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LOCAL WELL POSEDNESS, ASYMPTOTIC BEHAVIOR AND ASYMPTOTIC
BOOTSTRAPPING FOR A CLASS OF SEMILINEAR EVOLUTION EQUATIONS
OF THE SECOND ORDER IN TIME

A. N. CARVALHO
J. W. CHOLEWA

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Resumo

Neste artigo consideramos a classe de problemas de evolução semilineares de segunda ordem $u_{tt} + Au + \mu Au_t + Au_{tt} = f(u)$, onde $-A$ é o operador Laplaciano com condição de fronteira de Dirichlet, Ω é um domínio suave em $\mathbb{R}^N$ e $f \in C^1(\mathbb{R}, \mathbb{R})$. Provamos um resultado de boa colocação local nos espaços de Banach $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ quando f satisfaz condições de crescimento crítico apropriadas. No contexto de Hilbert ($p = 2$), se f satisfaz uma condição de dissipação, o semigrupo não linear das soluções globais possui um atrator global com estrutura gradiente. Um método de *bootstrapping* assintótico é desenvolvido para obter regularidade adicional do atrator no caso de crescimento subcrítico.

LOCAL WELL POSEDNESS, ASYMPTOTIC BEHAVIOR AND ASYMPTOTIC BOOTSTRAPPING FOR A CLASS OF SEMILINEAR EVOLUTION EQUATIONS OF THE SECOND ORDER IN TIME

A. N. CARVALHO AND J. W. CHOLEWA

ABSTRACT. A class of semilinear evolution equations of the second order in time of the form $u_{tt} + Au + \mu Au_t + Au_{tt} = f(u)$ is considered, where $-A$ is the Dirichlet Laplacian, Ω is a smooth bounded domain in $\mathbb{R}^N$ and $f \in C^1(\mathbb{R}, \mathbb{R})$. A local well posedness result is proved in the Banach spaces $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ when f satisfies appropriate critical growth conditions. In the Hilbert setting, if f satisfies an additional dissipativeness condition, the nonlinear semigroup of global solutions is shown to possess a gradient-like attractor. Existence and regularity of the global attractor are also investigated following the unified semigroup approach, bootstrapping and the interpolation-extrapolation techniques.

Mathematical Subject Classification 2000: 35G25, 35B33, 35B40, 35B41, 35B65.

Key words and phrases: Evolution equations of the second order in time; Existence, uniqueness and continuous dependence of solutions on initial conditions; Asymptotic behavior of solutions; Attractors, Regularity; Critical exponents.

1. INTRODUCTORY NOTES

In this article we consider a class of semilinear evolution equations of the second order in time of the form

$$u_{tt} + Au + \mu Au_t + Au_{tt} = f(u), \quad t > 0, \quad (1.1)$$

with the initial conditions

$$u(0) = u_0, \quad u_t(0) = v_0 \quad (1.2)$$

from a suitably chosen Banach space X^1 . Here $-A : D(A) \subset X \rightarrow X$ is the generator of an exponentially decaying analytic semigroup of bounded linear operators in X and X^1 is the domain of A with the graph norm.

The equations that fall into this class are known to represent a sort of ‘propagation problems’ (see [3, 5, 20]; also [16] and references therein); among which a specific problem is

$$\begin{cases} u_{tt} - \Delta u - \mu \Delta u_t - \Delta u_{tt} = f(u), & t > 0, \quad x \in \Omega, \\ u = 0, & t \geq 0, \quad x \in \partial\Omega, \\ u(0, x) = u_0(x), \quad u_t(0, x) = v_0(x), & x \in \Omega, \end{cases} \quad (1.3)$$

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where $\mu > 0$, Ω is a bounded smooth domain in $\mathbb{R}^N$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitz function satisfying suitable growth and dissipativeness conditions.

Denote by λ_1 the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions in Ω .

In [16] the problem (1.3) was considered in a Hilbert space $H_0^1(\Omega) \times H_0^1(\Omega)$, where the existence of a global attractor was proved under assumptions

$$\limsup_{|s| \rightarrow \infty} \frac{f(s)}{s} < \lambda_1, \quad (1.4)$$

and

$$|f(s)| \leq c(1 + |s|^\rho), s \in \mathbb{R}, \text{ where } \rho \in (1, \infty) \text{ is} \quad (1.5)$$

arbitrary if $N = 1, 2$ and $\rho \leq \frac{N+2}{N-2}$ if $N \geq 3$.

The difficulties connected with the consideration of the critical exponent $\rho^* = \frac{N+2}{N-2}$ ($N \geq 3$) have been successfully overcome following the ideas of [12, 19]. In this consideration the key steps was the existence of an absorbing set accompanied by the suitable estimates for the solutions, from which the following asymptotic compactness property could be inferred:

$$\begin{aligned} (\text{ACP}) \quad & \text{if } r > 0, \|u_{0n}\|_{H_0^1(\Omega)} + \|v_{0n}\|_{H_0^1(\Omega)} \leq r \text{ and } t_n \nearrow \infty, \text{ there is a subsequence } \{n_k\} \\ & \text{so that } \{u(t_{n_k}, u_{0n_k}, v_{0n_k})\} \text{ and } \{u_t(t_{n_k}, u_{0n_k}, v_{0n_k})\} \text{ are convergent in } H_0^1(\Omega). \end{aligned} \quad (1.6)$$

As for the existence-uniqueness results in [16], they have been quoted from previous references [17, 18], where the proof was based on the Faedo-Galerkin method.

In this paper we show that the semigroup approach, known from the geometric theory of semilinear problems (see [10, 13]) and the dissipative systems theory (see [8, 11]), is applicable to the problems that fall into the class (1.1)-(1.2) and provides the relevant information about their solutions.

Within this approach we obtain local well posedness result for (1.3) in the Banach space $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$, $p \in (1, \infty)$, provided that one of the following conditions holds

$$\begin{aligned} (\text{H1}) \quad & p > N \text{ and } f : \mathbb{R} \rightarrow \mathbb{R} \text{ is a locally Lipschitz continuous map,} \\ (\text{H2}) \quad & 1 < p < N \text{ and } f \text{ satisfies } (*), \text{ for some } c > 0 \text{ and } 1 < \rho \leq \frac{N+p}{N-p}, \\ & (*) \quad |f(s_1) - f(s_2)| \leq c|s_1 - s_2|(1 + |s_1|^{\rho-1} + |s_2|^{\rho-1}), \quad s_1, s_2 \in \mathbb{R}, \\ (\text{H3}) \quad & 1 < p = N \text{ and } f \text{ satisfies } (*) \text{ for some } c > 0 \text{ and } \rho \in (1, \infty). \end{aligned} \quad (1.7)$$

Within the same approach we then obtain, for $p = 2$ and assuming (1.4), the global well posedness result showing also that the associated semigroup of global solutions has a global attractor and is much like a gradient system (see [8]).

