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REGULARITY OF THE SOLUTIONS ON THE GLOBAL ATTRACTOR FOR A SEMILINEAR HYPERBOLIC DAMPED WAVE EQUATIONS

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Resumo

Consideramos os atratores $\mathbf{A}_\eta$, $\eta \in [0,1]$, associados à equação de ondas amortecidas singularmente perturbada

$$u_{tt} + 2\eta A^{\frac{1}{2}} u_t + a u_t + Au = f(u)$$

em $H_0^1(\Omega) \times L^2(\Omega)$, onde Ω é um domínio limitado com fronteira suave em $\mathbb{R}^3$. Para não linearidades dissipativas $f \in C^2(\mathbb{R}, \mathbb{R})$ que satisfazem $|f''(s)| \leq c(1 + |s|)$ para algum $c > 0$, provamos que a família de atratores $\{\mathbf{A}_\eta, \eta \geq 0\}$ é semicontinua superiormente em $\eta = 0$ nos espaços $H^{1+s}(\Omega) \times H^s(\Omega)$ para quaisquer $s \in (0,1)$. Para não linearidades dissipativas $f \in C^3(\mathbb{R}, \mathbb{R})$ que satisfazem $\lim_{|s|\to\infty} \frac{f''(s)}{s} = 0$ provamos que o atrator $\mathbf{A_0}$ para a equação de ondas amortecidas ($\eta = 0$) é limitado em $H^4(\Omega) \times H^3(\Omega)$ e portanto compacto em $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$ para todo $\mu \in (0, \frac{1}{2})$. Como conseqüência das limitações uniformes obtemos que a família de atratores $\{\mathbf{A}_\eta, \eta \in [0,1]\}$ é semicontinua superiormente e inferiormente em $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$ para todo $\mu \in (0, \frac{1}{2})$.

REGULARITY OF THE SOLUTIONS ON THE GLOBAL ATTRACTOR FOR A SEMILINEAR HYPERBOLIC DAMPED WAVE EQUATIONS

A. N. CARVALHO[1] AND J. W. CHOLEWA[2]

ABSTRACT. We consider attractors $\mathbf{A}_\eta$, $\eta \in [0,1]$, corresponding to singularly perturbed hyperbolic damped wave equations

$$u_{tt} + 2\eta A^{\frac{1}{2}} u_t + a u_t + Au = f(u)$$

in $H_0^1(\Omega) \times L^2(\Omega)$, where Ω is a bounded smooth domain in $\mathbb{R}^3$. For dissipative nonlinearity $f \in C^2(\mathbb{R},\mathbb{R})$ satisfying $|f''(s)| \le c(1+|s|)$ for some $c > 0$, we prove that the family of attractors $\{\mathbf{A}_\eta, \eta \ge 0\}$ is upper semicontinuous at $\eta = 0$ in $H^{1+s}(\Omega) \times H^s(\Omega)$ for any $s \in (0,1)$. For dissipative $f \in C^3(\mathbb{R},\mathbb{R})$ satisfying $\lim_{|s|\to\infty} \frac{f''(s)}{s} = 0$ we prove that the attractor $\mathbf{A}_0$ for the damped wave equation ($\eta = 0$) is bounded in $H^4(\Omega) \times H^3(\Omega)$ and thus is compact in the Hölder spaces $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$ for every $\mu \in (0, \frac{1}{2})$. As a consequence of the uniform bounds we obtain that the family of attractors $\{\mathbf{A}_\eta, \eta \in [0,1]\}$ is upper and lower semicontinuous in $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$ for every $\mu \in (0, \frac{1}{2})$.

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1. Introduction

We study a singularly perturbed damped wave equation

$$\begin{cases} u_{tt} + 2\eta A^{\frac{1}{2}} u_t + a u_t + Au = f(u), \ t > 0, \ x \in \Omega, \\ u(0,x) = u_0(x), \ u_t(0,x) = v_0(x), \ x \in \Omega, \\ u(t,x) = 0, \ t \ge 0, \ x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded subdomain of $\mathbb{R}^3$, A denotes the negative Laplacian in $L^2(\Omega)$ with the domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, a is a fixed positive number, $f : \mathbb{R} \to \mathbb{R}$ satisfies suitable regularity and growth conditions and $\eta \in [0,1]$ is a parameter.

It is known that for a dissipative $f \in C^2(\mathbb{R},\mathbb{R})$ with $\limsup_{|s|\to\infty} \frac{|f''(s)|}{1+|s|} < \infty$ there exists a C^0 semigroup $\{T_0(t)\}$ associated to the problem (1.1) with $\eta = 0$, which has a global attractor $\mathbf{A}_0$ in $H_0^1(\Omega) \times L^2(\Omega)$ (see e.g. [2, 3, 5, 11] and references therein). It is also known that for $\eta > 0$ problem (1.1) with dissipative $\lim_{|s|\to\infty} \frac{|f'(s)|}{1+|s|^4} = 0$ defines a C^0 nonlinear semigroup $\{T_\eta(t)\}$ in $H_0^1(\Omega) \times L^2(\Omega)$, which has a global attractor $\mathbf{A}_\eta$ (see [6, 7, 8]).

In [5] it was shown that, if dissipative f satisfies $\lim_{|s|\to\infty} \frac{f'(s)}{s^2} = 0$ ($N \ge 3$), and if

$$0 \notin \sigma(-A + f'(u_0)) \ \text{ whenever } \ u_0 \in H_0^1(\Omega) \ \text{ and } \ u_0 = A^{-1} f(u_0), \tag{1.2}$$

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(which is known to be a generic property, see [4]) then the family of attractors for the semigroups $\{T_\eta(t)\}$ is continuous at $\eta = 0$; that is

$$\sup_{a_\eta \in A_\eta} \inf_{a_0 \in A_0} \|a_0 - a_\eta\|_{H_0^1(\Omega)\times L^2(\Omega)} + \sup_{a_0 \in A_0} \inf_{a_\eta \in A_\eta} \|a_0 - a_\eta\|_{H_0^1(\Omega)\times L^2(\Omega)} \to 0 \text{ as } \eta \to 0^+. \quad (1.3)$$

The growth of f was limited in [5] to the one needed to obtain existence of the global attractor for the problem (1.1) with $\eta = 0$ and, simultaneously, to the one needed to ensure compactness of the Nemytskii map $f^e : H_0^1(\Omega) \to L^2(\Omega)$ associated with f, which condition was essentially used in the proof of the lower semicontinuity property.

In the present work upper semicontinuity result of [5] is proved for critically growing nonlinearities. This is based on the uniform bound on the approximating attractors in $H^2(\Omega) \times H^1(\Omega)$-norm obtained with the aid of *Alekseev's nonlinear variation of constants formula* (see [2, 7]).

If $f \in C^2(\mathbb{R}, \mathbb{R})$ is dissipative and $\lim_{|s|\to\infty} \frac{f'(s)}{s^2} = 0$, the approximating attractors are shown in the present work to be uniformly bounded in $H^3(\Omega) \times H^2(\Omega)$ and therefore compact in the space of Hölder functions $C^{1+\mu}(\bar{\Omega}) \times C^{\mu}(\bar{\Omega})$, $\mu \in (0, \frac{1}{2})$. This together with (1.3) guarantee compactness of $\mathbf{A_0}$ in $C^{1+\mu}(\bar{\Omega}) \times C^{\mu}(\bar{\Omega})$, $\mu \in (0, \frac{1}{2})$. If, in addition, $f \in C^3(\mathbb{R}, \mathbb{R})$ then $\mathbf{A_0}$ is shown to be bounded in $H^4(\Omega) \times H^3(\Omega)$ and compact in $C^{2+\mu}(\bar{\Omega}) \times C^{1+\mu}(\bar{\Omega})$, $\mu \in (0, \frac{1}{2})$.

Similar regularity properties of $\mathbf{A_0}$ are proved in [12, Theorem 2.21], where Galerkin method have been used to obtain sharp time and space regularity properties of $\mathbf{A_0}$. In particular, in [12] it is proved that, if $f^{(2)+\mu}$ satisfies suitable growth assumption, then $\mathbf{A_0}$ is bounded in $H^3(\Omega) \times H^2(\Omega)$. We prove that

$$\lim_{|s|\to\infty} \frac{f''(s)}{s} = 0 \text{ and } f \in C^{1+j}(\mathbb{R}, \mathbb{R}) \text{ implies } \mathbf{A_0} \subset H^{2+j}(\Omega) \times H^{1+j}(\Omega), \ j = 1, 2.$$

We remark that no Hölder continuity is required for f'' (or f''') to obtain the above mentioned regularity of $\mathbf{A}_0$.

The article is organized in the following way. In Section 2 we establish a suitable functional analytic framework to study nonlinear semigroups associated to (1.1) and properties of their attractors. In Section 3 we show that in the case when $f \in C^2(\mathbb{R}, \mathbb{R})$ is dissipative, and $|f''(s)| \leq c(1 + |s|)$, for some $c > 0$, the approximating attractors are uniformly bounded in $H^2(\Omega) \times H^1(\Omega)$. For such f we conclude upper semicontinuity of $\{\mathbf{A}_\eta, \eta \in [0, 1]\}$ at $\eta = 0$ in $H^{2^-}(\Omega) \times H^{1^-}(\Omega)$. Under the same assumptions as in Section 3, we obtain in Section 4 a uniform $H^3(\Omega) \times H^2(\Omega)$−bound for the approximating attractors and upper semicontinuity of the family of attractors in $H^{3^-}(\Omega) \times H^{2^-}(\Omega)$. If, in addition, we assume that $\lim_{|s|\to\infty} \frac{f^{(2)}(s)}{s} = 0$ we obtain $H^3(\Omega) \times H^2(\Omega) \cap C^{1+\frac{1}{2}^-}(\bar{\Omega}) \times C^{\frac{1}{2}^-}(\bar{\Omega})$−regularity of $\mathbf{A}_0$ and continuity (upper and lower semicontinuity) of the family of attractors in $H^{3^-}(\Omega) \times H^{2^-}(\Omega)$. In Section 5, assuming additionally that $f \in C^3(\mathbb{R}, \mathbb{R})$, we establish $H^4(\Omega) \times H^3(\Omega) \cap C^{2+\frac{1}{2}^-}(\bar{\Omega}) \times C^{1+\frac{1}{2}^-}(\bar{\Omega})$−regularity of $\mathbf{A_0}$ and continuity (upper and lower semicontinuity) of the family of attractors in $H^{4^-}(\Omega) \times H^{3^-}(\Omega)$.

2. Functional analytic framework and existence of attractors

In this section we establish the functional analytic framework to study (1.1) and indicate how one can obtain the existence of attractors for the nonlinear semigroup associated to (1.1).

