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Resumo

Neste artigo tratamos a semicontinuidade inferior de atratores para equações diferenciais semilineares (autônomas ou não autônomas) em espaços de Banach. Assumimos que o atrator do problema limite é dado pela união de variedades instáveis de soluções globais hiperbólicas estendendo os resultados anteriores que são apenas válidos para o caso em que estas soluções são equilíbrios. As ferramentas empregadas são o estudo da continuidade das variedades instáveis locais das soluções globais hiperbólicas e o estudo da continuidade da dicotomia exponencial da linearização em torno destas soluções.

LOWER SEMICONTINUITY OF ATTRACTORS FOR NON-GRADIENT DYNAMICAL SYSTEMS

ALEXANDRE N. CARVALHO†, JOSÉ A. LANGA‡, AND JAMES C. ROBINSON*

ABSTRACT. This paper is concerned with the lower semicontinuity of attractors for semilinear (autonomous and non-autonomous) differential equations in Banach spaces. We require the unperturbed attractor to be given as the union of unstable manifolds of potentially time-dependent hyperbolic solutions, unlike previous results only valid for gradient-like systems in which the hyperbolic solutions are equilibria. The tools employed are the study of the continuity of the local unstable manifolds of the hyperbolic solutions and a study of the continuity of the exponential dichotomy of the linearization around each of these solutions.

1. INTRODUCTION

Results on the upper semicontinuity of attractors with respect to perturbations ('no explosion') are relatively easy to prove, and are now essentially classical. Results on lower semicontinuity ('no collapse') are much more difficult: generally they involve assumptions on the structure of the unperturbed attractor, and one then tries to reproduce a similar structure inside the perturbed attractor. Currently these lower semicontinuity results are restricted to the class of autonomous dynamical systems that are gradient or 'gradient-like', i.e. systems for which the global attractor is given by the union of the unstable manifolds of a finite set of hyperbolic equilibria.

The study of lower semicontinuity of attractors under perturbation, given this gradient assumption, was set in motion by Hale & Raugel [12], who proved an abstract result and considered applications to partial differential equations. Their proof of this result relies on the continuity of the equilibria and lower semicontinuity of the local unstable manifolds under perturbation (see also Stuart & Humphries [17]). Several applications have been considered in [1, 2, 3, 4, 5] (including more complicated situations in which even the continuity of the equilibria is not straightforward) maintaining the hypothesis of gradient structure for the limit problem. In all these works a key step is to show the uniformity (with respect to the underlying parameter) of the exponential dichotomy of the linearization around each hyperbolic equilibrium.

One would expect, therefore, that any attempt to consider attractors with more general structure would have to take into account linearizations around solutions that are time-dependent, leading to problems concerning exponential dichotomies for non-autonomous linear equations and their roughness under perturbation. Until now this has been a barrier

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in the extension of lower semicontinuity results to more general attractors, and has meant, for example, that it has been impossible to treat any attractor containing even a simple time-dependent structure such as a periodic orbit.

In this paper we overcome this difficulty, and consider an unperturbed attractor that is given as the union of the unstable manifolds of a (possibly infinite) number of hyperbolic time-dependent solutions under regular perturbation. We conjecture that, in general, the attractors of autonomous differential equations are given in this form. This is also the natural starting point for the study of the effect of non-autonomous pertubations on autonomous differential equations (see Langa et al. [15] and Carvalho & Langa [6]).

Given this assumption on the structure of the unperturbed attractor, we use results adapted from Carvalho & Langa [6] on the continuity of stable and unstable manifolds under non-autonomous perturbations to show that the attractor behaves continuously under regular perturbation. A similar result for the pullback attractors of a family of non-autonomous problems also follows essentially from the autonomous argument.

Before we are able to state our results precisely we need to introduce some terminology which will be used throughout the paper. Let $\mathfrak{B} : D(\mathfrak{B}) \subset \mathcal{Z} \to \mathcal{Z}$ be the generator of a strongly continuous semigroup $\{e^{\mathfrak{B}t} : t \geq 0\} \subset L(\mathcal{Z})$ and consider the semilinear problem on a Banach space $\mathcal{Z}$

$$\begin{aligned} &\dot{y} = \mathfrak{B}y + f_0(y), \\ &y(\tau) = y_0 \end{aligned} \tag{1.1}$$

and a regular perturbation of it

$$\begin{aligned} &\dot{y} = \mathfrak{B}y + f_\eta(y), \\ &y(\tau) = y_0. \end{aligned} \tag{1.2}$$

If we assume that, for $\eta \in [0,1]$, $f_\eta : \mathcal{Z} \to \mathcal{Z}$ is Lipschitz continuous in bounded subsets of $\mathcal{Z}$ then the problems (1.1) and (1.2) are locally well posed. Assuming that for each $y_0 \in \mathcal{Z}$ the solution $y(s-\tau; y_0)$ of (1.2) exists for all $s \geq \tau$, we can define a nonlinear semigroup $\{T_\eta(t) : t \geq 0\}$ by $T_\eta(t)y_0 = y(t; y_0)$. In this case $T_\eta(t)y_0$ is given by the variation of constants formula

$$T_\eta(t)y_0 = e^{\mathfrak{B}t}y_0 + \int_0^t e^{\mathfrak{B}(t-s)} f_\eta(T_\eta(s)y_0)\, ds. \tag{1.3}$$

Assume also that, for $\eta \in [0,1]$, $f_\eta : \mathcal{Z} \to \mathcal{Z}$ is continuously differentiable. If $\xi^*_\eta(\cdot) : \mathbb{R} \to \mathcal{Z}$ is a global solution to (1.2), $\eta \in [0,1]$, we consider the linearization of (1.2) around $\xi^*_\eta(\cdot)$

$$\begin{aligned} &\dot{z} = \mathfrak{B}z + f'_\eta(\xi^*_\eta(t))z \\ &z(\tau) = z_0 \in \mathcal{Z} \end{aligned} \tag{1.4}$$

and the associated linear evolution process $\{U_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ which is given by

$$U_\eta(t,\tau)z_0 = e^{\mathfrak{B}(t-\tau)}z_0 + \int_\tau^t e^{\mathfrak{B}(t-s)} f'_\eta(\xi^*_\eta(s))U_\eta(s,\tau)z_0\, ds. \tag{1.5}$$

Definition 1.1. *We say that* (1.4) *has an* exponential dichotomy *with exponent* ω *and constant* M *if there exists a family of projections* $\{Q_\eta(t) : t \in \mathbb{R}\} \subset L(\mathcal{Z})$ *such that*

i) $Q_\eta(t)U_\eta(t,s) = U_\eta(t,s)Q_\eta(s)$, *for all* $t \geqslant s$;
ii) $U_\eta(t,s) : R(Q_\eta(s)) \to R(Q_\eta(t))$ *is an isomorphism; we denote its inverse by* $U_\eta(s,t) : R(Q_\eta(t)) \to R(Q_\eta(s))$.
iii)

$$\begin{aligned} \|U_\eta(t,s)(I - Q_\eta(s))\| &\leqslant M e^{-\omega(t-s)} \quad t \geqslant s \\ \|U_\eta(t,s)Q_\eta(s)\| &\leqslant M e^{\omega(t-s)}, \quad t \leqslant s. \end{aligned} \tag{1.6}$$

Definition 1.2. *A global solution* $\xi^*_\eta(\cdot) : \mathbb{R} \to \mathcal{Z}$ *of* (1.2), $\eta \in [0,1]$, *is said to be* hyperbolic *if the process* $\{U_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ *associated to* (1.4) *has an exponential dichotomy in the sense of Definition 1.1.*

We assume that $\{T_\eta(t) : t \geqslant 0\}$ has an *attractor* A_η for each $\eta \geqslant 0$. Also, we assume that A_0 is the union of the *unstable manifolds* $W^u(\xi^*_i(\cdot))$ of a (possibly infinite) collection of global hyperbolic solutions $\xi^*_i(\cdot)$, $i \in \mathcal{I}$, where $\mathcal{I} = \{1, \dots, n\}$ or $\mathcal{I} = \mathbb{N}$.