For $p = 2$ and in the subcritical case; that is, when

$$\begin{aligned} \text{(SC)} \quad & N = 1 \text{ and } f : \mathbb{R} \rightarrow \mathbb{R} \text{ is a locally Lipschitz continuous function,} \\ & \text{or } N = 2 \text{ and } (*) \text{ in (1.7) is satisfied with some } \rho \in (1, \infty), \ c > 0, \\ & \text{or } N \geq 3 \text{ and } (*) \text{ in (1.7) holds with some } 1 < \rho < \frac{N+2}{N-2}, \ c > 0, \end{aligned} \quad (1.8)$$

the mentioned approach provides, besides the existence, additional regularity of the global attractor $\mathbf{A}$ through an asymptotic bootstrapping procedure.

In the case involving the critical exponent $\rho^* = \frac{N+2}{N-2}$, and specifically when

$$\text{(CC)} \quad N \geq 3, \ f \in C^1(\mathbb{R}, \mathbb{R}) \text{ and } |f'(s)| \leq c(1 + |s|^{\frac{N+2}{N-2}-1}), \ s \in \mathbb{R}, \quad (1.9)$$

the existence of the global attractor is obtained with the aid of the Alekseev's nonlinear variation of constants formula (see [2]). In either case the attractor is gradient-like (see [9]); that is,

$$\mathbf{A} = W^u(\mathcal{E}), \quad (1.10)$$

where $\mathcal{E}$ denotes the set of equilibria of (1.3) and $W^u(\mathcal{E})$ is the unstable set of $\mathcal{E}$.

To be more specific about our technique and approach note first that in variables (t, v) , where $v = (I + A)u$, the problem (1.1)-(1.2) becomes

$$v_{tt} + \mu \Lambda v_t + \Lambda v = f^e(v), \ t > 0, \quad (1.11)$$

with the initial conditions

$$v(0) = v_0, \ v_t(0) = w_0 \quad (1.12)$$

in a Banach space X , where $\Lambda := I - (I + A)^{-1} \in L(X)$, $f^e := f \circ (I - \Lambda) : X \rightarrow X$ is locally Lipschitz continuous, and $X := X^0$ is defined as the extrapolation space of X^1 generated by the realization of A in X^1 (see [1]).

In the extrapolation space X the resemblance of (the counterpart (1.11)) of (1.1) with the classical strongly damped wave equation seems worthy to be mentioned and exploited. To exhibit these type properties it is then reasonable to substitute $w := v_t$, which leads to the first order problem

$$\frac{d}{dt} \begin{bmatrix} v \\ w \end{bmatrix} + \mathcal{A}_\mu \begin{bmatrix} v \\ w \end{bmatrix} = \mathcal{F} \left(\begin{bmatrix} v \\ w \end{bmatrix} \right), \ t > 0, \quad \begin{bmatrix} v \\ w \end{bmatrix}_{|t=0} = \begin{bmatrix} v_0 \\ w_0 \end{bmatrix} \quad (1.13)$$

in variables (t, v, w) with $\mathcal{A}_\mu \in L(X \times X)$ given in the matrix form as

$$\mathcal{A}_\mu := \begin{bmatrix} 0 & -I \\ \Lambda & \mu \Lambda \end{bmatrix} = \begin{bmatrix} 0 & -I \\ I - (I + A)^{-1} & \mu(I - (I + A)^{-1}) \end{bmatrix} \quad (1.14)$$

and a locally Lipschitz map $\mathcal{F} : X \times X \rightarrow X \times X$ being defined by

$$\mathcal{F} \left(\begin{bmatrix} v \\ w \end{bmatrix} \right) = \begin{bmatrix} 0 \\ f^e(v) \end{bmatrix}, \quad \begin{bmatrix} v \\ w \end{bmatrix} \in X \times X. \quad (1.15)$$

In the above setting the linear operator $\mathcal{A}_\mu$ defined in the product (extrapolation) space $X \times X$ essentially resembles the strongly damped wave operator appearing in the context

of the classical wave equations (see e.g. [4, 7, 15]). The implicit conclusion drawn from the above generalities is that the problem (1.1) should be well posed in the Banach spaces $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$, $p \in (1, \infty)$, and that its solutions should exhibit some smoothing properties at least at infinity and at least in the (subcritical) case (SC). Also the existence of an attractor should follow within the semigroup approach as for the problems enjoying a gradient-like structure.

To be more specific about our results let A be the negative Dirichlet Laplacian Δ in $L^p(\Omega)$, $p \in (1, \infty)$, with the domain $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and consider its closed extension¹ to $X = (W_0^{1,p'}(\Omega))' =: W^{-1,p}(\Omega)$.

As for the local well posedness of (1.1)-(1.2) we then obtain the following theorem.

Theorem 1.1. *For each $u_0, v_0 \in W_0^{1,p}(\Omega)$, there exists $\tau(u_0, v_0) > 0$ and a unique function $u(\cdot) \in C^2([0, \tau(u_0, v_0)), W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega))$ satisfying*

$$u(0) = u_0, \quad u_t(0) = v_0,$$

and

$$u_{tt}(t) + \mu A u_t(t) + A u(t) + A u_{tt}(t) = f(u(t)) \quad \text{for each } t \in [0, \tau(u_0, v_0))$$

provided that one of the conditions (H1), (H2) or (H3) in (1.7) holds.

Furthermore, functions $u(t, u_0, v_0)$ and $u_t(t, u_0, v_0)$ depend continuously on its variables $(t, u_0, v_0) \in [0, \tau(u_0, v_0)) \times W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ in the norm of $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ and either $\tau_{u_0, v_0} = \infty$ or $\|u(t, u_0, v_0)\|_{W_0^{1,p}(\Omega)} + \|u_t(t, u_0, v_0)\|_{W_0^{1,p}(\Omega)} \rightarrow \infty$ as $t \rightarrow \tau_{u_0, v_0}^-$.

As for the global well posedness result in $H_0^1(\Omega) \times H_0^1(\Omega)$, the nonlinear semigroup of global solutions associated to the problem (1.1)-(1.2) is shown to possess much of the properties of gradient systems (see [8]); namely,

Theorem 1.2. *Assume that (1.4) holds, and that one of the conditions (H1), (H2) or (H3) in (1.7) is satisfied with $p = 2$.*

Then

- i) *for each $u_0, v_0 \in H_0^1(\Omega)$ the solution $u(t, u_0, v_0)$ of (1.1) exists globally in time,*
- ii) *relation $S(t)(\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}) = \begin{bmatrix} u(t, u_0, v_0) \\ u_t(t, u_0, v_0) \end{bmatrix}$, $t \geq 0$, $u_0, v_0 \in H_0^1(\Omega)$, defines a C^0 semigroup on $H_0^1(\Omega) \times H_0^1(\Omega)$ with bounded orbits of bounded sets,*
- iii) *the set $\mathcal{E}$ consisting of all the equilibria of $\{S(t)\}$ has the form $\mathcal{E} = \{(\phi, 0), \phi \in \mathcal{E}_1\}$, where $\mathcal{E}_1 = \{\phi \in H_0^1(\Omega) : A\phi = f(\phi)\}$ is a bounded subset of $C^2(\bar{\Omega})$,*
- iv) *there is a continuous functional $\mathcal{V} : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$ such that*
 - a) *$\mathcal{V}(S(\cdot)(\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}))$ is bounded from below and non-increasing in $[0, \infty)$ for any $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in H_0^1(\Omega) \times H_0^1(\Omega)$,*
 - b) *$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in H_0^1(\Omega) \times H_0^1(\Omega)$ and $\mathcal{V}(S(\cdot)(\begin{bmatrix} u_0 \\ v_0 \end{bmatrix})) = \text{const.}$ in $[0, \infty)$ implies that $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in \mathcal{E}$,*