Let $X_p = L^p(\Omega)$, $p \in (1,\infty)$, and denote by $\{X_p^\alpha, \alpha \geq 0\}$ the fractional power scale generated by (A, X_p). It is known that, for suitably smooth Ω, A has bounded imaginary powers (see [15]) and spaces $X_p^\alpha =$, $\alpha \in (0,1)$, $p > 1$, coincide with $[L^p(\Omega), H^2_{p,\{I\}}(\Omega)]_\alpha = H^{2\alpha}_{p,\{I\}}(\Omega)$ (see [16]).

Consider damped wave operators

$$\mathcal{A}_\eta : D(\mathcal{A}_\eta) \subset X_2^{\frac{1}{2}} \times X_2 \to X_2^{\frac{1}{2}} \times X_2, \quad \mathcal{A}_\eta = \begin{bmatrix} 0 & -I \\ A & 2\eta A^{\frac{1}{2}} + aI \end{bmatrix}, \quad D(\mathcal{A}_\eta) = X_2^1 \times X_2^{\frac{1}{2}},\ \eta \geq 0.$$

Note that

$$\mathcal{A}_\eta^{-1} : X_2^{\frac{1}{2}} \times X_2 \to X_2^{\frac{1}{2}} \times X_2, \quad \mathcal{A}_\eta^{-1} = \begin{bmatrix} 2\eta A^{-\frac{1}{2}} + aA^{-1} & A^{-1} \\ -I & 0 \end{bmatrix} \quad \text{for } \eta \geq 0. \tag{2.1}$$

Recall from [5] that there is certain $d > 0$ such that

$$d^{-1} \| \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix} \right] \|_{X_2^1 \times X_2^{\frac{1}{2}}} \leq \| \mathcal{A}_\eta \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix} \right] \|_{X_2^{\frac{1}{2}} \times X_2} \leq d \| \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix} \right] \|_{X_2^1 \times X_2^{\frac{1}{2}}},\ \eta \in (0,1],\ \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix} \right] \in X_2^1 \times X_2^{\frac{1}{2}}. \tag{2.2}$$

Let $Y_2 = Y_2^0$ be the extrapolated space (see [1]); that is, the completion of the normed space $(X_2^{\frac{1}{2}} \times X_2, \|\mathcal{A}_\eta^{-1} \cdot \|_{X_2^{\frac{1}{2}} \times X_2})$. For some $\tilde{d} > 0$ (independent of $\eta \in [0,1]$), we have that

$$\tilde{d}^{-1} \| \mathcal{A}_\eta^{-1} \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix} \right] \|_{X_2^{\frac{1}{2}} \times X_2} \leq \| \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix} \right] \|_{X_2 \times X_2^{-\frac{1}{2}}} \leq \tilde{d} \| \mathcal{A}_\eta^{-1} \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix} \right] \|_{X_2^{\frac{1}{2}} \times X_2},\ \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix} \right] \in X_2^{\frac{1}{2}} \times X_2. \tag{2.3}$$

Thus the completions of $(X_2^{\frac{1}{2}} \times X_2, \|\mathcal{A}_\eta^{-1} \cdot \|_{X_2^{\frac{1}{2}} \times X_2})$ and $(X_2^{\frac{1}{2}} \times X_2, \| \cdot \|_{X_2 \times X_2^{-\frac{1}{2}}})$ coincide with equivalent norms independent of $\eta \in [0,1]$. Then $Y_2 = X_2 \times X_2^{-\frac{1}{2}}$.

The operator $\mathcal{A}_\eta$ has the closed extension to Y_2 with domain Y_2^1 and it generates (both in Y_2 and in Y_2^1) a compact C^0-semigroup of contractions $\{e^{-\mathcal{A}_\eta t} : t \geq 0\}$ which is uniformly exponentially decaying; that is, there are positive constants c and ω such that

$$\|e^{-\mathcal{A}_\eta t}\|_{L(X_2^{\frac{1}{2}} \times X_2)} \leqslant ce^{-\omega t},\ t \geq 0,\ \eta \in [0,1]. \tag{2.4}$$

Furthermore, this semigroup is analytic for $\eta \in (0,1]$ and the fractional power scale generated by $(\mathcal{A}_\eta, Y_2)$ is characterized as (see [6, 10])

$$Y_2^\alpha = X_2^{\frac{\alpha}{2}} \times X_2^{\frac{\alpha-1}{2}},\ \alpha \geq 0. \tag{2.5}$$

With this set-up we consider problem (1.1) in the form

$$\tfrac{d}{dt} \left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right] + \mathcal{A}_\eta \left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right] = F\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right] \right),\ t > 0, \quad \left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right]_{t=0} = \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix} \right], \tag{2.6}$$

where $\eta \in [0,1]$ and

$$F\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right] \right) = \left[\begin{smallmatrix} 0 \\ f^e(u) \end{smallmatrix} \right],\ f^e(u)(x) = f(u(x)) \ \text{ for all } \ u \in X_2^{\frac{1}{2}},\ x \in \Omega.$$

Since we are working in space dimension $N=3$, we assume that

$$f\in C^2(\mathbb{R},\mathbb{R}),\quad |f''(s)|\leqslant c(1+|s|),\ s\in\mathbb{R}, \tag{2.7}$$

and

$$\limsup_{|s|\to\infty}\frac{f(s)}{s}<\lambda_1, \tag{2.8}$$

where λ_1 is the first positive eigenvalue of A.

If (2.7) is satisfied, then $F:Y_2^1\to Y_2^1$ is continuously differentiable and Lipschitz continuous in bounded subsets of Y_2^1. Hence, the problem (2.6) is such that, for each $\left[\begin{smallmatrix}w_0\\z_0\end{smallmatrix}\right]\in Y_2^1$ there are constants $r>0$ and $\tau=\tau_{\left[\begin{smallmatrix}w_0\\z_0\end{smallmatrix}\right]}>0$ such that, for each $\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]$ in the ball $B_r(\left[\begin{smallmatrix}w_0\\z_0\end{smallmatrix}\right])\subset Y_2^1$ there is a unique mild solution

$$T_\eta(\cdot)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]:=\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right](\cdot,\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right],\eta)\in C([0,\tau],Y_2^1)$$

of (2.6) and the map

$$B_r(\left[\begin{smallmatrix}w_0\\z_0\end{smallmatrix}\right])\ni\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\mapsto T_\eta(\cdot)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\in C([0,\tau],Y_2^1)$$

is continuously differentiable. If, in addition to (2.7), the condition (2.8) is satisfied the solutions of (2.6) are defined for all $t\geq 0$ and are bounded, uniformly in bounded subsets of Y_2^1. That is a consequence of the properties of the functional $\mathcal{L}:Y_2^1\to\mathbb{R}$,

$$\mathcal{L}(\left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right])=\frac{1}{2}\|w_1\|^2_{X_2^{\frac12}}+\frac{1}{2}\|w_2\|^2_{X_2}-\int_\Omega\int_0^{w_1}f(s)dsdx,\ \left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right]\in Y_2^1. \tag{2.9}$$

Namely, for some $\varepsilon>0$ and $C_\varepsilon>0$, $\mathcal{L}$ satisfies under the mentioned assumptions

$$\mathcal{L}(\left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right])\geq\frac{\varepsilon}{2\lambda_1}\|w_1\|^2_{X_2^{\frac12}}+\frac{1}{2}\|w_2\|^2_{X_2}-C_\varepsilon,\ \left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right]\in Y_2^1, \tag{2.10}$$

decreases along solutions $\left[\begin{smallmatrix}u_\eta\\v_\eta\end{smallmatrix}\right]=\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right](t,\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right],\eta)$ of (2.6) and

$$\|\left[\begin{smallmatrix}u_\eta\\v_\eta\end{smallmatrix}\right]\|_{Y_2^1}\leqslant c_{\mathcal{L}}\sqrt{1+\mathcal{L}(\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])} \tag{2.11}$$

with a constant $c_{\mathcal{L}}$ independent of $\eta\in[0,1]$. Consequently, all solutions of (2.6) are globally defined and bounded uniformly for $t\geq 0$ and $\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]$ in bounded subsets of Y_2^1.

Hence we have a nonlinear semigroup $\{T_\eta(t):t\geq 0\}$ associated to (2.6) and next we outline the proof that this semigroup has a globall attractor. In fact, to obtain the existence of attractor for $\{T_\eta(t):t\geq 0\}$, we only need to prove that the set of equilibria for (2.6) is bounded and that $\{T_\eta(t):t\geq 0\}$ is asymptotically compact.

The set of equilibria $\mathcal{E}=\{\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\in Y_2^1:\ u_0=A^{-1}f^e(u_0),\ v_0=0\}$ is bounded in Y_2^2. Indeed, from (2.8) we have

$$\|A^{\frac12}u_0\|_{X_2}\leqslant M_0^{\mathcal{E}},\ \left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\in\mathcal{E} \tag{2.12}$$

and, since (2.7) ensures that f^e takes bounded subsets of $X_2^{\frac12}$ into bounded subsets of X_2,

$$\|Au_0\|_{X_2}=\|f^e(u_0)\|_{X_2}\leqslant\sup_{\|\phi\|_{X_2^{\frac12}}\leqslant M_0^{\mathcal{E}}}\|f^e(\phi)\|_{X_2}=M_1^{\mathcal{E}}. \tag{2.13}$$

The asymptotic compactness property of $\{T_\eta(t) : t \geq 0\}$ is proved in [2] for the case $\eta = 0$ and is an easy consequence of compactness of the linear semigroup $\{e^{-A_\eta t} : t \geq 0\}$ in the case $\eta > 0$ (see [7, 8] for even faster growth).

¿From this, if $\eta \in [0, 1]$ and (2.7), (2.8) hold, then there exists a C^1-semigroup $\{T_\eta(t)\}$ of global solutions of (2.6) in Y_2^1 which has a global attractor $\mathbf{A}_\eta$.

3. Upper semicontinuity result for critically growing f

In this section we obtain uniform Y_2^2-bound on the attractors when f satisfies (2.7) and (2.8). Under the same assumptions on f we prove upper semicontinuity of $\{\mathbf{A}_\eta,\ \eta \geq 0\}$ at $\eta = 0$.

We start establishing the Y_2^2-bound for $\cup_{\eta\in(0,1]}\boldsymbol{A}_\eta$. The proof will be divided into several lemmas.

