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UNIVERSIDADE DE SÃO PAULO
Instituto de Ciências Matemáticas e de Computação

ISSN 0103-2577

DAMPED WAVE EQUATIONS WITH FAST GROWING DISSIPATIVE NONLINEARITIES

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Nº 272

NOTAS

Série Matemática

São Carlos – SP
Mar./2007

Resumo

Sejam $a > 0$, $\Omega \subset \mathbb{R}^N$ um conjunto aberto, limitado e com fronteira regular e $-A$ o operador laplaciano com condição de fronteira de Dirichlet em Ω. Estudamos o problema de ondas amortecidas

$$\begin{cases} u_{tt} + au_t + Au = f(u), \ t > 0, \\ u(0) = u_0 \in H_0^1(\Omega), \quad u_t(0) = v_0 \in L^2(\Omega), \end{cases}$$

onde $f : \mathbb{R} \to \mathbb{R}$ é continuamente diferenciável, satisfaz a condição de crescimento $|f(s) - f(t)| \leq C|s-t|(1 + |s|^{\rho-1} + |t|^{\rho-1})$, $1 < \rho < \frac{N+2}{N-2}$, $(N \geq 3)$, e a condição de dissipatividade $\limsup_{|s|\to\infty} \frac{f(s)}{s} < \lambda_1$ (λ_1 é o primeiro auto-valor de A). Construímos soluções globais fracas para este problema como limite em $\eta = 0$ de soluções correspondentes a problemas de ondas amortecidas aos quais adicionamos o amortecimento forte $\eta A^{\frac{1}{2}} u$ com $\eta > 0$. Definimos uma subclasse $\mathcal{LS} \subset C([0,\infty), L^2(\Omega) \times H^{-1}(\Omega))$ de limites de soluções e construímos um conjunto fechado e limitado $\mathbf{A}$ de $H_0^1(\Omega) \times L^2(\Omega)$, que atrai qualquer conjunto limitado B de $H^1(\Omega) \times L^2(\Omega)$. Para $N = 3, 4, 5$ e dados iniciais regulares, mostramos que as soluções fracas globais da classe $\mathcal{LS}$ *localmente únicas.*

DAMPED WAVE EQUATIONS WITH FAST GROWING DISSIPATIVE NONLINEARITIES

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ABSTRACT. Let $a > 0$, $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain and $-A$ denote the Laplace operator with Dirichlet boundary conditions in $L^2(\Omega)$. We study the damped wave problem

$$\begin{cases} u_{tt} + au_t + Au = f(u), \ t > 0, \\ u(0) = u_0 \in H_0^1(\Omega), \quad u_t(0) = v_0 \in L^2(\Omega), \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying the growth condition $|f(s) - f(t)| \leq C|s-t|(1 + |s|^{\rho-1} + |t|^{\rho-1})$, $1 < \rho < \frac{N+2}{N-2}$, $(N \geq 3)$, and the dissipativeness condition $\limsup_{|s|\to\infty} \frac{f(s)}{s} < \lambda_1$ with λ_1 being the first eigenvalue of A. We construct the global weak solutions of this problem as the limits at $\eta = 0$ of the solutions corresponding to wave equations involving a strongly damping term $\eta A^{\frac{1}{2}} u$ with $\eta > 0$. We define a subclass $\mathcal{LS} \subset C([0,\infty), L^2(\Omega) \times H^{-1}(\Omega))$ of the 'limit' solutions and construct a closed bounded subset $\mathbf{A}$ of $H_0^1(\Omega) \times L^2(\Omega)$, which attracts any set B bounded in $H^1(\Omega) \times L^2(\Omega)$. For $N = 3, 4, 5$ and smooth initial data, the global weak solutions of the class $\mathcal{LS}$ are shown to be *locally* unique.

1. INTRODUCTION

In this paper we study damped wave equations of the form

$$\begin{cases} u_{tt} + au_t - \Delta u = f(u), \ t > 0, \ x \in \Omega, \\ u(0,x) = u_0(x), \ u_t(0,x) = v_0(x), \ x \in \Omega, \\ u(t,x) = 0, \ t \geq 0, \ x \in \partial\Omega, \end{cases} \tag{1.1}$$

[a]Research partially supported by CNPq # 305447/2005-0 and by FAPESP # 03/10042-0, Brazil

Key words and phrases. Damped wave equations, singular perturbations, supercritical exponents, dissipativeness, asymptotic behavior of solutions.

2000 Mathematics Subject Classification. Primary: 35L05; Secondary: 35B25, 35B33, 35B40.

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $N \geq 3$ and $a > 0$. We assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying for some $1 < \rho < \frac{N+2}{N-2}$ the growth condition

$$|f'(s)| \leq C(1 + |s|^{\rho-1}) \tag{1.2}$$

and the dissipativeness condition

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} =: \mu_1 < \lambda_1, \tag{1.3}$$

with λ_1 being the first eigenvalue of the negative Dirichlet Laplacian in Ω.

Assuming conditions (1.3) and (1.2) with $\rho \in (1, \frac{N+2}{N-2})$, we consider a *parabolic type perturbations* of (1.4) and, taking advantage of the smoothing properties of their solutions, we obtain *global weak solutions* of (1.1) for initial data $u_0 \in H_0^1(\Omega)$, $v_0 \in L^2(\Omega)$. We also construct a bounded (in $H_0^1(\Omega) \times L^2(\Omega)$) absorbing set for these weak solutions of (1.1).

Damped wave equations with fast growing nonlinearities have been considered before by many authors. In [13] the author studied the damped wave problem in $H^1(\Omega)\times L^2(\Omega)$, when Ω was a *N-dimensional Riemannian compact manifold without boundary* and f satisfied (1.2) with $\rho < \frac{N+2}{N-2}$, $N \geq 3$. In [14] the wave problems were locally well posed in the whole of $\mathbb{R}^N$, $N \geq 3$, for $u_0 \in H^1(\mathbb{R}^N)$, $v_0 \in L^2(\mathbb{R}^N)$ and f satisfying (1.2) with $\rho \leq \frac{N+2}{N-2}$ and $f(0) = 0$. The solutions were unique thanks to the *Strichartz inequality*, being the main tool to obtain suitable a priori estimates in case of these special domains. Such results have not been extended to bounded domains with Dirichlet of Neumann boundary conditions.

In [17] the author studied asymptotic behavior of weak energy solutions to (1.1) in the non-autonomous case. Some sort of the monotonicity assumption was considered for f but no growth restriction was used to prove existence of the solutions of (1.1). The approach of [17] comes back to the classical monograph [15], where the solutions were obtained for the problem (1.1) with $|f(s)|$ growing not faster than $|s|^{\frac{N}{N-2}}$ $(N \geq 3)$.

We remark that when the nonlinear term f satisfies (1.3) and $|f(s)| \leq c(1 + |s|^{\frac{N+2}{N-2}})$ for $s \in \mathbb{R}$, it is known that no further assumption is needed to construct (a suitable notion of) a global solution of (1.1). This can be found in consideration of [4, Theorem 4.3], where the approach of [15] was followed.

In [4] previous results of [5] concerning generalized semiflows in metric spaces were also applied to show the existence of an attractor for (1.1). For this an unproved condition (see [4, p. 46]) that *"every weakly continuous solution satisfies the energy equation"* was assumed.

Since the question whether weakly continuous solutions fulfil the energy equation has not been answered yet. We investigate the existence and asymptotic behavior of the solutions of (1.1) based on the regularity and dissipativeness properties of solutions of perturbed (parabolic type) problems replacing the Galerkin approximation technique by a type of *Viscosity Method* which (in this case) seems to lead to more interesting results.