¹To simplify the notation we will next use for the extensions (resp. restrictions) of the operator A the same symbol A .

v) for each $t \geq 0$ and $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ the following formula holds:

$$\begin{aligned} S(t)\left(\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\right) &= e^{-\mathcal{A}_\mu t} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} + U(t, \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}), \\ U(t, \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}) &:= \int_0^t e^{-\mathcal{A}_\mu(t-s)} \Phi_1 \mathcal{F}(\Phi_{-1} S(s)\left(\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\right)) ds, \end{aligned} \quad (1.16)$$

where

$$e^{-\mathcal{A}_\mu t} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^k \mathcal{A}_\mu^k$$

is the exponential function of the linear bounded operator $-\mathcal{A}_\mu$ given by (1.14) in $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ and $\Phi_{-1} : W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) \rightarrow W_0^{-1,p}(\Omega) \times W_0^{-1,p}(\Omega)$ is given by $\Phi_{-1} = \begin{bmatrix} (I+A) & 0 \\ 0 & (I+A) \end{bmatrix}$ and $\Phi_1 = (\Phi_{-1})^{-1}$.

As for the asymptotic properties of the solutions of (1.1)-(1.2) the following result is given.

Theorem 1.3. *If (1.4) and (1.8) or (1.9) hold, then the semigroup $\{S(t)\}$ associated in $H_0^1(\Omega) \times H_0^1(\Omega)$ to the evolution problem (1.1)-(1.2) has a global attractor $\mathbf{A}$ given by (1.10).*

When (1.8) holds, the global attractor $\mathbf{A}$ is bounded in $W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \times W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ for each $q \in (1, \infty)$ and thus compact in the space of Hölder functions $C^{1+\nu}(\overline{\Omega}) \times C^{1+\nu}(\overline{\Omega})$ for every $\nu \in (0, 1)$.

Theorems 1.1 and 1.2 are proved in Section 2. In Section 3 the asymptotic regularity properties of $\{S(t)\}$ are obtained and the proof of Theorem 1.3 is completed under the assumption (1.8). In Section 4 we give the proof of (1.6) under the assumption (1.9) using the Alekseev's nonlinear variation of constants formula (see [2, 4]).

An open question remains whether the global attractor regularizes in the critical case (1.9), that is when the critical exponent ρ^* is involved.

2. LOCAL AND GLOBAL WELL POSEDNESS RESULTS

In what follows we will essentially use the properties of the double sided fractional power scale $\{E^\alpha, \alpha \in \mathbb{R}\}$ generated by $(E, \tilde{A})$, where E is a reflexive Banach space and $\tilde{A}$ with the domain E^1 is a sectorial positive operator in E . In particular, we will use the properties of the scale corresponding to $\tilde{A} = -\Delta + I$ in $E = L^p(\Omega) =: E_p$ with the domain $E_p^1 = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, where $p \in (1, \infty)$ and Δ denotes the Dirichlet Laplacian.

Recall that the extrapolation space $E_p^{-\frac{1}{2}}$ of $E_p^{\frac{1}{2}}$ is the completion of $(E_p^{\frac{1}{2}}, \|(\tilde{A}_\frac{1}{2})^{-1} \cdot\|_{E_p^{\frac{1}{2}}})$.

We know that $E_p^{\frac{1}{2}}$ coincides with $W_0^{1,p}(\Omega)$ and from [1, Proposition V.1.4.3] we know that the (adjoint) space $((E_{p'})^{\frac{1}{2}})'$ coincides with $E_p^{-\frac{1}{2}}$ and with the completion of $(E_p, \|\tilde{A}^{-\frac{1}{2}} \cdot\|_{E_p})$.

Following [1, Proposition V.1.4.3] we remark that we identify the elements $\phi \in E_p^{-\frac{1}{2}}$ with the functionals from $(E_{p'}^{\frac{1}{2}})' = (W_0^{1,p'}(\Omega))'$ in such a way that the norm $\|\tilde{A}^{-\frac{1}{2}} \phi\|_{E_p} =: \|\phi\|_{E_p^{-\frac{1}{2}}}$

is equal to the norm $\|\phi\|_{L(E_{p'}^{\frac{1}{2}}, \mathbb{C})} = \|\phi\|_{(W_0^{1,p'}(\Omega))'} = \|\phi\|_{W^{-1,p}(\Omega)}$ whenever $\phi \in E_p$. For $\phi \in E_p^{-\frac{1}{2}} \setminus E_p$ we then have $\|\phi\|_{L(E_{p'}^{\frac{1}{2}}, \mathbb{C})} = \|\phi\|_{(W_0^{1,p'}(\Omega))'} = \|\phi\|_{W^{-1,p}(\Omega)} = \lim_{j \rightarrow \infty} \|\tilde{A}^{-\frac{1}{2}} \phi_j\|_{E_p}$ for any suitable (Cauchy) sequence $\{\phi_j\} \subset E$.

As for the representation of the functionals we obtain the following relation.

Lemma 2.1. *If $E_2 = L^2(\Omega)$ and $\tilde{A}$ is the negative Dirichlet Laplacian plus any multiple of identity such that $\tilde{A}$ is positive with the domain $E_2^1 = H^2(\Omega) \cap H_0^1(\Omega)$, then*

$$\langle \tilde{A}^{-\frac{1}{2}} \phi, \tilde{A}^{\frac{1}{2}} \psi \rangle_{E_2} = \int_{\Omega} \phi \psi ds, \quad \phi \in E_{\frac{2N}{N+2}}, \quad \psi \in E_2^{\frac{1}{2}}. \quad (2.1)$$

Proof: If $E_2^{\frac{1}{2}} \ni \phi_n \xrightarrow{E_{\frac{2N}{N+2}}} \phi \in E_{\frac{2N}{N+2}}$ then $\phi_n \xrightarrow{E_2^{-\frac{1}{2}}} \phi$ and for $\psi \in E_2^{\frac{1}{2}}$ we get

$$\langle \tilde{A}^{-\frac{1}{2}} \phi, \tilde{A}^{\frac{1}{2}} \psi \rangle_{E_2} \leftarrow \langle \tilde{A}^{-\frac{1}{2}} \phi_n, \tilde{A}^{\frac{1}{2}} \psi \rangle_{E_2} = \int_{\Omega} \phi_n \psi dx \rightarrow \int_{\Omega} \phi \psi ds,$$

which proves the result. $\square$

As for the properties of $\mathcal{A}_\mu$ defined by (1.14) in $E_p^{-\frac{1}{2}} \times E_p^{-\frac{1}{2}}$ we prove the following lemma.