Lemma 3.1. *If* (2.7) *and* (2.8) *are satisfied, then*

$$\sup_{\eta\in[0,1]} \sup_{\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\in \boldsymbol{A}_\eta} \mathcal{L}(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]) \leq \sup_{\left[\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right]\in \mathcal{E}} \mathcal{L}(\left[\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right]). \tag{3.1}$$

Consequently, $\bigcup_{\eta\in[0,1]} \boldsymbol{A}_\eta$ *is bounded in* Y_2^1-*norm.*

Proof: For $\eta \in [0, 1]$ and $\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right] \in \boldsymbol{A}_\eta$, there exists a precompact complete orbit through $\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]$. Denoting it by $\gamma_\eta = \{T_\eta(t)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right],\ t \in \mathbb{R}\}$ we have that $\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right] = T_\eta(k)T_\eta(-k)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]$ for arbitrary $k \in \mathbb{N}$. Sequence $\{T_\eta(-k)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\}$ has a subsequence $\{T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\}$ convergent strongly in Y_2^1 to certain element of the α-limit set $\alpha_{\gamma_\eta}(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right])$ of $\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]$. Since $\alpha_{\gamma_\eta}(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]) \subset \mathcal{E}$, from (2.11) and (2.12) we have that

$$\mathcal{L}(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]) \leqslant \mathcal{L}(T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]),$$

$$\mathcal{L}(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]) \leqslant \lim_{k_l\to\infty} \mathcal{L}(T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]) \leqslant \sup_{\left[\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right]\in\mathcal{E}} \mathcal{L}(\left[\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right]).$$

¿From this and (2.10) it follows that

$$\sup_{\eta\in(0,1]} \sup_{\left[\begin{smallmatrix} \phi_\eta \\ \psi_\eta \end{smallmatrix}\right]\in\mathbf{A}_\eta} \| \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \|_{Y_2^1} \leqslant c_{\mathcal{L}} \sup_{\left[\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right]\in\mathcal{E}} \sqrt{1 + \mathcal{L}(\left[\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right])} =: \zeta \tag{3.2}$$

and the result is proved. □

Note that, from (2.8), there exists $K_f > \max\{0, f'(0)\}$ such that

$$s^{-1}(f(s) - f(0)) < K_f \quad \text{for each } 0 \neq s \in \mathbb{R}. \tag{3.3}$$

Decompose f as $f(s) = \tilde{f}(s) + \hat{f}(s)$ where

$$\tilde{f}(s) = f(s) - f(0) - K_f s, \quad \hat{f}(s) = f(0) + K_f s, \quad s \in \mathbb{R},$$

and

$$s\tilde{f}(s) \leqslant 0 \text{ for } s \in \mathbb{R}. \tag{3.4}$$

Denote by $\tilde{T}_\eta(t)\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$ the solution of

$$\tfrac{d}{dt}\left[\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}\right] + A_\eta\left[\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}\right] = \begin{bmatrix} 0 \\ \tilde{f}^e(\tilde{u}) \end{bmatrix} =: \tilde{F}(\left[\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}\right]),\ t > 0, \quad \left[\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}\right]_{t=0} = \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]. \tag{3.5}$$

With this decomposition we write Alekseev's nonlinear variation of constants formula (see [7, Lemma 7]; also [2]),

$$T_\eta(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]=\tilde{T}_\eta(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]+\int_0^t(\partial\tilde{T}_\eta)(t-s,T_\eta(s)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])\hat{F}(T_\eta(s)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])ds,\quad t>0,\tag{3.6}$$

where $\partial\tilde{T}_\eta$ is partial derivative of the map $(0,\infty)\times Y_2^1\ni(t,V_0)\to\tilde{T}_\eta(t)V_0\in Y_2^1$ with respect to V_0 and

$$\hat{F}(\left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right])=\left[\begin{smallmatrix}0\\K_fw_1+f(0)\end{smallmatrix}\right],\quad\left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right]\in Y_2^1.$$

We recall from [13, Theorem 3.4.4] that $(\partial\tilde{T}_\eta)(\cdot,\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])\hat{F}(\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])$ is a solution of

$$\frac{d}{dt}W+A_\eta W=\tilde{F}'(\tilde{T}_\eta(t)V_0)W,\ t>0,\quad W(0)=\hat{F}(V_0).\tag{3.7}$$

In what follows we will derive suitable estimates for the solutions of (3.5) and (3.7).

Lemma 3.2. *If $\tilde{f}$ is as before, for each $r>0$ there are positive constants c_r and w_r such that*

$$\sup_{\eta\in(0,1]}\ \sup_{\|\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\|_{Y_2^1}\le r}\|\left[\begin{smallmatrix}\tilde{u}\\\tilde{v}\end{smallmatrix}\right](t,\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right],\eta)\|_{Y_2^1}\le c_re^{-\omega_rt},\quad t\ge 0,\tag{3.8}$$

where $\left[\begin{smallmatrix}\tilde{u}\\\tilde{v}\end{smallmatrix}\right](t,\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\eta)$ is the solution of (3.5).

Proof: As in [2, 7] we consider for $\delta\ge 0$ a suitable Lyapunov type functional

$$\tilde{\mathcal{L}}_\delta(\left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right])=\frac{1}{2}\|A^{\frac{1}{2}}w_1\|^2_{X_2}+\frac{1}{2}\|w_2\|^2_{X_2}-\int_\Omega\int_0^{w_1}\tilde{f}(s)dsdx+\delta\int_\Omega w_1w_2dx,\tag{3.9}$$

$\left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right]\in Y_2^1$. The integral $\int_\Omega\int_0^{w_1}\tilde{f}(s)dsdx$ is nonpositive and from the Poincaré's inequality $\lambda_1\|w_1\|^2_{X_2}\le\|A^{\frac{1}{2}}w_1\|^2_{X_2}$ we have

$$\frac{1}{4}(\|A^{\frac{1}{2}}w_1\|^2_{X_2}+\|w_2\|^2_{X_2})\le\tilde{\mathcal{L}}_\delta(\left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right]),\quad 0<\delta\le\delta_0:=\frac{1}{2}\min\{\lambda_1,1\},\ \left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right]\in Y_2^1.\tag{3.10}$$

Estimating in a standard way and using (3.4) we obtain, for a suitable choice of $\delta\in(0,\delta_0)$

$$\begin{aligned}\frac{d}{dt}\tilde{\mathcal{L}}_\delta(\left[\begin{smallmatrix}\tilde{u}\\\tilde{v}\end{smallmatrix}\right])&=-a\|\tilde{v}\|^2_{X_2}-2\eta\|A^{\frac{1}{4}}\tilde{v}\|^2_{X_2}+\delta\|\tilde{v}\|^2_{X_2}+\delta\int_\Omega\tilde{u}\left(-A\tilde{u}-a\tilde{v}-2\eta A^{\frac{1}{2}}\tilde{v}+\tilde{f}(\tilde{u})\right)dx\\&\le-\frac{\delta}{2}(\|A^{\frac{1}{2}}\tilde{u}\|^2_{X_2}+\|\tilde{v}\|^2_{X_2}),\quad\eta\in(0,1].\end{aligned}\tag{3.11}$$

In particular, for $\delta=0$ we get

$$\|A^{\frac{1}{2}}w_1\|^2_{X_2}+\|w_2\|^2_{X_2}\le 2\tilde{\mathcal{L}}_0(\left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right]),\quad\left[\begin{smallmatrix}w_1\\w_2\end{smallmatrix}\right]\in Y_2^1,\tag{3.12}$$

and

$$\frac{d}{dt}\tilde{\mathcal{L}}_0(\left[\begin{smallmatrix}\tilde{u}\\\tilde{v}\end{smallmatrix}\right])\le 0.\tag{3.13}$$

Since $|\tilde{f}(s)|\le(c+K_f)|s|(1+|s|^2)$, we have

$$|\int_\Omega\int_0^{\tilde{u}}\tilde{f}(s)dsdx|\le(c+K_f)(\|\tilde{u}\|^2_{X_2}+\|\tilde{u}\|^4_{X_4}).$$

Therefore, from (3.12), (3.13) and Sobolev embedding we infer that there is constant $c_0 > 1$

$$|\int_\Omega \int_0^{\tilde{u}} \tilde{f}(s)dsdx| \leq c_0 \left(1 + \tilde{\mathcal{L}}_0([\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}])\right) \|A^{\frac{1}{2}}\tilde{u}\|^2_{X_2}. \tag{3.14}$$

¿From (3.9), (3.11), (3.14) it follows that for each $r > 0$ there is a constant $\rho_r > 0$ such that

$$\frac{d}{dt}\tilde{\mathcal{L}}_\delta(t, [\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}](u_0, v_0, \eta) \leq -\rho_r \tilde{\mathcal{L}}_\delta(t, [\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}](u_0, v_0, \eta)), \quad t \geq 0,\ \eta \in [0,1],\ \|[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}]\|_{Y_2^1} \leq r.$$

With the estimate (3.10) condition (3.8) now follows easily. □

Below estimate solutions of (3.7) uniformly with respect to $\eta \in (0,1]$ and $V_0 \in \bigcup_{\eta\in(0,1]} \boldsymbol{A}_\eta$.

Lemma 3.3. *If* $\left[\begin{smallmatrix} \phi_\eta \\ \psi_\eta \end{smallmatrix}\right] \in \boldsymbol{A}_\eta$, $\eta \in (0,1]$ *and* $W = [\begin{smallmatrix} w \\ \chi \end{smallmatrix}]$ *is the solution of* (3.7) *with* $W(0) = \hat{F}(\left[\begin{smallmatrix} \phi_\eta \\ \psi_\eta \end{smallmatrix}\right]) = \left[\begin{smallmatrix} 0 \\ K_f\phi_\eta + f(0) \end{smallmatrix}\right]$, *there are positive constants* ϵ *and* L_ϵ *such that*

$$\|[\begin{smallmatrix} w \\ \chi \end{smallmatrix}]\|_{Y_2^1} \leqslant L_\epsilon e^{-\epsilon t}, \quad t \geq 0. \tag{3.15}$$

Proof: As a consequence (3.2) we have

$$\sup_{\eta\in(0,1]} \sup_{\left[\begin{smallmatrix} \phi_\eta \\ \psi_\eta \end{smallmatrix}\right]\in \boldsymbol{A}_\eta} \|\hat{F}(\left[\begin{smallmatrix} \phi_\eta \\ \psi_\eta \end{smallmatrix}\right])\|_{Y_2^1} \leq \tilde{\zeta}. \tag{3.16}$$

Note also that the equation for w can be rewritten as

$$w_{tt} + 2\eta A^{\frac{1}{2}} w_t + a w_t + Aw = (f'(\tilde{u}_\eta) - f'(0))w + (f'(0) - K_f)w \tag{3.17}$$

with

$$f'(0) - K_f < 0. \tag{3.18}$$

For $\delta \geq 0$ and $[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}] \in Y_2^1$ define

$$\hat{\mathcal{L}}_\delta([\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}]) = \frac{1}{2}\|w_2\|^2_{X_2} + \frac{1}{2}(K_f - f'(0))\|w_1\|^2_{X_2} + \frac{1}{2}\|A^{\frac{1}{2}} w_1\|^2_{X_2} + \delta \int_\Omega w_1 w_2 dx. \tag{3.19}$$