In order to explain better the procedure we introduce some terminology. Let $-A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ denote the Laplacian in $L^2(\Omega)$, with the domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Then A is a positive and self-adjoint operator with compact resolvent. The closed extension of A to $(H_0^1(\Omega))'$ with domain $H_0^1(\Omega)$ will still be denoted by A and $A^{\frac{1}{2}} : H_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ be the half fractional power of A.

We consider the family of problems

$$\begin{cases} u_{tt} + \eta A^{\frac{1}{2}} u_t + a u_t + Au = f(u), \\ u(0) = u_0 \in H_0^1(\Omega),\ u_t(0) = v_0 \in L^2(\Omega), \end{cases} \tag{1.4}$$

which we rewrite as a first order system

$$\tfrac{d}{dt}\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] + \mathcal{A}_\eta \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] = F\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right),\ t > 0, \quad \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]_{t=0} = \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in H_0^1(\Omega) \times L^2(\Omega), \tag{1.5}$$

where η is a non-negative parameter, $\mathcal{A}_\eta$ denotes the strongly damped wave operator;

$$\mathcal{A}_\eta = \begin{bmatrix} 0 & -I \\ A & 2\eta A^{\frac{1}{2}} + aI \end{bmatrix}, \quad D(\mathcal{A}_\eta) = H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega),$$

and

$$F\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 0 \\ f(u) \end{smallmatrix}\right], \ f(u)(x) = f(u(x)) \text{ for } u \in H_0^1(\Omega),\ x \in \Omega.$$

For f satisfying (1.2) with $1 < \rho \le \frac{N+2}{N-2}$, $N \ge 3$, it has been shown in [7] (exploiting the fact that $\mathcal{A}_\eta$ generates analytic semigroup) that (1.5) is locally well posed for any $\eta > 0$ and that its solutions are regular. Also, it is shown in [10] that bootstrapping methods can be used to obtain additional regularity properties of solutions of (1.5).

For $\eta > 0$ and f satisfying (1.3) and (1.2) with $1 < \rho < \frac{N+2}{N-2}$ it has been shown in [8] that (1.5) is globally well posed. In addition, it was shown in [8] that the nonlinear semigroup of global solutions associated to (1.5) has a compact global attractor, which results were extended in [9] to almost critical nonlinearities; that is maps $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfying

$$\limsup_{|s|\to\infty} s^{-\rho+1} f'(s) = 0 \ \text{ with } \rho = \frac{N+2}{N-2} \text{ for } N \ge 3. \tag{1.6}$$

In this paper, we obtain the existence of global weak solutions of (1.5) with $\eta = 0$ when the nonlinearity f satisfies (1.2) with $\rho \in (1, \frac{N+2}{N-2})$ and (1.3). The procedure is to show that the solutions of (1.5) with $\eta > 0$ converges along subsequences to a function which is, in a sense to be explained, a solution of (1.5) with $\eta = 0$. We also obtain a bounded absorbing set for such weak solutions without the need to assume any unproved fact. Here, if $\eta \ge 0$, we will say that

Definition 1.1. *A bounded function* $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](\cdot, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta) : [0,\infty) \to H^1(\Omega) \times L^2(\Omega)$ *is a* global weak solution *of* (1.5) *provided that maps* $[0,\infty) \ni t \mapsto \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](\cdot, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta) \in L^2(\Omega) \times H^{-1}(\Omega)$ *and* $[0,\infty) \ni t \mapsto F(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta)) \in L^2(\Omega) \times H^{-1}(\Omega)$ *are continuous and for each* $t \in [0,\infty)$

$$\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta) = e^{-A_\eta t}\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] + \int_0^t e^{-A_\eta (t-s)} F(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](s, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta)) ds. \tag{1.7}$$

To be more specific about our technique and results note first that, for any $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in H^1_0(\Omega) \times L^2(\Omega)$ and $\eta > 0$, the mild solution of the problem (1.5) with $\eta > 0$ is globally defined and, moreover, its $H^1_0(\Omega) \times L^2(\Omega)-$norm is bounded uniformly for $\eta > 0$ and $t \geq 0$. Hence, as a result of the limiting procedure, with the same f and for the same initial conditions there exists a global weak solution of (1.5) with $\eta = 0$.

To describe a large class of global weak solutions of (1.5) with $\eta = 0$ which are obtained as limits of the solutions of (1.5) with $\eta = \eta_n \to 0^+$ we will prove the following result.

Theorem 1.2. *Assume that* $f \in C^1(\mathbb{R}, \mathbb{R})$ *satisfies* (1.3) *and* (1.2) *with some* $\rho \in (1, \frac{N+2}{N-2})$. *Hence, if* $\eta_n \to 0^+$, $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$ *belongs to* $H^1_0(\Omega) \times L^2(\Omega)$, $\left\{\left[\begin{smallmatrix} u_0^n \\ v_0^n \end{smallmatrix}\right]\right\}_{n=1}^{\infty}$ *is a sequence in* $H^1_0(\Omega) \times L^2(\Omega)$, $\left\|\left[\begin{smallmatrix} u_0^n \\ v_0^n \end{smallmatrix}\right] - \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]\right\|_{H^1_0(\Omega)\times L^2(\Omega)} \to 0$, *and if for* $n \in \mathbb{N}$ *function* $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\left(\cdot, \left[\begin{smallmatrix} u_0^n \\ v_0^n \end{smallmatrix}\right], \eta_n\right)$ *is a mild solution of* (1.5) *with* $\eta = \eta_n$, *then there is a subsequence* $\{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\left(\cdot, \left[\begin{smallmatrix} u_0^{n_k} \\ v_0^{n_k} \end{smallmatrix}\right], \eta_{n_k}\right)\}$ *and a bounded function* $\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] : [0,\infty) \to H^1_0(\Omega) \times L^2(\Omega)$, *being a global weak solution of* (1.5) *with* $\eta = 0$, *to which this subsequence converges in* $L^2(\Omega) \times H^{-1}(\Omega)$ *uniformly on compact subsets of* $[0,\infty)$; *that is*

$$\sup_{t\in[0,\tau]} \left\|\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\left(t, \left[\begin{smallmatrix} u_0^{n_k} \\ v_0^{n_k} \end{smallmatrix}\right], \eta_{n_k}\right) - \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right](t)\right\|_{L^2(\Omega)\times H^{-1}(\Omega)} \longrightarrow 0 \quad \textit{for each } \tau > 0. \tag{1.8}$$

We then consider a following subclass $\mathcal{LS}$ of the global weak solutions of (1.5) with $\eta = 0$;

$$\begin{cases} \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \in \mathcal{LS} \text{ if and only if there exists an element } \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in H^1(\Omega) \times L^2(\Omega) \text{ such} \\ \text{that the function } \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \text{ is a global weak solution of (1.5) with } \eta = 0 \text{ for which} \\ \text{there is a sequence } \{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](\cdot, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta_n)\} \text{ of solutions of (1.5) with } \eta = \eta_n \to 0^+ \\ \text{and } \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta_n) \xrightarrow{Y} \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t) \text{ uniformly for } t \text{ in compact subsets of } [0,\infty). \end{cases} \tag{1.9}$$

As follows from Theorem 1.2 for each $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in H^1_0(\Omega) \times L^2(\Omega)$ there is (at least one) $\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \in \mathcal{LS}$ with $\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right](0) = \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$ and we will prove that

Theorem 1.3. *Under the assumptions of Theorem 1.2 there exists a bounded and closed subset* $\mathbf{A}$ *of* $H^1_0(\Omega) \times L^2(\Omega)$ *(compact in* $L^2(\Omega) \times H^{-1}(\Omega)$*) such that, for any set* B *bounded in* $H^1(\Omega) \times L^2(\Omega)$,

$$\sup_{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\in\mathcal{LS}, \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](0)\in B} \quad \inf_{\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\in\mathbf{A}} \|\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t) - \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\|_Y \overset{t\to\infty}{\to} 0. \tag{1.10}$$

We also obtain that

Theorem 1.4. *If* $N = 3, 4, 5$ *and* $f \in C^2(\mathbb{R}, \mathbb{R})$ *satisfies with certain* $\rho \in \left[2, \frac{N+2}{N-2}\right)$ *the condition*

$$|f''(s)| \le c(1 + |s|^{\rho-2}) \tag{1.11}$$

then, for each $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$ *and any* $\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \in \mathcal{LS}$ *with* $\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right](0) = \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$, *there exists* $\tau_0 > 0$ *such that* $\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]$ *restricted to the inerval* $[0, \tau_0)$ *coincides with a certain (uniquely defined) element of the space of continuous functions* $C([0, \tau_0), H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega))$.