Lemma 2.2. *Let $p \in (1, \infty)$ and $\mu > 0$. Suppose that A is the negative Dirichlet Laplacian in $L^p(\Omega)$ with the domain $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and consider its closed extension to $(W_0^{1,p'}(\Omega))' =: W^{-1,p}(\Omega)$. Then $\mathcal{A}_\mu$ is a bounded linear operator in $E_p^{-\frac{1}{2}} \times E_p^{-\frac{1}{2}}$ and the uniformly continuous semigroup $e^{-\mathcal{A}_\mu t} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^k \mathcal{A}_\mu^k$ generated by $-\mathcal{A}_\mu$ in $E_p^{-\frac{1}{2}} \times E_p^{-\frac{1}{2}}$ satisfies the estimate*

$$\|e^{-\mathcal{A}_\mu t}\|_{L(E_p^{-\frac{1}{2}} \times E_p^{-\frac{1}{2}})} \leq M_p e^{-\omega t}, \quad t \geq 0, \quad (2.2)$$

with certain positive constants M_p, ω .

Proof: If we write the equation $(\lambda I + \mathcal{A}_\mu) \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$ we will observe that

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} ((\lambda^2 + \mu\lambda + 1)I - (\mu\lambda + 1)(I + A)^{-1})^{-1} (\lambda\psi_1 + \mu(I - (I + A)^{-1})\psi_1 + \psi_2) \\ \lambda((\lambda^2 + \mu\lambda + 1)I - (\mu\lambda + 1)(I + A)^{-1})^{-1} (\lambda\psi_1 + \mu(I - (I + A)^{-1})\psi_1 + \psi_2) - \psi_1 \end{bmatrix}$$

whenever the inverse operator $((\lambda^2 + \mu\lambda + 1)I - (\mu\lambda + 1)(I + A)^{-1})^{-1}$ is well defined. From the spectral mapping theorem (see [6, p. 243]) we know that

$$\sigma((I + A)^{-1}) - \{0\} = \{(1 + \nu)^{-1}, \nu \in \sigma(A)\},$$

which shows that $\sigma((I + A)^{-1}) \subset [0, (1 + \lambda_1)^{-1}]$.

Since, for $\lambda \neq -\mu^{-1}$,

$$Im(\lambda^2(1 + \mu\lambda)^{-1} + 1) = \frac{\mu Im \lambda}{(1 + \mu Re \lambda)^2 + (\mu Im \lambda)^2} \left[(Im \lambda)^2 + (Re \lambda + \frac{1}{\mu})^2 - \frac{1}{\mu^2} \right]$$

we have that:

- the points in $\sigma(\mathcal{A}_\mu)$ with $\operatorname{Im}\lambda \neq 0$ must be of the form $\lambda = -\frac{1}{\mu} + \frac{1}{\mu}e^{i\theta}$. For $\lambda = -\frac{1}{\mu} + \frac{1}{\mu}e^{i\theta}$ we have that $\lambda^2(1 + \mu\lambda)^{-1} + 1 = 2\mu^{-2}(\cos\theta - 1) + 1$. Now, for $\mu^2 > 4\lambda_1^{-1}(1 + \lambda_1)$ and $\theta \in \mathbb{R}$ or for $\mu^2 \leq 4\lambda_1^{-1}(1 + \lambda_1)$ and $0 < |\theta| < \theta_0 := \arccos\left(\frac{2+(2-\mu^2)\lambda_1}{2+2\lambda_1}\right)$, we have that $\lambda = -\mu^{-1} + \mu^{-1}e^{i\theta} \in \rho(-\mathcal{A}_\mu)$. This shows that $\lambda \in \rho(-\mathcal{A}_\mu)$ whenever $\operatorname{Re}\lambda > -\frac{\mu\lambda_1}{2+2\lambda_1}$, $\operatorname{Im}\lambda \neq 0$.
- the points with $\operatorname{Im}\lambda = 0$ which satisfy the condition $\lambda^2(\mu\lambda + 1)^{-1} + 1 > (1 + \lambda_1)^{-1}$ are in $\rho(\mathcal{A}_\mu)$. Hence, we clearly have,

$$\rho(\mathcal{A}_\mu) \cap \mathbb{R} \supset (-\mu^{-1}, \infty).$$

Thus, $\rho(\mathcal{A}_\mu) \supset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -\min\{\mu^{-1}, \mu\lambda_1(2 + 2\lambda_1)^{-1}\}\}$ and (2.2) is proved. $\square$

As for the isometric properties of the operator $\tilde{A}$ we specifically mention the following result (see [1, Corollary 1.3.9]).

Lemma 2.3. *If $p \in (1, \infty)$, $s \in [0, \infty)$ and $\Phi_s : E_p^{-\frac{1}{2}} \times E_p^{-\frac{1}{2}} \rightarrow E_p^{s-\frac{1}{2}} \times E_p^{s-\frac{1}{2}}$ is given in the matrix form as*

$$\Phi_s = \begin{bmatrix} (I+A)^{-s} & 0 \\ 0 & (I+A)^{-s} \end{bmatrix}, \quad (2.3)$$

then Φ_s is an isometric isomorphism² from $E_p^r \times E_p^r$ onto $E_p^{r+s} \times E_p^{r+s}$ for each $r \geq -\frac{1}{2}$; in particular Φ_1 is an isometric isomorphism from $E_p^{-\frac{1}{2}} \times E_p^{-\frac{1}{2}}$ onto $E_p^{\frac{1}{2}} \times E_p^{\frac{1}{2}}$.

2.1. Local well posedness result and the proof of Theorem 1.1. We first prove the auxiliary lemma below.

Lemma 2.4. *Let $p \in (1, \infty)$, $f : \mathbb{R} \rightarrow \mathbb{R}$, and A be as in Lemma 2.2. Then*

$$f^e(\phi)(x) = f((I + A)^{-1}\phi(x)), \quad x \in \Omega,$$

defines an operator from $W^{-1,p}(\Omega)$ into $W^{-1,p}(\Omega)$ which is Lipschitz continuous in bounded sets provided that one of the conditions (H1), (H2) or (H3) in (1.7) hold.

Proof: Let B be bounded in $E_p^{-\frac{1}{2}}$ and choose arbitrary $\phi_1, \phi_2 \in B$.

If condition (H1) is satisfied, then $(I + A)^{-1}B$ is a bounded subset of $E_p^{\frac{1}{2}}$ and hence of $C^0(\bar{\Omega})$. Since f is Lipschitz continuous on bounded subsets of $\mathbb{R}^1$ we have

$$\begin{aligned} \|f^e(\phi_1) - f^e(\phi_2)\|_{E_p^{-\frac{1}{2}}} &\leq C\|f^e(\phi_1) - f^e(\phi_2)\|_{E_p^0} \\ &\leq CL\|(I + A)^{-1}(\phi_1 - \phi_2)\|_{E_p^0} \leq \tilde{C}L\|\phi_1 - \phi_2\|_{E_p^{-\frac{1}{2}}}. \end{aligned}$$

²Recall that we consider the double sided fractional power scale $\{E^\alpha, \alpha \in \mathbb{R}\}$ generated by $(E, \tilde{A})$ with $\tilde{A} = I + A$; in particular for $\phi \in E_p^{\frac{1}{2}}$ we have that $\|\phi\|_{E_p^{\frac{1}{2}}} = \|(I + A)^{\frac{1}{2}}\phi\|_{E_p}$ and $\|\phi\|_{E_p^{-\frac{1}{2}}} = \|(I + A)^{-\frac{1}{2}}\phi\|_{E_p}$.