Differentiating $\hat{\mathcal{L}}_\delta([\begin{smallmatrix} w \\ w_t \end{smallmatrix}])$ and using (3.17) we get

$$\frac{d}{dt}\hat{\mathcal{L}}_\delta([\begin{smallmatrix} w \\ \chi \end{smallmatrix}]) = -\epsilon \hat{\mathcal{L}}_\delta([\begin{smallmatrix} w \\ \chi \end{smallmatrix}]) + R_{\delta,\epsilon}, \tag{3.20}$$

where

$$\begin{aligned} R_{\delta,\epsilon} = {} & \frac{\epsilon}{2}\|w_t\|^2_{X_2} + \frac{\epsilon}{2}(K_f - f'(0))\|w\|^2_{X_2} + \frac{\epsilon}{2}\|A^{\frac{1}{2}} w\|^2_{X_2} + \epsilon\delta \int_\Omega w w_t dx \\ & \int_\Omega (f'(\tilde{u}_\eta) - f'(0)) w w_t dx + \delta \int_\Omega (f'(\tilde{u}_\eta) - f'(0)) w^2 dx \\ & + \delta \int_\Omega (f'(0) - K_f) w^2 dx - \delta\|A^{\frac{1}{2}} w\|^2_{X_2} - a\|w_t\|^2_{X_2} - a\delta \int_\Omega w w_t dx \\ & - 2\eta \|A^{\frac{1}{4}} w_t\|^2_{X_2} - 2\delta\eta \int_\Omega w_t A^{\frac{1}{2}} w dx + \delta\|w_t\|^2_{X_2}. \end{aligned} \tag{3.21}$$

Applying (2.7) we obtain from (3.21)

$$\begin{aligned}R_{\delta,\epsilon} \leqslant\ & \frac{\epsilon}{2}\|w_t\|^2_{X_2} + \frac{\epsilon}{2}(K_f - f'(0))\|w\|^2_{X_2} + \frac{\epsilon}{2}\|A^{\frac{1}{2}}w\|^2_{X_2} + \frac{\epsilon\delta}{2}\|w\|^2_{X_2} + \frac{\epsilon\delta}{2}\|w_t\|^2_{X_2} \\ & + \frac{a}{2}\|w_t\|^2_{X_2} + \frac{c^2}{2a}\|w\|^2_{X_6}\|\tilde{u}_\eta\|^2_{X_6}(1 + \|\tilde{u}_\eta\|_{X_6})^2 \\ & + \delta c^2\|w\|^2_{X_6}\|\tilde{u}_\eta\|_{X_6}(1 + \|\tilde{u}_\eta\|_{X_6}) + \delta(f'(0) - K_f)\|w\|^2_{X_2} - \delta\|A^{\frac{1}{2}}w\|^2_{X_2} \\ & - a\|w_t\|^2_{X_2} + \frac{\lambda_1\delta}{2}\|w\|^2_{X_2} + \frac{a^2\delta}{2\lambda_1}\|w_t\|^2_{X_2} + 4\delta\eta\|w_t\|^2_{X_2} + \frac{\delta\eta}{4}\|A^{\frac{1}{2}}w\|^2_{X_2} + \delta\|w_t\|^2_{X_2}.\end{aligned}$$

Therefore, from Lemma 3.1 and Lemma 3.2, there are positive constants r, $\bar{c}_r$ and ω_r such that

$$\begin{aligned}R_{\delta,\epsilon} \leqslant & -\left(\delta - \frac{\epsilon}{2}\right)(K_f - f'(0))\|w\|^2_{X_2} - \left(\delta - \frac{\epsilon}{2} - \frac{\epsilon\delta}{2\lambda_1} - \frac{\delta}{2} - \frac{\delta\eta}{4}\right)\|A^{\frac{1}{2}}w\|^2_{X_2} \\ & - \left(\frac{a}{2} - \frac{\epsilon}{2} - \delta - \frac{\epsilon\delta}{2} - \frac{\delta a^2}{2\lambda_1} - 4\delta\eta\right)\|w_t\|^2_{X_2} + \bar{c}_r e^{-\omega_r t}\|A^{\frac{1}{2}}w\|^2_{X_2},\ \delta \in (0,1),\ \epsilon > 0.\end{aligned} \tag{3.22}$$

¿From (3.18), (3.20), (3.22) we can choose $\epsilon > 0$ and $\delta > 0$ such that, with the aid of (3.8),

$$\frac{d}{dt}\hat{\mathcal{L}}_\delta\left(\left[\begin{smallmatrix} w \\ \chi \end{smallmatrix}\right]\right) \leqslant -\epsilon\hat{\mathcal{L}}_\delta\left(\left[\begin{smallmatrix} w \\ \chi \end{smallmatrix}\right]\right) + \bar{c}_r e^{-\omega_r t}\hat{\mathcal{L}}_\delta\left(\left[\begin{smallmatrix} w \\ \chi \end{smallmatrix}\right]\right), \quad t \geq 0,\ \eta \in (0,1]. \tag{3.23}$$

Solving differential inequality (3.23) and using (3.16), (3.19) we get the result. □

We will next strengthen (3.15) to $X_2^{\frac{1+\alpha}{2}} \times X_2^{\frac{\alpha}{2}}$–estimate with $\alpha \in (0, \frac{1}{2})$. Let us show first that there is a constant $\hat{d} \geq 1$ such that

$$\hat{d}^{-1}\|\cdot\|_{X_2^{\frac{1+\alpha}{2}} \times X_2^{\frac{\alpha}{2}}} \leq \|\mathcal{A}^\alpha_\eta \cdot\|_{Y_2^1} \leq \hat{d}\|\cdot\|_{X_2^{\frac{1+\alpha}{2}} \times X_2^{\frac{\alpha}{2}}},\ \forall \eta \in (0,1]. \tag{3.24}$$

For this we recall from [6, Proposition 1 (vi)] that

$$\|\mathcal{A}^{it}_\eta\|_{L(Y_2^1)} \leq e^{\frac{\pi}{2}|t|},\ t \in \mathbb{R},\ \eta \in (0,1]. \tag{3.25}$$

As a consequence

$$D(\mathcal{A}^\alpha_\eta) = [Y_2^1, D(\mathcal{A}_\eta)]_\alpha,\ \alpha \in (0,1),\ \eta \in (0,1],$$

and

$$\hat{d}^{-1}\|\mathcal{A}^\alpha_\eta \cdot\|_{Y_2^1} \leq \|\cdot\|_{[Y_2^1, D(\mathcal{A}_\eta)]_\alpha} \leq \hat{d}\|\mathcal{A}^\alpha_\eta \cdot\|_{Y_2^1},\ \alpha \in (0,1),\ \eta \in (0,1], \tag{3.26}$$

where $\hat{d}$ is independent of $\eta \in (0,1]$ (see [1, §I.2.9 (2.9.9)]). Since (2.2) ensures that $\|\mathcal{A}_\eta \cdot\|_{Y_2^1}$ and $\|\cdot\|_{Y_2^2}$ are equivalent norms uniformly for $\eta \in (0,1]$, we infer that $\|\cdot\|_{[Y_2^1, D(\mathcal{A}_\eta)]_\alpha}$ and $\|\cdot\|_{[Y_2^1, Y_2^2]_\alpha}$ are also equivalent uniformly for $\eta \in (0,1]$. Recalling that $[Y_2^1, Y_2^2]_\alpha = X^{\frac{1+\alpha}{2}} \times X^{\frac{\alpha}{2}} := Y_2^{1+\alpha}$, $\alpha \in (0,1)$, (see [6, Proposition 3]), we get (3.24).

Lemma 3.4. *If* $\left[\begin{smallmatrix} \phi_\eta \\ \psi_\eta \end{smallmatrix}\right] \in \boldsymbol{A}_\eta$, $\eta \in (0,1]$, $\alpha \in (0, \frac{1}{2})$ *and* $W = \left[\begin{smallmatrix} w \\ \chi \end{smallmatrix}\right]$ *is the solution of* (3.7) *with* $W(0) = \hat{F}(\left[\begin{smallmatrix} \phi_\eta \\ \psi_\eta \end{smallmatrix}\right])$ *there are positive constants* d'' *and* ω'' *such that*

$$\|\left[\begin{smallmatrix} w \\ \chi \end{smallmatrix}\right]\|_{X_2^{\frac{1+\alpha}{2}} \times X_2^{\frac{\alpha}{2}}} \leqslant d'' e^{-\omega'' t}, \quad t \geqslant 0. \tag{3.27}$$

Proof: For $\alpha \in (0, \frac{1}{2})$ and the solution $W = \left[\begin{smallmatrix} w \\ \chi \end{smallmatrix}\right]$ of (3.7) with $W(0) = \hat{F}(\left[\begin{smallmatrix} \phi_\eta \\ \psi_\eta \end{smallmatrix}\right])$ we obtain

$$\mathcal{A}_\eta^\alpha \left[\begin{smallmatrix} w \\ \chi \end{smallmatrix}\right] = e^{-\mathcal{A}_\eta t} \mathcal{A}_\eta^\alpha \hat{F}(\left[\begin{smallmatrix} \phi_\eta \\ \psi_\eta \end{smallmatrix}\right]) + \int_0^t e^{-\mathcal{A}_\eta (t-s)} \mathcal{A}_\eta^\alpha \left[\begin{smallmatrix} 0 \\ f'(\tilde{u}_\eta)w - K_f w \end{smallmatrix}\right] ds, \ t \geqslant 0$$

and from (2.4)

$$\|\mathcal{A}_\eta^\alpha \left[\begin{smallmatrix} w \\ \chi \end{smallmatrix}\right]\|_{Y_2^1} \leqslant ce^{-\omega t} \|\mathcal{A}_\eta^\alpha \hat{F}(\left[\begin{smallmatrix} \phi_\eta \\ \psi_\eta \end{smallmatrix}\right])\|_{Y_2^1} + c\int_0^t e^{-\omega(t-s)} \|\mathcal{A}_\eta^\alpha \left[\begin{smallmatrix} 0 \\ f'(\tilde{u}_\eta)w - K_f w \end{smallmatrix}\right]\|_{Y_2^1} ds, \ t \geqslant 0. \quad (3.28)$$

As a consequence of (3.2) and (3.24) for each $\alpha \in (0, \frac{1}{2})$ there exists $\zeta_\alpha > 0$ such that

$$\sup_{\eta \in (0,1]} \sup_{\left[\begin{smallmatrix} \phi_\eta \\ \psi_\eta \end{smallmatrix}\right] \in \mathbf{A}_\eta} \|\mathcal{A}_\eta^\alpha \hat{F}(\left[\begin{smallmatrix} \phi_\eta \\ \psi_\eta \end{smallmatrix}\right])\|_{Y_2^1} \leqslant \hat{d}\|K_f \phi_\eta + f(0)\|_{X_2^{\frac{\alpha}{2}}} \leqslant \zeta_\alpha \quad (3.29)$$

and

$$\begin{aligned} \|\mathcal{A}_\eta^\alpha \left[\begin{smallmatrix} 0 \\ f'(\tilde{u}_\eta)w - K_f w \end{smallmatrix}\right]\|_{Y_2^1} &\leqslant \hat{d}\|A^{\frac{\alpha}{2}}(f'(\tilde{u}_\eta)w - K_f w)\|_{X_2} \\ &= \hat{d}\|A^{\frac{\alpha-1}{2}} A^{\frac{1}{2}}(f'(\tilde{u}_\eta)w - K_f w)\|_{X_2}. \end{aligned} \quad (3.30)$$