The paper is organized as follows. In Section 2 we will establish global well posedness of $(1.5)_{\eta>0}$, show boundedness of the solutions of $(1.5)_{\eta>0}$ uniform for $\eta \in (0, 1]$ and prove Theorem 1.2. In Section 3 we will construct a bounded absorbing set for $(1.5)_{\eta>0}$ with the *entrance time* independent of $\eta \in (0, 1]$ and $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$ in bounded subsets of $H^1_0(\Omega) \times L^2(\Omega)$ and we will complete the proof of Theorem 1.3. In Section 4 we will discuss briefly local well posedness of (1.5) with $\eta = 0$ for $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$ and complete the proof of Theorem 1.4. In the closing Section 5 we will view some additional properties of the solutions resulting from Theorem 1.2, that come back to consideration of [4, 3, 15, 17].

Acknowledgement: This work has been carried out while the first author visited the Institute of Mathematics of the University of Silesia in Katowice, Poland. The authors wish to thank the Institute for supporting this cooperation. Also, the first author wishes to thank the people at the Institute for their great hospitality.

2. Existence results

Let X^α, $\alpha \in \mathbb{R}$, denote the fractional power spaces associated to A. Then, $X^\alpha = [H^2(\Omega) \cap H^1_0(\Omega), L^2(\Omega)]_\alpha =: H^{2\alpha}_{\{I\}}(\Omega)$ and $X^{-\alpha} = (H^{2\alpha}_{\{I\}}(\Omega))'$ for $\alpha \in [0, 1]$, where $[\cdot, \cdot]_\alpha$ denotes the complex interpolation functor (see [16]). In particular $X := X^0 = L^2(\Omega)$, $X^{\frac{1}{2}} = H^1_0(\Omega)$, $X^{-\frac{1}{2}} = (H^1_0(\Omega))'$ and $X^1 = H^2(\Omega) \cap H^1_0(\Omega)$.

Let $a > 0$ and $Y^1 = X^{\frac{1}{2}} \times X$ with the norm $\| \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \|_{Y^1} = \|\phi\|_{X^{\frac{1}{2}}} + \|\psi\|_X$. Let also $Y^2 = X^1 \times X^{\frac{1}{2}}$. For $\eta \ge 0$ consider a family of the damped wave operators

$$\mathcal{A}_\eta : D(\mathcal{A}_\eta) \subset Y^1 \to Y^1$$

and note that

$$\mathcal{A}_\eta^{-1} : Y^1 \to Y^1, \quad \mathcal{A}_\eta^{-1} = \begin{bmatrix} 2\eta A^{-\frac{1}{2}} + aA^{-1} & A^{-1} \\ -I & 0 \end{bmatrix} \quad \text{for } \eta \ge 0. \tag{2.1}$$

The completion of the normed space $(Y^1, \|\mathcal{A}_\eta^{-1} \cdot \|_{X^{\frac{1}{2}} \times X})$ is $Y =: Y^0 := X \times X^{-\frac{1}{2}}$. Indeed, since there is a constant $d > 0$ (independent of $\eta \in [0, 1]$) such that

$$d^{-1} \|\mathcal{A}_\eta^{-1} \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \|_{X^{\frac{1}{2}} \times X} \le \| \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \|_{X \times X^{-\frac{1}{2}}} \le d \|\mathcal{A}_\eta^{-1} \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \|_{X^{\frac{1}{2}} \times X}, \quad \left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \in X^{\frac{1}{2}} \times X, \tag{2.2}$$

the completions of $(X^{\frac{1}{2}}\times X, \|\mathcal{A}_\eta^{-1}\cdot\|_{X^{\frac{1}{2}}\times X})$ and $(X^{\frac{1}{2}}\times X, \|\cdot\|_{X\times X^{-\frac{1}{2}}})$ coincide with equivalent norms (independent of $\eta\in[0,1]$).

In what follows we will still denote by $\mathcal{A}_\eta$ the closed extension of the operator $\left[\begin{smallmatrix} 0 & -I \\ A & 2\eta A^{\frac{1}{2}}+aI\end{smallmatrix}\right]$ to Y with the domain Y^1.

For $\eta\geq 0$, $\mathcal{A}_\eta : Y^1\subset Y\to Y$ is a maximal accretive operator (see [7] and references therein). Therefore it has bounded imaginary powers, generates a semigroup of contractions $\{e^{-\mathcal{A}_\eta t} : t\geq 0\}$ and its fractional power spaces are given by $Y^\alpha = X^{\frac{\alpha}{2}}\times X^{\frac{\alpha-1}{2}}$, $\alpha\geq 0$. If $\eta>0$, $\mathcal{A}_\eta$ is a sectorial operator.

With this set-up we will consider problem (1.4) in the form (1.5).

Let $F : Y^1\to Y^\alpha$, $\alpha\geq 0$, be a locally Lipschitz continuous map. Recall that a mild solution of (1.5) on $[0,\tau]$ is a function $\left[\begin{smallmatrix} u \\ v\end{smallmatrix}\right](\cdot,\left[\begin{smallmatrix} u_0 \\ v_0\end{smallmatrix}\right],\eta)\in C([0,\tau],Y^1)$ which satisfies (1.7) for $t\in[0,\tau]$. We say that (1.5) is *locally well posed in* Y^1 if for any $\left[\begin{smallmatrix} u_0 \\ v_0\end{smallmatrix}\right]\in Y^1$ there is a unique mild solution $t\mapsto\left[\begin{smallmatrix} u \\ v\end{smallmatrix}\right](t,\left[\begin{smallmatrix} u_0 \\ v_0\end{smallmatrix}\right],\eta)$ of (1.5) defined on a maximal interval of existence $[0,\tau_{u_0,v_0})$ and depending continuously on the initial data $\left[\begin{smallmatrix} u_0 \\ v_0\end{smallmatrix}\right]$.

The following two propositions can be found in [7, 8, 9]. Their proof is included here for completeness.

Proposition 2.1. *The problem* (1.5) *with* $\eta>0$ *is locally well posed in* Y^1 *whenever* f *satisfies* (1.2) *for some* $1<\rho<\frac{N+2}{N-2}$.

