum_{j=0}^u h_R^{(1,0)}(r, j)$$

which for  $r, u \gg 0$  becomes a polynomial of degree  $d$  that can be written in the form

$$P_R^{(1,1)}(r, u) = \sum_{\substack{k, l \geq 0 \\ k+l \leq d}} a_{k,l}^{(1,1)}(R) \binom{r}{k} \binom{u}{l}$$

with  $a_{k,l+1}^{(1,1)}(R) = a_{k,l}^{(1,0)}(R)$  for all  $k+l \leq d-1$ .

**Definition 3.1.** For the coefficients of the terms of highest degree in  $P_R^{(1,1)}$  we introduce the symbols

$$c_k(R) := a_{k,d-k}^{(1,1)}(R),$$

for  $k = 0, \dots, d$ , and call it the **multiplicity sequence** of  $R$ .

The Hilbert-Samuel multiplicity of a  $\mathfrak{m}$ -primary ideal in a local ring  $(A, \mathfrak{m})$  has been generalized for arbitrary ideals by Achilles and Manaresi [AM]. They introduced a multiplicity sequence  $c_0(I, A), \dots, c_d(I, A)$  for an arbitrary ideal  $I$  in a  $d$ -dimensional local noetherian ring  $(A, \mathfrak{m})$  which when  $I$  is  $\mathfrak{m}$ -primary then  $c_0(I, A)$  coincides with the Hilbert-Samuel multiplicity  $e(I, A)$  of  $I$  and it is the only element of the sequence which is different from zero.

In this section and for the sake of completeness we describe their construction. Let  $(A, \mathfrak{m})$  be a local noetherian ring of dimension  $d$ ,  $I$  be an arbitrary ideal of  $A$  and  $G_I(A)$  be the associated graded ring of  $A$  with respect to  $I$ . It is well known that  $\dim G_I(A) = \dim A =: d$ . Put  $R = G_{\mathfrak{m}}(G_I(A))$ . Then  $R = \bigoplus_{r,u=0}^{\infty} R_{r,u}$  with

$$R_{r,u} = (\mathfrak{m}^r I^u + I^{u+1}) / (\mathfrak{m}^{r+1} I^u + I^{u+1})$$

is a bigraded algebra of (Krull-)dimension  $d$ , and  $R_{00} = A/\mathfrak{m}$  is a field.

**Definition 3.2.** ([AM], Definition 2.2) Following the notation of Definition 3.1 we call the sequence of nonnegative integers

$$c_k(I, A) := c_k(G_{\mathfrak{m}}(G_I(A))), \quad k = 0, \dots, d$$

the **Achilles-Manaresi multiplicity sequence** of the ideal  $I$  of  $A$ .

## 4 Mixed multiplicities of $\mathfrak{m}$ -primary ideals

In this section we recall the definition and some known facts about the mixed multiplicities of a set of  $\mathfrak{m}$ -primary ideals in a local ring as introduced by Teissier in [T].

Let  $(A, \mathfrak{m})$  be a local ring of dimension  $d$  and  $I_1, \dots, I_q \subset A$  be  $\mathfrak{m}$ -primary ideals of  $A$ . Let us consider the Bhattacharya function  $B : \mathbb{N}^q \rightarrow \mathbb{N}$  given by

$$B(u_1, \dots, u_q) = \ell(A/I_1^{u_1} \cdots I_q^{u_q}), \quad \text{for all } (u_1, \dots, u_q) \in \mathbb{N}^q.$$

It was proved by Bhattacharya [B] for  $q = 2$  and by Risler and Teissier [T], p. 302, in the general case that there exist a polynomial  $Q(u_1, \dots, u_q)$  of total degree  $d$  with rational coefficients such that  $B(u_1, \dots, u_q) = Q(u_1, \dots, u_q)$  for all  $u_1, \dots, u_q \gg 0$ . Suppose that the homogeneous term of degree  $d$  of  $Q(u_1, \dots, u_q)$  is written as

$$\sum_{\substack{l_1, \dots, l_q \geq 0 \\ l_1 + \dots + l_q = d}} \frac{a_{l_1, \dots, l_q}}{l_1! \cdots l_q!} u_1^{l_1} \cdots u_q^{l_q}.$$

The numbers  $e_{l_1, \dots, l_q}(I_1 \mid \dots \mid I_q) := a_{l_1, \dots, l_q}$ , where  $(l_1, \dots, l_q) \in \mathbb{N}^q$  and  $l_1 + \dots + l_q = d$ , are known as the **mixed multiplicities of  $I_1, \dots, I_q$** .

Here we recall an important result about mixed multiplicities which was proved by Teissier [T] and that plays a key role in Teissier's work on Whitney equisingularity for families of isolated hypersurfaces singularities:

**Theorem 4.1.** (Teissier, [T]) Let  $I_1, \dots, I_q$  be  $\mathfrak{m}$ -primary ideals of  $A$ . Then,

$$e(I_1 \cdots I_q, A) = \sum_{l_1 + \dots + l_q = d} \frac{d!}{l_1! \cdots l_q!} e_{l_1, \dots, l_q}(I_1 \mid \dots \mid I_q)$$

## 5 Mixed multiplicities of arbitrary ideals

In this section, inspired by the work of Achilles and Manaresi [AM], we extend the notion of mixed multiplicities to a set of arbitrary ideals in a local ring. In some sense, this notion generalizes the concept of mixed multiplicities for  $\mathfrak{m}$ -primary ideals (see Proposition 5.2, (2b)). We also generalize Teissier's Product Formula (see Theorem 4.1) for a set of arbitrary ideals (see Theorem 5.3). This is one of the important results we prove in this work.