When either (H2) or (H3) hold we use the Sobolev and Hölder inequalities to get

$$\begin{aligned}
\|f^e(\phi_1) - f^e(\phi_2)\|_{E_p^{-\frac{1}{2}}} &\leq C\|f^e(\phi_1) - f^e(\phi_2)\|_{E_{\frac{Np}{N+p}}} \\
&= C\|f((I+A)^{-1}\phi_1) - f((I+A)^{-1}\phi_2)\|_{E_{\frac{Np}{N+p}}} \\
&\leq \hat{C}\|(I+A)^{-1}(\phi_1 - \phi_2)\|_{E_p^{\frac{1}{2}}} \left(1 + \|(I+A)^{-1}\phi_1\|_{E_{\frac{N(\rho-1)}{2}}}^{\rho-1} + \|(I+A)^{-1}\phi_2\|_{E_{\frac{N(\rho-1)}{2}}}^{\rho-1}\right) \\
&\leq \tilde{C}\|(I+A)^{-1}(\phi_1 - \phi_2)\|_{E_p^{\frac{1}{2}}} \left(1 + \|(I+A)^{-1}\phi_1\|_{E_p^{\frac{1}{2}}}^{\rho-1} + \|(I+A)^{-1}\phi_2\|_{E_p^{\frac{1}{2}}}^{\rho-1}\right) \\
&\leq \tilde{C}\|\phi_1 - \phi_2\|_{E_p^{-\frac{1}{2}}} \left(1 + \|\phi_1\|_{E_p^{-\frac{1}{2}}}^{\rho-1} + \|\phi_2\|_{E_p^{-\frac{1}{2}}}^{\rho-1}\right) \\
&\leq \bar{C}\|\phi_1 - \phi_2\|_{E_p^{-\frac{1}{2}}}. \quad \square
\end{aligned}$$

The local well posedness result for (1.13) in $W^{-1,p}(\Omega) \times W^{-1,p}(\Omega)$ now follows.

Theorem 2.1. *If the assumptions of Lemma 2.4 are satisfied and one of the conditions (H1), (H2) or (H3) in (1.7) holds, then for each $v_0, w_0 \in W^{-1,p}(\Omega)$ the problem (1.13) has a unique solution $[\frac{v}{w}](\cdot, v_0, w_0) \in C([0, \tau_{v_0, w_0}), W^{-1,p}(\Omega) \times W^{-1,p}(\Omega))$ defined on the maximal interval of existence $[0, \tau_{v_0, w_0})$ and continuously depending on the initial data, where either $\tau_{v_0, w_0} = \infty$ or $\|v(t, v_0, w_0)\|_X + \|w(t, v_0, w_0)\|_X \rightarrow \infty$ as $t \rightarrow \tau_{v_0, w_0}^-$. The solution satisfies in $W^{-1,p}(\Omega)$ the variation of constants formula*

$$[\frac{v}{w}](\cdot, v_0, w_0) = e^{-A_\mu t} [\frac{v_0}{w_0}] + \int_0^t e^{-A_\mu(t-s)} [f^e(v(s, v_0, w_0))] ds, \quad t \in [0, \tau_{v_0, w_0}), \quad (2.4)$$

with $\{e^{-A_\mu t}\}$ defined in Lemma 2.2 and, in addition,

$$\begin{aligned}
[\frac{v}{w}](\cdot, v_0, w_0) &\in C^1([0, \tau_{v_0, w_0}), W^{-1,p}(\Omega) \times W^{-1,p}(\Omega)) \\
[\frac{v_t}{w_t}](\cdot, v_0, w_0) &\in C^1([0, \tau_{v_0, w_0}), W^{-1,p}(\Omega) \times W^{-1,p}(\Omega)).
\end{aligned} \quad (2.5)$$

In particular, $[\frac{v}{w}](\cdot, v_0, w_0)$ is a classical solution of (1.13) in $X \times X := W^{-1,p}(\Omega) \times W^{-1,p}(\Omega)$.

Proof: Let $p \in (1, \infty)$ and $X = E_p^{-\frac{1}{2}} = W^{-1,p}(\Omega)$. From Lemma 2.4, if B is a bounded subset of $X \times X$ and $[\frac{\phi_1}{\psi_1}], [\frac{\phi_2}{\psi_2}] \in B$, we obtain

$$\|\mathcal{F}([\frac{\phi_1}{\psi_1}]) - \mathcal{F}([\frac{\phi_2}{\psi_2}])\|_{X \times X} = \|f^e(\phi_1) - f^e(\phi_2)\|_{E_p^{-\frac{1}{2}}} \leq c \|[\frac{\phi_1}{\psi_1}] - [\frac{\phi_2}{\psi_2}]\|_{X \times X}.$$

Hence, $\mathcal{F}$ takes $X \times X$ into $X \times X$ and is Lipschitz continuous on bounded sets so that the semigroup theory applies (see [10, 13]) from which the results follow easily. $\square$

The proof of Theorem 1.1. The local well posedness of (1.1) in $W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ follows from Theorem 2.1 and Lemma 2.3. For this we define $\tau(u_0, v_0) := \tau_{(I+A)u_0, (I+A)v_0}$ and

$$u(t) = (I+A)^{-1}v(t), \quad u_t(t) = (I+A)^{-1}v_t(t), \quad (2.6)$$

where $[\frac{v}{v_t}]$ is the (uniquely defined on $[0, \tau(u_0, v_0))$ solution of (1.13) from Theorem 2.1. $\square$

2.2. Global solutions to (1.1) and the proof of Theorem 1.2. Assuming (1.4) we obtain below the global well posedness result for (1.1)-(1.2) and prove Theorem 1.2.

Lemma 2.5. *Under the assumptions and notation of Theorem 1.1 condition (1.4) implies the existence of a constant $C > 0$ such that if $u_0, v_0 \in W_0^{1,p}(\Omega)$ and $p \in [2, \infty)$ then the functions $u(t, u_0, v_0)$ and $u_t(t, u_0, v_0)$ fulfil the estimate*

$$\|u_t\|_{E_2^{-\frac{1}{2}}}^2 + \|u\|_{E_2^{-\frac{1}{2}}}^2 \leq C \left(1 + \|u_0\|_{E_2^{-\frac{1}{2}}}^{\rho+1} + \|v_0\|_{E_2^{-\frac{1}{2}}}^2 \right), \quad t \in [0, \tau(u_0, v_0)). \quad (2.7)$$

Proof: From Theorem 2.1 function $v = (I + A)u$ solves (1.11) in $E_2^{-\frac{1}{2}}$. We thus take the $E_2^{-\frac{1}{2}}(\Omega)$ product³ $\langle \cdot, \cdot \rangle_{E_2^{-\frac{1}{2}}}$ of each side of (1.11) with v_t to get

$$\begin{aligned} & \frac{d}{dt} \frac{1}{2} \left(\|v_t\|_{E_2^{-\frac{1}{2}}}^2 + \|v\|_{E_2^{-\frac{1}{2}}}^2 - \|(I + A)^{-\frac{1}{2}}v\|_{E_2^{-\frac{1}{2}}}^2 \right) - \langle f^e(v), v_t \rangle_{E_2^{-\frac{1}{2}}} \\ &= -\mu \left(\|v_t\|_{E_2^{-\frac{1}{2}}} - \|(I + A)^{-\frac{1}{2}}v_t\|_{E_2^{-\frac{1}{2}}} \right), \end{aligned} \quad (2.8)$$