Proof: F is Lipschitz continuous on bounded sets from Y^1 into $X^{\frac{\sigma}{2}}\times X^{\frac{\sigma-1}{2}} = Y^\sigma$ whenever $0<\sigma\leq\tilde\sigma$ and $\tilde\sigma=\min\{1,\frac{N}{2}+1-\rho(\frac{N}{2}-1)\}$. Indeed, if B is a bounded subset of Y^1 and $\left[\begin{smallmatrix} \phi_1 \\ \psi_1\end{smallmatrix}\right],\left[\begin{smallmatrix} \phi_2 \\ \psi_2\end{smallmatrix}\right]\in B$, we have

$$\begin{aligned}\left\|F\left(\left[\begin{smallmatrix} \phi_1 \\ \psi_1\end{smallmatrix}\right]\right)-F\left(\left[\begin{smallmatrix} \phi_2 \\ \psi_2\end{smallmatrix}\right]\right)\right\|_{Y^\sigma} &\leq c_1\|f(\phi_1)-f(\phi_2)\|_{X^{\frac{\sigma-1}{2}}}\leq c_2\|f(\phi_1)-f(\phi_2)\|_{L^{\frac{2N}{N+(1-\sigma)2}}(\Omega)}\\ &\leq c_3\|\phi_1-\phi_2\|_{X^{\frac{1}{2}}}\left(1+\|\phi_1\|^{\rho-1}_{L^{\frac{N(\rho-1)}{2-\sigma}}(\Omega)}+\|\phi_2\|^{\rho-1}_{L^{\frac{N(\rho-1)}{2-\sigma}}(\Omega)}\right)\leq c_4\left\|\left[\begin{smallmatrix} \phi_1 \\ \psi_1\end{smallmatrix}\right]-\left[\begin{smallmatrix} \phi_2 \\ \psi_2\end{smallmatrix}\right]\right\|_{Y^1}.\end{aligned}$$

The proof now follows from Theorem 3.3.3 in [12]. □

Proposition 2.2. *Under the assumptions of Proposition 2.1 and* (1.3) *we have that*

i) *the problem* (1.5) *is globally well posed in* Y^1,

ii) *positive orbits of bounded subsets of* Y^1 *are bounded in* Y^1 *uniformly for* $\eta\in(0,1]$.

Proof: The functional $\mathcal{L}_0 : Y^1\to\mathbb{R}$,

$$\mathcal{L}_0(\left[\begin{smallmatrix} w_1 \\ w_2\end{smallmatrix}\right]) = \frac{1}{2}\|w_1\|^2_{X^{\frac{1}{2}}}+\frac{1}{2}\|w_2\|^2_X-\int_\Omega\int_0^{w_1} f(s)dsdx,\ \left[\begin{smallmatrix} w_1 \\ w_2\end{smallmatrix}\right]\in Y^1, \tag{2.3}$$

satisfies for $\mu\in(\mu_1,\lambda_1)$

$$\mathcal{L}_0(\left[\begin{smallmatrix} w_1 \\ w_2\end{smallmatrix}\right])\geq\frac{\lambda_1-\mu}{2\lambda_1}\|w_1\|^2_{X^{\frac{1}{2}}}+\frac{1}{2}\|w_2\|^2_X-c_\mu,$$

where (1.3) and the Poincaré inequality $\lambda_1\|w_1\|_X^2 \le \|w_1\|^2_{X^{\frac{1}{2}}}$ were used to obtain that

$$\int_\Omega \int_0^{w_1} f(s)ds\,dx \le \frac{\mu}{2}\|w_1\|_X^2 + c_\mu.$$

If $\eta > 0$ and $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] = \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta)$ is a local solution of (1.5) through $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in Y^1$, then

$$\frac{d}{dt}\mathcal{L}_0(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]) = -\eta\|v\|^2_{X^{\frac{1}{4}}} - a\|v\|_X^2, \tag{2.4}$$

that is $\mathcal{L}_0$ decreases along $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta)$. Hence we have the estimate

$$\| \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta) \|_{Y^1} \leqslant c_{\mathcal{L}_0}\sqrt{1 + \mathcal{L}(\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right])}, \tag{2.5}$$

where $c_{\mathcal{L}_0}$ does not depend on $\eta > 0$. The proof now follows easily. □

The existence of mild solutions of (1.5) with $\eta = 0$ when $F : Y^1 \to Y^1$ is locally Lipschitz continuous is well known. However, in order F takes Y^1 into itself f should satisfy (1.2) with $1 \le \rho \le \frac{N}{N-2}$. We will next show existence of the *global weak solutions* of (1.5) with $\eta = 0$ when (1.2) holds with $1 < \rho < \frac{N+2}{N-2}$. For this the following two lemmas will be useful.

Lemma 2.3. ([6, Lemma 2.4]) *For* Re$\lambda > 0$ *we have that* λ *is in the resolvent set of* $\mathcal{A}_\eta$ *for each* $\eta \ge 0$. *If* $m = 0, 1, 2$, *there is a constant* c_λ *(independent of* $\eta \in [0, 1]$*), such that*

$$\|(\lambda + \mathcal{A}_0)^{-1} - (\lambda + \mathcal{A}_\eta)^{-1}\|_{L(Y^m)} \leqslant \eta c_\lambda d, \ \ \eta \in (0, 1], \tag{2.6}$$

where d *is given in* (2.2). *Furthermore, if* J *is a compact subset of* Y^m, *then*

$$\sup_{\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] \in J} \sup_{t\in[0,\tau]} \|e^{-\mathcal{A}_\eta t}\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right] - e^{-\mathcal{A}_0 t}\left[\begin{smallmatrix} \phi \\ \psi \end{smallmatrix}\right]\|_{Y^m} \to 0 \ \ as \ \ \eta \to 0^+. \tag{2.7}$$

Proof: If $\mathcal{R}_\eta = \begin{bmatrix} 0 & 0 \\ 0 & \eta A^{\frac{1}{2}} \end{bmatrix}$, then $\|\mathcal{R}_\eta\|_{L(Y^{m+1}, Y^m)} \leqslant \eta$, $\eta \in [0, 1]$. Consequently, we have

$$\lambda I + \mathcal{A}_\eta = (\lambda I + \mathcal{A}_0)(I + (\lambda I + \mathcal{A}_0)^{-1}\mathcal{R}_\eta) \tag{2.8}$$

for each $\lambda \in \mathbb{C}$ with Re$\lambda > 0$ and

$$\begin{aligned} \|(\lambda I + \mathcal{A}_0)^{-1} - (\lambda I + \mathcal{A}_\eta)^{-1}\|_{L(Y^m)} &= \|(\lambda I + \mathcal{A}_0)^{-1}\mathcal{R}_\eta(\lambda I + \mathcal{A}_\eta)^{-1}\|_{L(Y^m)} \\ &\le \|(\lambda I + \mathcal{A}_0)^{-1}\|_{L(Y^m)}\|\mathcal{R}_\eta\|_{L(Y^{m+1}, Y^m)}\|(\lambda I + \mathcal{A}_\eta)^{-1}\|_{L(Y^m, Y^{m+1})}. \end{aligned}$$

From (2.2),

$$\|(\lambda I + \mathcal{A}_\eta)^{-1}\|_{L(Y^m, Y^{m+1})} \le d\|\mathcal{A}_\eta(\lambda I + \mathcal{A}_\eta)^{-1}\|_{L(Y^m)} = d\|I - \lambda(\lambda I + \mathcal{A}_\eta)^{-1}\|_{L(Y^m)}.$$

This proves (2.6), where

$$c_\lambda = \sup_{\eta\in[0,1]} \|(\lambda I + \mathcal{A}_0)^{-1}\|_{L(Y^m)}\|I - \lambda(\lambda I + \mathcal{A}_\eta)^{-1}\|_{L(Y^m)}.$$

To prove (2.7) fix $\delta > 0$, $\tau > 0$. Choose $\epsilon = \frac{\delta}{4}$, $n \in \mathbb{N}$ and $\left[\begin{smallmatrix}\phi_i\\ \psi_i\end{smallmatrix}\right]$, $1 \le i \le n$, such that $\mathcal{J} \subset \cup_{i=1}^n \mathcal{B}_\epsilon^i$ where B_ϵ^i is the ball of radius ϵ centered in $\left[\begin{smallmatrix}\phi_i\\ \psi_i\end{smallmatrix}\right]$, $1 \le i \le n$. From Trotter-Kato theorem, there exists $\eta_\delta = \eta(\delta) > 0$ such that

$$\|e^{-\mathcal{A}_\eta t}\left[\begin{smallmatrix}\phi_i\\ \psi_i\end{smallmatrix}\right] - e^{-\mathcal{A}_0 t}\left[\begin{smallmatrix}\phi_i\\ \psi_i\end{smallmatrix}\right]\|_{Y^m} < \frac{\delta}{2} \text{ for all } t \in [0,\tau],\ \eta \in (0,\eta_\delta),\ i = 1,\dots,n. \tag{2.9}$$