Let  $(A, \mathfrak{m})$  be a local ring of dimension  $d$  and  $I_1, \dots, I_q \subset A$  be ideals of  $A$  of positive height. Consider the multi-form ring

$$\begin{aligned} S &:= A[I_1 T_1, \dots, I_q T_q] / (I_1 \cdots I_q) \\ &= \bigoplus_{u_1, \dots, u_q=0}^{\infty} \left( I_1^{u_1} \cdots I_q^{u_q} / I_1^{u_1+1} \cdots I_q^{u_q+1} \right) \\ &=: \bigoplus_{u_1, \dots, u_q=0}^{\infty} S_{u_1, \dots, u_q}. \end{aligned}$$

Then  $S$  has dimension  $d+q-1$ . Let  $R := G_{\mathfrak{m}}(S) =: \bigoplus_{r, u_1, \dots, u_q=0}^{\infty} R_{r, u_1, \dots, u_q}$ , where

$$R_{r, u_1, \dots, u_q} = (\mathfrak{m}^r I_1^{u_1} \cdots I_q^{u_q} + I_1^{u_1+1} \cdots I_q^{u_q+1}) / (\mathfrak{m}^{r+1} I_1^{u_1} \cdots I_q^{u_q} + I_1^{u_1+1} \cdots I_q^{u_q+1}).$$

Then  $R$  is a  $(q+1)$ -graded algebra of dimension  $d+q-1$ .

**Definition 5.1.** For every  $k = 0, \dots, d-1$  we define the  $k^{\text{th}}$ -mixed multiplicity sequence  $c_k^{l_1, \dots, l_q}(I_1 \mid \dots \mid I_q)$  of the ideals  $I_1, \dots, I_q$  by

$$c_k^{l_1, \dots, l_q}(I_1 \mid \dots \mid I_q) := c_k^{l_1, \dots, l_q}(R), \quad l_1 + \dots + l_q = d-1-k.$$

The next result shows the relationship between the above definition of mixed multiplicities and the one introduced by Teissier for  $\mathfrak{m}$ -primary ideals.

**Proposition 5.2.** *Let  $(A, \mathfrak{m})$  be a local ring of dimension  $d$  and let  $I_1, \dots, I_q$  be ideals of  $A$  of positive height. Then we have that*

1.  $c_k^{l_{\sigma(1)}, \dots, l_{\sigma(q)}}(I_{\sigma(1)} \mid \dots \mid I_{\sigma(q)}) = c_k^{l_1, \dots, l_q}(I_1 \mid \dots \mid I_q)$  for all permutation  $\sigma \in S_q$  of  $\{1, \dots, q\}$  and for all  $k = 0, \dots, d+q-1$ ;
2. If  $I_1, \dots, I_q$  are  $\mathfrak{m}$ -primary ideals then
  - (a)  $c_k^{l_1, \dots, l_q}(I_1 \mid \dots \mid I_q) = 0$ ,  $k = 1, \dots, d-1$ ,  $l_1 + \dots + l_q = d-1-k$ ,

$$(b) \ c_0^{l_1, \dots, l_q}(I_1 \mid \dots \mid I_q) = \sum_{j=1}^q e_{l_1, \dots, l_{j-1}, l_j+1, l_{j+1}, \dots, l_q}(I_1 \mid \dots \mid I_q)$$

where  $l_1 + \dots + l_q = d - 1$ .

*Proof.* The proof of (1) is straightforward from the definition of mixed multiplicities.

Let us then prove the assertion (2). In order to compute the mixed multiplicities  $c_k^{l_1, \dots, l_q}(I_1 \mid \dots \mid I_q)$  we have to consider the length

$$h(r, u_1, \dots, u_q) := \ell \left( \frac{\mathfrak{m}^r I_1^{u_1} \dots I_q^{u_q} + I_1^{u_1+1} \dots I_q^{u_q+1}}{\mathfrak{m}^{r+1} I_1^{u_1} \dots I_q^{u_q} + I_1^{u_1+1} \dots I_q^{u_q+1}} \right).$$

Since  $I_1, \dots, I_q$  are  $\mathfrak{m}$ -primary ideals, we have that the product  $I_1 \dots I_q$  also is a  $\mathfrak{m}$ -primary ideal. Hence, for  $r \gg 0$  we have that

$$\mathfrak{m}^{r+1} I_1^{u_1} \dots I_q^{u_q} \subseteq I_1^{u_1+1} \dots I_q^{u_q+1}.$$

