UNIVERSIDADE DE SÃO PAULO
Instituto de Ciências Matemáticas e de Computação
ISSN 0103-2577

Finite Determinacy and Whitney Equisingularity of Map Germs from $\mathbb{C}^n$ to $\mathbb{C}^{2n-1}$

V. H. Jorge Pérez
J. J. Nuño-Ballesteros

Nº 263

NOTAS

Série Matemática

São Carlos – SP
Mai./2006

FINITE DETERMINACY AND WHITNEY EQUISINGULARITY OF MAP GERMS FROM $\mathbb{C}^n$ TO $\mathbb{C}^{2n-1}$

V. H. JORGE PÉREZ AND J. J. NUÑO-BALLESTEROS

ABSTRACT. We show that a holomorphic map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$ is finitely determined if and only if the double point scheme $D(f)$ is a curve germ with isolated singularity. In this case, we have that $\mu(D(f)) = 2\mu(f(D(f))) + C(f) - 1$, where $\mu(X)$ denotes the Milnor number of a curve germ X with isolated singularity and $C(f)$ is the number of cross-caps that appear in a stable deformation of f. In the second part, we consider 1-parameter unfoldings $F(t, x) = (t, f_t(x))$ of f. We show that if $\mu(D(f_t))$ is constant, then F is excellent in the sense of Gaffney [8]. In the last part, we study the polar multiplicities of the strata induced by stable types in f. We find a minimal set of invariants whose constancy in a 1-parameter unfolding F is equivalent to the Whitney equisingularity of F. In the corank 1 case, this is equivalent to the Whitney equisingularity of the families of double point curves $D(f_t)$ and $f_t(D(f_t))$. We also give an example of an unfolding which is μ-constant, but it is not Whitney equisingular.

Resumo. Demonstramos que um germe de aplicação holomorfa $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$ é finitamente determinado se e somente se o conjunto de pontos duplos $D(f)$ é uma curva com singularidade isolada. Em este caso, nos temos que $\mu(D(f)) = 2\mu(f(D(f))) + C(f) - 1$, onde $\mu(X)$ denota o número de Milnor da curva X com singularidade isolada e $C(f)$ é o número de cross-caps que aparecem na deformação estável de f. Na segunda parte, considereamos desdobramento a 1-parâmetro $F(t, x) = (t, f_t(x))$ de f. Demonstramos que se $\mu(D(f_t))$ é constante, então F é excelente no sentido de Gaffney [8]. Na ultima parte, estudamos as multiplicidades polares sobre os tipos estáveis sobre f. Encontramos o conjunto minimal de invariantes cuja constância desdobramento a 1-parâmetro F é equivalente a Whitney equisingularidade de F. equisingular.

1. INTRODUCTION

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a holomorphic map germ, with $n \leq p$. We denote by $D(f)$ the double point set germ in $(\mathbb{C}^n, 0)$, which is defined as the closure of points x such that $f^{-1}(f(x)) \neq \{x\}$. We take an appropriate analytic structure in $D(f)$ so that it is well behaved with respect to deformations. This means that if $F(t, x) = (t, f_t(x))$ is a 1-parameter unfolding of f, with $f_0 = f$, then $D(F) \cap \{t = 0\} = D(f)$. However, this implies that $D(f)$ may be non reduced, in general.

In the case that $p = 2n - 1$, it follows from Whitney's work [24] that a holomorphic map f is stable if and only if it an immersion with normal crossings, except at isolated points, where f is $\mathcal{A}$-equivalent to the generalized Whitney umbrella or cross-cap. As a consequence, we deduce that f is finitely determined if and only if $D(f)$ is a complex curve germ in $(\mathbb{C}^n, 0)$ with isolated singularity. When $n = 2$, we have a map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ and this is a known fact (see [17]). Moreover, since $D(f)$ is a plane

2000 *Mathematics Subject Classification.* Primary 32S015; Secondary 32S030, 58K040.

Key words and phrases. Whitney equisingularity, polar multiplicities, Milnor number.

The first author has been partially supported by CAPES - Grant 3131/04-1. The second author has been partially supported by DGICYT Grant BFM2003- 02037.

curve, we can consider here the reduced structure. There is also another related result in [19], for a map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with $n \leq p$ and corank 1. In this case, f is finitely determined if and only if all the multiple point schemes $D^k(f)$, for $k \geq 1$, are isolated complete intersection singularities (ICIS) of dimension $p - k(p-n)$ or empty.

In our case, since $D(f)$ may have embedded points, we take the Milnor number $\mu(D(f))$ as defined in [6]. Then we show that if $n \geq 3$, then

$$\mu(D(f)) = 2\mu(f(D(f))) + C(f) - 1,$$

where $C(f)$ is the number of cross-caps that appear near the origin in a stable deformation of f. When $n = 2$, there is also a similar formula [19], although we must take into account the number of triple points $T(f)$.

In the second part of the paper, we consider μ-constant unfoldings F of f. This means that $\mu(D(f_t))$ is constant in the family $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$. The main result is that any μ-constant unfolding is excellent in the sense of Gaffney [8]. An unfolding F is said to be excellent if there is a representative $F : D \times U \to D \times \mathbb{C}^{2n-1}$, where D, U are open neighbourhoods of 0 in $\mathbb{C}$, $\mathbb{C}^n$ respectively, such that for any $t \in D$, $f_t^{-1}(0) = \{0\}$ and $f_t : U \setminus \{0\} \to \mathbb{C}^{2n-1}$ is an immersion with only transverse double points. Moreover, we also deduce that any topologically trivial unfolding is μ-constant.

Finally, we consider the problem of determine whether an unfolding F of f is Whitney equisingular (i.e., there is a representative of F which admits a regular stratification with the parameter axes as strata). We apply the main theorem of [8], which says that F is Whitney equisingular if and only if F is excellent and all the polar multiplicities of the strata induced by stable types in F are constant. The problem is that there is a big amount of invariants to consider. However, most of the times these invariants are related and it is possible to reduce this number. This is done in [8] in the case of a map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. Gaffney shows that the Whitney equisingularity of the unfolding is equivalent to the constancy of $\mu(D(f_t))$, the multiplicity $m_0(f_t(D(f_t)))$ and the first polar multiplicity $m_1(f_t(\mathbb{C}^2))$. Moreover, if f has corank 1, then the constancy of $m_1(f_t(\mathbb{C}^2))$ is not necessary.

For $n \geq 3$, we extend all the results of Gaffney and show that an unfolding F is Whitney equisingular if and only if

(1) $\mu(D(f_t))$, $m_0(D(f_t))$, $m_0(f_t(D(f_t)))$ are constant, and
(2) the odd polar multiplicities $m_{2i-1}(f_t(\mathbb{C}^n))$, $i = 1, 2, \dots$ are constant.

Note that condition (1) is equivalent to the Whitney equisingularity of the families of double point curves $D(f_t)$ and $f_t(D(f_t))$. If f has corank 2, condition (2) can be reduced to the constancy of just $m_1(f_t(\mathbb{C}^n))$. Moreover, if f has corank 1, then condition (2) can be dropped out.

We finish the paper with an example of an unfolding F which is μ-constant, but it is not Whitney equisingular. Therefore, we have the following situation:

$$\text{Whitney equising.} \Rightarrow \text{top. trivial} \Rightarrow \mu\text{-constant} \Rightarrow \text{excellent} \not\Rightarrow \text{Whitney equising.}$$

The Whitney equisingularity problem for unfoldings of map germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ has been previously studied in the following cases: $p = 1$ [22], $n = p = 2$ and $n = 2$, $p = 3$ [8], $n = p = 3$ and corank 1 [10], $n = p$ and corank 1 [11], $n < p$ and corank 1 [12], $n \geq 3$ and $p = 2$ [9].

2. Finite determinacy and multiple point schemes

Let $f : U \to \mathbb{C}^p$ be a holomorphic map, where $U \subset \mathbb{C}^n$ is an open subset and $n \leq p$. Given $k \geq 1$ we define the *k-multiple point set*, $D^k(f)$, as the closure in U^k of the set

$$\{(x_1, \dots, x_k) \in U^k : \ f(x_i) = f(x_j),\ x_i \neq x_j, \text{ for any } i \neq j\}.$$

Assume $p = 2n - 1$ and $n \geq 3$. Then it is well known (see [24]) that f is a stable map if and only if there is a discrete subset $\Sigma \subset U$ such that:

(1) $f : U \setminus \Sigma \to \mathbb{C}^{2n-1}$ is an immersion with normal crossings.

(2) For each $x_0 \in \Sigma$, $f^{-1}(f(x_0)) = \{x_0\}$ and the map germ $f : (\mathbb{C}^n, x_0) \to (\mathbb{C}^{2n-1}, f(x_0))$ is $\mathcal{A}$-equivalent to the *Whitney umbrella*, which is the map germ $(\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$ given by

$$(x_1, \dots, x_n) \longmapsto (x_1, \dots, x_{n-1}, x_n^2, x_1 x_n, \dots, x_{n-1} x_n).$$

It is not difficult to see that these conditions imply that $D^2(f)$ is a 1-dimensional complex submanifold of U^k, and that $D^k(f)$ is empty for $k \geq 3$.