¿From [1, Theorem 1.3.8 and 1.4.12], we have

$$\begin{aligned} \|A^{\frac{\alpha-1}{2}} A^{\frac{1}{2}}(f'(\tilde{u}_\eta)w - K_f w)\|_{X_2} &= \|A^{\frac{1}{2}}(f'(\tilde{u}_\eta)w - K_f w)\|_{X_2^{\frac{\alpha-1}{2}}} \\ = \|A^{\frac{1}{2}}(f'(\tilde{u}_\eta)w - K_f w)\|_{\left(X_2^{\frac{1-\alpha}{2}}\right)^*} &\leqslant K_\alpha \|A^{\frac{1}{2}}(f'(\tilde{u}_\eta)w - K_f w)\|_{X_{\frac{6}{5-2\alpha}}}. \end{aligned} \quad (3.31)$$

As a consequence of (3.30), (3.31) we thus get

$$\begin{aligned} \|\mathcal{A}_\eta^\alpha \left[\begin{smallmatrix} 0 \\ f'(\tilde{u}_\eta)w - K_f w \end{smallmatrix}\right]\|_{Y_2^1} &\leqslant \tilde{K}_\alpha \|\nabla(f'(\tilde{u}_\eta)w - K_f w)\|_{X_{\frac{6}{5-2\alpha}}} \\ &\leqslant \tilde{K}_\alpha \|f''(\tilde{u}_\eta)\nabla\tilde{u}_\eta w + f'(\tilde{u}_\eta)\nabla w - K_f \nabla w)\|_{X_{\frac{6}{5-2\alpha}}}. \end{aligned} \quad (3.32)$$

Since $\alpha \in (0, \frac{1}{2})$, from (2.7) we have

$$\begin{aligned} \|f'(\tilde{u}_\eta)\nabla w - K_f \nabla w\|_{X_{\frac{6}{5-2\alpha}}} &\leqslant \|c(1 + |\tilde{u}_\eta|^2)\nabla w\|_{X_{\frac{6}{5-2\alpha}}} + K_f \|\nabla w\|_{X_{\frac{6}{5-2\alpha}}} \\ &\leqslant (c + K_f)\|\nabla w\|_{X_{\frac{6}{5-2\alpha}}} + c\|\tilde{u}_\eta\|_{X_6}^2 \|\nabla w\|_{X_{\frac{6}{3-2\alpha}}} \\ &\leqslant K' \left(\|A^{\frac{1}{2}} w\|_{X_2} + \|A^{\frac{1}{2}}\tilde{u}_\eta\|_{X_2}^2 \|A^{\frac{1+\alpha}{2}} w\|_{X_2}\right) \end{aligned} \quad (3.33)$$

and

$$\begin{aligned} \|f''(\tilde{u}_\eta)\nabla\tilde{u}_\eta w\|_{X_{\frac{6}{5+2\alpha}}} &\leqslant \|c(1 + |\tilde{u}_\eta|)\|_{X_6} \|\nabla\tilde{u}_\eta\|_{X_2} \|w\|_{X_{\frac{6}{1-2\alpha}}} \\ &\leqslant K'' \left(1 + \|A^{\frac{1}{2}}\tilde{u}_\eta\|_{X_2}\right) \|A^{\frac{1}{2}}\tilde{u}_\eta\|_{X_2} \|A^{\frac{1+\alpha}{2}} w\|_{X_2}. \end{aligned} \quad (3.34)$$

By (3.8), (3.15) and (3.32)-(3.34) there are positive numbers d' and w' such that

$$\|\mathcal{A}_\eta^\alpha \left[\begin{smallmatrix} 0 \\ f'(\tilde{u}_\eta)w - K_f w \end{smallmatrix}\right]\|_{Y_2^1} \leqslant d' e^{-\omega' t} \left(1 + \|w\|_{X_2^{\frac{1+\alpha}{2}}}\right), \ t \geqslant 0. \quad (3.35)$$

Inserting (3.29), (3.35) into the right hand side of (3.28) and using (3.24) we then have

$$\|\left[\begin{smallmatrix} w \\ \chi \end{smallmatrix}\right]\|_{X_2^{\frac{1+\alpha}{2}}\times X_2^{\frac{\alpha}{2}}} \leqslant c\hat{d}^2 e^{-\omega t}\zeta_\alpha + c\hat{d}d'\int_0^t e^{-\omega(t-s)}e^{-\omega' s}\left(1+\|\left[\begin{smallmatrix} w \\ \chi \end{smallmatrix}\right]\|_{X_2^{\frac{1+\alpha}{2}}\times X_2^{\frac{\alpha}{2}}}\right)ds,\ t\geqslant 0$$

and therefore (3.27) follows from Gronwall's inequality. □

As a consequence of Lemma 3.4 we obtain the following result

Lemma 3.5. *For every $\alpha\in(0,\frac{1}{2})$, $\bigcup_{\eta\in(0,1]}\boldsymbol{A}_\eta$ is a bounded subset of $X_2^{\frac{1+\alpha}{2}}\times X_2^{\frac{\alpha}{2}}$.*

Proof: It follows from the proof of [5, Lemma 3.5] that any complete orbit $\gamma_\eta(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right])$, $\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\in\boldsymbol{A}_\eta$, is precompact in Y_2^2. Hence, there exists a subsequence $\{T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\}$ convergent in Y_2^2 to an element of the α-limit set $\alpha_\eta(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right])\subset\mathcal{E}$. Letting $T_\eta(t)T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right] =: \left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right](t,T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right],\eta)$, from (3.5) we then have

$$\left[\begin{smallmatrix} \tilde{u}_\eta \\ \tilde{v}_\eta \end{smallmatrix}\right](t,T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right],\eta) = e^{-\mathcal{A}_\eta t}T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right] + \int_0^t e^{-\mathcal{A}_\eta(t-s)}\tilde{F}(\left[\begin{smallmatrix} \tilde{u}_\eta \\ \tilde{v}_\eta \end{smallmatrix}\right](s,T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right],\eta))ds,$$

and, since $\mathcal{E}$ is bounded in Y_2^2 (see (2.13)), choosing $\eta>0$ we find certain $N_\eta\in\mathbb{N}$ such that for all $k_l\geqslant N_\eta(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right])$

$$\|T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\|_{Y_2^2}\leqslant M_1^{\mathcal{E}}+1. \tag{3.36}$$

Therefore, we get

$$\begin{aligned}
\|\mathcal{A}_\eta\left[\begin{smallmatrix} \tilde{u}_\eta \\ \tilde{v}_\eta \end{smallmatrix}\right](t,T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right],\eta)\|_{Y_2^1} &\leqslant \|e^{-\mathcal{A}_\eta t}\|_{L(Y_2^1)}\|\mathcal{A}_\eta T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\|_{Y_2^1} \\
&+\int_0^t \|e^{-\mathcal{A}_\eta(t-s)}\|_{L(Y_2^1)}\|\mathcal{A}_\eta\left(\tilde{F}(\left[\begin{smallmatrix} \tilde{u}_\eta \\ \tilde{v}_\eta \end{smallmatrix}\right](s,T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right],\eta)\right)\|_{Y_2^1}ds \\
&\leq ce^{-\omega t}(M_1^{\mathcal{E}}+1)+\int_0^t ce^{-\omega(t-s)}\|\mathcal{A}_\eta\left(\tilde{F}(\left[\begin{smallmatrix} \tilde{u}_\eta \\ \tilde{v}_\eta \end{smallmatrix}\right](s,T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right],\eta)\right)\|_{Y_2^1}ds.
\end{aligned}\tag{3.37}$$

¿From (2.7), (2.2) and (3.8) we have, for $\eta\in(0,1]$,

$$\begin{aligned}
\|\mathcal{A}_\eta\tilde{F}\left(\left[\begin{smallmatrix} \tilde{u}_\eta \\ \tilde{v}_\eta \end{smallmatrix}\right]\right)\|_{Y_2^1} &= \|\mathcal{A}_\eta\left[\begin{smallmatrix} 0 \\ \tilde{f}(\tilde{u}_\eta) \end{smallmatrix}\right]\|_{Y_2^1}\leqslant(1+2\eta+\frac{a}{\lambda_1})\|A^{\frac{1}{2}}(f(\tilde{u}_\eta)-f(0)-K_f\tilde{u}_\eta)\|_{X_2} \\
&\leqslant C(\|(1+|\tilde{u}_\eta|^2)\nabla\tilde{u}_\eta\|_{X_2}+\|\nabla\tilde{u}_\eta\|_{X_2})\leqslant(C+1)\|\nabla\tilde{u}_\eta\|_{X_2}+C\||\tilde{u}_\eta|^2\nabla\tilde{u}_\eta\|_{X_2} \\
&\leqslant(C+1)c_\varsigma e^{-\omega_\varsigma t}+C\|\tilde{u}_\eta\|_{X_6}^2\|\nabla\tilde{u}_\eta\|_{X_6}\leqslant(C+1)c_\varsigma e^{-\omega_\varsigma t}+\tilde{C}\|A^{\frac{1}{2}}\tilde{u}_\eta\|_{X_2}^2\|A\tilde{u}_\eta\|_{X_2} \\
&\leqslant(C+1)c_\varsigma e^{-\omega_\varsigma t}+d\tilde{C}c_\varsigma^2e^{-2\omega_\varsigma t}\|\mathcal{A}_\eta\left[\begin{smallmatrix} \tilde{u}_\eta \\ \tilde{v}_\eta \end{smallmatrix}\right]\|_{Y_2^1}.
\end{aligned}\tag{3.38}$$

Joining (3.37), (3.38), solving the resulting integral inequality and using (2.2) we conclude that there exists a positive constant $\hat{c}$ such that

$$\forall_{\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right]\in\mathbf{A}_\eta}\ \forall_{k_l\geq N_\eta(\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right])}\ \|\left[\begin{smallmatrix} \tilde{u}_\eta \\ \tilde{v}_\eta \end{smallmatrix}\right](t,T_\eta(-k_l)\left[\begin{smallmatrix} u_{0\eta} \\ v_{0\eta} \end{smallmatrix}\right],\eta)\|_{Y_2^2}\leqslant\hat{c},\ t\geq 0,\ \eta\in(0,1]. \tag{3.39}$$

We now use (3.6) to conclude the proof. First note that, from Lemma 3.4, we have

$$\begin{aligned}\sup_{\eta\in(0,1]}\sup_{\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\in\boldsymbol{A}_\eta}&\int_0^t\|(\partial\tilde{T}_\eta)(t-s,T_\eta(s)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])\hat{F}(T_\eta(s)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])\|_{X_2^{\frac{1+\alpha}{2}}\times X_2^{\frac{\alpha}{2}}}ds\\ &\leqslant\int_0^t\sup_{\eta\in(0,1]}\sup_{\left[\begin{smallmatrix}\phi_\eta\\\psi_\eta\end{smallmatrix}\right]\in\boldsymbol{A}_\eta}\|(\partial\tilde{T}_\eta)(t-s,\left[\begin{smallmatrix}\phi_\eta\\\psi_\eta\end{smallmatrix}\right])\hat{F}(\left[\begin{smallmatrix}\phi_\eta\\\psi_\eta\end{smallmatrix}\right])\|_{X_2^{\frac{1+\alpha}{2}}\times X_2^{\frac{\alpha}{2}}}ds\\ &\leqslant\frac{d''}{\omega''}\left(1-e^{-\omega''t}\right)\leqslant\frac{d''}{\omega''},\ t\geq 0.\end{aligned}\tag{3.40}$$

Connecting (3.40), (3.39) and (3.6) we get the result. □

Using Lemma 3.5 we derive the uniform bound on the attractors in Y_2^2−norm.