For the regular perturbation we assume that

$$\lim_{\eta \to 0} \sup_{z \in B(0,r)} \|f_\eta(z) - f_0(z)\|_{\mathcal{Z}} + \|f'_\eta(z) - f'_0(z)\|_{L(\mathcal{Z})} = 0, \tag{1.7}$$

for all $r > 0$.

Under (1.7) it is easy to see that, for each $T > 0$,

$$\sup_{t \in [0,T]} \sup_{\|z\|_{\mathcal{Z}} \leqslant r} \|T_\eta(t)z - T_0(t)z\|_{\mathcal{Z}} \xrightarrow{\eta \to 0} 0. \tag{1.8}$$

For a global hyperbolic solution $\xi^*_{i,0}(\cdot)$ for (1.1) (i.e. for which $\{U_0(t,\tau) : t \geq \tau \in \mathbb{R}\}$ has an exponential dichotomy), we prove that, given $\epsilon > 0$, there exists η_0 such that, for all $\eta \leqslant \eta_0$, there is a unique global solution $\xi^*_{i,\eta}(\cdot)$ of (1.2) with

$$\sup_{t \in \mathbb{R}} \|\xi^*_{i,\eta}(t) - \xi^*_{i,0}(t)\|_{\mathcal{Z}} \xrightarrow{\eta \to 0} 0.$$

It follows from (1.7) that, for each $T > 0$,

$$\sup_{\tau \in \mathbb{R}} \sup_{t \in [0,T]} \|U_\eta(t+\tau,\tau)z - U_0(t+\tau,\tau)\|_{L(\mathcal{Z})} \xrightarrow{\eta \to 0} 0 \tag{1.9}$$

and from Theorem 7.6.10 in Henry's monograph [13] that, for η_0 sufficiently small, $\{U_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ has an exponential dichotomy, and consequently that $\xi^*_{i,\eta}(\cdot)$ is hyperbolic for all η.

Definition 1.3. *The* unstable manifold *of a hyperbolic solution* ξ^*_η *to* (1.2) *is the set*

$$\begin{aligned} W^u_\eta(\xi^*_\eta) = \{ (\tau,\zeta) \in \mathbb{R} \times \mathcal{Z} : \ & \text{there is a backward solution } z(t,\tau,\zeta) \text{ of (1.2)} \\ & \text{satisfying } z(\tau,\tau,\zeta) = \zeta \text{ and such that } \lim_{t \to -\infty} \|z(t,\tau,\zeta) - \xi^*_\eta(t)\| = 0 \}. \end{aligned}$$

The section of the unstable manifold at time τ *is denoted by* $W^u_\eta(\xi^*_\eta)(\tau) = \{z : (\tau, z) \in W^u_\eta(\xi^*_\eta)\}$. *In the case of autonomous differential equations all sections are the same and we will, in this case, identify* $W^u_\eta(\xi^*_\eta)$ *with one of its sections.*

Under assumption (1.7) we prove in Section 2 that for each ξ_0^* there is a $\delta > 0$ and an $\eta_\delta > 0$ such that, if

$$V_\delta(\xi_0^*) = \{(\tau, z) \in \mathbb{R} \times \mathcal{Z} : \|\xi_0^*(\tau) - z\|_{\mathcal{Z}} < \delta, \ \tau \in \mathbb{R}\}$$

and

$$W_\eta^{u,\delta}(\xi_\eta^*) := W_\eta^u(\xi_\eta^*) \cap V_\delta(\xi_0^*),$$

then

$$\mathrm{dist}_{\mathrm{H}}(W_\eta^{u,\delta}(\xi_\eta^*), W_0^{u,\delta}(\xi_0^*)) \to 0 \qquad \text{as} \qquad \eta \to 0,$$

where $\mathrm{dist}_{\mathrm{H}}$ is the symmetric Hausdorff distance, $\mathrm{dist}_{\mathrm{H}}(B, C) = \max(\mathrm{dist}(B, C), \mathrm{dist}(C, B))$.

As a consequence of this, we are able to prove in Section 3 that, if (1.1) has an attractor $\mathcal{A}_0$ that is the closure of the union of the unstable manifolds of a (possibly infinite) number of global hyperbolic solutions, and if the equations (1.2) have global attractors $\mathcal{A}_\eta$ such that $\cup_{0\leq\eta\leq\eta_0}\mathcal{A}_\eta$ is compact, then the family $\{\mathcal{A}_\eta : 0 \leq \eta \leq \eta_0\}$ is upper and lower semicontinuous at $\eta = 0$.

We emphasize that we require neither that the attractor is a union of unstable manifolds of equilibria, nor that there are only a finite number of these unstable manifolds. We discuss some properties of our assumed structure of the unperturbed attractor in Section 4, and in particular we show by means of a simple example that the limiting attractor may be the closure of an infinite union of unstable manifolds of global hyperbolic solutions and the perturbed attractors may be a finite union of unstable manifolds, while still maintaining the lower semicontinuity.

We note here, however, that while there is a strong connection between lower semicontinuity results and knowledge of the structure of the unperturbed attractor, here we are not able to characterize the whole attractor of the perturbed problem.

Finally, in Section 5 we obtain a similar continuity result for the pullback attractors of non-autonomous equations, assuming an analogous structure for the attractor of the unperturbed equation.

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2. Existence and continuity of hyperbolic solutions and associated unstable manifolds

In this section we show the existence of perturbed hyperbolic global solutions and associated unstable manifolds by following, step by step, a procedure similar to that used by Carvalho & Langa in [6], which treated non-autonomous perturbations of manifolds associated with hyperbolic equilibria.

We start by proving that near a global hyperbolic solution $\xi_0^*(\cdot) : \mathbb{R} \to \mathcal{Z}$ of (1.1) there is a unique global hyperbolic solution of (1.2).

If $y(t)$ is a solution to (1.1) and $z(t) = y(t) - \xi_0^*(t)$, then z satisfies

$$\begin{aligned} \dot{z} &= \mathcal{A}_0(t) z + h_0(t, z) \\ z(\tau) &= z_0 \end{aligned} \tag{2.1}$$

where $\mathcal{A}_0(t) = \mathfrak{B} + f_0'(\xi_0^*(t))$ and $h_0(t,z) = f_0(\xi_0^*(t)+z) - f_0(\xi_0^*(t)) - f_0'(\xi_0^*(t))z$. Then $\mathcal{A}_0(t)$ generates a linear evolution process $\{U_0(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$. Moreover, 0 is an equilibrium solution for (2.1) and $h_0(t,0) = 0$, $(h_0)_z(t,0) = 0 \in L(\mathcal{Z})$.

Theorem 2.1. *Let $\xi_0^*(\cdot)$ be a bounded hyperbolic solution for* (1.1) *and assume that* (1.7) *holds. Then, there exists an $\eta_0 > 0$ such that, for each $0 < \eta < \eta_0$ there is a unique bounded solution $\xi_\eta^*(\cdot) : \mathbb{R} \to \mathcal{Z}$ of* (1.2) *that satisfies*

$$\lim_{\eta\to 0} \sup_{t\in\mathbb{R}} \|\xi_\eta^*(t) - \xi_0^*(t)\|_{\mathcal{Z}} = 0.$$

Furthermore, $\xi_\eta^*(\cdot) : \mathbb{R} \to \mathcal{Z}$ is hyperbolic for all η sufficiently small.