We would like to have a converse of this assertion. To do this, we need to choose a convenient analytic structure for the double point set $D^2(f)$. We follow the construction of [18] which is also valid for holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^p$ with $n \leq p$. Let us denote the diagonals of $\mathbb{C}^n \times \mathbb{C}^n$ and $\mathbb{C}^p \times \mathbb{C}^p$ by Δ_n, Δ_p and denote the sheaves of ideals defining them by $\mathcal{I}_n$, $\mathcal{I}_p$. We write the points of $\mathbb{C}^n \times \mathbb{C}^n$ as (x, x'). Then, for each $i = 1, \dots, p$, it is clear that

$$f_i(x) - f_i(x') \in \mathcal{I}_n,$$

so there exist $\alpha_{ij}(x, x')$, $1 \leq i \leq p$, $1 \leq j \leq n$, such that

$$f_i(x) - f_i(x') = \sum_{j=1}^{n} \alpha_{ij}(x, x')(x_j - x_j').$$

If $f(x) = f(x')$ and $x \neq x'$, then clearly every $n \times n$ minor of the matrix $\alpha = [\alpha_{ij}]$ must vanish at (x, x'). We denote by $\mathcal{R}_n(\alpha)$ the ideal in $\mathcal{O}_{\mathbb{C}^{2n}}$ generated by the $n \times n$ minors of α. Then we define the *double point ideal* as

$$\mathcal{I}^2(f) = (f \times f)^* \mathcal{I}_p + \mathcal{R}_n(\alpha).$$

It is easy to see that $V(\mathcal{I}^2(f)) = D^2(f)$ and we call this complex space the *double point locus* of f. At a non-diagonal point (x, x'), $\mathcal{I}^2(f)$ is generated by the functions $f_i(x) - f_i(x')$. Moreover, the restriction of $\mathcal{I}^2(f)$ to the diagonal Δ_n is the ideal generated by the $n \times n$ minors of the jacobian matrix of f, so that $\Delta_n \cap D^2(f)$ is just the singular locus of f. The following property of the double point locus is a consequence of [2].

Lemma 2.1. *The codimension of $D^2(f)$ is $\leq p$. Moreover, if the codimension is p, then $D^2(f)$ is Cohen-Macaulay.*

Now we come back to the case $p = 2n - 1$ and $n \geq 3$. Next proposition has been showed in [19] for map germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with $n \leq p$ and corank 1.

Proposition 2.2. *Let $f : U \to \mathbb{C}^{2n-1}$ be a holomorphic map, where $U \subset \mathbb{C}^n$ is an open subset and $n \geq 3$. Then f is stable if and only if $D^2(f)$ is a smooth curve and $D^3(f) = \emptyset$.*

Proof. If f is stable, then obviously $D^3(f) = \emptyset$. Given a point $(x_0, x_0') \in D^2(f)$ it is either a singular point of Whitney umbrella type f if $x_0 = x_0'$ or a transverse double point of f if $x_0 \neq x_0'$. In both cases we can take normal forms for the corresponding germ or bi-germ and verify easily that $D^2(f)$ is smooth at (x_0, x_0').

Suppose now that $D^2(f)$ is smooth and $D^3(f) = \emptyset$. Given a singular point of f, $(x_0, x_0) \in D^2(f)$, we can assume without loss of generality that $x_0 = 0$. Let k be the corank of f and assume that after a linear change of coordinates the germ of f at 0 is given by

$$f(x_1, \dots, x_n) = (x_1, \dots, x_{n-k}, g_1(x), \dots, g_{n+k-1}(x)),$$

with $g_i \in m_n^2$. Then for each $i = 1, \dots, n+k-1$ we can write

$$g_i(x) - g_i(x') = \sum_{j=n-k+1}^{n} \beta_{ij}(x, x')(x_j - x_j'),$$

with $\beta_{ij} \in m_{2n}$. Thus, it follows that $\mathcal{I}^2(f)$ is generated at $(0,0)$ by $x_j - x_j'$, $j = 1, \dots, n-k$, the function germs $g_i(x) - g_i(x')$, $i = 1, \dots, n+k-1$, and the $k \times k$ minors of the matrix $\beta = [\beta_{ij}]$. If $D^2(f)$ is smooth at $(0,0)$, this implies necessarily that $k = 1$ and the result is a consequence of [19]. □

Now we use this proposition in order to characterize finitely determined map germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$. Again, next result has been showed in [19] for map germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with $n \le p$ and corank 1.

Proposition 2.3. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$ be a holomorphic map germ, where $n \ge 3$. Then f is finitely determined if and only if $D^2(f)$ is a germ of reduced curve and $D^3(f) = \emptyset$.*

Proof. By the Mather-Gaffney criterion for finite determinacy [23], f is finitely determined if and only if there is a representative $f : U \to \mathbb{C}^{2n-1}$ defined on some open neighbourhood U of 0 in $\mathbb{C}^n$ such that $f : U \setminus \{0\} \to \mathbb{C}^{2n-1}$ is stable and $f^{-1}(0) = \{0\}$. By Proposition 2.2, we have that $D^3(f) = \emptyset$ and $D^2(f) \setminus \{0\}$ is a smooth curve in U. Thus, $D^2(f)$ has dimension 1 and is Cohen-Macaulay by Lemma 2.1. In particular, $D^2(f)$ is pure dimensional and hence reduced.

Conversely, suppose that there is a representative $f : U \to \mathbb{C}^{2n-1}$ defined on some open neighbourhood U of 0 in $\mathbb{C}^n$ such that $D^2(f)$ is a reduced curve and $D^3(f) = \emptyset$. By shrinking the neighbourhood U if necessary we can assume that $D^2(f) \setminus \{0\}$ is a smooth curve and $f^{-1}(0) = \{0\}$. Thus, $f : U \setminus \{0\} \to \mathbb{C}^{2n-1}$ is stable by Proposition 2.2 and $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$ is finitely determined. □

We consider now the image of $D^2(f)$ through the projection $p_1 : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ onto the first factor. If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$ is finite, then the restriction of p_1 to $D^2(f)$ is also finite and hence, the image $D(f) = p_1(D^2(f)) \subset \mathbb{C}^n$ is a germ of analytic set. However, we need to take some appropriate analytic structure so that it behaves well under deformation. We define

$$D(f) = V(\mathcal{F}_0(p_{1*}\mathcal{O}_{D^2(f)})),$$

where $\mathcal{F}_0(p_{1*}\mathcal{O}_{D^2(f)})$ is the 0th Fitting ideal sheaf of the $\mathcal{O}_{\mathbb{C}^n}$-module $p_{1*}\mathcal{O}_{D^2(f)}$. The underlying set germ of $D(f)$ is obviously $p_1(D^2(f))$ but, in general, it is not reduced and may have embedded points even in the case that f is finitely determined.

Corollary 2.4. *Let $U \subset \mathbb{C}^n$ be an open subset with $n \ge 3$.*

(1) *A holomorphic map $f : U \to \mathbb{C}^{2n-1}$ is stable if and only if $D(f)$ is a smooth curve.*
(2) *A holomorphic map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$ is finitely determined if and only if $D(f)$ is the germ of a curve with isolated singularity.*

Proof. The first part follows from Proposition 2.2. In fact, if f is stable, then $D^2(f)$ is a smooth curve and $D^3(f)=\emptyset$, which implies that $p_1 : D^2(f) \to D(f)$ is one-to-one. Moreover, by taking normal forms is obvious that p_1 is also regular, so that it is in fact biholomorphic and hence, $D(f)$ is also smooth. Conversely, if $D(f)$ is smooth, then $D^2(f)$ is also smooth and $p_1 : D^2(f) \to D(f)$ is one-to-one. This implies that $D^3(f)=\emptyset$ and f is stable.

The second part follows from Proposition 2.3 by using similar arguments. □

To complete the setup, we define two more double point schemes. Assume that G is a finite group which acts linearly on $\mathbb{C}^N$. This action induces an analytic structure on the quotient $\mathbb{C}^N/G$ so that the local ring at a point $z \in \mathbb{C}^N$ is given by

$$\mathcal{O}^G_{N,z} = \{h \in \mathcal{O}_{N,z} : gh = h, \forall g \in G\}.$$

Assume now that $I \subset \mathcal{O}_{N,z}$ is a G-invariant ideal. Then G acts also on the germ of analytic set $X = V(I) \subset (\mathbb{C}^N, z)$ and gives again an analytic structure on X/G with local ring

$$\mathcal{O}^G_X = \{h \in \mathcal{O}_X : gh = h, \forall g \in G\},$$

where $\mathcal{O}_X = \mathcal{O}_{N,z}/I$, in such a way that X/G embeds naturally in $(\mathbb{C}^N/G, z)$. If I is generated by G-invariant functions $a_1, \dots, a_r \in \mathcal{O}_{N,z}$, then

$$\mathcal{O}^G_X \equiv \mathcal{O}^G_X/I^G,$$

where I^G is the ideal in $\mathcal{O}^G_{N,z}$ generated by the same functions $a_1, \dots, a_r$. Since $\mathcal{O}^G_X$ is in fact a subring of $\mathcal{O}_X$, we have that if X is reduced, then X/G is also reduced.