Therefore, for  $r \gg 0$  the sum transform becomes,

$$\begin{aligned} h^{(1,0_q)}(r, u_1, \dots, u_q) &:= \sum_{i=0}^r h(i, u_1, \dots, u_q) \\ &= \ell \left( I_1^{u_1} \dots I_q^{u_q} / \mathfrak{m}^{r+1} I_1^{u_1} \dots I_q^{u_q} + I_1^{u_1+1} \dots I_q^{u_q+1} \right) \\ &= \ell \left( I_1^{u_1} \dots I_q^{u_q} / I_1^{u_1+1} \dots I_q^{u_q+1} \right) \\ &= \ell \left( A / I_1^{u_1+1} \dots I_q^{u_q+1} \right) - \ell \left( A / I_1^{u_1} \dots I_q^{u_q} \right). \quad (1) \end{aligned}$$

But by definition of the mixed multiplicities of a set of  $\mathfrak{m}$ -primary ideals due to Teissier we have that, for  $u_1, \dots, u_q \gg 0$ , the highest degree term of the polynomial  $\ell(A / I_1^{u_1} \dots I_q^{u_q})$  could be written as

$$\sum_{l_1 + \dots + l_q = d} \frac{e_{l_1, \dots, l_q}(I_1 \mid \dots \mid I_q)}{l_1! \dots l_q!} u_1^{l_1} \dots u_q^{l_q}.$$

Hence, from equation (1) above we have that the highest degree term of  $h^{(1,0_q)}(r, u_1, \dots, u_q)$  could be written as



$$\begin{aligned}
& \sum_{l_1+\dots+l_q=d} \frac{e_{l_1,\dots,l_q}(I_1|\dots|I_q)}{l_1!\dots l_q!} \left[ (u_1+1)^{l_1} \dots (u_q+1)^{l_q} - u_1^{l_1} \dots u_q^{l_q} \right] \\
&= \sum_{l_1+\dots+l_q=d} \sum_{j=1}^q \frac{e_{l_1,\dots,l_q}(I_1|\dots|I_q)}{l_1!\dots l_q!} l_j u_1^{l_1} \dots u_j^{l_j-1} \dots u_q^{l_q} \\
&= \sum_{l_1+\dots+l_q=d} \sum_{j=1}^q \frac{e_{l_1,\dots,l_q}(I_1|\dots|I_q)}{l_1!\dots(l_j-1)!\dots l_q!} u_1^{l_1} \dots u_j^{l_j-1} \dots u_q^{l_q} \\
&= \sum_{t_1+\dots+t_q=d-1} \frac{\sum_{j=1}^q e_{t_1,\dots,t_j+1,\dots,t_q}(I_1|\dots|I_q)}{t_1!\dots t_q!} u_1^{t_1} \dots u_q^{t_q}. \quad (2)
\end{aligned}$$

But, by definition of the  $k^{\text{th}}$ -mixed multiplicity sequence (Definition 5.1), we have that the highest degree term of  $h^{(1,0_q)}(r, u_1, \dots, u_q)$  could be written as

$$\sum_{k=0}^{d-1} \sum_{t_1+\dots+t_q=d-1-k} \frac{c_k^{t_1,\dots,t_q}(I_1|\dots|I_q)}{t_1!\dots t_q!} r^k u_1^{t_1} \dots u_q^{t_q}, \quad (3)$$

Thus, by comparing both terms (2) and (3) of the above equations which describe the highest degree term of  $h^{(1,0_q)}(r, u_1, \dots, u_q)$ , we have that the claims (a) and (b) holds.  $\square$

**Theorem 5.3.** (Product Formula) For all  $k = 0, \dots, d-1$  we have that

$$c_k(I_1 \dots I_q, A) = \sum_{l_1+\dots+l_q=d-1-k} \frac{(d-1-k)!}{l_1!\dots l_q!} c_k^{l_1,\dots,l_q}(I_1|\dots|I_q).$$

*Proof.* Let  $S^\bullet := G_{I_1 \dots I_q}(A) := \bigoplus_{s=0}^\infty (I_1 \dots I_q)^s / (I_1 \dots I_q)^{s+1} =: \bigoplus_{s=0}^\infty S_s^\bullet$ . Let  $R^\bullet := G_m(S^\bullet) := \bigoplus_{s,t=0}^\infty (R_{rs}^\bullet)$ . We know that  $\dim R^\bullet = d$ . Then the function  $h_{R^\bullet}(r, s) := \ell_{R_{00}^\bullet}(R_{rs}^\bullet)$  is a polynomial of total degree  $d-2$  for all large  $r$  and  $s$  which can be written in the form

$$P_{R^\bullet}(r, s) = \sum_{\substack{k, l \geq 0 \\ k+l \leq d-2}} b_{k,l} \binom{r}{k} \binom{s}{l}.$$

Define now  $h_{R^\bullet}^{(1,0)}(r, s) := \sum_{u=0}^r h_{R^\bullet}(u, s)$  and  $h_{R^\bullet}^{(1,1)}(r, s) := \sum_{v=0}^s h_{R^\bullet}^{(1,0)}(r, v)$ . Then,  $h_{R^\bullet}^{(1,0)}(r, s)$  is a polynomial of total degree at most  $d-1$  for all large  $r$

and  $s$  which can be written in the form

$$\begin{aligned}
P_{R^\bullet}^{(1,0)}(r, s) &= \sum_{\substack{k, l \geq 0 \\ k+l \leq d-1}} b_{k,l}^{(1,0)} \binom{r}{k} \binom{s}{l} \\
&= \sum_{k=0}^{d-1} \frac{b_{k,d-1-k}^{(1,0)}}{k!(d-1-k)!} r^k s^{d-1-k} + (\text{lower degree terms})
\end{aligned}$$

with  $b_{k+1,l}^{(1,0)} = b_{k,l}^{(1,0)}$  for  $k, l \geq 0$ ,  $k+l \leq d-2$ .