Proof. The proof of this result is completely analogous to the proof of Theorem 2.1 in [6] and we include its main steps here for completeness only. Denote by $y(t,\tau;y_0)$ the solution of the initial value problem (1.2). Then, if we define $\phi(t) = y(t) - \xi_0^*(t)$, we have

$$\phi(t) = U_0(t,\tau)\phi(\tau) + \int_\tau^t U_0(t,s) g_\eta(s,(\phi(s)))\,\mathrm{d}s, \tag{2.2}$$

where $g_\eta(t,\phi) = f_\eta(\phi + \xi_0^*(t)) - f_\eta(\xi_0^*(t)) - f_0'(\xi_0^*(t))\phi$. Hence, if we project by $Q_0(t)$ and $I - Q_0(t)$ and take limits as $\tau \to +\infty$ and $\tau \to -\infty$ respectively, we get

$$Q_0(t)\phi(t) = \int_\infty^t U_0(t,s)Q_0(s)g_\eta(s,(\phi(s)))\,\mathrm{d}s$$

and

$$(I - Q_0(t))\phi(t) = \int_{-\infty}^t U_0(t,s)(I - Q_0(s))g_\eta(s,(\phi(s)))\,\mathrm{d}s.$$

Consequently, a global bounded solution for (2.2) exists in a small neighborhood of $\phi = 0$ if and only if

$$\mathcal{T}(\phi)(t) = \int_\infty^t U_0(t,s)Q_0(s)g_\eta(s,(\phi(s)))ds + \int_{-\infty}^t U_0(t,s)(I - Q_0(s))g_\eta(s,(\phi(s)))\,\mathrm{d}s$$

has a unique fixed point in the set

$$\{\phi : \mathbb{R} \to \mathcal{Z} : \sup_{t\in\mathbb{R}} \|\phi(t)\|_{\mathcal{Z}} \leqslant \epsilon\}$$

for some ϵ sufficiently small. The existence of such a fixed point follows from the hyperbolicity of $\xi_0^* : \mathbb{R} \to \mathcal{Z}$ using a standard contraction mapping argument.

It follows in addition that $\xi_\eta^*(\cdot)$ is uniformly close to ξ_0^* and tends to ξ_0^* as $\eta \to 0$.

Now, proceeding as before we change variables $z(t) = y(t) - \xi_\eta^*(t)$ in (1.2) and rewrite it as

$$\begin{aligned} \dot z &= (\mathcal{A}_0(t) + B_\eta(t))z + h_\eta(t,z) \\ z(\tau) &= z_0 \end{aligned} \tag{2.3}$$

where $\mathbb{R} \ni t \mapsto B_\eta(t) = f_\eta'(\xi_\eta^*(t)) - f_0'(\xi_0^*(t)) \in L(\mathcal{Z})$ and $h_\eta(t,z) = f_\eta(\xi_\eta^*(t)+z) - f_\eta(\xi_\eta^*(t)) - f_\eta'(\xi_\eta^*(t))z$. Hence, 0 is an equilibrium for (2.3), $h_\eta(t,0) = 0$ and $(h_\eta)_z(t,0) = 0 \in L(\mathcal{Z})$.

The exponential dichotomy for (1.4) follows from Theorem 7.6.11 in Henry's monograph [13].

□

Now we are ready to study the unstable manifold of a hyperbolic solution $\xi_\eta^*(\cdot)$. We will show that the unstable manifold of ξ_η^* is given by a map

$$\mathbb{R}\times\mathcal{Z}\ni(t,z)\mapsto\Sigma_\eta^u(t,Q_\eta(t)z)\in(I-Q_\eta(t))\mathcal{Z}.$$

The points in the unstable manifold will be those of the form

$$(t,Q_\eta(t)z+\Sigma_\eta^u(t,Q_\eta(t)z))\in\mathbb{R}\times\mathcal{Z}\text{ with }(t,z)\in\mathbb{R}\times\mathcal{Z}\text{ and }z\text{ small.}$$

Note that the way we obtained (2.3) allows us to concentrate on the existence of invariant manifolds of global hyperbolic solutions around the zero stationary solution. Thus, if z is a solution of (2.3) we write $z^+(t)=Q_\eta(t)z$ and $z^-(t)=z-z^+(t)$. Then we have

$$\begin{aligned}\dot z^+&=\mathcal{A}_\eta^+(t)z^++H(t,z^+,z^-),\\ \dot z^-&=\mathcal{A}_\eta^-(t)z^-+G(t,z^+,z^-),\end{aligned}\tag{2.4}$$

where

$$\mathcal{A}_\eta^+(t)=(\mathcal{A}_0(t)+B_\eta(t))Q_\eta(t),$$

$$\mathcal{A}_\eta^-(t)=(\mathcal{A}_0(t)+B_\eta(t)))(I-Q_\eta(t)),$$

$$H(t,z^+,z^-)=Q_\eta(t)h_\eta(t,z^++z^-),$$

and

$$G(t,z^+,z^-)=(I-Q_\eta(t))h_\eta(t,z^++z^-).$$

Since at $(t,0,0)$ the functions H and G are zero with zero derivatives (with respect to z^+ and z^-), from the continuous differentiability of H and G, uniform with respect to t, we obtain that given $\rho>0$ there exists $\delta>0$ such that if $\|z\|_\mathcal{Z}=\|(z^++z^-)\|_\mathcal{Z}<\delta$, then

$$\begin{aligned}&\|H(t,z^+,z^-)\|_\mathcal{Z}\leqslant\rho,\\ &\|G(t,z^+,z^-)\|_\mathcal{Z}\leqslant\rho,\\ &\|H(t,z^+,z^-)-H(t,\tilde z^+,\tilde z^-)\|_\mathcal{Z}\leqslant\rho(\|z^+-\tilde z^+\|_\mathcal{Z}+\|z^--\tilde z^-\|_\mathcal{Z}),\\ &\|G(t,z^+,z^-)-G(t,\tilde z^+,\tilde z^-)\|_\mathcal{Z}\leqslant\rho(\|z^+-\tilde z^+\|_\mathcal{Z}+\|z^--\tilde z^-\|_\mathcal{Z}).\end{aligned}\tag{2.5}$$

Firstly we can obtain the existence of the unstable manifold under the assumption that (2.5) holds for all $z=(z^+,z^-)\in\mathcal{Z}$ with some suitably small $\rho>0$. Also assuming that (2.5) holds for all $z=(z^+,z^-)\in\mathcal{Z}$, we get continuity of the unstable and stable manifolds. Finally, we conclude the existence and continuity of local unstable and stable manifolds for the case when h_η only satisfies (2.5) for $\|z\|_\mathcal{Z}=\|z^++z^-\|_\mathcal{Z}<\delta$ with $\delta>0$ suitably small.

In order to show existence of the function $\Sigma_\eta^{*,u}(\tau,\cdot)$ we will use the Banach contraction principle. For this we fix $D>0$, $L>0$, $0<\vartheta<1$ and choose $\rho>0$ such that

$$\begin{aligned}&\tfrac{\rho M}{\omega}\leqslant D, && \tfrac{\rho M}{\omega}(1+L)\leqslant\vartheta<1,\\ &\frac{\rho M^2(1+L)}{\omega-\rho M(1+L)}\leqslant L, && \rho M+\tfrac{\rho^2M^2(1+L)(1+M)}{2\omega-\rho M(1+L)}<\omega.\end{aligned}\tag{2.6}$$

Definition 2.2. *Given* e0, *denote by* $\mathcal{LB}(D,L)$ *the complete metric space of all bounded and globally Lipschitz continuous functions* $\mathbb{R}\times\mathcal{Z}\to\mathcal{Z}$ *defined as* $(\tau,z)\mapsto\Sigma(\tau,Q_\eta(\tau)z)\in(I-Q_\eta(\tau))\mathcal{Z}$ *satisfying*

$$
\begin{aligned}
&\sup\{\|\Sigma(\tau,Q_\eta(\tau)z)\|_{\mathcal{Z}},\ (\tau,z)\in\mathbb{R}\times\mathcal{Z}\}\leqslant D,\\
&\|\Sigma(\tau,Q_\eta(\tau)z)-\Sigma(\tau,Q_\eta(\tau)\tilde{z})\|_{\mathcal{Z}}\leqslant L\|Q_\eta(\tau)z-Q_\eta(\tau)\tilde{z}\|_{\mathcal{Z}},\ \forall(\tau,z,\tilde{z})\in\mathbb{R}\times\mathcal{Z}\times\mathcal{Z},
\end{aligned}
\tag{2.7}
$$

where the distance between $\Sigma,\tilde{\Sigma}\in\mathcal{LB}(D,L)$ *is defined as*

$$\|\!|\Sigma(\cdot,\cdot)-\tilde{\Sigma}(\cdot,\cdot)|\!\|:=\sup\{\|\Sigma(\tau,Q_\eta(\tau)z)-\tilde{\Sigma}(\tau,Q_\eta(\tau)z)\|_{\mathcal{Z}},\ (\tau,z)\in\mathbb{R}\times\mathcal{Z}\}.$$

First of all, we need to prove a result on the continuity of projections $Q_\eta(t)$ and $Q_0(t)$.