Theorem 3.1. *If (2.7) and (2.8) hold, then $\bigcup_{\eta\in(0,1]}\boldsymbol{A}_\eta$ is a bounded subset of Y_2^2; that is, there is a positive constant* $\mathbf{C}$ *such that*

$$\sup_{\eta\in(0,1]}\sup_{\left[\begin{smallmatrix}\phi_\eta\\\psi_\eta\end{smallmatrix}\right]\in\boldsymbol{A}_\eta}\|\left[\begin{smallmatrix}\phi_\eta\\\psi_\eta\end{smallmatrix}\right]\|_{Y_2^2}\leq\mathbf{C}.\tag{3.41}$$

Proof: Since a complete orbit $\gamma_\eta(\left[\begin{smallmatrix}u_{0\eta}\\v_{0\eta}\end{smallmatrix}\right])$, $\left[\begin{smallmatrix}u_{0\eta}\\v_{0\eta}\end{smallmatrix}\right]\in\boldsymbol{A}_\eta$, is precompact in Y_2^2, there exists a subsequence $\{T_\eta(-k_l)\left[\begin{smallmatrix}u_{0\eta}\\v_{0\eta}\end{smallmatrix}\right]\}$ convergent in Y_2^2 to an element of the α-limit set $\alpha_{\gamma_\eta}(\left[\begin{smallmatrix}u_{0\eta}\\v_{0\eta}\end{smallmatrix}\right])\subset\mathcal{E}$. Recall that $\mathcal{E}$ is bounded in Y_2^2, let $T_\eta(t)T_\eta(-k_l)\left[\begin{smallmatrix}u_{0\eta}\\v_{0\eta}\end{smallmatrix}\right]=:\left[\begin{smallmatrix}u_\eta\\v_\eta\end{smallmatrix}\right]$ and write the integral formula

$$\left[\begin{smallmatrix}u_\eta\\v_\eta\end{smallmatrix}\right]=e^{-\mathcal{A}_\eta t}T_\eta(-k_l)\left[\begin{smallmatrix}u_{0\eta}\\v_{0\eta}\end{smallmatrix}\right]+\int_0^t e^{-\mathcal{A}_\eta(t-s)}F(\left[\begin{smallmatrix}u_\eta\\v_\eta\end{smallmatrix}\right])ds,\ t\geq 0,$$

to obtain with the aid of (2.4) and (3.36) that

$$\begin{aligned}\|\mathcal{A}_\eta\left[\begin{smallmatrix}u_\eta\\v_\eta\end{smallmatrix}\right]\|_{Y_2^1}&\leqslant\|e^{-\mathcal{A}_\eta t}\|_{L(Y_2^1)}\|\mathcal{A}_\eta T_\eta(-k_l)\left[\begin{smallmatrix}u_{0\eta}\\v_{0\eta}\end{smallmatrix}\right]\|_{Y_2^1}+\|\left(I-e^{-\mathcal{A}_\eta t}\right)F(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right])\|_{Y_2^1}\\ &\quad+\int_0^t\|e^{-\mathcal{A}_\eta(t-s)}\|_{L(Y_2^1)}\|\mathcal{A}_\eta\left(F(\left[\begin{smallmatrix}u_\eta\\v_\eta\end{smallmatrix}\right])-F(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right])\right)\|_{Y_2^1}ds\\ &\leq ce^{-\omega t}(M_1^{\mathcal{E}}+1)+(1+ce^{-\omega t})\|F(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right])\|_{Y_2^1}\\ &\quad+\int_0^t ce^{-\omega(t-s)}\|\mathcal{A}_\eta\left(F(\left[\begin{smallmatrix}u_\eta\\v_\eta\end{smallmatrix}\right])-F(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right])\right)\|_{Y_2^1}ds.\end{aligned}\tag{3.42}$$

Then, for each $\eta\in(0,1]$, we have

$$\begin{aligned}&\|\mathcal{A}_\eta\big(F\left(\left[\begin{smallmatrix}u_\eta\\v_\eta\end{smallmatrix}\right]\right)-F\left(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\right)\big)\|_{Y_2^1}=\|\mathcal{A}_\eta\left[\begin{smallmatrix}0\\f(u_\eta)-f(0)\end{smallmatrix}\right]\|_{Y_2^1}\\ &\quad\leqslant C\|\nabla(f(u_\eta)-f(0))\|_{X_2}+a\|f(u_\eta)-f(0)\|_{X_2}=C\|f'(u_\eta)\nabla u_\eta\|_{X_2}+a\|f(u_\eta)-f(0)\|_{X_2}\end{aligned}$$

so that, using (2.7) and Lemma 3.5 with $\alpha\in[\frac{1}{3},\frac{1}{2})$, we get

$$\begin{aligned}&\|\mathcal{A}_\eta\big(F\left(\left[\begin{smallmatrix}u_\eta\\v_\eta\end{smallmatrix}\right]\right)-F\left(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\right)\big)\|_{Y_2^1}\leqslant\tilde{C}(\|(1+|u_\eta|^2)\nabla u_\eta\|_{X_2}+1)\\ &\quad\leqslant\hat{C}(\||u_\eta|^2\nabla u_\eta\|_{X_2}+1)\leqslant\hat{C}(\|u_\eta\|^2_{X_{\frac{6}{\alpha}}}\|\nabla u_\eta\|_{X_{\frac{6}{3-2\alpha}}}+1)\leqslant\hat{C}'(\|u_\eta\|^3_{X_2^{\frac{1+\alpha}{2}}}+1)\leqslant\bar{C}.\end{aligned}\tag{3.43}$$

¿From (3.42), (3.43) and (2.2) we obtain (3.41). The proof is thus complete. □

Theorem 3.2. *Under the assumptions of Theorem 3.1 the family of attractors $\{\mathbf{A}_\eta,\ \eta \in (0,1]\}$, is upper semicontinuous at $\eta = 0$ in Y_2^1; that is*

$$\sup_{a_\eta \in \mathbf{A}_\eta} \inf_{a_0 \in \mathbf{A}_0} \|a_0 - a_\eta\|_{Y_2^1} \to 0 \quad as \quad \eta \to 0^+.$$

Proof. If $\eta_n \to 0^+$ and $\left[\begin{smallmatrix} u_{0\eta_n} \\ v_{0\eta_n} \end{smallmatrix}\right] \in \mathbf{A}_{\eta_n}$, then as a consequence of Theorem 3.1 and compactness of the embedding $Y_2^2 \hookrightarrow Y_2^1$ there exists a subsequence $\left\{\left[\begin{smallmatrix} u_{0\eta_{n_k}} \\ v_{0\eta_{n_k}} \end{smallmatrix}\right]\right\}$ convergent in Y_2^1 to certain $\left[\begin{smallmatrix} \hat{u}_0 \\ \hat{v}_0 \end{smallmatrix}\right] \in Y_2^1$. Whenever $\left[\begin{smallmatrix} \phi_n \\ \psi_n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]$ in Y_2^1, from [5, Lemma 3.1] we have for every $\tau > 0$ that

$$\sup_{t\in[0,\tau]} \|T_{\eta_n}(t) \left[\begin{smallmatrix} \phi_n \\ \psi_n \end{smallmatrix}\right] - T_0(t) \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \|_{Y_2^1} \to 0 \quad \text{as} \quad \eta_n \to 0^+.$$

Therefore, it is possible to construct a complete bounded orbit of the limit semigroup $\{T_0(t)\}$ through $\left[\begin{smallmatrix} \hat{u}_0 \\ \hat{v}_0 \end{smallmatrix}\right]$. Since such orbit is contained in $\mathbf{A}_0$, the proof is complete. □

As a consequence of the compactness of embedding $Y_2^2 \hookrightarrow Y_2^{1+\alpha}$, $\alpha \in [0,1)$, we have that

Corollary 3.1. *The family of attractors $\{\mathbf{A}_\eta,\ \eta \in (0,1]\}$ is upper semicontinuous at $\eta = 0$ in $X_2^{\frac{1+\alpha}{2}} \times X_2^{\frac{\alpha}{2}}$ for each $\alpha \in [0,1)$; that is, $\sup_{a_\eta \in \mathbf{A}_\eta} \inf_{a_0 \in \mathbf{A}_0} \|a_0 - a_\eta\|_{X_2^{\frac{1+\alpha}{2}} \times X_2^{\frac{\alpha}{2}}} \to 0$ as $\eta \to 0^+$, for each $\alpha \in [0,1)$.*

4. $H^3(\Omega) \times H^2(\Omega) \cap C^{1+\frac{1}{2}^-}(\bar{\Omega}) \times C^{\frac{1}{2}^-}(\bar{\Omega})$–REGULARITY OF $\mathbf{A}_0$

In this section uniform $H^3(\Omega) \times H^2(\Omega)$–bound for the attractors is established. We also obtain $H^3(\Omega) \times H^2(\Omega) \cap C^{1+\frac{1}{2}^-}(\bar{\Omega}) \times C^{\frac{1}{2}^-}(\bar{\Omega})$–regularity of the limit attractor $\mathbf{A}_0$.