Also,  $h_{R^\bullet}^{(1,1)}(r, s)$  is a polynomial of total degree  $d$  for all large  $r$  and  $s$  which can be written in the form

$$\begin{aligned}
P_{R^\bullet}^{(1,1)}(r, s) &= \sum_{\substack{k, l \geq 0 \\ k+l \leq d}} b_{k,l}^{(1,1)} \binom{r}{k} \binom{s}{l} \\
&= \sum_{k=0}^d \frac{b_{k,d-k}^{(1,1)}}{k!(d-k)!} r^k s^{d-k} + (\text{lower degree terms}) \quad (1)
\end{aligned}$$

with  $b_{k,l+1}^{(1,1)} = b_{k,l}^{(1,0)}$  for  $k, l \geq 0$ ,  $k+l \leq d-1$ .

Hence, by definition of the Achilles-Manaresi multiplicity sequence, we have that

$$b_{k,d-k}^{(1,1)} = c_k(I_1 \cdots I_q, A), \quad k = 0, \dots, d.$$

On the other hand, let

$$S := \oplus_{u_1, \dots, u_q=0}^{\infty} (I_1^{u_1} \cdots I_q^{u_q} / I_1^{u_1+1} \cdots I_q^{u_q+1})$$

and let  $R := G_{\mathfrak{m}}(S)$ . Notice that  $S_s^\bullet = S_{s, \dots, s}$ , where

$$S_{u_1, \dots, u_q} := (I_1^{u_1} \cdots I_q^{u_q} / I_1^{u_1+1} \cdots I_q^{u_q+1}).$$

Hence we have that

$$\begin{aligned}
h_{R^\bullet}^{(1,0)}(r, s) &= h_R^{(1,0)}(r, s, \dots, s) \\
&= \sum_{\substack{k, l_1, \dots, l_q \geq 0 \\ k + l_1 + \dots + l_q \leq d-1}} a_{k, l_1, \dots, l_q}^{(1,0_q)} \binom{r}{k} \binom{s}{l_1} \cdots \binom{s}{l_q} \\
&= \sum_{k=0}^{d-1} \sum_{l_1 + \dots + l_q = d-1-k} \frac{a_{k, l_1, \dots, l_q}^{(1,0_q)}}{k! l_1! \cdots l_q!} r^k s^{d-1-k} + (\text{lower deg. terms}). \quad (2)
\end{aligned}$$

Therefore, by comparing the coefficients of  $r^k s^{d-1-k}$  in both equations (1) and (2) we get that

$$\frac{b_{k, d-1-k}^{(1,0)}}{k!(d-1-k)!} = \sum_{l_1 + \dots + l_q = d-1-k} \frac{a_{k, l_1, \dots, l_q}^{(1,0_q)}}{k! l_1! \cdots l_q!}$$

or, equivalently

$$b_{k, d-1-k}^{(1,0)} = \sum_{l_1 + \dots + l_q = d-1-k} \frac{(d-1-k)!}{l_1! \cdots l_q!} a_{k, l_1, \dots, l_q}^{(1,0_q)}$$

Hence, if  $k = 0, \dots, d-1$ , then

$$\begin{aligned}
c_k(I_1 \cdots I_q, A) &= b_{k, d-k}^{(1,1)} \\
&= b_{k, d-k-1}^{(1,0)} \\
&= \sum_{l_1 + \dots + l_q = d-1-k} \frac{(d-1-k)!}{l_1! \cdots l_q!} a_{k, l_1, \dots, l_q}^{(1,0_q)} \\
&= \sum_{l_1 + \dots + l_q = d-1-k} \frac{(d-1-k)!}{l_1! \cdots l_q!} c_k^{l_1, \dots, l_q}(I_1 \mid \cdots \mid I_q).
\end{aligned}$$

□

As a consequence of this theorem we recover Teissier's product formula given in [T] (see also Theorem 4.1).

**Corollary 5.4.** (Teissier [T]) Let  $I_1, \dots, I_q \subset A$  be  $\mathfrak{m}$ -primary ideals of  $A$ . Then,

$$e(I_1 \cdots I_q, A) = \sum_{l_1 + \dots + l_q = d} \frac{d!}{l_1! \cdots l_q!} e_{l_1, \dots, l_q}(I_1 \mid \dots \mid I_q).$$

*Proof.* Since  $I_1, \dots, I_q \subset A$  are  $\mathfrak{m}$ -primary ideals of  $A$  we have that  $I_1 \cdots I_q$  is also a  $\mathfrak{m}$ -primary ideal of  $A$ . Hence, by [AM] Corollary 2.4(i), we have that  $c_k(I_1 \cdots I_q, A) = 0$  for all  $k = 1, \dots, d$  and  $c_0(I_1 \cdots I_q, A) = e(I_1 \cdots I_q, A)$ . Thus, by the above theorem we have that

$$\begin{aligned} e(I_1 \cdots I_q, A) &= c_0(I_1 \cdots I_q, A) \\ &= \sum_{t_1 + \dots + t_q = d-1} \frac{(d-1)!}{t_1! \cdots t_q!} c_0^{t_1, \dots, t_q}(I_1 \mid \dots \mid I_q). \end{aligned} \quad (1)$$