Proposition 2.3. *Let* $Q_\eta(\cdot),Q_0(\cdot)$, *be the projections associated to the linear dichotomies introduced in Section 2. Then for any* $\tau\in\mathbb{R}$

$$\lim_{\eta\to0}\sup_{s\leqslant\tau}\|Q_\eta(s)-Q_0(s)\|_{L(\mathcal{Z})}\to0 \tag{2.8}$$

and

$$\lim_{\eta\to0}\sup_{s\geqslant\tau}\|Q_\eta(s)-Q_0(s)\|_{L(\mathcal{Z})}\to0. \tag{2.9}$$

Proof. The proof of this result is completely analogous to the proof of Lemma 5.1 in [6] with the obvious changes needed to cope with the fact that ξ_0 need not be a stationary solution. We include an abridged version of the proof here for completeness. For $\zeta\in\mathcal{Z}$, let $\mathbb{R}\ni t\mapsto z_\eta(t)\in\mathcal{Z}$ be the unique solution of

$$
\begin{aligned}
\dot{z}&=(\mathcal{A}_0(t)+B_\eta(t))z\\
z(\tau)&=Q_\eta(\tau)\zeta.
\end{aligned}
$$

Writing the variation of constants formula, projecting with $(I-Q_0(t))$, and noting that this solution is bounded in $(-\infty,\tau]$, we have

$$(I-Q_0(\tau))Q_\eta(\tau)\zeta=\int_{-\infty}^{\tau}U_0(\tau,s)(I-Q_0(s))B_\eta(s)z(s)\,\mathrm{d}s,\ \text{ for all }\zeta\in\mathcal{Z}.$$

¿From this it follows that

$$\lim_{\eta\to0}\sup_{s\leqslant\tau}\|(I-Q_0(s))Q_\eta(s)\|_{L(\mathcal{Z})}\to0. \tag{2.10}$$

Considering $\xi\in\mathcal{Z}$, let $\mathbb{R}\ni t\mapsto x(t)\in\mathcal{Z}$ be the unique solution of

$$
\begin{aligned}
\dot{x}&=\mathcal{A}_0(t)x=(\mathcal{A}_0(t)+B_\eta(t))x-B_\eta(t)x\\
x(\tau)&=Q_0(\tau)\xi.
\end{aligned}
$$

and proceeding in a similar manner we obtain

$$(I-Q_\eta(\tau))Q_0(\tau)\xi=-\int_{-\infty}^{\tau}U_\eta(\tau,s)(I-Q_\eta(s))B_\eta(s)z(s)\,\mathrm{d}s,\ \text{ for all }\xi\in\mathcal{Z}.$$

Consequently,

$$\lim_{\eta\to0}\sup_{s\leqslant\tau}\|(I-Q_\eta(s))Q_0(s)\|_{L(\mathcal{Z})}\to0. \tag{2.11}$$

The proof of (2.8) now follows from the identity

$$Q_\eta(s) - Q_0(s) = (I - Q_0(s))Q_\eta(s) - Q_0(s)(I - Q_\eta(s)). \tag{2.12}$$

To prove (2.9) we proceed in a similar way. For $\zeta \in \mathcal{Z}$, let $\mathbb{R} \ni t \mapsto z_\eta(t) \in \mathcal{Z}$ be the unique solution of

$$\dot{z} = (\mathcal{A}_0(t) + B_\eta(t))z$$
$$z(\tau) = (I - Q_\eta(\tau))\zeta.$$

Writing the variation of constants formula, projecting with $Q_0(t)$, and noting that $z_\eta(t)$ is bounded in $[\tau, \infty)$ we obtain

$$Q_0(\tau)(I - Q_\eta(\tau))\zeta = \int_\infty^\tau U_0(t,s)Q_0(s)B_\eta(s)z(s)\,\mathrm{d}s, \text{ for all } \zeta \in \mathcal{Z}$$

and

$$\lim_{\eta\to 0} \sup_{s\geq \tau} \|Q_0(s)(I - Q_\eta(s))\|_{L(\mathcal{Z})} \to 0. \tag{2.13}$$

Similarly, for $\xi \in \mathcal{Z}$, let $\mathbb{R} \ni t \mapsto x(t) \in \mathcal{Z}$ be the unique solution of

$$\dot{x} = \mathcal{A}_0(t)x = (\mathcal{A}_0(t) + B_\eta(t))x - B_\eta(t)x$$
$$x(\tau) = (I - Q_0(\tau))\xi.$$

¿From this we obtain

$$Q_\eta(\tau)(I - Q_0(\tau))\xi = -\int_\infty^\tau U_\eta(\tau,s)Q_\eta(s)B_\eta(s)x(s)\,\mathrm{d}s, \text{ for all } \xi \in \mathcal{Z}$$

and

$$\lim_{\eta\to 0} \sup_{s\geq \tau} \|Q_\eta(s)(I - Q_0(s))\|_{L(\mathcal{Z})} \to 0, \tag{2.14}$$

and (2.9) follows from (2.12), (2.13) and (2.14). □

With the above results and notation, the existence and continuity of unstable manifolds is now a consequence of the results of Carvalho & Langa in [6].

Theorem 2.4. *(Theorems 3.1 and 3.5 in [6]) Suppose that all the conditions in (2.6) are satisfied. Then, there exists a function $\Sigma_\eta^{*,u}(\tau,\cdot) \in \mathcal{LB}(D,L)$, such that the unstable manifold $W_\eta^u(0,0)$ for (2.4) is given by*

$$W_\eta^u(0,0) = \{(\tau,w) \in \mathbb{R}\times\mathcal{Z} : w = (Q_\eta(\tau)w, \Sigma_\eta^{*,u}(\tau, Q_\eta(\tau)w))\}. \tag{2.15}$$

If in addition

$$\left[\frac{\rho M}{\omega} + \frac{\rho^2 M^2(1+L)}{\omega(2\omega - \rho M(1+L))}\right] \leqslant \frac{1}{2}$$

then, for any $r > 0$,

$$\sup_{t\leqslant\tau} \sup_{\substack{z\in\mathcal{Z}\\ \|z\|_{\mathcal{Z}}\leqslant r}} \{\|Q_\eta(t)(z) - Q_0(t)(z)\|_{\mathcal{Z}} + \|\Sigma_\eta^{*,u}(t, Q_\eta(t)(z)) - \Sigma_0^{*,u}(t, Q_0(t)(z))\|_{\mathcal{Z}}\} \xrightarrow{\eta\to 0} 0.$$

As a consequence of the previous results, we can conclude the existence and continuity of local unstable manifolds when h, h_η only satisfy (2.5) for $\|z\|_{\mathcal{Z}} = \|z^+ + z^-\|_{\mathcal{Z}} < \delta$ with $\delta > 0$ sufficiently small:

Theorem 2.5. *(Theorem 6.1 in [6]) For $\eta \in [0,1]$ suppose that $h_\eta : \mathbb{R} \times \mathcal{Z} \to \mathcal{Z}$ is differentiable, and consider the initial value problem*

$$\dot{z} = \mathcal{A}_0(t)z + h_\eta(t,z), \quad z(\tau) = z_0 \in \mathcal{Z}. \tag{2.16}$$