In our case, if f is a holomorphic map or map germ from $\mathbb{C}^n$ to $\mathbb{C}^p$, with $n \le p$, then the double point ideal $\mathcal{I}^2(f)$ is S_2-invariant, where we consider the action of the group S_2 on $\mathbb{C}^n \times \mathbb{C}^n$ given by $\tau(x,x') = (x',x)$. In this way, we can define the quotient complex space or complex space germ $D^2(f)/S_2$.

If $f : U \subset \mathbb{C}^n \to \mathbb{C}^{2n-1}$ is stable, then $D^2(f)/S_2$ is a smooth curve and the quotient map $\pi : D^2(f) \to D^2(f)/S_2$ is a 2-fold branched covering, where the branching points correspond to the Whitney umbrellas of f and have order 2. For a finitely determined map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$, $D^2(f)/S_2$ is the germ of a reduced curve and $\pi : D^2(f) \to D^2(f)/S_2$ is a finite map germ, which is generically 2-1.

Finally, we also define $f(D(f)) \subset \mathbb{C}^{2n-1}$ by using Fitting ideals as we did with $D(f)$. If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$ is a finite map, then the induced map $\tilde{f}$ from $D^2(f)/S_2$ to $\mathbb{C}^{2n-1}$ is also finite. We set

$$f(D(f)) = V(\mathcal{F}_0(\tilde{f}_* \mathcal{O}_{D^2(f)/S_2})),$$

where $\mathcal{F}_0(\tilde{f}_* \mathcal{O}_{D^2(f)/S_2})$ is the 0th Fitting ideal sheaf of the $\mathcal{O}_{\mathbb{C}^{2n-1}}$-module $\tilde{f}_* \mathcal{O}_{D^2(f)/S_2}$.

If $f : U \subset \mathbb{C}^n \to \mathbb{C}^{2n-1}$ is stable and $n \ge 3$, then $f(D(f))$ is a smooth curve and the induced map $\tilde{f} : D^2(f)/S_2 \to f(D(f))$ is biholomorphic. For a finitely determined map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$, $f(D(f))$ is the germ of a curve with isolated singularity (possibly with embedded points) and $\tilde{f} : D^2(f)/S_2 \to f(D(f))$ is 1-1.

We have the following commutative diagram

$$\begin{array}{ccc} D^2(f) & \longrightarrow & D^2(f)/S_2 \\ \downarrow & & \downarrow \\ D(f) & \longrightarrow & f(D(f)), \end{array} \tag{1}$$

where the rows are 1-1. In the case that $f : U \to \mathbb{C}^{2n-1}$ is stable, then the four curves are smooth and the rows are in fact biholomorphic.

Definition 2.5. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$ be a finitely determined map germ. A 1-parameter *unfolding* of f is a map germ $F : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C} \times \mathbb{C}^{2n-1}, 0)$ of the form $F(t, x) = (t, f_t(x))$ such that $f_0 = f$. We say that an unfolding F is a *stabilization* of f if there is a representative $F : D \times U \to D \times \mathbb{C}^{2n-1}$, where D, U are open neighbourhoods of 0 in $\mathbb{C}$, $\mathbb{C}^n$ respectively such that $f_t : U \to \mathbb{C}^{2n-1}$ is stable for any $t \in D \setminus \{0\}$.

Since we are in the range of nice dimensions in the sense of Mather, it is well known that a stabilization of a finitely determined map germ always exist.

Given an unfolding F of f, we can also define the double point locus $D^2(F)$ which is considered in $(\mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n, 0)$ instead of $(\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}, 0)$ and the other complex space germs $D^2(F)/S_2$ in $(\mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n/S_2, 0)$, $D(F)$ in $(\mathbb{C} \times \mathbb{C}^n, 0)$ and $F(D(F))$ in $(\mathbb{C} \times \mathbb{C}^{2n-1}, 0)$.

Proposition 2.6. *Let $F : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C} \times \mathbb{C}^{2n-1}, 0)$ be an unfolding of a finitely determined map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$. Let $\mathfrak{X}$ be one of the double point germs $D^2(F)$, $D^2(F)/S_2$, $D(F)$ or $F(D(F))$ and let X be the corresponding germ $D^2(f)$, $D^2(f)/S_2$, $D(f)$ or $f(D(f))$, respectively. Then, $\pi : \mathfrak{X} \to (\mathbb{C}, 0)$ is a flat deformation of X. Moreover, if F is a stabilization, then it is a smoothing.*

Proof. Note that the second part of the proposition follows from the first one, since for each $t \neq 0$, in all the cases $X_t = \pi^{-1}(t)$ is smooth if f_t is stable.

Let us show that $\pi : D^2(F) \to (\mathbb{C}, 0)$ is a flat deformation of $D^2(f)$. Note that $D^2(F)$ is reduced and has pure dimension 2 and $D^2(f)$ is also reduced and has pure dimension 1 (since both are Cohen-Macaulay). We only need to show that $D^2(f) = D^2(F) \cap \{t = 0\}$ as germs of complex spaces. This is obvious, since the defining ideal of $D^2(F)$ is

$$\mathcal{I}^2(F) = (f_t \times f_t)^* \mathcal{I}_p + \mathcal{R}_n(\alpha_t),$$

which restricted to $t = 0$ gives exactly $\mathcal{I}^2(f)$, the defining ideal of $D^2(f)$.

The proof for $D^2(F)/S_2$ is analogous. Suppose that $\mathcal{I}^2(F)$ is generated by S_2-invariant functions $H_1, \dots, H_r \in \mathcal{O}_{2n+1}$. Then $h_1, \dots, h_r \in \mathcal{O}_{2n}$ are also S_2-invariant, where $h_i(x, x') = H_i(0, x, x')$, and generate $\mathcal{I}^2(f)$. Hence, $\mathcal{I}^2(F)^{S_2}$ restricted to $t = 0$ is equal to $\mathcal{I}^2(f)^{S_2}$. Finally, the flatness for $D(F)$ and $F(D(F))$ follows from the properties of the Fitting ideals. $\square$

We recall here the definition and main properties of the Milnor number of a space curve germ with isolated singularity, which generalizes the notion of Milnor number of a plane curve. For full details and proofs we refer to [3] in the reduced case, and [6] if the curve has embedded components.

Definition 2.7. Let X be a space curve germ in $(\mathbb{C}^m, 0)$ with isolated singularity. If X is reduced, we define the *Milnor number* as

$$\mu(X) = \dim_{\mathbb{C}} \operatorname{coker}(\mathrm{cl} \circ d),$$

where $d : \mathcal{O}_X \to \Omega_X$ is the differential operator, Ω_X is the module of holomorphic 1-forms on X, $\mathrm{cl} : \Omega_X \to \omega_X$ is the class map and ω_X is the dualizing module of Grothendieck.

If X is not reduced, we can write $X = X^{\mathrm{red}} \cup E$, where X^{red} is the reduced curve germ given by the 1-dimensional part of X, and E is the 0-dimensional part which contains the embedded points of X. We define the *Milnor number* as

$$\mu(X) = \mu(X^{\mathrm{red}}) - 2\varepsilon(X),$$

where $\varepsilon(X) = \dim_{\mathbb{C}} N$ and N is the nilradical of $\mathcal{O}_X$.

In the case that X is smoothable (which will be always our case) the Milnor number can be seen as the number which measures the *vanishing cohomology* of X. More exactly, let $\pi : \mathfrak{X} \to D$ be a good representative of a smoothing of X. Then, for $t \in D \setminus \{0\}$,

$$\mu(X) = 1 - \chi(X_t),$$

where $X_t = \pi^{-1}(t)$ and $\chi(X_t)$ is the Euler characteristic of X_t.

If X is reduced, it follows that X_t is connected, so that

$$\mu(X) = \dim_{\mathbb{C}} H^1(X_t; \mathbb{C}) \geq 0,$$

for any $t \neq 0$. Moreover, $\mu(X) = 0$ if and only if X is smooth. In the non reduced case, this is not true in general and we can have negative Milnor number.

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$ be a finitely determined map germ. Then the double point curves $D^2(f), D^2(f)/S_2, D(f), f(D(f))$ have isolated singularity and we can consider their Milnor numbers, $\mu(D^2(f)), \mu(D^2(f)/S_2), \mu(D(f)), \mu(f(D(f)))$, which in fact give $\mathcal{A}$-invariants of f. In next definition, we recall the construction of one more invariant that we will need.

Definition 2.8. Let $F : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C} \times \mathbb{C}^{2n-1}, 0)$ be a stabilization of a finitely determined map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$. We consider a small enough representative $F : D \times U \to D \times \mathbb{C}^{2n-1}$, where D, U are open neighbourhoods of 0 in $\mathbb{C}$, $\mathbb{C}^n$ respectively such that:

(1) For any $t \in D \setminus \{0\}$, $f_t : U \to \mathbb{C}^{2n-1}$ is stable .
(2) For $t = 0$, $f : U \setminus \{0\} \to \mathbb{C}^{2n-1}$ is a stable immersion and $f^{-1}(0) = \{0\}$.

Given $t \in D \setminus \{0\}$, we define

$$C(f) = \text{ number of singularities of } f_t \text{ in } U,$$

Since f_t and f_s are $\mathcal{A}$-equivalent for $t, s \in D \setminus \{0\}$, it is clear that $C(f)$ does not depend on the parameter t. Moreover, it is well known it does not depend also on the stabilization F. In fact, this number can be computed algebraically as

$$C(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{J_n(f)},$$

where $J_n(f)$ is the ideal generated by the $n \times n$ minors of the jacobian matrix of f (see [17] for the case $n = 2$ or [20] for the general case).