On the other hand, by Proposition 5.2(2b), we have that

$$c_0^{l_1, \dots, l_q}(I_1 \mid \dots \mid I_q) = \sum_{j=1}^q e_{l_1, \dots, l_{j-1}, l_j+1, l_{j+1}, \dots, l_q}(I_1 \mid \dots \mid I_q). \quad (2)$$

Thus, by equation (1) and (2) we have that

$$\begin{aligned} e(I_1 \cdots I_q, A) &= \sum_{t_1 + \dots + t_q = d-1} \frac{(d-1)!}{t_1! \cdots t_q!} \left( \sum_{j=1}^q e_{t_1, \dots, t_{j-1}, t_j+1, \dots, t_q}(I_1 \mid \dots \mid I_q) \right) \\ &= \sum_{l_1 + \dots + l_q = d} e_{l_1, \dots, l_q}(I_1 \mid \dots \mid I_q) \left( \sum_{j=1}^q \frac{(d-1)!}{l_1! \cdots (l_{j-1}!) \cdots (l_q!)} \right) \\ &= \sum_{l_1 + \dots + l_q = d} \frac{e_{l_1, \dots, l_q}(I_1 \mid \dots \mid I_q)}{l_1! \cdots l_q!} (d-1)! \left( \sum_{j=1}^q l_j \right) \\ &= \sum_{l_1 + \dots + l_q = d} \frac{d!}{l_1! \cdots l_q!} e_{l_1, \dots, l_q}(I_1 \mid \dots \mid I_q). \end{aligned}$$

□

## 6 Buchsbaum-Rim multiplicity

In this section we recall the definition and some known facts about the Buchsbaum-Rim multiplicity of a submodule, of finite colength, of a free module (see also [BR] and [Ga])

Let  $M$  be a submodule of the free  $A$ -module  $A^p$  of rank  $p$  such that  $A^p/M$  has finite length. The symmetric algebra  $S(A^p) = \bigoplus S_n(A^p)$  of  $A^p$  is



a polynomial ring  $A[t_1, \dots, t_p]$ . If  $h = (h_1, \dots, h_p) \in A^p$ , then we define the element  $w(h) = h_1 t_1 + \dots + h_p t_p \in S(A^p)$ . We denote by  $\mathcal{R}(M) := \oplus \mathcal{R}_n(M)$  the subalgebra of  $S(A^p)$  generated by  $\{w(h) : h \in M\}$  and call it *the Rees algebra* of  $M$ . Then  $\mathcal{R}(M)$  has dimension  $d + p$ .

**Theorem 6.1.** ([BR]) *Consider the function  $H(n) = \ell_A(S_n(A^p)/\mathcal{R}_n(M))$ . Then there exist a polynomial  $Q(n)$  in  $n$  with rational coefficients and degree  $d + p - 1$  such that  $H(n) = Q(n)$ , for all  $n \gg 0$ .*

In the conditions of the above theorem, if  $a_{d+p-1} \in \mathbb{Q}$  denotes the coefficient of  $n^{d+p-1}$  in the expression of  $Q(n)$ , we define the *Buchsbaum-Rim multiplicity* of  $M$ ,  $e_{BR}(M)$ , as

$$e_{BR}(M) := (d + p - 1)! a_{d+p-1}.$$

It is proved that, if  $M \neq A^p$ , then  $e_{BR}(M) > 0$  (see [BR], p. 214). If  $I$  is an ideal of  $A$  of finite colength, then the Buchsbaum-Rim multiplicity of  $I$ , with  $I$  being considered as a submodule of  $A$ , is equal to the Samuel multiplicity of  $I$ .

The next result has been proved by Kirby and Rees in [KR2], p. 444, considering the notion of *joint reduction* of a set of ideals (see [R]) and it was reproved by C. Bivià-Ausina in [BA], Theorem 4.9 who used a more direct method based on a combinatorial lemma of J. Verma [V2], p. 221.

**Theorem 6.2.** *Let  $I_1, \dots, I_p$  be ideals of  $A$  of finite colength and let  $M$  be the submodule of  $A^p$  given by  $M = I_1 \oplus \dots \oplus I_p$ . Then  $M$  also has finite colength and*

$$e_{BR}(M) = \sum_{\substack{l_1, \dots, l_p \geq 0 \\ l_1 + \dots + l_p = d}} e_{l_1, \dots, l_p}(I_1 \mid \dots \mid I_p),$$

where  $e_{l_1, \dots, l_p}(I_1 \mid \dots \mid I_p)$  denotes the mixed multiplicities of the ideals  $I_1, \dots, I_p$  as defined in Section 4.

## 7 A generalized Buchsbaum-Rim multiplicities

Let  $(A, \mathfrak{m})$  be a local ring of dimension  $d$  and let  $M$  be a submodule of the free  $A$ -module  $A^p$  of rank  $p$ . In this section we introduce a multiplicity

sequence  $e_{BR}^0(M), \dots, e_{BR}^{d+p-1}(M)$ , which in the ideal case coincides with the multiplicity sequence  $c_0(I, A), \dots, c_d(I, A)$  defined for an arbitrary ideal  $I$  of  $A$  by Achilles and Manaresi [AM].