Consider

$$\frac{d}{dt} \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] + \mathcal{A}_\eta \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right],\ t > 0, \quad \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]_{t=0} = \left[\begin{smallmatrix} u_0 - A^{-1}f(0) \\ v_0 \end{smallmatrix}\right], \tag{4.1}$$

and the solution $\left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right])$ of

$$\frac{d}{dt} \left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right] + \mathcal{A}_\eta \left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ f(u_\eta) \end{smallmatrix}\right],\ t > 0, \quad \left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right]_{t=0} = \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in Y_2^1. \tag{4.2}$$

Define

$$\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right](t, u_\eta) := \left[\begin{smallmatrix} u_\eta \\ v_\eta \end{smallmatrix}\right] - e^{-\mathcal{A}_\eta t} \left[\begin{smallmatrix} u_0 - A^{-1}f(0) \\ v_0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} A^{-1}f(0) \\ 0 \end{smallmatrix}\right]. \tag{4.3}$$

Then $\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right](t, u_\eta)$ solves

$$\frac{d}{dt} \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] + \mathcal{A}_\eta \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ f(u_\eta) - f(0) \end{smallmatrix}\right],\ t > 0, \quad \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]_{t=0} = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \tag{4.4}$$

and fulfils the integral equation

$$\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right](t, u_\eta) = \int_0^t e^{-\mathcal{A}_\eta (t-s)} \left[\begin{smallmatrix} 0 \\ f(u_\eta(s)) - f(0) \end{smallmatrix}\right] ds,\ t \geq 0, \tag{4.5}$$

whereas from (2.4) we have that, for all $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in Y_2^1$

$$\|e^{-\mathcal{A}_\eta t} \left[\begin{smallmatrix} u_0 - A^{-1}f(0) \\ v_0 \end{smallmatrix}\right] \|_{Y_2^1} \leq c e^{-\omega t} \| \left[\begin{smallmatrix} u_0 - A^{-1}f(0) \\ v_0 \end{smallmatrix}\right] \|_{Y_2^1},\ t \geq 0,\ \eta \in (0,1]. \tag{4.6}$$

It is easy to see that there is a constant $d_1 > 0$, independent of $\eta \in (0,1]$, such that

$$d_1^{-1} \| \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \|_{Y_2^3} \leq \|\mathcal{A}_\eta^2 \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \|_{Y_2^1} \leq d_1 \| \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \|_{Y_2^3},\ \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \in Y_2^3. \tag{4.7}$$

Lemma 4.1. *If (2.7) and (2.8) hold, then* $\tilde{\mathbf{A}} := \begin{bmatrix} -A^{-1}f(0) \\ 0 \end{bmatrix} + \bigcup_{\eta\in(0,1]} \boldsymbol{A}_\eta$ *is a bounded subset of* Y_2^3 *and the union of attractors* $\bigcup_{\eta\in(0,1]} \boldsymbol{A}_\eta$ *is bounded in* $H^3(\Omega)\times H^2(\Omega)$*−norm.*

Proof: From (3.41) and

$$
\begin{aligned}
\|\mathcal{A}_\eta{}^2 \begin{bmatrix} 0 \\ f(\phi_\eta)-f(0) \end{bmatrix}\|_{Y_2^1} &\leq d_1 \| \begin{bmatrix} 0 \\ f(\phi_\eta)-f(0) \end{bmatrix}\|_{Y_2^3} = d_1 \| \begin{bmatrix} 0 \\ f(\phi_\eta)-f(0) \end{bmatrix}\|_{X_2^{\frac{3}{2}}\times X_2^1} \\
&= d_1 \|A(f(\phi_\eta)-f(0))\|_{X_2} = d_1\|f''(\phi_\eta)|\nabla\phi_\eta|^2 + f'(\phi_\eta)\Delta\phi_\eta\|_{X_2},
\end{aligned}
$$

we have that

$$
\sup_{\eta\in(0,1]} \sup_{\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}\in\boldsymbol{A}_\eta} \|\mathcal{A}_\eta{}^2 \begin{bmatrix} 0 \\ f(\phi_\eta)-f(0) \end{bmatrix}\|_{Y_2^1} \leq \tilde{\mathbf{C}}. \tag{4.8}
$$

If $\begin{bmatrix} u_\eta \\ v_\eta \end{bmatrix}$ is a solution of (4.2) with $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \in \mathbf{A}_\eta$, then for $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}(t,u_\eta)$ in (4.5) we have[1]

$$
\begin{aligned}
&\sup_{\eta\in(0,1]} \sup_{\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}\in\boldsymbol{A}_\eta} \|\mathcal{A}_\eta^2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}(t,u_\eta)\|_{Y_2^1} \\
&\leq \sup_{\eta\in(0,1]} \sup_{\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}\in\boldsymbol{A}_\eta} \int_0^t \|e^{-\mathcal{A}_\eta(t-s)}\|_{L(Y_2^1)} \|\mathcal{A}_\eta^2(\begin{bmatrix} 0 \\ f(u_\eta(s))-f(0) \end{bmatrix})\|_{Y_2^1} ds \leq \frac{c}{\omega}\tilde{\mathbf{C}}(1-e^{-\omega t}), \quad t\geq 0.
\end{aligned}
$$

This together with (4.7) implies that

$$
\sup_{t\geq 0} \sup_{\eta\in(0,1]} \sup_{\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}\in\boldsymbol{A}_\eta} \| \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}(t,u_\eta)\|_{Y_2^3} \leq \frac{cd_1}{\omega}\tilde{\mathbf{C}}. \tag{4.9}
$$

Fix $\eta\in(0,1]$, $\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \in \boldsymbol{A}_\eta$ and let $\gamma_\eta = \{T_\eta(t)\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}, t\in\mathbb{R}\}$ be the double-sided orbit through $\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}$. If $k_n\to\infty$ and $\begin{bmatrix} u_{0n\eta} \\ v_{0n\eta} \end{bmatrix} = T_\eta(-k_n)\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}$, then (4.3) reads

$$
\begin{aligned}
\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} &= T_\eta(k_n)\begin{bmatrix} u_{0n\eta} \\ v_{0n\eta} \end{bmatrix} = \begin{bmatrix} u_\eta^n \\ v_\eta^n \end{bmatrix}(k_n) \\
&= \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}(k_n, u_\eta^n) + e^{-\mathcal{A}_\eta k_n}\begin{bmatrix} u_{0n\eta}-A^{-1}f(0) \\ v_{0n\eta} \end{bmatrix} + \begin{bmatrix} A^{-1}f(0) \\ 0 \end{bmatrix}, \; n\in\mathbb{N},
\end{aligned}
$$

where $\begin{bmatrix} u_\eta^n \\ v_\eta^n \end{bmatrix}$ is a solution of (4.2) with $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} u_{0n\eta} \\ v_{0n\eta} \end{bmatrix}$. Since $\begin{bmatrix} u_{0n\eta} \\ v_{0n\eta} \end{bmatrix} = T_\eta(-k_n)\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \in \mathbf{A}_\eta$, from (3.2) and (4.6), taking the limit in Y_2^1 we obtain that

$$
\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} - \begin{bmatrix} A^{-1}f(0) \\ 0 \end{bmatrix} = \lim_{k_n\to\infty} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}(k_n, u_\eta). \tag{4.10}
$$

---

[1] We remark that $\begin{bmatrix} u_\eta \\ v_\eta \end{bmatrix}(t, \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}) \in \boldsymbol{A}_\eta$ for all $t\geq 0$, $\eta\in(0,1]$.

With the aid of (4.9) we infer that quantity $\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right] - \left[\begin{smallmatrix}A^{-1}f(0)\\ 0\end{smallmatrix}\right]$ is simultaneously a limit of a sequence convergent weakly in Y_2^3, which translates into the bound

$$\sup_{\eta\in(0,1]} \sup_{\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right]\in \boldsymbol{A}_\eta} \| \left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right] - \left[\begin{smallmatrix}A^{-1}f(0)\\ 0\end{smallmatrix}\right] \|_{Y_2^3} \leq \frac{cd_1}{\omega}\tilde{\mathbf{C}}. \tag{4.11}$$

This in the light of (2.5) and the elliptic regularity theory completes the proof.[2] □

Corollary 4.1. *Under the assumptions of Lemma 4.1 the union of attractors* $\bigcup_{\eta\in(0,1]}\boldsymbol{A}_\eta$ *is precompact in* $C^{1+\mu}(\bar\Omega)\times C^\mu(\bar\Omega)$ *for each* $\mu\in(0,\frac{1}{2})$.

Below we adress regularity properties of the limit attractor $\mathbf{A_0}$.

Theorem 4.1. *Suppose that* (1.2), (2.7), (2.8) *hold and that* $\lim_{|s|\to\infty}\frac{f'(s)}{s^2}=0$. *Then the attractor* $\boldsymbol{A}_0$ *for the semigroup* $\{T_0(t)\}$ *associated to* (2.6) *with* $\eta=0$ *in* $H_0^1(\Omega)\times L^2(\Omega)$ *is bounded in* $H^3(\Omega)\times H^2(\Omega)$ *and compact in* $C^{1+\mu}(\bar\Omega)\times C^\mu(\bar\Omega)$, $\mu\in(0,\frac{1}{2})$.

Proof: As shown in [5], the family of attractors $\boldsymbol{A}_\eta$ is lower semicontinuous at $\eta=0$ with respect to the Hausdorff semidistance in Y_2^1. Therefore, if $\left[\begin{smallmatrix}\phi_0\\ \psi_0\end{smallmatrix}\right]\in\boldsymbol{A}_0$ then $\left[\begin{smallmatrix}\phi_0\\ \psi_0\end{smallmatrix}\right]$ is a limit in Y_2^1 of a sequence $\{\left[\begin{smallmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{smallmatrix}\right]\}$ as $\eta_n\to0^+$, where $\left[\begin{smallmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{smallmatrix}\right]\in\boldsymbol{A}_{\eta_n}$. As a consequence of Lemma 4.1 the sequence $\{\left[\begin{smallmatrix}\phi_{\eta_n}\\ \psi_{\eta_n}\end{smallmatrix}\right]\}$ is bounded in $H^3(\Omega)\times H^2(\Omega)-$norm by certain positive constant $D>0$ uniformly with respect to $n\in\mathbb{N}$. The latter ensures that $\left[\begin{smallmatrix}\phi_0\\ \psi_0\end{smallmatrix}\right]$ is a weak limit of a subsequence bounded in $H^3(\Omega)\times H^2(\Omega)-$norm and thus is bounded by the same constant D. As a consequence we get

$$\sup_{\left[\begin{smallmatrix}\phi_0\\ \psi_0\end{smallmatrix}\right]\in\boldsymbol{A}_0} \| \left[\begin{smallmatrix}\phi_0\\ \psi_0\end{smallmatrix}\right] \|_{H^3(\Omega)\times H^2(\Omega)} < \infty,$$

which completes the proof. □

Corollary 4.2. *Under the assumptions of Theorem 4.1, the family of attractors* $\{\mathbf{A}_\eta,\ \eta\in(0,1],\}$ *is upper and lower semicontinuous at* $\eta=0$ *in* $C^{1+\mu}(\bar\Omega)\times C^\mu(\bar\Omega)$ *for each* $\mu\in(0,\frac{1}{2})$; *that is*

$$\sup_{a_\eta\in\boldsymbol{A}_\eta}\inf_{a_0\in\boldsymbol{A}_0}\|a_0-a_\eta\|_{C^{1+\mu}(\bar\Omega)\times C^\mu(\bar\Omega)}+\sup_{a_0\in\boldsymbol{A}_0}\inf_{a_\eta\in\boldsymbol{A}_\eta}\|a_0-a_\eta\|_{C^{1+\mu}(\bar\Omega)\times C^\mu(\bar\Omega)}\to0\quad as\quad \eta\to0^+,$$

for each $\mu\in(0,\frac{1}{2})$.