Theorem 2.9. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$ be a finitely determined map germ, with $n \geq 3$. Then, $\mu(D^2(f)) = \mu(D(f))$, $\mu(D^2(f)/S_2) = \mu(f(D(f)))$ and*

$$\mu(D(f)) = 2\mu(f(D(f))) + C(f) - 1.$$

Proof. Let $F : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C} \times \mathbb{C}^{2n-1}, 0)$ be a stabilization of f and consider a small enough representative as in previous definition and so that $\pi : \mathfrak{X} \to D$ is a good representative for $\mathfrak{X}$ being any of the four double point spaces. Given $t \in D \setminus \{0\}$, we have for f_t the commutative diagram (1), where the four curves $D^2(f_t)$, $D^2(f_t)/S_2$, $D(f_t)$ and $f_t(D(f_t))$ are smooth and the rows are biholomorphic. Hence, $\chi(D^2(f_t)) = \chi(D(f_t))$ and $\chi(D^2(f_t)/S_2) = \chi(f_t(D(f_t)))$.

Moreover, $f : D(f_t) \to f_t(D(f_t))$ is a 2-fold branched covering, where the branching points correspond to the Whitney umbrellas of f_t and have order 2. Thus, the Riemann-Hurwitz formula gives

$$\chi(D(f_t)) = 2\chi(f_t(D(f_t))) + C(f),$$

and the result follows from $\chi(X_t) = 1 - \mu(X)$. □

Remark 2.10. When $n = 2$, the set of triple points of a stable map $f : U \subset \mathbb{C}^2 \to \mathbb{C}^3$ has dimension 0 and for a finitely determined map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ we have that $D^3(f) = \{0\}$. This gives another invariant, namely, the number $T(f)$ of triple points that appear in a stabilization of f. The above formulas have to be adapted in this case taking into account this number (see [17] or [19] for details).

3. μ-Constant unfoldings

In this and next sections, we will assume that all the unfoldings $F : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C} \times \mathbb{C}^{2n-1}, 0)$ are origin preserving, which means that $f_t(0) = 0$ for any t and thus, we have a 1-parameter family of map germs $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$.

Definition 3.1. Let $F : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C} \times \mathbb{C}^{2n-1}, 0)$ be an unfolding of a finitely determined map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$. We will say that F is *μ-constant* if $\mu(D(f_t))$ is constant.

Note that the five invariants $\mu(D^2(f_t))$, $\mu(D^2(f_t)/S_2)$, $\mu(D(f_t))$, $\mu(f_t(D(f_t)))$ and $C(f_t)$ are upper semi-continuous (this is true for the Milnor number of a family of reduced curves [3] and for $C(f)$ it is showed in [8]). As a consequence of this fact, together with the formulas in Theorem 2.9, it follows that for $n \geq 3$, they are equivalent:

(1) $\mu(D(f_t))$ is constant,
(2) $\mu(D^2(f_t))$ is constant,
(3) both $\mu(f_t(D(f_t)))$ and $C(f_t)$ are constant,
(4) both $\mu(D^2(f_t)/S_2)$ and $C(f_t)$ are constant.

Definition 3.2. Let $F : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C} \times \mathbb{C}^{2n-1}, 0)$ be an unfolding of a finitely determined map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$. We say that F is *topologically trivial* if there are homeomorphism map germs:

$$\Phi : (\mathbb{C} \times \mathbb{C}^n, (0,0)) \to (\mathbb{C} \times \mathbb{C}^n, (0,0)), \qquad \Phi(t,x) = (t, \phi_t(x)),$$
$$\Psi : (\mathbb{C} \times \mathbb{C}^{2n-1}, (0,0)) \to (\mathbb{C} \times \mathbb{C}^{2n-1}, (0,0)), \qquad \Psi(t,y) = (t, \psi_t(y)),$$

such that $F = \Psi \circ G \circ \phi$, where $G(t,x) = (t, f(x))$ is the trivial unfolding of f.

Since double points are preserved under homeomorphism, it follows that if F is topologically trivial, then $\pi : D^2(F) \to (\mathbb{C}, 0)$ is a topologically trivial deformation of $D^2(f)$, and hence, the Milnor number $\mu(D^2(f_t))$ is constant (see [3]). In particular, $C(f)$ is also a topological invariant of the family. This gives the following immediate consequence.

Corollary 3.3. *Let $F : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C} \times \mathbb{C}^{2n-1}, 0)$ be an unfolding of a finitely determined map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$. If F is topologically trivial, then it is μ-constant.*

We recall now from [8] the definitions of good and excellent unfoldings, where they are introduced, in general, for map germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. The importance of this definition is that they are necessary in order to have Whitney equisingularity.

Definition 3.4. Let $F : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C} \times \mathbb{C}^{2n-1}, 0)$ be an unfolding of a finitely determined map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$. We say that F is *good* if there is a small enough representative $F : D \times U \to D \times \mathbb{C}^{2n-1}$, where D, U are open neighbourhoods of 0 in $\mathbb{C}$, $\mathbb{C}^n$ respectively, such that $f_t^{-1}(0) = \{0\}$ and $f_t : U \setminus \{0\} \to \mathbb{C}^{2n-1}$ is stable for any $t \in D$.

We say that F is *excellent* if it is good and $C(f_t)$ is constant. This condition guarantees there is no coalescing of cross-caps in the family f_t. With the notation of the above paragraph, this means that F is excellent if and only if there is a representative $F : D \times U \to D \times \mathbb{C}^{2n-1}$ such that $f_t^{-1}(0) = \{0\}$ and $f_t : U \setminus \{0\} \to \mathbb{C}^{2n-1}$ is an immersion with only transverse double points, for any $t \in D$.

Theorem 3.5. *Let $F : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C} \times \mathbb{C}^{2n-1}, 0)$ be an unfolding of a finitely determined map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$ with $n \geq 3$. If F is μ-constant, then it is excellent.*

Proof. We choose a small enough representative $F : D \times U \to D \times \mathbb{C}^{2n-1}$, where D, U are open neighbourhoods of 0 in $\mathbb{C}$, $\mathbb{C}^n$ respectively such that $\pi : D^2(F) \to D$ and $\pi : D(F) \to D$ are good representatives. We denote

$$\mu_t = \sum_{x \in D(f_t)} \mu(D(f_t), x), \quad \tilde{\mu}_t = \sum_{(x,x') \in D^2(f_t)} \mu(D^2(f_t), (x, x')),$$

and

$$\chi_t = \chi(D(f_t)), \quad \tilde{\chi}_t = \chi(D^2(f_t)).$$

Then it follows (see [3] and [6]) that

$$\mu_0 - \mu_t = 1 - \chi_t, \quad \tilde{\mu}_0 - \tilde{\mu}_t = 1 - \tilde{\chi}_t.$$

Since $D^2(f)$ is reduced, $D^2(f_t)$ is connected and thus

$$\tilde{\mu}_0 - \tilde{\mu}_t = 1 - \tilde{\chi}_t = \dim_{\mathbb{C}} H^1(D^2(f_t); \mathbb{C}) \geq 0.$$

Therefore,

$$\mu(D^2(f), (0,0)) = \tilde{\mu}_0 \geq \tilde{\mu}_t = \sum_{(x,x') \in D^2(f_t)} \mu(D^2(f_t), (x, x')) \geq \mu(D^2(f_t), (0,0)),$$

and the constancy of this number implies that $D^2(f_t) \setminus \{(0,0)\}$ is smooth (since any other singular point should have Milnor number ≥ 1).

Note also that $f_t^{-1}(0) = \{0\}$ in U, for any $t \in D$. In fact, let $x \in U$ be such that $f_t(x) = 0$. Then $(x, 0)$ is a singular point of $D^2(f_t)$, and hence, $x = 0$. To finish the proof, we only need to show that $D^3(f_t) = \emptyset$, or equivalently, that $D(f_t) \setminus \{0\}$ is also smooth.

We use the same argument as before. Note that $D(f)$ is not reduced, but since $D(f_t) = p_1(D^2(f_t))$ it is also connected (by the continuity of p_1). Hence,

$$\mu_0 - \mu_t = 1 - \chi_t = \dim_{\mathbb{C}} H^1(D(f_t); \mathbb{C}) \geq 0$$

and

$$\mu(D(f), 0) = \mu_0 \geq \mu_t = \sum_{x \in D(f_t)} \mu(D(f_t), x).$$

Again this sum includes the Milnor number at the origin $\mu(D(f_t), 0) = \mu(D(f), 0)$ and we just need to show that any other singular point $x \in D(f_t)$, $x \neq 0$, should have Milnor number ≥ 1.

Since $D^2(f_t) \setminus \{(0,0)\}$ is smooth and p_1 is regular on this set, the only possibility for x to be a singular point of $D(f_t)$ is that $p_1^{-1}(x) = \{x_1, \ldots, x_k\}$, with $k \geq 2$. But this implies that the germ of $D(f_t)$ at x is a union of k smooth curves and hence, the Milnor number is always ≥ 1. □

Another consequence of the theorem is that if $\mu(D(f_t))$ is constant, then the deformations $D^2(f_t)$ and $D^2(f_t)/S_2$ are topologically trivial [3]. This should suggest the following stronger result.