Poincaré inequality reads

$$\|(I + A)^{\frac{1}{2}}\phi\|_{E_2}^2 \geq (1 + \lambda_1)\|\phi\|_{E_2}^2, \quad \phi \in E_2^{\frac{1}{2}}, \quad (2.9)$$

which for $\phi = (I + A)^{-1}\psi$ translates into the estimates

$$\|\psi\|_{E_2^{-\frac{1}{2}}}^2 \geq (1 + \lambda_1)\|(I + A)^{-\frac{1}{2}}\psi\|_{E_2^{-\frac{1}{2}}}^2, \quad \psi \in E_2^{-\frac{1}{2}}, \quad (2.10)$$

and implies

$$\begin{aligned} \frac{\lambda_1}{1 + \lambda_1} \|v\|_{E_2^{-\frac{1}{2}}}^2 &\leq \|v\|_{E_2^{-\frac{1}{2}}}^2 - \|(I + A)^{-\frac{1}{2}}v\|_{E_2^{-\frac{1}{2}}}^2 \\ \frac{\lambda_1}{1 + \lambda_1} \|v_t\|_{E_2^{-\frac{1}{2}}}^2 &\leq \|v_t\|_{E_2^{-\frac{1}{2}}}^2 - \|(I + A)^{-\frac{1}{2}}v_t\|_{E_2^{-\frac{1}{2}}}^2. \end{aligned} \quad (2.11)$$

From Lemma 2.1 we get

$$\langle (I + A)^{-\frac{1}{2}}f^e(v), (I + A)^{\frac{1}{2}}(I + A)^{-1}v_t \rangle_{E_2} = \int_{\Omega} f^e(v)(I + A)^{-1}v_t dx.$$

and hence

$$\langle f^e(v), v_t \rangle_{E_2^{-\frac{1}{2}}} = \langle (I + A)^{-\frac{1}{2}}f^e(v), (I + A)^{-\frac{1}{2}}v_t \rangle_{E_2} = \int_{\Omega} f((I + A)^{-1}v)(I + A)^{-1}v_t dx.$$

If F is the primitive function of f in $\mathbb{R}^1$ we then have

$$\int_{\Omega} f((I + A)^{-1}v)(I + A)^{-1}v_t dx = \frac{d}{dt} \int_{\Omega} F((I + A)^{-1}v) dx. \quad (2.12)$$

³Since $\{E^\alpha, \alpha \in \mathbb{R}\}$ is generated by $(E, \tilde{A})$ with $\tilde{A} = I + A$, then the product $\langle \phi, \psi \rangle_{E_2^{-\frac{1}{2}}}$ of $\phi, \psi \in E_2^{-\frac{1}{2}}$ is given by $\langle (I + A)^{-\frac{1}{2}}\phi, (I + A)^{-\frac{1}{2}}\psi \rangle_{E_2}$; $\langle \cdot, \cdot \rangle_{E_2}$ being the (original) scalar product in E_2 .

We now remark that (1.4) implies the existence of constants $\epsilon, C > 0$, for which

$$F(t) = \int_0^t f(s)ds \leq \frac{1}{2}(\lambda_1 - \epsilon)t^2 + C, \quad t \in \mathbb{R}^1.$$

As a consequence we infer

$$\langle F((I + A)^{-1}v), 1 \rangle_{E_2} \leq \frac{1}{2}(\lambda_1 - \epsilon)\|(I + A)^{-\frac{1}{2}}v\|_{E_2^{-\frac{1}{2}}}^2 + C|\Omega|,$$

which with the aid of (2.10) reads

$$\int_{\Omega} F((I + A)^{-1}v)dx \leq \frac{\lambda_1 - \epsilon}{2(1 + \lambda_1)}\|v\|_{E_2^{-\frac{1}{2}}}^2 + C|\Omega|. \quad (2.13)$$

Connecting (2.8)-(2.13) we get for

$$\mathcal{L}(v, v_t) := \frac{1}{2}\|v_t\|_{E_2^{-\frac{1}{2}}}^2 + \frac{1}{2}\|v\|_{E_2^{-\frac{1}{2}}}^2 - \frac{1}{2}\|(I + A)^{-\frac{1}{2}}v\|_{E_2^{-\frac{1}{2}}}^2 - \int_{\Omega} F((I + A)^{-1}v)dx \quad (2.14)$$

that

$$\frac{d}{dt}(\mathcal{L}(v, v_t)) = -\mu \left(\|v_t\|_{E_2^{-\frac{1}{2}}} - \|(I + A)^{-\frac{1}{2}}v_t\|_{E_2^{-\frac{1}{2}}} \right) \leq -\frac{\mu\lambda_1}{1 + \lambda_1}\|v_t\|_{E_2^{-\frac{1}{2}}}^2 \leq 0$$

and hence

$$\frac{1}{2}\|v_t\|_{E_2^{-\frac{1}{2}}}^2 + \frac{\epsilon}{2(1 + \lambda_1)}\|v\|_{E_2^{-\frac{1}{2}}}^2 - C|\Omega| \leq \mathcal{L}(v, v_t) \leq \mathcal{L}(v(0), v_t(0)), \quad (2.15)$$

as long as the solution v exists.

In variables (t, u) we then have

$$\|u_t\|_{E_2^{\frac{1}{2}}}^2 + \frac{\epsilon}{1 + \lambda_1}\|u\|_{E_2^{\frac{1}{2}}}^2 \leq \|v_0\|_{E_2^{\frac{1}{2}}}^2 + \|(I + A)^{\frac{1}{2}}u_0\|_{E_2}^2 - \|u_0\|_{E_2}^2 - 2 \int_{\Omega} F(u_0)dx + 2C|\Omega|, \quad (2.16)$$

where

$$\|F(u_0)\|_{L^1(\Omega)} \leq \text{const.} \left(1 + \|u_0\|_{E_2^{\frac{1}{2}}}^{\rho+1} \right), \quad (2.17)$$

since (1.7) implies that $|F(s)| \leq \text{const.}(1 + |s|^{\rho+1})$ for $s \in \mathbb{R}$. Thus (2.7) follows from (2.16) and (2.17) via Poincaré inequality (2.9). $\square$

Proof of Theorem 1.2. Parts i) and ii) follows from Lemma 2.5 and Theorem 1.1.