If $\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right] \in \mathcal{J}$ and i is such that $\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right] \in \mathcal{B}_\epsilon^i$, we conclude that

$$\begin{aligned}\sup_{t\in[0,\tau]} \|e^{-\mathcal{A}_\eta t}\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right] - e^{-\mathcal{A}_0 t}\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]\|_{Y^m} &\leqslant \sup_{t\in[0,\tau]} \|e^{-\mathcal{A}_\eta t}(\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right] - \left[\begin{smallmatrix}\phi_i\\ \psi_i\end{smallmatrix}\right])\|_{Y^m}\\ &+ \sup_{t\in[0,\tau]} \|(e^{-\mathcal{A}_\eta t} - e^{-\mathcal{A}_0 t})\left[\begin{smallmatrix}\phi_i\\ \psi_i\end{smallmatrix}\right]\|_{Y^m} + \sup_{t\in[0,\tau]} \|e^{-\mathcal{A}_0 t}(\left[\begin{smallmatrix}\phi_i\\ \psi_i\end{smallmatrix}\right] - \left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right])\|_{Y^m} < 2\epsilon + \frac{\delta}{2} = \delta.\end{aligned}$$ □

Lemma 2.4. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies (1.2) with some $\rho \in (1, \frac{N+2}{N-2})$ and $c > 0$. Then for $\theta = \frac{(\rho-1)(N-2)}{4}$, certain $\bar{c} > 0$ and all $u_1, u_1 \in X^{\frac{1}{2}}$ we have that*

$$\|f(u_1) - f(u_2)\|_{X^{-\frac{1}{2}}} \le \bar{c}\,\|u_1 - u_2\|_{X^{-\frac{1}{2}}}^{1-\theta}\,\|u_1 - u_2\|_{X^{\frac{1}{2}}}^{\theta}\left(1 + \|u_1\|_{X^{\frac{1}{2}}}^{\rho-1} + \|u_2\|_{X^{\frac{1}{2}}}^{\rho-1}\right). \tag{2.10}$$

Proof: Using Hölder inequality and Sobolev embedding we obtain

$$\begin{aligned}\|f(u_1) - f(u_2)\|_{X^{-\frac{1}{2}}} &\le \|f(u_1) - f(u_2)\|_{L^{\frac{2N}{N+2}}(\Omega)}\\ &\le \tilde{c}\,\|u_1 - u_2\|_{L^{\frac{2Nr}{N+2}}(\Omega)}\left(1 + \|u_1\|^{\rho-1}_{L^{\frac{2Nr'(\rho-1)}{N+2}}(\Omega)} + \|u_2\|^{\rho-1}_{L^{\frac{2Nr'(\rho-1)}{N+2}}(\Omega)}\right)\\ &\le \tilde{c}\,\|u_1 - u_2\|_{X^{\frac{s}{2}}}\left(1 + \|u_1\|_{X^{\frac{1}{2}}}^{\rho-1} + \|u_2\|_{X^{\frac{1}{2}}}^{\rho-1}\right),\end{aligned}$$

where $r = \frac{N+2}{N+2-(\rho-1)(N-2)}$ and $s = \frac{(\rho-1)(N-2)-2}{2}$. Since the *moment inequality* implies that

$$\|u_1 - u_2\|_{X^{\frac{s}{2}}} \le \hat{c}\|u_1 - u_2\|_{X^{\frac{1}{2}}}^{\frac{s+1}{2}}\,\|u_1 - u_2\|_{X^{-\frac{1}{2}}}^{\frac{1-s}{2}},$$

the result now follows easily. □

Proof of Theorem 1.2. Let B be a bounded subset of Y^1. Then, for $\left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right] \in B$, $\eta \in (0,1]$ and $t \ge 0$ we have, from Proposition 2.2, that there is a constant $C_B > 0$ such that

$$\|\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](t, \left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right], \eta)\|_{Y^1} \le C_B. \tag{2.11}$$

From this and from (1.4), there is $C_B' > 0$ such that, for $t \ge 0$, $\eta \in (0,1]$ and $\left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right] \in B$

$$\|\left[\begin{smallmatrix}u_t\\ v_t\end{smallmatrix}\right](t, \left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right], \eta)\|_{Y} \le C_B'. \tag{2.12}$$

It follows from the Arzela-Ascoli theorem (see [11]) that for each sequence $\{\eta_n\}_{n=1}^\infty$ convergent to 0 and $\left\{\left[\begin{smallmatrix}u_0^n\\ v_0^n\end{smallmatrix}\right]\right\}_{n=1}^\infty \subset B$ convergent in Y to $\left[\begin{smallmatrix}u_0\\ v_0\end{smallmatrix}\right]$ there is a subsequence (which we denote the same) and a function $\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right] \in C([0,\infty),Y)$ such that (1.8) holds. Furthermore, as a consequence of (2.11) and weak convergence

$$\sup_{t\in[0,\infty)} \left\| \left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right](t)\right\|_{Y^1} \leq C_B. \tag{2.13}$$

It remains to prove that $\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right]$ satisfies (1.7) with $\eta = 0$. Note that

$$\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](t, \left[\begin{smallmatrix}u_0^n\\ v_0^n\end{smallmatrix}\right], \eta_n) = e^{-\mathcal{A}_{\eta_n} t}\left[\begin{smallmatrix}u_0^n\\ v_0^n\end{smallmatrix}\right] + \int_0^t e^{-\mathcal{A}_{\eta_n}(t-s)} F(\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](s, \left[\begin{smallmatrix}u_0^n\\ v_0^n\end{smallmatrix}\right], \eta_n))ds. \tag{2.14}$$

Hence, using Lemma 2.3, it is enough to prove that

$$\int_0^t e^{-\mathcal{A}_{\eta_n}(t-s)} F(\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](s, \left[\begin{smallmatrix}u_0^n\\ v_0^n\end{smallmatrix}\right], \eta_n))ds \text{ converges in } Y \text{ to } \int_0^t e^{-\mathcal{A}_0(t-s)} F(\left[\begin{smallmatrix}\phi(s)\\ \psi(s)\end{smallmatrix}\right])ds$$

uniformly for $t \in [0,\tau]$. For that, using again Lemma 2.3, it is sufficient that

$$F(\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](s, \left[\begin{smallmatrix}u_0^n\\ v_0^n\end{smallmatrix}\right], \eta_n)) \text{ converges in } Y \text{ to } F(\left[\begin{smallmatrix}\phi(s)\\ \psi(s)\end{smallmatrix}\right])$$

uniformly for $s \in [0,\tau]$. That is obtained in the following manner. If $r > 0$ is such that

$$\sup_{n\in\mathbb{N}} \sup_{s\in[0,\tau]} \max\left\{ \left\| \left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](s, \left[\begin{smallmatrix}u_0^n\\ v_0^n\end{smallmatrix}\right], \eta_n)\right\|_{Y^1}, \left\| \left[\begin{smallmatrix}\phi(s)\\ \psi(s)\end{smallmatrix}\right]\right\|_{Y^1} \right\} \leq r,$$

then with the aid of Lemma 2.4 for $\theta = \frac{(\rho-1)(N-2)}{4}$ we get

$$\left\| F(\left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](s, \left[\begin{smallmatrix}u_0^n\\ v_0^n\end{smallmatrix}\right], \eta_n)) - F(\left[\begin{smallmatrix}\phi(s)\\ \psi(s)\end{smallmatrix}\right])\right\|_Y = \|f(u(s, \left[\begin{smallmatrix}u_0^n\\ v_0^n\end{smallmatrix}\right], \eta_n) - f(\phi(s))\|_{X^{-\frac{1}{2}}}$$