Assume that $h_0 : \mathbb{R} \times \mathcal{Z} \to \mathcal{Z}$ is such that $h_0(t,0) = 0$, $(h_0)_z(t,0) = 0 \in L(\mathcal{Z})$, and that $\dot{z} = \mathcal{A}_0(t)z$ has an exponential dichotomy with projections $\{Q_0(t) : t \in \mathbb{R}\}$. Suppose that

$$\lim_{\eta\to 0} \sup_{z\in B(0,r)} \|h_\eta(t,z) - h_0(t,z)\|_{\mathcal{Z}} + \|(h_\eta)_z(t,z) - (h_0)_z(t,z)\|_{L(\mathcal{Z})} = 0 \tag{2.17}$$

for some $r > 0$. Under these assumptions:

(1) *For each η sufficiently small, there exists a globally defined solution of (2.16) $\xi_\eta^* : \mathbb{R} \to \mathcal{Z}$ with $\lim_{\eta\to 0} \sup_{t\in\mathbb{R}} \|\xi_\eta^*(t)\|_{L(\mathcal{Z})} = 0$ and such that*

$$\dot{z} = (\mathcal{A}_0(t) + (h_\eta)_z(t, \xi_\eta^*(t)))z \tag{2.18}$$

has an exponential dichotomy; that is, there is a a family of projections $\{Q_\eta(t) : t \in \mathbb{R}\}$ such that the conditions in Definition 1.1 are satisfied and $U_\eta(t,\tau)$ is the solution operator associated to (2.18).

(2) *For any $T > 0$*

$$\lim_{\eta\to 0} \sup_{t\in[0,T]} \sup_{\tau\in\mathbb{R}} \|U_\eta(t+\tau,\tau) - U_0(t+\tau,\tau)\|_{L(\mathcal{Z})} \to 0.$$

(3) *The family of projections $\{Q_\eta(t) : t \in \mathbb{R}\}$ satisfies*

$$\lim_{\eta\to 0} \sup_{t\in\mathbb{R}} \|Q_\eta(t) - Q_0(t)\|_{L(\mathcal{Z})} = 0.$$

(4) *There exists an $\eta_0 > 0$, a neighborhood V of $z = 0$ in $\mathcal{Z}$ (independent of η) with $\xi_\eta^*(t) \in V$, $\forall t \in \mathbb{R}$, $\eta \in [0, \eta_0]$ and, for each $0 \leqslant \eta \leqslant \eta_0$, a function $(\tau, z) \mapsto \Sigma_\eta^{*,u}(\tau, Q(\tau)z) : \mathbb{R} \times V \to \mathcal{Z}$, such that the local unstable manifold $W^u_{\mathrm{loc},\eta}(\xi_\eta^*) = W^u_\eta(\xi_\eta^*) \cap V$ for (2.16) is given by*

$$W^u_{\mathrm{loc},\eta}(\xi_\eta^*) = \{(\tau, w) \in V : w = Q_\eta(\tau)w + \Sigma_\eta^{*,u}(\tau, Q(\tau)w)\}.$$

(5) *Finally, the unstable manifolds behave continuously at $\eta = 0$ in the sense that*

$$\sup_{t\leqslant\tau} \sup_{z\in V} \left\{\|Q_\eta(t)z - Q_0(t)z\|_{\mathcal{Z}} + \|\Sigma_\eta^{*,u}(t, Q_\eta(t)z) - \Sigma_0^{*,u}(t, Q_0(t)z)\|_{\mathcal{Z}}\right\} \xrightarrow{\eta\to 0} 0.$$

Note that it would be possible to prove a similar result concerning the existence and stability of local stable manifolds (as in [6]), but we do not need such a result here.

3. Lower semicontinuity of global attractors

The upper semicontinuity of attractors for autonomous dynamical systems is a relatively simple matter depending only on uniform bounds on the attractors and continuity of the nonlinear semigroups (see the paper by Hale, Lin, & Raugel [10], or the books by Hale [11], Robinson [16], or Temam [18]). On the other hand, results on the lower semicontinuity of attractors are much more difficult. The results on lower semicontinuity of attractors, in general, rely on reproducing the structures present in the limiting attractor inside the perturbed attractors. Because of this, a gradient-like structure for the attractors of the limiting problem has always been taken as the starting point (for autonomous problems by Hale & Raugel [12], Stuart & Humphries [17], and various authors in [1, 2, 3, 4, 5]; and for non-autonomous perturbations of autonomous systems by Langa et al. [15] and Carvalho et al. [7]). The following result replaces this very restrictive assumption and requires only that the limiting attractor is the closure of the union of a (possibly infinite) number of unstable manifolds of hyperbolic global solutions.

Theorem 3.1. *Consider the family $\{T_\eta(t) : t \in \mathbb{R}\}$, $\eta \in [0,1]$, of nonlinear semigroups and assume that (1.8) is satisfied. Suppose that, for each $\eta \in [0,1]$ the nonlinear semigroup $\{T_\eta(t) : t \geq 0\}$ has a global attractor A_η, that $\overline{\cup_{\eta\in[0,\eta_0]} A_\eta}$ is compact, and that A_0 is given by the closure of the union of the unstable manifolds of a collection of global hyperbolic solutions $\{\xi_j^*(\cdot)\}_{j=1}^\infty$,*

$$A_0 = \overline{\bigcup_{j=1}^{\infty} W^u(\xi_j^*(\cdot))}. \tag{3.1}$$

Assume further that, for each $j \in \mathbb{N}$,

- *given $\epsilon > 0$, there exists an $\eta_{j,\epsilon}$ such that for all $0 < \eta < \eta_{j,\epsilon}$ there exists a global hyperbolic solution $\xi_{j,\eta}^*(\cdot)$ for (1.2) and*
$$\sup_{t\in\mathbb{R}} \|\xi_{j,\eta}^*(t) - \xi_j^*(t)\|_Z < \epsilon,$$
- *the local unstable manifold of $\xi_{j,\eta}^*$ behaves continuously as $\eta \to 0$; that is, there exists a $\delta_j > 0$ such that and*
$$\mathrm{dist}_{\mathrm{H}}(W_\eta^{u,\delta_j}(\xi_{j,\eta}^*), W_0^{u,\delta_j}(\xi_j^*)) \xrightarrow{\eta\to 0} 0.$$

Then the family $\{A_\eta, 0 \leq \eta \leq \eta_0\}$ is upper and lower semicontinuous at $\eta = 0$, namely

$$\mathrm{dist}_{\mathrm{H}}(A_\eta, A_0) \xrightarrow{\eta\to 0} 0. \tag{3.2}$$

Proof. First note that upper semicontinuity is a direct consequence of the continuity of the nonlinear semigroups $\{T_\eta(t) : t \geq 0\}$ and of compactness of $\overline{\cup_{\eta\in[0,\eta_0]} A_\eta}$; for the standard argument see for example the books by Hale [11], Robinson [16], or Temam [18]. Here we prove that the attractor is also lower semicontinuous.

To prove that $\mathrm{dist}(A_0, A_\eta) \xrightarrow{\eta\to 0} 0$ it suffices to show that for each $x \in A_0$ there are sequences $\eta_k \to 0^+$ and $x_k \in A_{\eta_k}$ such that $x_k \to x$.