Let  $u^1, \dots, u^s$  be a system of generators of  $M$ , where  $u^i = (u_1^i, \dots, u_p^i)$ ,  $i = 1, \dots, s$ . Let  $[M]$  be the matrix

$$[M] := \begin{pmatrix} u_1^1 & \dots & u_1^s \\ \vdots & \ddots & \vdots \\ u_p^1 & \dots & u_p^s \end{pmatrix}$$

We denote by  $I_k(M)$  the ideal generated by the  $k \times k$  minors of  $[M]$ . This is the same as the  $(p-k)$ -th Fitting ideal of  $A^p/M$ , hence is independent of the choice of generators of  $M$ . From now on we assume that  $I_p(M) \neq 0$ .

For each  $q = 1, \dots, p$  and each sequence  $1 \leq i_1 < \dots < i_q \leq p$  define the projection  $\pi_{i_1, \dots, i_q} : M \rightarrow A^q$  by  $\pi_{i_1, \dots, i_q}(a_1, \dots, a_p) := (a_{i_1}, \dots, a_{i_q})$ . Denote by  $M_{i_1, \dots, i_q} \subset A^q$  the image of  $M$  under this projection

Let

$$G(M_{i_1, \dots, i_q}) := \mathcal{R}(M_{i_1, \dots, i_q}) / I_q(M_{i_1, \dots, i_q}) \mathcal{R}(M_{i_1, \dots, i_q}).$$

Then  $G(M_{i_1, \dots, i_q})$  is a  $q$ -graded  $A$ -algebra of dimension  $d + q - 1$ .

Let now

$$A(M_{i_1, \dots, i_q}) := G_m(G(M_{i_1, \dots, i_q})) =: \bigoplus_{r, u_{i_1}, \dots, u_{i_q} \geq 0} A(M)_{r, u_{i_1}, \dots, u_{i_q}}.$$

Then  $A(M)$  is a  $(q+1)$ -graded algebra of dimension  $d + q - 1$ .

We define

$$S(M_{i_1, \dots, i_q}) := {}^q A(M_{i_1, \dots, i_q}) := \bigoplus_{r, u=0}^{\infty} \left( \bigoplus_{u_{i_1} + \dots + u_{i_q} = u} A(M)_{r, u_{i_1}, \dots, u_{i_q}} \right).$$

Then  $S(M_{i_1, \dots, i_q})$  is a bigraded algebra of dimension  $d + q - 1$ .

Let

$$a_k(M_{i_1, \dots, i_q}) = \begin{cases} c_k(S(M_{i_1, \dots, i_q})) & \text{if } k = 0, \dots, d + q - 1 \\ 0 & \text{if } k = d + q, \dots, d + p - 1 \end{cases}$$

where  $c_k(S(M_{i_1, \dots, i_q}))$ ,  $k = 0, \dots, d + q - 1$ , is the multiplicity sequence of the bigraded algebra  $S(M_{i_1, \dots, i_q})$ , as in Definition 3.1.

**Definition 7.1.** We call the sequence of nonnegative integers

$$e_{BR}^k(M) := \sum_{q=1}^p (p-q)!(q-1)! \sum_{1 \leq i_1 < \dots < i_q \leq p} a_k(M_{i_1, \dots, i_q}), \quad k = 0, \dots, d+p-1$$

the **Buchsbaum-Rim multiplicity sequence**, or just BR-multiplicity sequence, of the  $A$ -module  $M$ .

It is clear from the construction that if  $I$  is an ideal of  $A$  then the BR-multiplicity sequence of  $I$  just defined, with  $I$  being considered as a submodule of  $A$ , coincides with the multiplicity sequence  $c_k(I, A)$  of the ideal  $I$  of  $A$  as defined by Achilles and Manaresi in [AM]. Precisely, we have that

$$e_{BR}^k(I) = c_k(I, A), \quad \text{for all } k = 0, \dots, d.$$

Our next goal is to relate the BR-multiplicity sequence just defined with the usual BR-multiplicity for modules of finite colength of a special kind. For doing so, we need some preliminary results which we now develop.

**Lemma 7.2.** *For all  $k = 0, \dots, d-1$ ,  $1 \leq q \leq p$  and all choice of indexes  $1 \leq i_1 < \dots < i_q \leq p$  we have that*

$$a_k(M_{i_1, \dots, i_q}) = \sum_{l_{i_1} + \dots + l_{i_q} = d-1-k} c_k^{l_{i_1}, \dots, l_{i_q}}(A(M_{i_1, \dots, i_q}))$$

where  $c_k^{l_{i_1}, \dots, l_{i_q}}(A(M_{i_1, \dots, i_q}))$ , denotes the Mixed Multiplicity sequence of  $A(M_{i_1, \dots, i_q})$  as defined in Definition 2.1.