Conjecture 3.6. *Let $F : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C} \times \mathbb{C}^{2n-1}, 0)$ be an unfolding of a finitely determined map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$. If F is μ-constant, then it is topologically trivial.*

Remark 3.7. This conjecture has been proved recently by Bedregal, Houston and Ruas [1], in the case of a map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ of corank 1. There is also a proof by Damon [7] in the particular case that f has type $\Sigma^{1,0}$, i.e., $f(x, y, z) = (x, y^2, yp(x, y^2))$, where $p : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ defines a curve with isolated singularity.

4. Polar multiplicities and Whitney equisingularity

In this section we recall from [8] the notion of Whitney equisingular unfolding $F : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C} \times \mathbb{C}^p, 0)$ and the theorem which characterizes it in terms of the polar multiplicities of the strata defined by stable types.

Definition 4.1. Let $f : \mathbb{C}^n \to \mathbb{C}^p$ be a complex analytic map and let $A \subset \mathbb{C}^n$, $A' \subset \mathbb{C}^p$ be subsets such that $f(A) \subset A'$. A *stratification* of $f : A \to A'$ is a pair $(\mathcal{A}, \mathcal{A}')$ of stratifications of A, A' respectively such that f maps strata submersively to strata.

The stratification $(\mathcal{A}, \mathcal{A}')$ is said to be *regular* if $\mathcal{A}, \mathcal{A}'$ satisfy the Whitney regularity conditions and if any stratum $Y \in \mathcal{A}$ satisfies Thom's condition A_f over any other stratum $X \in \mathcal{A}$.

Definition 4.2. Let $F : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C} \times \mathbb{C}^p, 0)$ be an unfolding of a finitely determined map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. We say that F is *Whitney equisingular* if there is a representative of F which admits a regular stratification so that the parameter axes $S = \mathbb{C} \times \{0\} \subset \mathbb{C} \times \mathbb{C}^n$ and $T = \mathbb{C} \times \{0\} \subset \mathbb{C} \times \mathbb{C}^p$ are strata.

By using an appropriate version of Thom's second isotopy lemma for complex analytic maps, it follows that any Whitney equisingular unfolding is topologically trivial (see [8] for details).

If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is a finitely determined germ of discrete stable type, then we have a natural stratification of a representative of f by stable types. The same can be done if $F : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C} \times \mathbb{C}^p, 0)$ is a good unfolding of f. Moreover, the parameter axes S and T are strata and the stratification is regular outside S, T. In order to control the regularity of the strata with respect to S, T, Gaffney uses the polar multiplicities of the strata, a notion introduced by Teissier [21] in the study of Whitney equisingularity of hypersurfaces.

Definition 4.3. Let $\pi : (X, x) \to (\mathbb{C}, 0)$ be a flat deformation with a section σ such that the fibres are of pure dimension $d \geq 1$ and reduced except perhaps at $\sigma(t)$, $t \in \mathbb{C}$. We take an embedding $X \subset \mathbb{C} \times \mathbb{C}^N$ of a representative of (X, x) so that π is the restriction of the projection onto the first factor. For each k with $0 \leq k \leq d - 1$, we choose a generic linear projection p: $\mathbb{C}^N \to \mathbb{C}^{d-k+1}$. We define the *relative polar variety* $P_k(X, \pi)$ as the closure in X of the critical locus of the restriction $(\pi, p)|_{X^0}$ of (π, p) to the nonsingular part X^0 of X. This is an analytic subset of X of pure codimension k or empty.

The *relative polar multiplicity*, denoted by $m_k(X, \pi)$, is the multiplicity at x of $P_k(X, \pi)$. It follows that $m_k(X, \pi)$ is an invariant of $\pi : (X, x) \to (\mathbb{C}, 0)$ which is independent of the choice of the generic linear projection p and of the embedding.

Finally, let (X,x) be a complex space germ which is of pure dimension and reduced except perhaps at x. Then we define the (absolute) *polar varieties* and *polar multiplicities*, denoted by $P_k(X)$ and $m_k(X)$ respectively, by taking the trivial deformation $\pi : (\mathbb{C}\times X,(0,x)) \to (\mathbb{C},0)$. In this case, $P_k(X)$ is defined as $\overline{\Sigma(p|_{X^0})}$ so that it is a subset of X instead of $\mathbb{C}\times X$.

Remark 4.4. (1) The original statement by Teissier [21] assumes that the fibres of π are of pure dimension and reduced everywhere, but the construction can be done also in our case (see [8]).

(2) Note that when $k=0$, $P_0(X)=X$ and $m_0(X)$ is the usual multiplicity. However, when $k=d$, the polar multiplicity has no sense, since $P_k(X)$ is 0-dimensional. In the relative case, $P_d(X,\pi)$ is 1-dimensional and we can define the dth *polar multiplicity*, $m_d(X,\pi)$, as the local degree at x of the restriction $\pi|_{P_d(X,\pi)} : P_d(X,\pi) \to \mathbb{C}$.

(3) In order to define polar varieties and multiplicities, it is possible to choose a generic family of linear forms $\ell_1,\dots,\ell_{d+1} : \mathbb{C}^N \to \mathbb{C}$, such that if $p_k = (\ell_1,\dots,\ell_{d-k+1})$, then $P_k(X)=\overline{\Sigma(p_k|_{X^0})}$ and $m_k(X)$ is the local degree at x of $p_{k+1}|_{P_k(X)} : P_k(X) \to \mathbb{C}^{n-k}$.

We apply now this polar varieties and multiplicities to the strata of the stratification defined by stable types in a finitely determined map germ of discrete stable type $f : (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$. We refer to [8] for full details.

Definition 4.5. Let $f:(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ be a finitely determined map germ of discrete stable type and let X be one of the strata induced by stable types either in the source $\mathbb{C}^n$ or the target $\mathbb{C}^p$. The closure $\overline{X}$ has an analytic structure in such a way that it is reduced and of pure dimension except perhaps at 0, and it is well behaved under deformation. Thus, if X has dimension $d\geq 1$, we can consider the *polar multiplicities* $m_k(\overline{X})$, for $k=0,\dots,d-1$.

Moreover, we take a stabilization $F:(\mathbb{C}\times\mathbb{C}^n,0)\to(\mathbb{C}\times\mathbb{C}^p,0)$ of f and denote by Y the corresponding stratum in F defined by the same stable type. The restriction of the projection onto the first factor $\pi : (\overline{Y},0)\to(\mathbb{C},0)$ gives a flat deformation of $\overline{X}$. Then we define the dth *stable multiplicity* as $m_d(\overline{X}) = m_d(\overline{Y},\pi)$. It follows that $m_d(\overline{X})$ is also an invariant of f, which does not depend on the stabilization F.

We are able now to state the main theorem of [8], which characterizes Whitney equisingularity in terms of the constancy of all the polar multiplicities (including the dth stable multiplicity) of all the strata defined by stable types. For the definitions of good and excellent unfolding in the general case see [8].

Theorem 4.6. *Suppose that $F:(\mathbb{C}\times\mathbb{C}^n,0)\to(\mathbb{C}\times\mathbb{C}^p,0)$ is a good unfolding of a finitely determined map germ $f:(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ with (n,p) in the range of nice dimensions. Then F is Whitney equisingular if and only if it is excellent and the polar and stable multiplicities of all the strata of the stratification given by stable types are constant.*

Let us return now to our case $p=2n-1$, with $n\geq 3$. Given a finitely determined map germ $f:(\mathbb{C}^n,0)\to(\mathbb{C}^{2n-1},0)$, we take a representative $f:U\to\mathbb{C}^{2n-1}$ such that $f^{-1}(0)=\{0\}$ and $f:U\setminus\{0\}\to\mathbb{C}^{2n-1}$ is a stable immersion. The stratification induced by stable types has the following strata: In the source, we have $U\setminus D(f)$, $D(f)\setminus\{0\}$ and $\{0\}$. In the target, we have $\mathbb{C}^{2n-1}\setminus f(U)$, $f(U)\setminus f(D(f))$, $f(D(f))\setminus\{0\}$ and $\{0\}$.

Thus, in order to apply the above theorem, we have to control $n+6$ invariants: $m_0(D(f))$, $m_1(D(f))$, $m_0(f(D(f)))$, $m_1(f(D(f)))$, $C(f)$ and $m_i(f(\mathbb{C}^n))$, with $0 \leq i \leq n$. Moreover, we need that the unfolding is good, which will be controlled by the μ-constant condition. In the remainder of this section, we will see that some of these invariants are related, so that it is possible to reduce the number of invariants. In fact, we will obtain the corresponding formulas to the ones that appear in [8] for the case $n=2$ and $p=3$.

We will use the following formula due to Lê and Greuel [5, 13] which gives the Milnor number of an ICIS. Let us introduce some notation. Let $f:(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ be a complex analytic map germ, then $I(f)$ is the ideal in $\mathcal{O}_n$ generated by the components of f. We also denote by $J(f)$ the ideal in $\mathcal{O}_n$ generated by the $p\times p$ minors of the jacobian matrix of f, if $p\leq n$, or $\{0\}$ otherwise.