Given $x \in A_0$ and $\epsilon > 0$ there exists $x_\epsilon \in \cup_{j=1}^\infty W^u(\xi_j^*)$ such that

$$\|x - x_\epsilon\|_Z < \frac{\epsilon}{2}. \tag{3.3}$$

Let $j \in \mathbb{N}$ such that $(t_\epsilon, x_\epsilon) \in W^u(\xi_j^*)$ and choose $\tau > 0$ such that $T_0(-\tau)x_\epsilon \in W^{u,\delta_j}(\xi_j^*)$.
Choose $r > 0$ such that

$$\sup_{\|z\|_Z \le r} \sup_{0 \le \eta \le \eta_0} \sup\{\|T_\eta(\tau)(z+h) - T_\eta(\tau)z\|_Z : z \in \cup_{0 \le \eta \le \eta_0} A_\eta\} < \frac{\epsilon}{4}. \tag{3.4}$$

Let $\eta_\epsilon < \eta_0$ be such that

$$\sup_{0 \le \eta \le \eta_\epsilon} \operatorname{dist}(W_0^{u,\delta_j}(\xi_j^*), W_\eta^{u,\delta_j}(\xi_{j,\eta}^*)) < r \tag{3.5}$$

and (using the convergence of T_η to T_0 in (1.8))

$$\sup\{\|T_\eta(\tau)z - T_0(\tau)z\|_Z : z \in \cup_{0 \le \eta \le \eta_\epsilon} A_\eta\} < \frac{\epsilon}{4}. \tag{3.6}$$

¿From (3.5) we choose $x_{\epsilon,\eta} \in W_\eta^{u,\delta_j}(\xi_{j,\eta}^*)$ such that

$$\|x_\epsilon - x_{\epsilon,\eta}\|_Z < r$$

and from (3.6) and (3.4) it follows that if $y_{\epsilon,\eta} = T_\eta(\tau)x_{\epsilon,\eta}$ then $T_\eta(\tau)x_{\epsilon,\eta} \in A_\eta$ and

$$\|x_\epsilon - y_{\epsilon,\eta}\|_Z = \|T_0(\tau)T_0(-\tau)x_\epsilon - T_\eta(\tau)x_{\epsilon,\eta}\|_Z$$

$$\le \|T_0(\tau)T_0(-\tau)x_\epsilon - T_\eta(\tau)T_0(-\tau)x_\epsilon\|_Z + \|T_\eta(\tau)T_0(-\tau)x_\epsilon - T_\eta(\tau)x_{\epsilon,\eta}\|_Z < \tfrac{\epsilon}{2}.$$

¿From (3.3) we have $y_{\epsilon,\eta} \in A_\eta$ and $\|x - y_{\epsilon,\eta}\|_Z < \epsilon$. This concludes the proof. □

Observe that the perturbed bounded global solutions and their associated unstable manifolds have to be part of the global attractors for the perturbed problems (e.g. Theorem 3.3 in Chapter VII of Temam [18]), i.e.,

$$\bigcup_{i=0}^\infty W_\eta^u(\xi_\eta^i(\cdot)) \subseteq A_\eta. \tag{3.7}$$

Thus, we are still able to describe some part of the structure of the perturbed attractors, and in particular we can obtain lower bounds on their dimension (see below). It would be very interesting to prove that we have equality in (3.7), but currently this is only known when the unperturbed attractor has a gradient structure (see Carvalho et al. [7]).

4. The structure of the unperturbed attractor

We now discuss the structure (3.1) that we have imposed on the unperturbed attractor A_0. In all previous results on lower semicontinuity, the 'hyperbolic solutions' ξ_j^* were in fact equilibria, and it is relatively simple to prove that there can only be a finite number of such hyperbolic equilibria (cf. Theorem 3.8.5 in Hale [11]).

However, it is possible to have an infinite number of hyperbolic solutions in the attractor, as the following simple example shows. Consider the ordinary differential equation in $\mathbb{R}^2$ given in polar coordinates

$$\begin{aligned} \dot{r} &= \begin{cases} \pi^{-1}(1-r)^3 \sin\frac{\pi}{1-r}, & r<1 \\ -(1-r)^2, & r\geq 1 \end{cases} \\ \dot{\theta} &= 1 \end{aligned} \tag{4.1}$$

We observe that $r=1$ and $r=1-\frac{1}{k}$, $k\in\mathbb{N}$, correspond to periodic solutions with period 2π. The periodic solution ξ_0 with $r=1$ is stable but not asymptotically stable, and is not hyperbolic. The periodic solutions ξ_k^*, $k\in\mathbb{N}$, with $r=1-\frac{1}{k}$ are global hyperbolic solutions such that, if k is even the orbit is unstable (with a two-dimensional unstable manifold) and if k is odd the orbit is stable (with a two-dimensional stable manifold). In this case, it is easy to see that the attractor A_0 - precisely the unit disc - is the closure of the union of the unstable manifolds of the countably infinite collection of global hyperbolic solutions $\{\xi_k^*:\ k\in\mathbb{N}\}$.

Nevertheless, a small C^1 perturbation of (4.1) can produce a situation in which the attractor is a finite union of unstable manifolds of hyperbolic solutions. Indeed, for each $m\in\mathbb{N}$, the problem

$$\begin{aligned} \dot{r} &= \begin{cases} \pi^{-1}(1-\frac{1}{2m+1}-r)^3 \sin\frac{\pi}{1-r}, & r<1-\frac{1}{2m+1} \\ -(1-\frac{1}{2m+1}-r)^2, & r\geq 1-\frac{1}{2m+1} \end{cases} \\ \dot{\theta} &= 1 \end{aligned} \tag{4.2}$$

has an attractor $A_m=\{|r|\leq 1-\frac{1}{2m+1}\}$, which is the union of the unstable manifolds of ξ_j^*, $1\leq j\leq 2m+1$.

Note that the nonlinearities in (4.2) converge to those of (4.1) in such a way that the conditions of Theorem 3.1 are fulfilled, and so our result suffices to ensure the lower semicontinuity of the attractors in this example (although here we can see this lower semicontinuity explicitly).

It is nevertheless of interest to consider situations in which we can guarantee that the attractor is given as the union of a finite number of such solutions. In this direction we have the following lemma:

Lemma 4.1. *Let $\xi_0^*(\cdot)$ be an hyperbolic solution of problem* (1.1). *Then $\xi_0^*(\cdot)$ is isolated (uniformly in t).*

Proof. The proof follows closely the argument used for Theorem 2.1 since, if we define $\phi(t)=y(t)-\xi_0^*(t)$ then

$$\phi(t)=U_0(t,\tau)\phi(\tau)+\int_\tau^t U_0(t,s)g_0(s,(\phi(s)))\,\mathrm{d}s,$$

with $g_0(t,\phi)=f_0(\phi+\xi_0^*(t))-f_0(\xi_0^*(t))-f_0'(\xi_0^*(t))\phi$ and

$$\mathcal{T}(\phi)(t)=\int_\infty^t U_0(t,s)Q_0(s)g_\eta(s,(\phi(s)))\,\mathrm{d}s+\int_{-\infty}^t U_0(t,s)(I-Q_0(s))g_\eta(s,(\phi(s)))\,\mathrm{d}s$$

has a unique fixed point in the set

$$\{\phi : \mathbb{R} \to \mathcal{Z} : \sup_{t\in\mathbb{R}} \|\phi(t)\|_{\mathcal{Z}} \leqslant \epsilon\}$$

for ϵ sufficiently small. This follows from the hyperbolicity of $\xi_0^* : \mathbb{R} \to \mathcal{Z}$ via a contraction mapping argument. □

One situation for which we can ensure that there are only a finite number of global hyperbolic solutions is given in the following theorem:

Theorem 4.2. *Let A_0 be the attractor of the problem* (1.1), *and assume that A_0 is the union of the unstable manifolds of a (possibly infinite) number of hyperbolic solutions. Let $C_B(\mathbb{R}, A_0)$ denote the space of continuous functions from $\mathbb{R}$ into A_0 equipped with the supremum norm. If $\mathfrak{A} := \{\xi \in C_B(\mathbb{R}, A_0) : \xi$ is a global hyperbolic solution of $(1.1)\}$ is a compact subset of $C_B(\mathbb{R}, A_0)$ then there are only a finite number of global hyperbolic solutions in A_0.*

Proof. Suppose that $\xi_n^* : \mathbb{R} \to A_0$, $n \in \mathbb{N}$ are a countably infinite number of distinct hyperbolic solutions. From the compactness of $\mathfrak{A}$ we may assume that there is a $\xi \in C_B(\mathbb{R}, A_0)$ such that $\xi_n^* \to \xi$; i.e., $\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \|\xi_\eta^*(t) - \xi(t)\| = 0$. Clearly ξ is a global bounded solution of (1.1). Since a hyperbolic solution is necessarily isolated, ξ is not hyperbolic. On the other hand, since A_0 is the union of unstable manifolds of hyperbolic solutions, there is a hyperbolic solution ξ^* such that $\xi \in W^u(\xi^*)$.