To prove boundedness of $\mathcal{E}_1$ in $H_0^1(\Omega)$ we first infer from Lemma 2.1 that

$$\langle A^{-\frac{1}{2}}f(u), A^{\frac{1}{2}}u \rangle_{E_2} = \int_{\Omega} f(u)udx. \quad (2.18)$$

Since $u \in \mathcal{E}_1$ implies that $u \in E_2^{\frac{1}{2}}$ and $A^{\frac{1}{2}}u = A^{-\frac{1}{2}}f(u)$, then

$$\langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}u \rangle_{E_2} = \int_{\Omega} f(u)udx, \quad u \in \mathcal{E}_1. \quad (2.19)$$

With the aid of (1.4) and Poincaré inequality we then get

$$\|A^{\frac{1}{2}}u\|_{E_2}^2 \leq (\lambda_1 - \epsilon)\|u\|_{E_2}^2 + C|\Omega| \leq (1 - \epsilon\lambda_1^{-1})\|A^{\frac{1}{2}}u\|_{E_2}^2 + C|\Omega| \quad (2.20)$$

for some $\epsilon, C > 0$ and hence

$$\sup_{u \in \mathcal{E}_1} \|A^{\frac{1}{2}} u\|_{E_2}^2 \leq C |\Omega| \lambda_1 \epsilon^{-1} =: M_1. \quad (2.21)$$

For the proof part iv) we come back to (2.14) defining

$$\mathcal{V}(S(t)(\begin{bmatrix} u_0 \\ v_0 \end{bmatrix})) := \mathcal{L}(\Phi_{-1} S(t)(\begin{bmatrix} u_0 \\ v_0 \end{bmatrix})) = \frac{1}{2} \|u_t\|_{E_2^{\frac{1}{2}}}^2 + \frac{1}{2} \|u\|_{E_2^{\frac{1}{2}}}^2 - \frac{1}{2} \|u\|_{E_2}^2 - \int_{\Omega} F(u) dx. \quad (2.22)$$

Following the assumptions on f for $\phi, \tilde{\phi} \in E_2^{\frac{1}{2}}$ we have

$$\|F(\phi) - F(\tilde{\phi})\|_{L^1(\Omega)} \leq \tilde{c} \|\phi - \tilde{\phi}\|_{E_2^{\frac{1}{2}}} (1 + \|\phi\|_{E_2^{\frac{1}{2}}}^{\rho} + \|\tilde{\phi}\|_{E_2^{\frac{1}{2}}}^{\rho}).$$

Properties a) and b) follow then from Lemma 2.5.

For the proof of v) we combine (1.15), (2.4) and (2.6) to get

$$\begin{bmatrix} (I+A)u \\ (I+A)u_t \end{bmatrix} (t) = e^{-\mathcal{A}_{\mu} t} \begin{bmatrix} (I+A)u_0 \\ (I+A)v_0 \end{bmatrix} + \int_0^t e^{-\mathcal{A}_{\mu}(t-s)} \mathcal{F} \left(\begin{bmatrix} (I+A)u(s) \\ (I+A)u_t(s) \end{bmatrix} \right) ds, \quad t \geq 0, \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in E_2^{\frac{1}{2}}. \quad (2.23)$$

We also know that $\Phi_1 \in L(E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}})$ and $\mathcal{A}_{\mu}$ commutes with Φ_1 . Therefore, $e^{-\mathcal{A}_{\mu} t}$ commutes with Φ_1 and application of Φ_1 to both sides of (2.23) gives the result. $\square$

3. ASYMPTOTIC REGULARITY PROPERTIES OF $\{S(t)\}$ AND REGULARITY OF THE GLOBAL ATTRACTOR IN THE SUBCRITICAL CASE

In this section we prove the asymptotic regularity properties of $\{S(t)\}$ assuming condition (1.8). We then obtain the existence and regularity properties of the global attractor proving Theorem 1.3 under assumption (1.8). The existence of attractor for the case (1.9) will be proved in Section 4.

Before we proceed with the proof of Theorem 1.3 we need the following two lemmas.

Lemma 3.1. *If condition (1.8) holds, then $\tilde{f}^e = f^e \circ (I + A)$ takes sets bounded in $E_2^{\frac{1}{2}}$ into sets bounded in $E_{\frac{2N}{N+2(1-\tilde{\sigma})}}$ and thus also in $E_2^{\frac{\tilde{\sigma}-1}{2}}$ with $\tilde{\sigma} = \min\{1, \frac{N+2}{2} - \rho \frac{N-2}{2}\}$. In particular, if $N = 1$ or $N = 2$, then $\tilde{f}^e$ takes bounded subsets of $E_2^{\frac{1}{2}}$ into bounded subsets of E_q for every $q \in (1, \infty)$.*

Proof: Let $N \geq 3$ and recall that $E_2^{\frac{1-\sigma}{2}} \hookrightarrow E_{\frac{2N}{N-2(1-\sigma)}}$ for $\sigma \in [0, 1]$. Recall also that $E_2^{\frac{\sigma-1}{2}} = (E_2^{\frac{1-\sigma}{2}})'$. If $1 < \rho < \frac{N+2}{N-2}$, then (1.7) leads for $\phi \in E_2^{\frac{1}{2}}$ to the condition

$$\|f^e((I+A)\phi)\|_{E_2^{\frac{\tilde{\sigma}-1}{2}}} \leq \tilde{c} \|f(\phi)\|_{E_{\frac{2N}{N+2(1-\tilde{\sigma})}}} \leq \tilde{c} \|1 + |\phi|^{\rho}\|_{E_{\frac{2N}{N+2(1-\tilde{\sigma})}}} \leq \hat{c} \left(1 + \|\phi\|_{E_2^{\frac{1}{2}}}^{\rho}\right). \quad (3.1)$$

If $N = 1$ or $N = 2$, then

$$\|f(\phi)\|_{E_q} \leq \tilde{c} \|1 + |\phi|^{\rho}\|_{E_q} \leq \tilde{c} \left(\|1\|_{E_q} + \|\phi\|_{E_{\rho q}}^{\rho}\right) \leq \hat{c} \left(1 + \|\phi\|_{E_2^{\frac{1}{2}}}^{\rho}\right), \quad \phi \in E_2^{\frac{1}{2}}, \quad (3.2)$$

since in this case $E_2^{\frac{1}{2}} \hookrightarrow E_r$ with any $r \in (1, \infty)$. $\square$

Lemma 3.2. *Suppose that $p \in (1, \infty)$, that A is as in Lemma 2.2 and (1.4), (1.8) hold. Then, for any set B bounded in $E_2^{\frac{1}{2}} \times E_2^{\frac{1}{2}}$ and $M_p, \omega, \hat{c}$ as in (2.2), (3.1) we have*

(α) *in the case $N \geq 3$*

$$\begin{aligned} & \sup_{\left[\frac{u_0}{v_0}\right] \in B} \sup_{t \geq 0} \left\| \int_0^t e^{-\mathcal{A}_\mu(t-s)} \Phi_1 \mathcal{F}(\Phi_{-1} S(s) \left(\left[\frac{u_0}{v_0} \right] \right)) ds \right\|_{E_2^{\frac{1+\tilde{\sigma}}{2}} \times E_2^{\frac{1+\tilde{\sigma}}{2}}} \\ & \leq \frac{M_2 \hat{c}}{\omega} \left(1 + \sup_{\left[\frac{u_0}{v_0}\right] \in \gamma^+(B)} \|u_0\|_{E_2^{\frac{1}{2}}}^\rho \right) \end{aligned}$$

with $\tilde{\sigma} := \min\{1, \frac{N}{2} + 1 - \rho(\frac{N}{2} - 1)\}$

and

(β) *in the case $N = 1$ or $N = 2$*

$$\begin{aligned} & \sup_{\left[\frac{u_0}{v_0}\right] \in B} \sup_{t \geq 0} \left\| \int_0^t e^{-\mathcal{A}_\mu(t-s)} \Phi_1 \mathcal{F}(\Phi_{-1} S(s) \left(\left[\frac{u_0}{v_0} \right] \right)) ds \right\|_{E_p^1 \times E_p^1} \\ & \leq \frac{M_p \hat{c}}{\omega} \left(1 + \sup_{\left[\frac{u_0}{v_0}\right] \in \gamma^+(B)} \|u_0\|_{E_2^{\frac{1}{2}}}^\rho \right). \end{aligned}$$