Theorem 4.7. *Let $f_1,\dots,f_k\in\mathcal{O}_n$ and assume that $X_1=V(f_1,\dots,f_{k-1})$ and $X=V(f_1,\dots,f_k)$ are ICIS. Then,*

$$\mu(X)+\mu(X_1)=\dim_{\mathbb{C}}\frac{\mathcal{O}_n}{\langle f_1,\dots,f_{k-1}\rangle+J(f_1,\dots,f_k)}.$$

Remark 4.8. In the case that $X=V(f_1,\dots,f_n)$ is a 0-dimensional ICIS, we have

$$\mu(X)=\dim_{\mathbb{C}}\frac{\mathcal{O}_n}{\langle f_1,\dots,f_n\rangle}-1.$$

Theorem 4.9. *Let $f:(\mathbb{C}^n,0)\to(\mathbb{C}^{2n-1},0)$ be a finitely determined map germ. Then*

$$\sum_{i=0}^{n}(-1)^i m_i(f(\mathbb{C}^n))=1.$$

Proof. Choose a generic family of linear forms $\ell_1,\dots,\ell_{n+1}:\mathbb{C}^{2n-1}\to\mathbb{C}$, such that if $p_k=(\ell_1,\dots,\ell_{n-k+1})$, then $P_k(f(\mathbb{C}^n))=\overline{\Sigma(p_k|_{f(\mathbb{C}^n)\setminus f(D(f))})}$ and $m_k(f(\mathbb{C}^n))$ is the local degree at 0 of

$$p_{k+1}|_{P_k(f(\mathbb{C}^n))}:P_k(f(\mathbb{C}^n))\to\mathbb{C}^{n-k},$$

for $k=0,\dots,n-1$. Moreover, let $F:(\mathbb{C}\times\mathbb{C}^n,0)\to(\mathbb{C}\times\mathbb{C}^p,0)$ be a stabilization of f and assume that $P_n(F(\mathbb{C}^{n+1}),\pi)=\overline{\Sigma((\pi,p_n)|_{F(\mathbb{C}^{n+1})\setminus F(D(F))})}$.

For each $k=0,\dots,n-1$, we have that $f(V(J(p_k\circ f)))\subset P_k(f(\mathbb{C}^n))$ and that the restriction $f|_{V(J(p_k\circ f))}:V(J(p_k\circ f))\to P_k(f(\mathbb{C}^n))$ is generically 1-1. In particular,

$$m_k(f(\mathbb{C}^n))=\deg(p_{k+1}|_{P_k(f(\mathbb{C}^n))})=\deg((p_{k+1}\circ f)|_{V(J(p_k\circ f))}).$$

But the local ring $\mathcal{O}_n/J(p_k\circ f)$ is Cohen-Macaulay of dimension $n-k$ (in fact, it is a determinantal ring), and the classes of $\ell_1\circ f,\dots,\ell_{n-k}\circ f$ in $\mathcal{O}_n/J(p_k\circ f)$ are a system of parameters. Thus, the degree of $(p_{k+1}\circ f)|_{V(J(p_k\circ f)}$ is computed as

$$m_k(f(\mathbb{C}^n))=\dim_{\mathbb{C}}\frac{\mathcal{O}_n}{I(p_{k+1}\circ f)+J(p_k\circ f)}.$$

On the other hand, let $X_k=(p_k\circ f)^{-1}(0)$ for $k=1,\dots,n$. These sets are ICIS and the Lê-Greuel formula gives

$$\mu(X_k)+\mu(X_{k+1})=\dim_{\mathbb{C}}\frac{\mathcal{O}_n}{I(p_{k+1}\circ f)+J(p_k\circ f)}=m_k(f(\mathbb{C}^n)).$$

For $k=0$, we have that $J(p_k\circ f)=\{0\}$ and

$$m_0(f(\mathbb{C}^n))=\dim_{\mathbb{C}}\frac{\mathcal{O}_n}{I(p_1\circ f)}=1+\mu(X_1),$$

since X_1 is 0-dimensional.

For $k=n$, we use a similar argument. We have $F(V(J((\pi,p_n)\circ F)))\subset P_n(F(\mathbb{C}^{n+1}),\pi))$ and $F|_{V(J((\pi,p_n)\circ F))}: V(J((\pi,p_n)\circ F))\to P_n(F(\mathbb{C}^{n+1}),\pi))$ is generically 1-1. Hence,

$$m_n(f(\mathbb{C}^n))=\deg(\pi|_{P_n(F(\mathbb{C}^{n+1}),\pi))})=\deg((\pi\circ F)|_{V(J((\pi,p_n)\circ F))}).$$

Now the local ring $\mathcal{O}_{n+1}/J((\pi,p_n)\circ F)$ is also Cohen-Macaulay of dimension 1, and the class of t in $\mathcal{O}_{n+1}/J((\pi,p_n)\circ F)$ is regular. Then,

$$m_n(f(\mathbb{C}^n))=\dim_{\mathbb{C}}\frac{\mathcal{O}_{n+1}}{\langle t\rangle+J((\pi,p_n)\circ F)}=\dim_{\mathbb{C}}\frac{\mathcal{O}_n}{J(p_n\circ f)}=\mu(X_n).$$

Finally, we put the formulas altogether and obtain

$$\sum_{i=0}^{n}(-1)^i m_i(f(\mathbb{C}^n))=1+\mu(X_1)-\mu(X_1)+\mu(X_2)-\mu(X_2)+\cdots=1.$$

□

Remark 4.10. Let $f:(\mathbb{C}^n,0)\to(\mathbb{C}^{2n-1},0)$ be a finitely determined map germ. Then

$$m_0(f(\mathbb{C}^n))=\dim_{\mathbb{C}}\frac{\mathcal{O}_n}{I(p\circ f)}=e(I(f),\mathcal{O}_n),$$

where $p:\mathbb{C}^{2n-1}\to\mathbb{C}^n$ is a generic linear projection and $e(I,R)$ denotes the multiplicity of an ideal of definition I in a Noetherian local ring R. In the case that f has corank 1, we can choose coordinates so that $I(f)$ is generated by n elements in $\mathcal{O}_n$ and hence,

$$m_0(f(\mathbb{C}^n))=\dim_{\mathbb{C}}\frac{\mathcal{O}_n}{I(f)}.$$

Note also that in this case, we have $m_i(f(\mathbb{C}^n))=0$, $i=2,\dots,n$ and $m_0(f(\mathbb{C}^n))-1=m_1(f(\mathbb{C}^n))$.

If f has corank 2, then $m_i(f(\mathbb{C}^n))=0$, $i=3,\dots,n$ and $m_2(f(\mathbb{C}^n))+m_0(f(\mathbb{C}^n))-1=m_1(f(\mathbb{C}^n))$.

In order to study the polar multiplicities of $D(f)$ and $f(D(f))$, which are space curves, we will use the notion of Milnor number of a function on a space curve, as defined by Mond and van Straten [18].

Definition 4.11. Let X be a reduced space curve germ in $(\mathbb{C}^m,0)$, and let $f:X\to\mathbb{C}$ be a function on X which is non-constant on each branch of X. Then $\mathcal{O}_X$ is a finite and free $\mathcal{O}_{\mathbb{C}}$-module via f. The *Milnor number* of the function f is defined as

$$\mu(f)=\dim_{\mathbb{C}}\operatorname{coker}(\mathrm{cl}\circ df),$$

where $df:\mathcal{O}_X\to\Omega_X$ and $\mathrm{cl}:\Omega_X\to\omega_X$ is the class map. The main properties of this Milnor number are:

(1) It is preserved under simultaneous deformation of f and X.
(2) If X is smooth, it coincides with the usual Milnor number.

Note that if X is smoothable, then these two properties are enough in order to determine the Milnor number of any function. In fact, we have the following formula.

Lemma 4.12. *Let X be a reduced space curve germ in $(\mathbb{C}^m,0)$, and let $f:X\to\mathbb{C}$ be a finite function germ on X. If X is smoothable,*

$$\mu(f)=\mu(X)+\deg(f)-1.$$

Proof. Let $\pi : \mathfrak{X} \to D$ be a good representative of a stabilization of X and denote by $X_t = \pi^{-1}(t)$ and $f_t : X_t \to \mathbb{C}$ the corresponding deformation of f. For each $t \in D \setminus \{0\}$, X_t is smooth and the Riemann-Hurwitz formula gives

$$\chi(X_t) = \deg(f_t)\chi(\mathbb{C}) - \sum_{x \in X_t} (r(f_t, x) - 1),$$

where χ denotes the Euler characteristic and $r(f_t, x)$ is the ramification index of f_t at x. It follows that $r(f_t, x) - 1 = \mu(f_t, x)$ and the sum $\sum_{x \in X_t} \mu(f_t, x) = \mu(f)$, since μ is preserved under deformations. To conclude the formula, just note that $\chi(X_t) = 1 - \mu(X)$ and $\deg(f_t) = \deg(f)$. □

Remark 4.13.
(1) Note that when X is smooth then we have the obvious relation $\mu(f) = \deg(f) - 1$.
(2) In the case that X is a 1-dimensional ICIS, the above formula can be obtained easily as a consequence of the Lê-Greuel formula.
(3) In general, if X is a smoothable reduced space curve germ in $(\mathbb{C}^m, 0)$, then $m_1(\mathfrak{X}, \pi)$ does not depend on the smoothing $\pi : \mathfrak{X} \to \mathbb{C}$ and we denote it by $m_1(X)$. It follows that
$$m_1(X) = \mu(p|_X) = \mu(X) + m_0(X) - 1,$$
where $p : \mathbb{C}^m \to \mathbb{C}$ is a generic linear projection. Moreover, we can rewrite the above formula as
$$m_0(X) - m_1(X) = 1 - \mu(X) = \chi(X_t),$$
for $t \neq 0$.