¿From this we can deduce a contradiction. Indeed, observe that there is an $\epsilon > 0$ such that, any solution ζ that stays in an ϵ neighbourhood of ξ^* for all t in an interval of the form $(-\infty, \tau]$ must be in the unstable manifold of ξ^*. However, it follows from the convergence of ξ_n^* to ξ in $C_B(\mathbb{R}, A_0)$ that there is a $k \in \mathbb{N}$ such that ξ_k^* stays in this ϵ neighbourhood for all t in an interval of the form $(-\infty, \tau]$. It follows that in fact $\xi_k^* \in W^u(\xi^*)$ for all k sufficiently large, and in particular that $\|\xi_k^*(t) - \xi^*(t)\| \to 0$ as $t \to -\infty$. But ξ^* is isolated (uniformly in t), which is a contradiction. □

The main hypothesis of the above Theorem 4.2 (compactness of $\mathfrak{A}$) is, admittedly, somewhat difficult to check in applications, but here we give a corollary of the above result that is of independent interest.

Corollary 4.3. *Let A_0 be the attractor of the problem* (1.1). *Assume that A_0 is the union of unstable manifolds of its global hyperbolic solutions (possibly infinite). Given $T > 0$, there are finitely many hyperbolic periodic solutions of* (1.1).

Proof. Observe that the set

$\mathfrak{A} := \{\xi \in C_B(\mathbb{R}, A_0) : \xi$ is a global hyperbolic periodic solution of (1.1) with period not exceeding $T\}$

is compact in $C_B(\mathbb{R}, A_0)$. The proof now follows that of Theorem 4.2. □

We conjecture that the structure

$$A_0 = \bigcup_{j=1}^{n} W^u(\xi_j^*(\cdot)), \tag{4.3}$$

is generic, at least in sufficiently smooth finite-dimensional systems. The Kupka–Smale Theorem ensures that generically there are only a finite number of periodic orbits and that these are all hyperbolic [9], but to our knowledge there is no result along the lines of (4.3) currently available. If the attractor is given in this form then we are able to give an exact expression for the Hausdorff dimension of the attractor:

$$\dim_H(A_0) = \max_{j=1,\dots,n} \dim_H(W^u(\xi_j^*)(t)) = \max_{j=1,\dots,n} \operatorname{rank}(Q_j(t)),$$

where Q_j is the projection from the definition of the exponential dichotomy about the hyperbolic trajectory ξ_j^* for $\eta = 0$ (note that $\operatorname{rank}(Q_j(t))$ is independent of t). The simple proof can be found in the paper by Carvalho et al. [7].

5. Continuity and Characterization of Attractors for Non-autonomous Perturbed Systems

We now show that, with straightforward modifications, we can adapt our previous results to deduce continuity of pullback attractors (assuming a structure analogous to that in (3.1)) in perturbations of non-autonomous equations. In particular, let $\mathfrak{B} : D(\mathfrak{B}) \subset \mathcal{Z} \to \mathcal{Z}$ be the generator of a strongly continuous semigroups $\{e^{\mathfrak{B}t} : t \geq 0\} \subset L(\mathcal{Z})$ and consider the semilinear non-autonomous problem on a Banach space $\mathcal{Z}$

$$\begin{aligned} \dot{y} &= \mathfrak{B}y + f_0(t, y), \\ y(\tau) &= y_0 \end{aligned} \tag{5.1}$$

and a regular perturbation of it

$$\begin{aligned} \dot{y} &= \mathfrak{B}y + f_\eta(t, y), \\ y(\tau) &= y_0. \end{aligned} \tag{5.2}$$

Assuming that, for $\eta \in [0, 1]$, $f_\eta : \mathbb{R} \times \mathcal{Z} \to \mathcal{Z}$ is continuous and Lipschitz continuous in the second variable, uniformly on bounded subsets of $\mathcal{Z}$ the problems (5.1) and (5.2) are locally well posed. Assuming that the solutions of (5.2) exist in $[\tau, \infty)$ for any $\tau \in \mathbb{R}$ and $y_0 \in \mathcal{Z}$ we can define

$$T_\eta(t, \tau)y_0 = e^{\mathfrak{B}t}y_0 + \int_\tau^t e^{\mathfrak{B}(t-s)} f_\eta(s, T_\eta(s, \tau)y_0)\, ds. \tag{5.3}$$

The family $\{T_\eta(t, \tau) : t \geqslant \tau\}$ defined in this way is a nonlinear process.

If $\xi_\eta^*(\cdot) : \mathbb{R} \to \mathcal{Z}$ is a solution to (5.2), $\eta \in [0, 1]$, we consider the linearization of (5.2) around $\xi_\eta^*(\cdot)$

$$\begin{aligned} \dot{z} &= \mathfrak{B}z + (f_\eta)_z(t, \xi_\eta^*(t))z \\ z(\tau) &= z_0 \in \mathcal{Z}. \end{aligned} \tag{5.4}$$

and the evolution process $\{U_\eta(t, \tau) : t \geq \tau \in \mathbb{R}\}$ given by

$$U_\eta(t, \tau)y_0 = e^{\mathfrak{B}(t-\tau)}y_0 + \int_\tau^t e^{\mathfrak{B}(t-s)} (f_\eta)_z(s, \xi_\eta^*(s)) U_\eta(s, \tau)y_0\, ds. \tag{5.5}$$

We define exponential dichotomy for (5.4) as in Definition 1.1 and hyperbolicity of ξ_η^* as in Definition 1.2.

One of the most useful concepts for describing the asymptotic dynamics of non-autonomous differential equations is the pullback attractor (this concept originated with Chepyzhov & Vishik (see [8] for example), who termed the sets $A_\eta(t)$ below '*kernel sections*'; the term 'pullback attractor' appears in the paper by Kloeden & Schmalfuss [14]).

Definition 5.1. *A family* $\{A_\eta(t) : t \in \mathbb{R}\}$ *of subsets of* $\mathcal{Z}$ *attracts a bounded set* $B \subset \mathcal{Z}$ *under* $\{T_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ in the pullback sense *if*

$$\lim_{\tau\to-\infty} \operatorname{dist}(T_\eta(t,\tau)B, A_\eta(t)) = 0, \ \forall t \in \mathbb{R}.$$

A family of compact sets $\{A_\eta(t) \subset \mathcal{Z} : t \in \mathbb{R}\}$ *is a* pullback attractor *for* $\{T_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ *if it is invariant and attracts all bounded subsets of* $\mathcal{Z}$ *under* $\{T_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ *in the pullback sense.*

Here we assume $\{T_\eta(t,\tau) : t \geqslant \tau\}$ has a pullback attractor $\{A_\eta(t) : t \geq 0\}$ for each $\eta \geqslant 0$, and that (5.1) has hyperbolic global solutions ξ_i^* such that $A_0(t)$ is the union of the unstable manifolds $W^u(\xi_i^*)$ of them,

$$A_0(t) = \overline{\bigcup_{j=1}^{\infty} W^u(\xi_j^*(\cdot))(t)}. \tag{5.6}$$

Since a hyperbolic global solution is the generalization to the non-autonomous setting of a hyperbolic stationary point, assuming the pullback attractor to have this structure is very natural, but none of the previous results in the literature suffice to prove lower semicontinuity for this kind of attractor (although the papers by Langa et al. [15] and Carvalho & Langa [6] contain results for non-autonomous perturbations of *autonomous* gradient systems).