*Proof.* Notice that, by Proposition 2.2, for each  $k = 0, \dots, d-1$  we have that

$$\begin{aligned} a_k(M_{i_1, \dots, i_q}) &= c_k(S(M_{i_1, \dots, i_q})) \\ &= a_{k, d-k}^{(1,1)}({}^q A(M_{i_1, \dots, i_q})) \\ &= a_{k, d-1-k}^{(1,0)}({}^q A(M_{i_1, \dots, i_q})) \\ &= \sum_{l_{i_1} + \dots + l_{i_q} = d-1-k} a_{k, l_{i_1}, \dots, l_{i_q}}^{(1,0_q)}(A(M_{i_1, \dots, i_q})) \\ &= \sum_{l_{i_1} + \dots + l_{i_q} = d-1-k} c_k^{l_{i_1}, \dots, l_{i_q}}(A(M_{i_1, \dots, i_q})). \end{aligned}$$

□

For the rest of this section we will assume that all modules are of the form  $M = I_1 \oplus \dots \oplus I_p$ , where  $I_1, \dots, I_p$  are ideals of  $A$  of positive height. In this case,  $I_q(M_{i_1, \dots, i_q}) = I_{i_1} \cdots I_{i_q}$ . Thus

$$G(M_{i_1, \dots, i_q}) = A[I_{i_1}T_{i_1}, \dots, I_{i_q}T_{i_q}]/(I_{i_1} \cdots I_{i_q}).$$

Therefore

$$c_k^{l_{i_1}, \dots, l_{i_q}}(A(M_{i_1, \dots, i_q})) = c_k^{l_{i_1}, \dots, l_{i_q}}(I_{i_1} \mid \dots \mid I_{i_q}).$$

Hence, by Lemma 7.2, we get the following description of the BR-multiplicity sequence for modules of the form  $M = I_1 \oplus \dots \oplus I_p$ :

**Proposition 7.3.** *For all  $k = 0, \dots, d-1$  we have that*

$$e_{BR}^k(M) := \sum_{q=1}^p (p-q)!(q-1)! \sum_{1 \leq i_1 < \dots < i_q \leq p} \sum_{l_{i_1} + \dots + l_{i_q} = d-1-k} c_k^{l_{i_1}, \dots, l_{i_q}}(I_{i_1} \mid \dots \mid I_{i_q}).$$

Next we prove a combinatorial lemma which will play a crucial role in the sequel.

**Lemma 7.4.** *Let  $p$  and  $d$  be any nonnegative integers. Let*

$$\{X_{t_1, \dots, t_p}; \quad t_1 + \dots + t_p = d\}$$

*be a set of independent variables. Then,*

$$\begin{aligned} & \sum_{q=1}^p \lambda_q \sum_{1 \leq i_1 < \dots < i_q \leq p} \sum_{l_{i_1} + \dots + l_{i_q} = d-1} \sum_{j=1}^q X_{0, \dots, 0, l_{i_1}, 0, \dots, 0, l_{i_j}+1, 0, \dots, 0, l_{i_q}, 0, \dots, 0} \\ &= p! \sum_{t_1 + \dots + t_p = d} X_{t_1, \dots, t_p} \end{aligned} \quad (1)$$

where  $\lambda_q = (p-q)!(q-1)!$

*Proof.* Let  $X_{t_1, \dots, t_p}$ , with  $t_1 + \dots + t_p = d$ , be one of the variables. We want to prove that the coefficient of  $X_{t_1, \dots, t_p}$  which appears in the left hand side of equation (1) is  $p!$ . Let  $s := \#\{j \in \{1, \dots, p\}; \quad t_j \neq 0\}$ . Since both sides of equation (1) are invariant by permutations, we may assume that  $X_{t_1, \dots, t_p} = X_{t_1, \dots, t_s, 0, \dots, 0}$ , with  $t_i \neq 0$  for  $i = 1, \dots, s$  and  $t_1 + \dots + t_s = d$ . Notice that among all terms of the form  $X_{0, \dots, 0, l_{i_1}, 0, \dots, 0, l_{i_j}+1, 0, \dots, 0, l_{i_q}, 0, \dots, 0}$ , with  $s \leq q \leq p$  and  $l_{i_1} + \dots + l_{i_q} = d-1$ , the only ones that give rise to



$X_{t_1, \dots, t_s, 0, \dots, 0}$  are the ones of the forms  $X_{l_1, \dots, l_j+1, \dots, l_s, 0, \dots, 0}$ ,  $j = 1, \dots, s$ , with  $l_i = t_i$  if  $i \neq j$  and  $l_j = t_j - 1$ , which amount in  $s$  of such a kind. On the other hand, for each  $s \leq q \leq p$ , we can choose  $q - s$  indexes  $t_{i_{s+1}}, \dots, t_{i_q}$ , all equal to zero, among the remaining  $p - s$  indexes in such a way that  $t_1 + \dots + t_s + t_{i_{s+1}} + \dots + t_{i_q} = d$ . Summing up, we have that the number of times in which  $X_{t_1, \dots, t_s, 0, \dots, 0}$  appears in the sum

$$\sum_{1 \leq i_1 < \dots < i_q \leq p} \sum_{l_{i_1} + \dots + l_{i_q} = d-1} \sum_{j=1}^q X_{0, \dots, 0, l_{i_1}, 0, \dots, 0, l_{i_j}+1, 0, \dots, 0, l_{i_q}, 0, \dots, 0},$$

with  $s \leq q \leq p$ , is  $s \cdot \binom{p-s}{q-s}$ .