Now we apply lema 4.12 in order to compute $m_1(D(f))$ and $m_1(f(D(f)))$.

Theorem 4.14. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$ be a finitely determined map germ. Then,*
(1) $m_1(D(f)) = m_0(D(f)) + \mu(D(f)) - 1$,
(2) $m_1(f(D(f))) = m_0(f(D(f))) + \mu(f(D(f))) - 1$.

Proof. We see that $m_1(D(f))$ is the Milnor number of $p \circ p_1|_{D^2(f)}$, where $p : \mathbb{C}^n \to \mathbb{C}$ is a generic linear projection. Hence, the first formula follows from lema 4.12 and theorem 2.9. In fact, let F be a stabilization of f and choose p in such a way that $P_1(D(F), \pi) = \overline{\Sigma((\pi, p)|_{D(F)^0})}$. We have that $m_1(D(f))$ is just the number of critical points of $p|_{D(f_t)}$ for $t \neq 0$. But since $p_1 : D^2(f_t) \to D(f_t)$ is biholomorphic, this is also equal to the number of critical points of $(p \circ p_1)|_{D^2(f_t)}$ for $t \neq 0$, which is nothing but $\mu(p \circ p_1)|_{D^2(f)}$.

For the second formula, we use a similar argument, by showing that $m_1(f(D(f)))$ is the Milnor number of $p \circ \tilde{f}|_{D^2(f)/S_2}$, where $p : \mathbb{C}^{2n-1} \to \mathbb{C}$ is a generic linear projection and $\tilde{f}$ is the induced map from $D^2(f)/S_2$ to $f(D(f))$. □

From the main Theorem of Gaffney, we need the constancy of $n + 6$ invariants to guarantee the Whitney equisingularity of an unfolding F of f. We reduce this number to $[n/2] + 3$.

Theorem 4.15. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$ be a finitely determined map germ, with $n \geq 3$ and let F be a 1-parameter unfolding of f. Then F is Whitney equisingular if and only if $\mu(D(f_t))$, $m_0(D(f_t))$, $m_0(f_t(D(f_t)))$ and $m_{2i-1}(f_t(\mathbb{C}^n))$, $i = 1, 2, \ldots$ are constant.*

Proof. By 4.9 and 4.14, the constancy of $\mu(D(f_t))$, $m_0(D(f_t))$, $m_0(f_t(D(f_t)))$ and the odd polar multiplicities $m_{2i-1}(f_t(\mathbb{C}^n))$ imply also the constancy of the remainder of invariants

$$C(f_t),\ m_1(f_t(D(f_t))),\ m_1(D(f_t)),\ m_{2i}(f_t(\mathbb{C}^n)),$$

for $i = 0, 1, \dots$ Moreover, the unfolding is excellent by 3.5 and F is Whitney equisingular by 4.6.

Conversely, if F is Whitney equisingular, then it is topologicaly trivial and it is μ-constant and hence excellent by 3.3 and 3.5. Thus, we can apply the convers of 4.6 and therefore, all the invariants are constant. □

In the case of corank one or two, we only need 3 or 4 invariants respectively for Whitney equisingularity.

Corollary 4.16. *Suppose $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$ is a finitely determined map germ of corank ≤ 2 and F is a 1-parameter unfolding of f. Then F is Whitney equisingular if and only if $\mu(D(f_t))$, $m_0(D(f_t))$ and $m_0(f_t(D(f_t)))$ are constant and either*

(1) *f has corank 1, or*

(2) *f has corank 2 and $m_1(f_t(\mathbb{C}^n))$ is also constant.*

Proof. The first part is a consequence of theorem 4.15 and remark 4.10. For the second part, we only need to show that if f has corank 1 and $\mu(D(f_t))$ is constant, then $m_0(f_t(\mathbb{C}^n))$ is also constant.

We can take coordinates such that $f_t(x, y) = (x, g_t(x, y))$, where $x \in \mathbb{C}^{n-1}$, $y \in \mathbb{C}$ and $g_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. Then

$$m_0(f_t(\mathbb{C}^n)) = m_0(h_t(\mathbb{C})) = \dim_{\mathbb{C}} \frac{\mathcal{O}_1}{I(h_t)},$$

where $h_t : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ is given by $h_t(y) = g_t(0, y)$. Since $\mu(D(f_t))$ is constant, the unfolding F is good by 3.5. In particular, there is an open neighbourhood U of 0 in $\mathbb{C}^n$ such that $f_t^{-1}(0) = \{0\}$ in U for any $t \in \mathbb{C}$. Then, there is an open neighbourhood U_1 of 0 in $\mathbb{C}$ such that $h_t^{-1}(0) = \{0\}$ in U_1 for any $t \in \mathbb{C}$. This implies that h_t has constant multiplicity at 0 and hence, $m_0(f_t(\mathbb{C}^n))$ is constant. □

Note that the constancy of $\mu(D(f_t))$, $m_0(D(f_t))$ and $m_0(f_t(D(f_t)))$ is equivalent to the Whitney equisingularity of the families of curves $D(f_t)$ and $f_t(D(f_t))$. When $n = 2$, $D(f_t)$ is a family of plane curves and the constancy of $m_0(D(f_t))$ follows from the constancy of $\mu(D(f_t))$. Therefore, this condition is not necessary (see [8]). However, for $n \geq 3$, the constancy of $\mu(D(f_t))$ does not imply the constancy of $m_0(D(f_t))$, as we will see in the example at the end of the paper.

Corollary 4.17. *Suppose $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$ is a finitely determined map germ and F is 1-parameter unfolding of f. If $m_0(D(f_t))$ is constant and $\pi : F(\mathbb{C} \times \mathbb{C}^n) \to \mathbb{C}$ is a Whitney equisingular deformation of $f(\mathbb{C}^n)$, then F is Whitney equisingular.*

Proof. Since $\pi : F(\mathbb{C} \times \mathbb{C}^n) \to \mathbb{C}$ is a Whitney equisingular deformation, it follows that $T = \mathbb{C} \times \{0\}$ is the only 1-dimensional stratum. Then, there is no coalescing of cross-caps in F and hence, $C(f_t)$ must be constant (see proposition 3.6 of [8]).

Now we have that $(F(\mathbb{C} \times \mathbb{C}^n)^0, T)$ satisfy the Whitney conditions and then $m_i(F(\mathbb{C} \times \mathbb{C}^n), \pi)_t$, for $i = 0, \dots, n-1$, are constant [21]. Moreover, by theorem 5.6 [8], $m_i(F(\mathbb{C} \times \mathbb{C}^n), \pi)_t = m_i(f_t(\mathbb{C}^n))$, $i = 0, \dots, n-1$.

Finally, note that $\pi : F(D(F)) \to \mathbb{C}$ is also a Whitney equisingular deformation of the curve $f(D(f))$. This is equivalent to the constancy of $\mu(f_t(D(f_t)))$ and $m_0(f_t(D(f_t)))$ (see [3]). By adding the condition that $m_0(D(f_t))$ is constant we get that F is Whitney equisingular by theorem 4.15. $\square$

In [8], Gaffney introduces a new invariant $e_D(f)$ for a map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ which has the property that $e_D(f)$ is constant in the family if and only if $m_0(f(D(f)))$ and $\mu(D(f))$ are constant. Then, it is possible to rewrite theorem 4.15 and corollary 4.16 in terms of this invariant. This can be done also in our case.

Definition 4.18. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$ be a finitely determined map germ and let $p : \mathbb{C}^{2n-1} \to \mathbb{C}$ be a generic linear projection so that the degree of $p|_{f(\mathbb{C}^n)}$ is $m_0(f(D(f)))$. We define

$$e_D(f) = \mu(p \circ f \circ p_1|_{D^2(f)}),$$

where $p_1 : \mathbb{C}^n \times \mathbb{C}^n$ is the projection onto the first factor. It will become clear from next proposition that $e_D(f)$ is an invariant of f which is independent of the chosen generic projection p.