$$\leq \bar{c}\left\|u(s, \left[\begin{smallmatrix}u_0^n\\ v_0^n\end{smallmatrix}\right], \eta_n) - \phi(s)\right\|_{X^{-\frac{1}{2}}}^{1-\theta} \left\|u(s, \left[\begin{smallmatrix}u_0^n\\ v_0^n\end{smallmatrix}\right], \eta_n) - \phi(s)\right\|_{X^{\frac{1}{2}}}^{\theta} \left(1 + \left\|u(s, \left[\begin{smallmatrix}u_0^n\\ v_0^n\end{smallmatrix}\right], \eta_n)\right\|_{X^{\frac{1}{2}}}^{\rho-1} + \|\phi(s)\|_{X^{\frac{1}{2}}}^{\rho-1}\right)$$

$$\leq \bar{c}\left\| \left[\begin{smallmatrix}u\\ v\end{smallmatrix}\right](s, \left[\begin{smallmatrix}u_0^n\\ v_0^n\end{smallmatrix}\right], \eta_n) - \left[\begin{smallmatrix}\phi(s)\\ \psi(s)\end{smallmatrix}\right]\right\|_Y^{1-\theta} (2r)^\theta \left(1 + 2r^{\rho-1}\right).$$

Note that $\theta \in (0,1)$ if and only if $\rho \in (1, \frac{N+2}{N-2})$. In the light of (1.8), the result is proved. □

3. Uniformly absorbing set and proof of Theorem 1.3

In this section, following a procedure adapted from [8], we prove the existence of a bounded absorbing set for (1.5) with the *entrance time* independent of $\eta \in (0,1]$ and initial data in bounded subsets of Y^1. We then complete the proof of Theorem 1.3.

Let $\mathcal{L}_0 : Y^1 \to \mathbb{R}$ be the functional given by (2.3). For $\delta \geq 0$ we define $\mathcal{L}_\delta : Y^1 \to \mathbb{R}$ by

$$\mathcal{L}_\delta(\left[\begin{smallmatrix}w_1\\ w_2\end{smallmatrix}\right]) = \mathcal{L}_0(\left[\begin{smallmatrix}w_1\\ w_2\end{smallmatrix}\right]) + \delta\int_\Omega w_1 w_2 dx, \quad \text{for all } \left[\begin{smallmatrix}w_1\\ w_2\end{smallmatrix}\right] \in Y^1. \tag{3.1}$$

Choosing $\delta \leq \min\{\frac{\lambda_1-\mu}{2}, \frac{1}{2}\}$ we have

$$\mathcal{L}_\delta(\left[\begin{smallmatrix}w_1\\ w_2\end{smallmatrix}\right]) \geq \frac{\lambda_1-\mu}{4\lambda_1}\|w_1\|_{X^{\frac{1}{2}}}^2 + \frac{1}{4}\|w_2\|_X^2 - c_\mu, \quad \left[\begin{smallmatrix}w_1\\ w_2\end{smallmatrix}\right] \in Y^1. \tag{3.2}$$

Writing $f(s) = f_0(s) + f(0)$ (see [2, 1, 8]) with

$$|f_0(s)| \leq c(|s|^\rho + |s|), \ s \in \mathbb{R},$$

for some $c > 0$ we find a constant $\bar{c} > 1$ such that

$$-\int_\Omega \int_0^{w_1} f_0(s) ds\, dx \leq \bar{c} \|w_1\|^2_{X^{\frac{1}{2}}} (1 + \|w_1\|^{\rho-1}_{X^{\frac{1}{2}}})$$

and

$$-d \int_\Omega \int_0^{w_1} f_0(s) ds\, dx \leq \|w_1\|^2_{X^{\frac{1}{2}}} \tag{3.3}$$

for $\|w_1\|_{X^{\frac{1}{2}}} \leq r$ and $d = \frac{1}{\bar{c}(1+r^{\rho-1})}$.

These properties of $\mathcal{L}_\delta$ lead to the following result.

Theorem 3.1. *Suppose that f satisfies (1.2) with some $\rho \in (1, \frac{N+2}{N-2})$ and (1.3) holds. Then there is a bounded set $B_0 \subset Y^1$ with the property that given a bounded set $B \subset Y^1$ there is a time $\tau_B > 0$ such that*

$$\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta) \in B_0 \ \ \textit{for all} \ \ t \geq t_B, \ \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B, \ \eta \in (0, 1].$$

In particular, each weak solution of (1.5) with $\eta = 0$ from the class $\mathcal{LS}$ and starting at $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in B$ enters B_0 no later than in time τ_B and stays in B_0 for all $t \geq \tau_B$.

Proof: Let $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta)$ ($\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ for short) be the solution of (1.5) with $\eta > 0$. Then

$$\frac{d}{dt} \int_\Omega u\, v dx = \|v\|^2_X - a \int_\Omega u\, v dx - \eta \int_\Omega A^{\frac{1}{2}} u\, v dx - \|u\|^2_{X^{\frac{1}{2}}} + \int_\Omega u\, f(u) dx. \tag{3.4}$$

Fix $\mu \in (\mu_1, \lambda_1)$. Using Young's inequality, for any $\epsilon > 0$, we have that

$$\frac{d}{dt} \int_\Omega u\, v dx \leq (1 + \frac{a}{2\epsilon} + \frac{\eta}{2\epsilon}) \|v\|^2_X - (1 - \frac{\mu}{\lambda_1} - \frac{\epsilon a}{2\lambda_1} - \frac{\epsilon \eta}{2}) \|u\|^2_{X^{\frac{1}{2}}} + \bar{c}_\mu, \tag{3.5}$$

where we have used the the dissipativeness condition (1.3) to obtain that

$$\int_\Omega u\, f(u) dx \leq \mu \|u\|^2_X + \bar{c}_\mu$$

and the Poincaré inequality $\lambda_1 \|u\|^2_X \leq \|u\|^2_{X^{\frac{1}{2}}}$. From (2.4) and (3.5) we have that

$$\begin{aligned} \frac{d}{dt} \mathcal{L}_\delta(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]) &\leq -(a - \delta - \frac{\delta a}{2\epsilon} - \frac{\delta \eta}{2\epsilon}) \|v\|^2_X - \delta(\frac{\lambda_1 - \mu}{\lambda_1} - \frac{a\epsilon}{2\lambda_1} - \frac{\eta \epsilon}{2}) \|u\|^2_{X^{\frac{1}{2}}} + \delta \bar{c}_\mu \\ &\leq -\frac{a}{2} \|v\|^2_X - \delta \frac{\lambda_1 - \mu}{2\lambda_1} \|u\|^2_{X^{\frac{1}{2}}} + \delta \bar{c}_\mu \leq -\omega \| \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \|^2_{Y^1} + \delta \bar{c}_\mu, \end{aligned} \tag{3.6}$$

where $\omega = \frac{1}{2} \min\{a, \delta \frac{\lambda_1 - \mu}{\lambda_1}\}$ and we have chosen ϵ such that $\frac{\lambda_1 - \mu}{2\lambda_1} \geq \frac{a\epsilon}{2\lambda_1} + \frac{\eta\epsilon}{2}$ and then δ such that $\frac{a}{2} \geq \delta + \frac{\delta a}{2\epsilon} + \frac{\delta\eta}{2\epsilon}$. Hence, for $\| \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \|_{Y^1} \leq r$, using (2.5), (3.3) and (3.6), we have

$$\frac{d}{dt} \mathcal{L}_\delta(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]) \leq -\frac{\omega}{2} \| \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \|^2_{Y^1} + \frac{d\omega}{2} \int_\Omega \int_0^u f(s) ds\, dx + \bar{C}_\mu \leq -\bar{\omega} \mathcal{L}_\delta(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]) + \bar{C}_\mu, \tag{3.7}$$

where $\bar{\omega} = \frac{d\omega}{2}$. From (3.2) and (3.7) the result follows easily. □