Therefore, the coefficient of  $X_{t_1, \dots, t_p}$  in the left hand side of equation (1) is

$$\sum_{q=s}^p (p-q)!(q-1)! s \binom{p-s}{q-s}$$

which is equal to

$$s!(p-s)! \sum_{q=s}^p \binom{q-1}{q-s}.$$

Now, repeatedly using the well-known formula

$$\binom{m}{n} = \binom{m-1}{n-1} + \binom{m-1}{n},$$

we get

$$\sum_{q=s}^p \binom{q-1}{q-s} = \binom{p}{s}.$$

Hence,

$$\begin{aligned} \sum_{q=s}^p (p-q)!(q-1)! s \binom{p-s}{q-s} &= s!(p-s)! \sum_{q=s}^p \binom{q-1}{q-s} \\ &= s!(p-s)! \binom{p}{s} \\ &= p!. \end{aligned}$$

□

**Lemma 7.5.** *If  $I_1, \dots, I_p$  are  $\mathfrak{m}$ -primary ideals of  $A$  then*

$$\begin{aligned} & \sum_{q=1}^p (p-q)!(q-1)! \sum_{1 \leq i_1 < \dots < i_q \leq p} \sum_{l_{i_1} + \dots + l_{i_q} = d-1} c_0^{l_{i_1}, \dots, l_{i_q}} (I_{i_1} \mid \dots \mid I_{i_q}) \\ &= p! \sum_{t_1 + \dots + t_p = d} e_{t_1, \dots, t_p} (I_1 \mid \dots \mid I_p) \end{aligned}$$

where  $e_{t_1, \dots, t_p} (I_1, \dots, I_p)$  denotes the mixed multiplicity sequence of the ideals  $I_1, \dots, I_p$ , as defined in [T].

*Proof.* According to Teissier [T], the number  $e_{t_{i_1}, \dots, t_{i_q}} (I_{i_1} \mid \dots \mid I_{i_q})$  is equal to the colength of the parameter ideal generated by  $t_{i_j}$  generic  $A$ -linear combinations of a given generating system of  $I_{i_j}$ ,  $j = 1, \dots, q$ . Therefore,

$$e_{t_{i_1}, \dots, t_{i_q}} (I_{i_1} \mid \dots \mid I_{i_q}) = e_{0, \dots, 0, t_{i_1}, 0, \dots, 0, t_{i_q}, 0, \dots, 0} (I_1 \mid \dots \mid I_p).$$

Hence the result follows directly from Proposition 5.2(2b) and the above Lemma.  $\square$

The next result shows the relationship between the above definition of BR-multiplicity sequence and the Buchsbaum-Rim multiplicity introduced by Buchsbaum and Rim in [BR] for submodules of finite colength of  $A^p$ .

**Proposition 7.6.** *If  $M = I_1 \oplus \dots \oplus I_p \subset A^p$  has finite colength then  $e_{BR}^0(M) = p! (e_{BR}(M))$  and  $e_{BR}^k(M) = 0$  for  $k = 1, \dots, d-1$ .*

*Proof.* Since  $M = I_1 \oplus \dots \oplus I_p$  has finite colength in  $A^p$  the ideals  $I_1, \dots, I_p$  have finite colength in  $A$ . Thus, by Proposition 5.2 (2a) and Proposition 7.3, we have that  $e_{BR}^k(M) = 0$  for  $k = 1, \dots, d-1$  and

$$e_{BR}^0(M) = \sum_{q=1}^p (p-q)!(q-1)! \sum_{1 \leq i_1 < \dots < i_q \leq p} \sum_{l_{i_1} + \dots + l_{i_q} = d-1} c_0^{l_{i_1}, \dots, l_{i_q}} (I_{i_1} \mid \dots \mid I_{i_q}).$$

Hence, by Lemma 7.5, we have that

$$e_{BR}^0(M) = p! \sum_{t_1 + \dots + t_p = d} e_{t_1, \dots, t_p} (I_1 \mid \dots \mid I_p)$$

where  $e_{t_1, \dots, t_p} (I_1, \dots, I_p)$  denotes the mixed multiplicity sequence of ideals as defined in [T]. Now, by a theorem of Kirby and Rees [KR2] (see Theorem 6.2), we have that

$$\sum_{t_1 + \dots + t_p = d} e_{t_1, \dots, t_p} (I_1, \dots, I_p) = e_{BR}(M).$$

Therefore,

$$e_{BR}^0(M) = p ! (e_{BR}(M)).$$

□

As a consequence of Proposition 7.3 and Lemma 7.5 we get the following version of Theorem 6.2.

**Corollary 7.7.** *Let  $I_1, \dots, I_p$  be ideals of  $A$  of finite colength and let  $M$  be the submodule of  $A^p$  given by  $M = I_1 \oplus \dots \oplus I_p$ . Then  $M$  also has finite colength and*

$$e_{BR}^0(M) = p ! \sum_{l_1 + \dots + l_p = d} e_{l_1, \dots, l_p}(I_1 \mid \dots \mid I_p),$$

where  $e_{l_1, \dots, l_p}(I_1 \mid \dots \mid I_p)$  denotes the mixed multiplicities of the ideals  $I_1, \dots, I_p$  as defined in Section 4.

Thus, the result obtained in Proposition 7.3 could be seen as a generalization of Theorem 6.2, obtained by Kirby and Rees in [KR2].

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