Proposition 4.19. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$ be a finitely determined map germ, with $n \geq 3$. Then,*

$$e_D(f) = 2m_0(f(D(f))) + \mu(D(f)) - 1 = 2m_1(f(D(f))) + C(f).$$

Proof. If the degree of $p|_{f(D(f))}$ is $m_0(f(D(f)))$, then $p \circ f \circ p_1|_{D^2(f)}$ has degree $2m_0(f(D(f)))$ and the first formula is a consequence of lema 4.12 and theorem 2.9. The second equality follows also from 2.9 and 4.14. $\square$

We now apply our results for a finitely determined map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{2n-1}, 0)$ of type $\Sigma^{1,0}$. The assumption on the Boardman symbol implies that we can choose coordinates such that

$$f(x, y) = (x, y^2, yg(x, y^2)),$$

where $x \in \mathbb{C}^{n-1}$, $y \in \mathbb{C}$ and $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n-1}, 0)$. The computations of the numerical invariants in this case are very simple, since

$$D(f) = \{(x, y) : g(x, y^2) = 0\}, \quad f(D(f)) = \{(X, Y, Z) : Z = 0,\ g(X, Y) = 0\},$$

where $X \in \mathbb{C}^{n-1}$, $Y \in \mathbb{C}$ and $Z \in \mathbb{C}^{n-1}$. It follows that f is finitely determined if and only if $D(f)$ (and hence $f(D(f))$) is a 1-dimensional ICIS. Thus, it is possible to compute $\mu(D(f))$ by using Lê-Greuel formula (see 4.7). By remark 4.10, we also have $m_0(f(\mathbb{C}^2)) = 2$, $m_1(f(\mathbb{C}^2)) = 1$ and $m_i(f(\mathbb{C}^2)) = 0$, for $i = 2, \dots, n$.

We finish with the following example, which shows that, in general, a μ-constant unfolding is not Whitney equisingular. This example is inspired in one of the examples given in [3] of a family of space curves in $(\mathbb{C}^3, 0)$ with constant Milnor number and non constant multiplicity.

Example 4.20. Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^5, 0)$ be the map germ given by $f(x, y, z) = (x, y, z^2, z(xy), z(z^{12} + x^{15} + y^{10}))$ and consider the 1-parameter unfolding $F = (t, f_t)$, where

$$f_t(x, y, z) = (x, y, z^2, z(xy - tz^2), z(z^{12} + x^{15} + y^{10})).$$

We have that the double point curves of f_t are

$$D(f_t) = V(xy - tz^2, z^{12} + x^{15} + y^{10})$$

and

$$f_t(D(f_t)) = V(XY - tZ, Z^6 + X^{15} + Y^{10}, U, V).$$

We compute the Milnor number and multiplicities of these curves and obtain: $\mu(D(f_t)) = 276$ for all t, $m_0(D(f_0)) = 22$ and $m_0(D(f_t)) = 20$ for $t \neq 0$. Moreover, $\mu(f_t(D(f_t))) = 126$ for all t, $m_0(f_0(D(f_0))) = 12$ and $m_0(f_t(D(f_t))) = 10$ for $t \neq 0$.

Thus, F is μ constant but it is not Whitney equisingular. We have also that $C(f_t) = 25$ for all t, which also shows that the constancy of the zero-stable invariants does not imply Whitney equisingularity. Note also that f is weighted homogeneous and that the unfolding F only adds terms of same degree. This implies that F is topologically trivial by [7].

References

[1] Bedregal, R.; Houston, K.; Ruas, M.A.S., *Topological triviality of families of singular surfaces*, Preprint.

[2] Buchsbaum, D.A.; Eisenbud, D., *Generic free resolutions and a family of generically perfect ideals.* Advances in Math. **18** (1975), no. 3, 245–301.

[3] Buchweitz, R.O.; Greuel, G.M., *The Milnor number and deformations of complex curve singularities.* Invent. Math. **58** (1980), no. 3, 241–281.

[4] Briançon, J.; Speder, J.P., *Les conditions de Whitney impliquent "$\mu(^*)$ constant"*, Ann. Inst. Fourier (Grenoble) **26** (1976), no. 2, xi, 153–163.

[5] Brieskorn, E.; Greuel, G.M., *Singularities of complete intersections.* Manifolds—Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973), pp. 123–129. Univ. Tokyo Press, Tokyo, 1975.

[6] Brücker, C.; Greuel, G.M., *Deformationen isolierter Kurvensingularitäten mit eingebetteten Komponenten.* Manuscripta Math. **70** (1990), no. 1, 93–114.

[7] Damon, J., *Topological triviality and versality for subgroups of $\mathcal{A}$ and $\mathcal{K}$. II. Sufficient conditions and applications*, Nonlinearity **5** (1992), no. 2, 373–412.

[8] Gaffney, T., *Polar multiplicities and equisingularity of map germs*, Topology **32** (1993), no. 1, 185–223.

[9] Gaffney, T.; Vohra, R., *A numerical characterization of equisingularity for map germs from n-space, ($n \geq 3$), to the plane*, J. Dyn. Syst. Geom. Theor. **2** (2004), 43-55.

[10] Jorge Pérez, V.H., *Polar multiplicities and equisingularity of map germs from $\mathbb{C}^3$ to $\mathbb{C}^3$*, Houston J. Math. **29** (2003), no.4, 901-924.

[11] Jorge Pérez, V.H.; Levcovitz, D.; Saia, M.J., *Invariants, equisingularity and Euler obstruction of map germs from $\mathbb{C}^n$ to $\mathbb{C}^n$*, J. Reine Angew. Math. **587** (2005), 145-167.

[12] Jorge Pérez, V.H.; Saia, M.J., *Euler obstruction, polar multiplicities and equisingularity of map germs from in $\mathcal{O}(n,p)$, $n < p$*, to appear in Int. J. Math.

[13] Lê Dung Tráng, *Computation of the Milnor number of an isolated singularity of a complete intersection*, Funkcional. Anal. i Priložen. **8** (1974), no. 2, 45–49.

[14] Looijenga, E.J.N., *Isolated singular points on complete intersections.* London Mathematical Soc., Lecture Note series, 77, (1984).

[15] Mather, J. N., *Stratifications and mappings.* Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971), pp. 195–232, Academic Press, New York, 1973

[16] Matsumura, H. *Commutative algebra.* Second edition. Mathematics Lecture Note Series, 56. Benjamin/Cummings Publishing Co., Inc., Reading, Mass., 1980.

[17] Mond, D., *Some remarks on the geometry and classification of germs of map from surfaces to 3-space.* Topology **26** , 1987, 3, 361–383.

[18] Mond, D.; van Straten, D., *Milnor number equals Tjurina number for functions on space curves*, J. London Math. Soc. (2) **63** (2001), no. 1, 177–187.

[19] Marar, W.L.; Mond, D., *Multiple point schemes for corank 1 maps*, J. London Math. Soc. (2) **39** (1989), no. 3, 553–567

[20] Nuño Ballesteros, J. J.; Saia, M. J.,*Multiplicity of Boardman strata and deformations of map germs*, Glasgow Math. J. **40** (1998), no. 1, 21–32.

[21] Tessier, B., *Variétés Polaires 2: Multiplicités Polaires, Sections Planes, et Conditions de Whitney*, Algebraic Geometry, Proc. Int. Conf. La Rábida/Spain 1981, Sringer Lecture Notes, 961 (1982), pp.314-491.
[22] Teissier, B., *Variétés polaires. I: Invariants polaires des singularites d'hypersurfaces*, Invent. Math. **40** (1977), 267-292.
[23] Wall; C. T. C.,*Finite determinacy of smooth map-germs*, Bull. London Math. Soc. **13** (1981), no. 6, 481–539.
[24] H. Whitney, *The singularities of a smooth n-manifold in $(2n-1)$-space*, Ann. of Math. (2), **45**, (1944), 247–293.

Universidade de São Paulo - ICMC, Av. do Trabalhador São Carlense, Centro, Caixa Postal 668, 13560-970, São Carlos-SP BRAZIL
E-mail address: vhjperez@icmc.usp.br

Departament de Geometria i Topologia, Universitat de València, Campus de Burjassot, 46100 Burjassot SPAIN
E-mail address: Juan.Nuno@uv.es

NOTAS DO ICMC

SÉRIE MATEMÁTICA

262/2006 CALLEJAS-BEDREGAL, R; PÉREZ, V.H.J. – Mixed multiplicities for arbitrary ideals and generalized buchsbaum-Rim multiplicities.

261/2006 PÉREZ, V.H.J.; RIZZIOLI, E.C.; SAIA, M.J. – Numerical control of equisingularity for map germs from C^n to C^3, $n \geq 3$.

260/2006 HERNADES, M.E.; RODRIGUES, H.; RUAS, M.A.S – A_e – codimension of germs of analytic monomial curves.

259/2006 CARBINATTO, M.C.; RYBAKOWSKI, R.P. – Continuation of the connection matrix for singularly perturbed hyperbolic equations.

258/2006 RIZZIOLLI, E.C.; SAIA, M.J .– Polar multiplicities and Euler obstruction of the types in weighted homogeneus map germs from C^n to C^3, $n \geq 3$.

257/2006 ARRIETA, J.M.; CARVALHO, A.N.; LOZADA-CRUZ, G. – Dynamics in Dumbbell domains I. Continuity of the set of equilibria.

256/2006 SAIA, M.J.; SOARES JR., C.H. – Modified $C^{\square}$ - trivialization of real germs of functions and maps.

255/2006 GIMENES, L.P.; FEDERSON, M.; TÁBOAS, P. - Impulsive stability of systems of second order retarded differential equations.

254/2006 NABARRO, A.C.; RUAS, M.A.S. – Vector fields in R^2.with maximal index.

253/2005 GUTIERREZ, C.; PIRES, B.; RABANAL, R.. – Asymptotic stability at infinity for differentiable vector fields of the plane.