Proof of Theorem 1.3. Let $\mathcal{LS}$ be a class of maps defined in (1.9) and B_0 be the absorbing set from Theorem 3.1. Denote by $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta)$ the global solution of (1.5) and define

$$\mathbf{A} := \{\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] : \exists_{t_n \nearrow \infty}\ \exists_{\left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right] \subset B_0}\ \exists_{\eta_n \searrow 0}\ \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t_n, \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right], \eta_n) \xrightarrow{Y} \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right]\}. \tag{3.8}$$

If $\{\left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right]\} \subset \mathbf{A}$, then there exist sequences $t_n \to \infty$, $\{\left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right]\} \subset B_0$, $\eta_n \searrow 0$ such that $\| \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t_n, \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right], \eta_n) - \left[\begin{smallmatrix} w_n \\ z_n \end{smallmatrix}\right] \|_Y \to 0$. Since almost all elements of the sequence $\{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t_n, \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right], \eta_n)\}$ are (via Theorem 3.1) in B_0 and Y^1 is compactly embedded in Y, then there are subsequences $\{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t_{n_k}, \left[\begin{smallmatrix} u_{n_k} \\ v_{n_k} \end{smallmatrix}\right], \eta_{n_k})\}$, $\{\left[\begin{smallmatrix} w_{n_k} \\ z_{n_k} \end{smallmatrix}\right]\}$ both convergent in Y to certain $\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \in Y$. Recalling definition (3.8) we obtain that $\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \in \mathbf{A}$; i.e. $\mathbf{A}$ is compact in Y. Using next weak convergence property and boundedness of B_0 in Y^1 we observe that $\mathbf{A}$ is a bounded subset of Y^1. Closedness of $\mathbf{A}$ in Y^1 is straightforward.

Suppose now that (1.10) fails and choose $t_n \nearrow \infty$, $\{\left[\begin{smallmatrix} \phi_n \\ \psi_n \end{smallmatrix}\right]\} \subset \mathcal{LS}$ with $\{\left[\begin{smallmatrix} \phi_n \\ \psi_n \end{smallmatrix}\right](0)\} \subset B$ such that

$$\inf_{\left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \in \mathbf{A}} \| \left[\begin{smallmatrix} \phi_n \\ \psi_n \end{smallmatrix}\right](t_n) - \left[\begin{smallmatrix} w \\ z \end{smallmatrix}\right] \|_Y > \epsilon, \tag{3.9}$$

where B is a certain bounded subset of Y^1 and ϵ is some positive number. By (1.9) there are sequences $\{\left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right]\}$ bounded in Y^1 and $\eta_n \searrow 0$ such that

$$\sup_{t \in [0, t_n]} \| \left[\begin{smallmatrix} \phi_n \\ \psi_n \end{smallmatrix}\right](t) - \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right], \eta_n) \|_Y \leq \frac{1}{n} \quad \text{for} \quad n \in \mathbb{N}. \tag{3.10}$$

From Theorem 3.1 we know that, for certain $t_B > 0$, $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t, \left[\begin{smallmatrix} u_n \\ v_n \end{smallmatrix}\right], \eta_n) \in B_0$ for $t \geq t_B$. Therefore, via compactness of the embedding $Y^1 \hookrightarrow Y$, we can choose a subsequence $\{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t_{n_k}, \left[\begin{smallmatrix} u_{n_k} \\ v_{n_k} \end{smallmatrix}\right], \eta_{n_k})\}$ convergent in Y to certain $\left[\begin{smallmatrix} \bar{w} \\ \bar{z} \end{smallmatrix}\right] \in Y$. Recalling definition (3.8) and writing $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t_{n_k}, \left[\begin{smallmatrix} u_{n_k} \\ v_{n_k} \end{smallmatrix}\right], \eta_{n_k}) = \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t_{n_k} - t_B, \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](t_B, \left[\begin{smallmatrix} u_{n_k} \\ v_{n_k} \end{smallmatrix}\right], \eta_{n_k}), \eta_{n_k}))$ we then have that $\left[\begin{smallmatrix} \bar{w} \\ \bar{z} \end{smallmatrix}\right] \in \mathbf{A}$ whereas, as a consequence of (3.10), $\left[\begin{smallmatrix} \phi_n \\ \psi_n \end{smallmatrix}\right](t_n) \to \left[\begin{smallmatrix} \bar{w} \\ \bar{z} \end{smallmatrix}\right]$, which contradicts (3.9). □

4. Local uniqueness with regular initial data

In this section we consider (1.5) with $\eta = 0$ and $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in X^1 \times X^{\frac{1}{2}}$. We prove Theorem 4.1 below, from which Theorem 1.4 follows.

Theorem 4.1. *Suppose that the assumptions of Theorem 1.4 hold and* $\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right] \in X^1 \times X^{\frac{1}{2}}$. *Then there exists* $\tau > 0$ *such that, for each* $\eta \in [0, 1]$, *the problem* (1.5) *has a unique solution* $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](\cdot, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta) \in C([0, \tau], X^1 \times X^{\frac{1}{2}})$. *Furthermore,*

$$\sup_{\eta \in [0,1]} \| \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](\cdot, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta) \|_{C([0,\tau], X^1 \times X^{\frac{1}{2}})} < \infty \tag{4.1}$$

and

$$\| \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](\cdot, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], \eta) - \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](\cdot, \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right], 0) \|_{C([0,\tau], X^1 \times X^{\frac{1}{2}})} \to 0 \quad as \quad \eta \to 0^+. \tag{4.2}$$

Proof: Using Hölder inequality and Sobolev type embedding, we obtain that

$$\begin{aligned}
\|f(u_1)-f(u_2)\|_{X^{\frac12}} &\le \|(f'(u_1)-f'(u_2))\nabla u_1\|_X+\|f'(u_2)\nabla(u_1-u_2)\|_X\\
&\le \|f'(u_1)-f'(u_2)\|_{L^N(\Omega)}\|\nabla u_1\|_{L^{\frac{2N}{N-2}}(\Omega)}+\|f'(u_2)\|_{L^N(\Omega)}\|\nabla(u_1-u_2)\|_{L^{\frac{2N}{N-2}}(\Omega)}\\
&\le \hat{c}\|u_1-u_2\|_{X^1}(1+\|u_1\|_{X^1}^{\rho-2}+\|u_2\|_{X^1}^{\rho-2})\|u_1\|_{X^1}+\hat{c}\|u_1-u_2\|_{X^1}(1+\|u_2\|_{X^1}^{\rho-1}).
\end{aligned}$$

Consequently, for some $\bar{c}>0$ and all $u_1, u_1\in X^1$ we have

$$\|f(u_1)-f(u_2)\|_{X^{\frac12}}\le \bar{c}\,\|u_1-u_2\|_{X^1}\left(1+\|u_1\|_{X^1}^{\rho-1}+\|u_2\|_{X^1}^{\rho-1}\right), \tag{4.3}$$

which ensures that $F: Y^2\to Y^2$ is Lipschitz continuous in bounded subsets of Y^2.

Note that the mild solutions of (1.5) are fixed points of a map constructed in accordance to (1.7). Since the Lipschitz constant of F is independent of $\eta\ge 0$ and $\|e^{-\mathcal{A}_\eta t}\|_{L(Y^2)}\le 1$ for all $\eta\ge 0$, the time of existence of these solutions can be obtained independently of η. In particular (4.1) is proved.