For the regular perturbation of the nonlinearity we now assume that

$$\lim_{\eta\to 0} \sup_{t\in\mathbb{R}} \sup_{z\in B(0,r)} \|f_\eta(t,z) - f_0(t,z)\|_{\mathcal{Z}} + \|(f_\eta)_z(t,z) - f_0'(t,z)\|_{L(\mathcal{Z})} = 0, \tag{5.7}$$

for all $r > 0$.

Under (5.7) it is easy to see that, for each $T > 0$, and compact subset K of $\mathcal{Z}$

$$\sup_{t\in[0,T]} \sup_{z\in K} \|T_\eta(t,\tau)z - T_0(t,\tau)z\|_{\mathcal{Z}} \xrightarrow{\eta\to 0} 0. \tag{5.8}$$

Then, by Theorem 2.1, for each hyperbolic global solution solution $\xi_i^*(\cdot)$ for (5.1) there is a unique hyperbolic equilibrium solution $\xi_{i,\eta}^*$ of (5.2) such that

$$\sup_{t\in\mathbb{R}} \|\xi_{i,\eta}^*(t) - \xi_i^*(t)\|_{\mathcal{Z}} \xrightarrow{\eta\to 0} 0.$$

As in Carvalho & Langa [6] we can prove the existence of the unstable manifolds associated to the $\xi_{i,\eta}^*(t)$. Moreover, these manifolds are locally given as graphs of Lipschitz functions; that is, if V is a suitably small neighbourhood of $z = 0$ in $\mathcal{Z}$ (independent of η), with $\xi_{i,\eta}^*(t) \in V$, there exist $(\tau, z) \mapsto \Sigma_\eta^{*,u}(\tau, Q_\eta^i(\tau)z) : \mathbb{R} \times V \to \mathcal{Z}$, such that the local unstable manifold $W_{\mathrm{loc},\eta}^u(\xi_{i,\eta}^*)$, is given by

$$W_{\mathrm{loc},\eta}^u(\xi_{i,\eta}^*)(\tau) = \{\xi_{i,\eta}^*(\tau) + w \in \xi_{i,\eta}^*(\tau) + V : w = Q_\eta^i(\tau)w + \Sigma_\eta^{*,u}(\tau, Q_\eta^i(\tau)w)\}.$$

Furthermore one can prove that these manifolds behave continuously as a function of η, and in particular

$$\sup_{t\leqslant\tau}\sup_{z\in V}\left\{\|Q^i_\eta(t)z-Q^i_0(t)z\|_{\mathcal{Z}}+\|\Sigma^{*,u}_\eta(t,Q^i_\eta(t)z)-\Sigma^{*,u}_0(t,Q^i_0(t)z)\|_{\mathcal{Z}}\right\}\overset{\eta\to 0}{\longrightarrow}0.$$

As a consequence of this and of the continuity of the nonlinear semigroups associated to our equations, it follows as in Section 3 that

$$\sup_{t\in\mathbb{R}}\mathrm{dist}_{\mathrm{H}}(A_\eta(t),A_0(t))\to 0\qquad\text{as}\qquad \eta\to 0.$$

As in the autonomous case, we know that the perturbed unstable manifolds are elements of the perturbed attractors, and this gives us both a lower bound on their dimension and on their 'structure'. In fact, for each $n\in\mathbb{N}$, there exists $\eta_n>0$ such that

$$\bigcup_{i=0}^{n}W^u_\eta(\xi^i_\eta(\cdot))(t)\subseteq A_\eta(t),\ \text{ for all }\eta\leq\eta_n. \tag{5.9}$$

As in the autonomous case, it would be very interesting to prove equality here (this is only known when the unperturbed attractor is autonomous and gradient, see Carvalho et al. [7]).

It is worth remarking that in place of (5.7) we could have considered the following weaker hypothesis: there exists $\tau\in\mathbb{R}$ such that

$$\lim_{\eta\to 0}\ \sup_{t\in(-\infty,\tau]}\ \sup_{z\in B(0,r)}\ \|f_\eta(t,z)-f_0(t,z)\|_{\mathcal{Z}}+\|(f_\eta)_z(t,z)-f_0'(t,z)\|_{L(\mathcal{Z})}=0, \tag{5.10}$$

for all $r>0$. This would imply the existence of global solutions close to $\xi_i(t)$ that are hyperbolic in $(-\infty,\tau]$, and would therefore allow for lower semicontinuity of a family pullback attractors that are not bounded as $t\to+\infty$ (i.e., $\bigcup_{[T,+\infty)}A_\eta(t)$ unbounded in $\mathcal{Z}$). This reinforces the idea that in a non-autonomous differential equations the pullback behaviour determines crucial properties of the future dynamics of the system.

Finally, as in Section 4, we briefly consider when we can guarantee that there are only a finite number of global hyperbolic solutions in the attractor. First note that the analogue of Theorem 4.2 holds with identical proof, that is:

Theorem 5.2. *Assume that* (5.1) *has a pullback attractor* $\{A_0(t):t\in\mathbb{R}\}$ *that is the union of the unstable manifolds of its global hyperbolic solutions. Denote by* $C_B(\mathbb{R},\{A_0(t):t\in\mathbb{R}\})$ *the space of continuous and bounded functions* $\mathbb{R}\ni t\mapsto\xi(t)\in A_0(t)$ *and assume that the set* $\mathfrak{A}:=\{\xi\in C_B(\mathbb{R},A_0):\xi$ *is a global hyperbolic solution of* (1.1)$\}$ *is a compact subset of* $C_B(\mathbb{R},A_0)$*; then* $\mathfrak{A}$ *is finite.*

In general the characterization of non-autonomous attractors is a difficult problem. The only current result in this direction is due to Carvalho et al. [7], and guarantees that the pullback attractor resulting from a uniformly small non-autonomous perturbation of an autonomous gradient system in which there are a finite number of equilibria, all of them hyperbolic, has a similar structure, namely

$$A(t)=\bigcup_{j=1}^{n}W^u(\xi_j(\cdot))(t), \tag{5.11}$$

where the $\xi_j(\cdot)$ are global hyperbolic trajectories. Even in the case of a system that is asymptotically autonomous as $t \to -\infty$ to such a simple autonomous system, it is an open problem whether the pullback attractor can be written in the form (5.11) where the $\xi_j(\cdot)$ are *globally* hyperbolic.

6. Final remarks and conclusions

We have proved the lower semicontinuity of global attractors given as the union of unstable manifolds of hyperbolic global solutions. In particular, this includes the case of attractors containing periodic, quasi periodic or almost periodic hyperbolic solutions. The presence of time-dependent hyperbolic solutions has been a crucial barrier to obtaining lower semicontinuity results for such attractors for more than fifteen years. Here we have advanced the theory by appealing to the recent results of Carvalho & Langa [6] that guarantee the existence and continuity of hyperbolic global solutions and their associated unstable manifolds under regular perturbations of dynamical systems in Banach spaces.

We conjecture that such a structure, requiring the union of only a finite number of global hyperbolic solutions, is generic for autonomous problems, at least for sufficiently smooth finite-dimensional systems. Whether this is reasonable for general non-autonomous systems is not clear, but it seems plausible for systems that are asymptotically autonomous as $t \to -\infty$.

On the other hand, these results suggest a new set of problems for further study. Lower semicontinuity results under singular perturbations of dynamical systems should be possible, although the results on the robustness of exponential dichotomies that would be required are yet to be proved (see, for example, the papers [1, 2, 3, 4, 5] for lower semicontinuity of singularly perturbed gradient systems). It would also be very interesting to be able to describe the geometrical structure of the perturbed attractors; the results of Langa et al. [15] or Carvalho et al. [7] are not applicable here. Finally, the fact that our perturbations are uniform in time prevents us from considering more general non-autonomous terms, or even the important case of random dynamical systems. We intend to follow these lines of inquiry in the near future.

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