5. $H^4(\Omega)\times H^3(\Omega)\cap C^{2+\frac{1}{2}^-}(\bar\Omega)\times C^{1+\frac{1}{2}^-}(\bar\Omega)-$REGULARITY OF $\mathbf{A_0}$

In this section uniform $H^4(\Omega)\times H^3(\Omega)-$bound for approximating attractors is derived and $H^4(\Omega)\times H^3(\Omega)\cap C^{2+\frac{1}{2}^-}(\bar\Omega)\times C^{1+\frac{1}{2}^-}(\bar\Omega)-$regularity of the limit attractor $\mathbf{A_0}$ is concluded.

Lemma 5.1. *Suppose that the assumptions of Lemma 4.1 hold and, in addition,* $f\in C^3(\mathbb{R},\mathbb{R})$. *Then, the union of attractors* $\bigcup_{\eta\in(0,1]}\mathbf{A}_\eta$ *is bounded in* $H^4(\Omega)\times H^3(\Omega)$.

[2] The idea of decomposing the solution as in this proof comes back to [14, Theorem 2.4] however our choice of the initial data in (4.1) is different than in [14].

Proof: We differentiate (4.4) and substitute $w_3 = \frac{d}{dt}w_2$ to get

$$\frac{d}{dt}\begin{bmatrix} w_2 \\ w_3 \end{bmatrix} + \mathcal{A}_\eta \begin{bmatrix} w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ f'(u_\eta)v_\eta \end{bmatrix}, \ t > 0, \quad \begin{bmatrix} w_2 \\ w_3 \end{bmatrix}_{t=0} = \begin{bmatrix} 0 \\ f(u_0)-f(0) \end{bmatrix}. \tag{5.1}$$

We then have

$$\begin{bmatrix} w_2 \\ w_3 \end{bmatrix}(t,u_\eta) = e^{-\mathcal{A}_\eta t}\begin{bmatrix} 0 \\ f(u_0)-f(0) \end{bmatrix} + \int_0^t e^{-\mathcal{A}_\eta(t-s)}\begin{bmatrix} 0 \\ f'(u_\eta(s))v_\eta(s) \end{bmatrix} ds, \ t \geq 0, \tag{5.2}$$

and from (2.4)

$$\begin{aligned} \|\mathcal{A}_\eta^{\frac{1}{2}}\begin{bmatrix} w_2 \\ w_3 \end{bmatrix}(t,u_\eta)\|_{Y_2^1} &\leq ce^{-\omega t}\|\mathcal{A}_\eta^{\frac{1}{2}}\begin{bmatrix} 0 \\ f(u_0)-f(0) \end{bmatrix}\|_{Y_2^1} \\ &+ \int_0^t ce^{-\omega(t-s)}\|\mathcal{A}_\eta^{\frac{1}{2}}\begin{bmatrix} 0 \\ f'(u_\eta(s))v_\eta(s) \end{bmatrix}\|_{Y_2^1} ds, \ t \geq 0. \end{aligned} \tag{5.3}$$

For $\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \in \boldsymbol{A}_\eta$ condition (4.11) ensures that $\psi_\eta \in X_2^1$ and

$$\sup_{\eta\in(0,1]} \sup_{\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}\in \boldsymbol{A}_\eta} \|\psi_\eta\|_{X_2^1} \leq \frac{cd_1}{\omega}\tilde{\mathbf{C}}, \tag{5.4}$$

whereas Lemma 4.1 implies that $\phi_\eta \in X_6^1 \cap C^{1+\frac{1}{2}^-}(\bar{\Omega})$ and

$$\sup_{\eta\in(0,1]} \sup_{\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}\in \boldsymbol{A}_\eta} \|\phi_\eta\|_{X_6^1\cap C^{1+\frac{1}{2}^-}(\bar{\Omega})} < \infty. \tag{5.5}$$

Since norms $\|\mathcal{A}_\eta^{\frac{1}{2}} \cdot \|_{Y_2^1}$ and $\|\cdot\|_{X_2^{\frac{3}{2}}\times X_2^{\frac{1}{2}}}$ are equivalent uniformly for $\eta \in (0,1]$ (see (4.7)) and

$$\begin{aligned} \|f'(\phi_\eta)\psi_\eta\|_{X_2^1} &= \|\Delta(f'(\phi_\eta)\psi_\eta)\|_{X_2} = \|f'''(\phi_\eta)|\nabla\phi_\eta|^2\psi_\eta + f''(\phi_\eta)\Delta\phi_\eta\psi_\eta \\ &+ (f''(\phi_\eta) + f'(\phi_\eta))\nabla\phi_\eta\nabla\psi_\eta + f'(\phi_\eta)\Delta\psi_\eta\|_{X_2} \ \text{ for } \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \in \boldsymbol{A}_\eta, \end{aligned}$$

then (5.4)-(5) lead to the estimate

$$\sup_{\eta\in(0,1]} \sup_{\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}\in \boldsymbol{A}_\eta} \|f'(\phi_\eta)\psi_\eta\|_{X_2^1} \leq \mathbf{G}. \tag{5.6}$$

If $\begin{bmatrix} u_\eta \\ v_\eta \end{bmatrix}$ is a solution of (4.2) through $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix} \in \mathbf{A}_\eta$, then (4.8), (5.3) imply

$$\|\mathcal{A}_\eta^{\frac{1}{2}}\begin{bmatrix} w_2 \\ w_3 \end{bmatrix}(t,u_\eta)\|_{Y_2^1} \leq c\tilde{\mathbf{C}} + \frac{c}{\omega}\sup_{s\geq 0}\|\mathcal{A}_\eta^{\frac{1}{2}}\begin{bmatrix} 0 \\ f'(u_\eta(s))v_\eta(s) \end{bmatrix}\|_{Y_2^1}, \ t \geq 0,$$

which via (4.7) and (5.6) leads to inequality

$$\sup_{t\geq 0} \sup_{\eta\in(0,1]} \sup_{\begin{bmatrix} \phi_\eta \\ \psi_\eta \end{bmatrix}\in \boldsymbol{A}_\eta} \|\begin{bmatrix} w_2 \\ w_3 \end{bmatrix}(t,u_\eta)\|_{X_2^{\frac{3}{2}}\times X_2^{\frac{1}{2}}} \leq cd_1\left(\tilde{\mathbf{C}} + \frac{d_1}{\omega}\mathbf{G}\right) =: \mathbf{D}. \tag{5.7}$$

For any fixed $t \geq 0$, $\eta \in (0,1])$, function $z := w_1(t,u_\eta)$ can be viewed as the solution of

$$Az = -w_3(t,u_\eta) - aw_2(t,u_\eta) - A^{\frac{1}{2}}w_2(t,u_\eta) + (f(u_\eta) - f(0)) =: \mathbf{H}(t,u_\eta). \tag{5.8}$$

As a consequence of (5.5), (5.7),

$$\sup_{t\geq 0}\sup_{\eta\in(0,1]}\sup_{\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right]\in \boldsymbol{A}_\eta}\|\mathbf{H}(t,u_\eta)\|_{X^1_{\frac{1}{2}}}<\infty.$$

Hence

$$\sup_{t\geq 0}\sup_{\eta\in(0,1]}\sup_{\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right]\in \boldsymbol{A}_\eta}\|w_1(t,u_\eta)\|_{X^2_{\frac{1}{2}}}\leq \tilde{\mathbf{D}},$$

which together with (5.7) proves that

$$\sup_{t\geq 0}\sup_{\eta\in(0,1]}\sup_{\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right]\in \boldsymbol{A}_\eta}\|\left[\begin{smallmatrix}w_1\\ w_2\end{smallmatrix}\right](t,u_\eta)\|_{Y^4_{\frac{1}{2}}}\leq \hat{\mathbf{D}}. \tag{5.9}$$

As in the proof of Lemma 4.1 for $\eta\in(0,1]$ and $\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right]\in \boldsymbol{A}_\eta$ let $\gamma_\eta=\{T_\eta(t)\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right],t\in\mathbb{R}\}$ be the orbit through $\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right]$. If $k_n\to\infty$ and $\left[\begin{smallmatrix}u_{0n\eta}\\ v_{0n\eta}\end{smallmatrix}\right]=T_\eta(-k_n)\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right]$, then (4.3) leads to (4) and (4.10) holds with the limit taken in $Y^1_{\frac{1}{2}}$. Thanks to (5.9) and weak convergence property

$$\sup_{\eta\in(0,1]}\sup_{\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right]\in \boldsymbol{A}_\eta}\|\left[\begin{smallmatrix}\phi_\eta\\ \psi_\eta\end{smallmatrix}\right]-\left[\begin{smallmatrix}A^{-1}f(0)\\ 0\end{smallmatrix}\right]\|_{Y^4_{\frac{1}{2}}}\leq \hat{\mathbf{D}}. \tag{5.10}$$

The proof is complete. □

Theorem 5.1. *If the assumptions of Theorem 4.1 hold and, in addition, $f\in C^3(\mathbb{R},\mathbb{R})$, then $\mathbf{A}_0$ is bounded in $H^4(\Omega)\times H^3(\Omega)$ and compact in $C^{2+\mu}(\bar\Omega)\times C^{1+\mu}(\bar\Omega)$ for each $\mu\in(0,\frac{1}{2})$.*

Proof: The result follows from property (1.3), Lemma 5.1 and Sobolev type embedding. □

Corollary 5.1. *Under the assumptions of Theorem 5.1, the family of attractors $\{\mathbf{A}_\eta,\ \eta\in(0,1],\}$ is upper and lower semicontinuous at $\eta=0$ in $C^{2+\mu}(\bar\Omega)\times C^{1+\mu}(\bar\Omega)$ for each $\mu\in(0,\frac{1}{2})$; that is*

$$\sup_{a_\eta\in\boldsymbol{A}_\eta}\inf_{a_0\in\boldsymbol{A}_0}\|a_0-a_\eta\|_{C^{2+\mu}(\bar\Omega)\times C^{1+\mu}(\bar\Omega)}+\sup_{a_0\in\boldsymbol{A}_0}\inf_{a_\eta\in\boldsymbol{A}_\eta}\|a_0-a_\eta\|_{C^{2+\mu}(\bar\Omega)\times C^{1+\mu}(\bar\Omega)}\to 0\quad as\quad \eta\to 0^+,$$

for each $\mu\in(0,\frac{1}{2})$.

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