It can be seen from Lemma 2.3 that $e^{-\mathcal{A}_\eta t}$ converges to $e^{-\mathcal{A}_0 t}$ in the strong topology of $L(Y^2)$ uniformly in bounded time intervals and on compact subsets of Y^2. Therefore, to conclude the proof, we need to ensure that the set $K=\{F\left(\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right](\cdot,\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right],0)\right): t\in[0,\tau]\}$ is compact in Y^2, which comes from continuity of the map $[0,T]\ni t\mapsto F\left(\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right](\cdot,\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right],0)\right)\in Y^2$.

The proof of (4.2) now follows from Gronwall's Inequality. □

Remark 4.1. *If $\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\in Y^1$ and $\rho\le\frac{N}{N-2}$ the weak solutions of (1.5) with $\eta=0$ will belong to $C([0,\tau],Y^1)$, for with this growth $F: Y^1\to Y^1$ is Lipschitz continuous in bounded sets. If, in addition to the assumptions of Theorem 4.1, $\eta>0$ and f satisfies (1.3), then (following [8]) the solutions $\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right](\cdot,\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right],\eta)$, $\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\in Y^2$, $\eta>0$, obtained in Theorem 4.1 are globally defined. However, one can hardly assure that they are bounded in Y^2 uniformly for $\eta\in(0,1]$ so that the global weak solutions of (1.5) with $\eta=0$ starting in $\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\in Y^2$ may have values in Y^2 only for $t\in[0,\tau]$. Note that, although the problem (1.5) with $\eta=0$ is locally well posed in Y^2, its global well posedness in Y^2 is unknown.*

5. Other properties of weak solutions

In this section we view the global weak solutions of (1.5) with $\eta=0$ from another perspective.

Recall that A (resp. $A^{\frac12}$) is an isomorphism from $X^{\frac12}$ onto $X^{-\frac12}$ (resp. from X onto $X^{-\frac12}$) and denote by $\langle\cdot,\cdot\rangle_H$ the scalar product in a Hilbert space H. Suppose that the assumptions of Theorem 1.2 are satisfied and let $\left[\begin{smallmatrix}u^{\eta_n}\\v^{\eta_n}\end{smallmatrix}\right]=\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right]\left(\cdot,\left[\begin{smallmatrix}u_0^n\\v_0^n\end{smallmatrix}\right],\eta_n\right)$ be the solution of (1.5) with $\eta_n\in(0,1]$, $\left[\begin{smallmatrix}u_0^n\\v_0^n\end{smallmatrix}\right]\in B$; B being a bounded subset of Y^1. Note that, by (2.11)-(2.12), we have

$$\begin{gathered}
\{u^{\eta_n}\}\ \text{ is bounded in } L^\infty((0,\infty),X^{\frac12}),\\
\{u_t^{\eta_n}\}\ \text{ is bounded in } L^\infty((0,\infty),X^0),\ \ \{u_{tt}^{\eta_n}\}\ \text{ is bounded in } L^\infty((0,\infty),X^{-\frac12}).
\end{gathered}\tag{5.1}$$

Fix arbitrary $\tau > 0$. If $\eta_n \to 0^+$ and $\left[\begin{smallmatrix} u_{0n} \\ v_{0n} \end{smallmatrix}\right] \overset{Y}{\to} \left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]$, then there is a subsequence, which we denote the same, and elements u, v, w of an appropriate Banach spaces (as below) such that for $p \in (1,\infty)$ and suitably small $\epsilon > 0$ (see [15, Chapter 1, Theorem 5.1])

$$u^{\eta_n} \overset{L^p((0,\tau),X^{\frac{1}{2}-\epsilon})}{\to} u, \qquad u_t^{\eta_n} \overset{L^p((0,\tau),X^{-\epsilon})}{\to} v.$$

Also, by the property of weak convergence,

$$u_t^{\eta_n} \overset{\mathrm{w}-L^2((0,\tau),X^0)}{\to} v, \qquad u^{\eta_n} \overset{\mathrm{w}-L^2((0,\tau),X^{\frac{1}{2}})}{\to} u, \qquad u_{tt}^{\eta_n} \overset{\mathrm{w}-L^2((0,\tau),X^{-\frac{1}{2}})}{\to} w.$$

We remark that there exist $X^{-\frac{1}{2}}$-valued distribution derivatives $u_t = v$, $u_{tt} = w$ and

$$\begin{gathered} u \in C([0,\tau], X^{\frac{1}{2}-\epsilon}) \cap B((0,\tau), X^{\frac{1}{2}}), \\ u_t \in C([0,\tau], X^{-\frac{1}{2}}) \cap B((0,\tau), X^0), \quad u_{tt} \in B((0,\tau), X^{-\frac{1}{2}}). \end{gathered} \tag{5.2}$$

These properties and properties of A and f allow us to show that for each $t > 0$, $\phi \in X$

$$\langle A^{-\frac{1}{2}}[f(u^{\eta_n}) - f(u)], \phi\rangle_X \to 0, \quad \langle A^{-\frac{1}{2}}[\eta_n A^{\frac{1}{2}} u_t^{\eta_n}], \phi\rangle_X = \eta_n \langle u_t^{\eta_n}, \phi\rangle_X \to 0,$$

$$\langle A^{-\frac{1}{2}}[a u_t^{\eta_n} - a u_t], \phi\rangle_X \to 0, \quad \langle A^{-\frac{1}{2}}[A u^{\eta_n} - A u], \phi\rangle_X \to 0, \quad \langle A^{-\frac{1}{2}}[u_{tt}^{\eta_n} - u_{tt}], \phi\rangle_X \to 0,$$

as $\eta_n \to 0^+$. We can thus pass to the limit in the equation

$$\langle A^{-\frac{1}{2}}[u_{tt}^{\eta_n} + \eta_n A^{\frac{1}{2}} u_t^{\eta_n} + a u_t^{\eta_n} + A u^{\eta_n}], \phi\rangle_X = \langle A^{-\frac{1}{2}}[f(u^{\eta_n})], \phi\rangle_X, \quad \phi \in X,$$

and obtain that

$$\langle A^{-\frac{1}{2}}[u_{tt} + a u_t + A u], \phi\rangle_X = \langle A^{-\frac{1}{2}} f(u), \phi\rangle_X, \quad \phi \in X.$$

In particular, we have that for every $\psi \in X^{-\frac{1}{2}}$

$$\langle u_{tt} + a u_t + A u, \psi\rangle_{X^{-\frac{1}{2}}} = \langle f(u), \psi\rangle_{X^{-\frac{1}{2}}},$$

or equivalently that u fulfils $u_{tt} + a u_t + A u = f(u)$ in $X^{-\frac{1}{2}}$ for a. e. $t > 0$, or that

$$\langle u_{tt} + a u_t + A u, \chi\rangle_{X^{-\frac{1}{2}}, X^{\frac{1}{2}}} = \langle f(u), \chi\rangle_{X^{-\frac{1}{2}}, X^{\frac{1}{2}}} \text{ for each } \chi \in X^{\frac{1}{2}}.$$

With our growth we have justified that $u_t \in X$ and $u_{tt} + Au - f(u) = -a u_t$ in $X^{-\frac{1}{2}}$ for all $t > 0$. Therefore we conclude that

$$u_{tt} + Au - f(u) = -a u_t \quad \text{in } X,\ t > 0,$$

and thus

$$\langle u_{tt} + Au - f(u), u_t\rangle_X = -a\|u_t\|_X^2,\ t > 0.$$

Nevertheless, it remains unknown whether the energy equation (as in [4, (E)])

$$\mathcal{L}_0(\left[\begin{smallmatrix} u \\ u_t \end{smallmatrix}\right]) + a\int_0^t \|u_t(s)\|_X^2 ds = \mathcal{L}_0(\left[\begin{smallmatrix} u_0 \\ v_0 \end{smallmatrix}\right]), \quad t \geq 0,$$

holds for the global weak solutions discussed above.

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