Proof: Recalling (1.16) and Lemma 2.3 we have

$$\begin{aligned} \|U(t, \left[\frac{u_0}{v_0}\right])\|_{E_2^{\frac{1+\tilde{\sigma}}{2}} \times E_2^{\frac{1+\tilde{\sigma}}{2}}} &= \left\| \Phi_{-1-\frac{\tilde{\sigma}}{2}} \Phi_1 \int_0^t e^{-\mathcal{A}_\mu(t-s)} \mathcal{F}(\Phi_{-1} S(s) \left(\left[\frac{u_0}{v_0} \right] \right)) ds \right\|_{E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}}} \\ &= \left\| \Phi_{-\frac{\tilde{\sigma}}{2}} \int_0^t e^{-\mathcal{A}_\mu(t-s)} \mathcal{F}(\Phi_{-1} S(s) \left(\left[\frac{u_0}{v_0} \right] \right)) ds \right\|_{E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}}}. \end{aligned} \quad (3.3)$$

We remark that $\Phi_{\frac{\tilde{\sigma}}{2}} \in L(E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}})$ and $\Phi_{\frac{\tilde{\sigma}}{2}}$ commutes with $\mathcal{A}_\mu$ and hence with $e^{-\mathcal{A}_\mu t}$. Thus $\Phi_{\frac{\tilde{\sigma}}{2}}^{-1} = \Phi_{-\frac{\tilde{\sigma}}{2}}$ commutes with $e^{-\mathcal{A}_\mu t}$ on $E_2^{\frac{\tilde{\sigma}-1}{2}} \times E_2^{\frac{\tilde{\sigma}-1}{2}}$ and from (3.3) we get

$$\begin{aligned} \|U(t, \left[\frac{u_0}{v_0}\right])\|_{E_2^{\frac{1+\tilde{\sigma}}{2}} \times E_2^{\frac{1+\tilde{\sigma}}{2}}} &= \left\| \int_0^t e^{-\mathcal{A}_\mu(t-s)} \Phi_{-\frac{\tilde{\sigma}}{2}} \mathcal{F}(\Phi_{-1} S(s) \left(\left[\frac{u_0}{v_0} \right] \right)) ds \right\|_{E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}}} \\ &\leq \int_0^t \|e^{-\mathcal{A}_\mu(t-s)}\|_{L(E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}})} \left\| \Phi_{-\frac{\tilde{\sigma}}{2}} \mathcal{F}(\Phi_{-1} S(s) \left(\left[\frac{u_0}{v_0} \right] \right)) \right\|_{E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}}} ds. \end{aligned} \quad (3.4)$$

Since via Theorem 1.2 ii) the positive orbit $\gamma^+(B)$ is bounded in $E_2^{\frac{1}{2}} \times E_2^{\frac{1}{2}}$, Lemma 3.1 implies

$$\begin{aligned} \left\| \Phi_{-\frac{\tilde{\sigma}}{2}} \mathcal{F}(\Phi_{-1} S(s) \left(\left[\frac{u_0}{v_0} \right] \right)) \right\|_{E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}}} &= \left\| \mathcal{F}(\Phi_{-1} S(s) \left(\left[\frac{u_0}{v_0} \right] \right)) \right\|_{E_2^{\frac{\tilde{\sigma}-1}{2}} \times E_2^{\frac{\tilde{\sigma}-1}{2}}} \\ &\leq \hat{c} \left(1 + \sup_{\left[\frac{u_0}{v_0}\right] \in \gamma^+(B)} \|u_0\|_{E_2^{\frac{1}{2}}}^\rho \right), \quad \left[\frac{u_0}{v_0}\right] \in B, \quad s \geq 0. \end{aligned} \quad (3.5)$$

With (3.5) and with the exponential estimate of Lemma 2.2 part (α) now follows easily.

For the proof of part (β) we take $\tilde{\sigma} = 1$, which is possible thanks to the estimate (3.2). We then replace symbol E_2 with E_p in (3.3)-(3.5) to get

$$\begin{aligned} \|U(t, [\frac{u_0}{v_0}])\|_{E_p^1 \times E_p^1} &= \left\| \int_0^t e^{-A_\mu(t-s)} \Phi_{-\frac{1}{2}} \mathcal{F}(\Phi_{-1} S(s)([\frac{u_0}{v_0}])) ds \right\|_{E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}}} \\ &\leq \int_0^t \|e^{-A_\mu(t-s)}\|_{L(E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}})} \|\Phi_{-\frac{1}{2}} \mathcal{F}(\Phi_{-1} S(s)([\frac{u_0}{v_0}]))\|_{E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}}} ds, \end{aligned} \quad (3.6)$$

where, thanks to (3.2), for $[\frac{u_0}{v_0}] \in B$ and $s \geq 0$ we have

$$\|\Phi_{-\frac{1}{2}} \mathcal{F}(\Phi_{-1}^{-1} S(s)([\frac{u_0}{v_0}]))\|_{E_p^{-\frac{1}{2}} \times E_p^{-\frac{1}{2}}} = \|\mathcal{F}(\Phi_{-1}^{-1} S(s)([\frac{u_0}{v_0}]))\|_{E_p \times E_p} \leq \hat{c} \left(1 + \sup_{[\frac{u_0}{v_0}] \in \gamma^+(B)} \|u_0\|_{E_2^{\frac{1}{2}}}^\rho\right).$$

The result now follows via (2.2). $\square$

Corollary 3.1. *Under the assumptions of Lemma 3.2 the semigroup $S(t) : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow H_0^1(\Omega) \times H_0^1(\Omega)$, $t \geq 0$, is asymptotically compact (equivalently asymptotically smooth).*

Proof: Since the embedding $E_2^{\frac{1+\tilde{\sigma}}{2}} \hookrightarrow E_1^{\frac{1}{2}}$ is compact, thanks to (1.16) and Lemmas 2.2, 3.2 the semigroup $\{S(t)\}$ is the sum of an exponentially decaying linear semigroup and of a family of maps $U(t)$ compact for $t > 0$. The result thus follows from [8, Lemma 3.2.3]. $\square$

Remark 3.1. *Theorem 1.2 and Corollary 3.1 ensure the existence of a global attractor $\mathbf{A}$ in the presence of the assumption (1.8).*

A bootstrapping procedure for the solutions in the attractor will lead to the regularity result of Theorem 1.3. The next four lemmas contain a proof of this fact.

Lemma 3.3. *Suppose that the assumptions of Lemma 3.2 hold and let $\mathbf{A}$ denote a global attractor for $\{S(t)\}$ in $H_0^1(\Omega) \times H_0^1(\Omega)$ (see Remark 3.1). Then*

- (a) $\mathbf{A}$ is a bounded subset of $H^{1+\tilde{\sigma}}(\Omega) \cap H_0^1(\Omega) \times H^{1+\tilde{\sigma}}(\Omega) \cap H_0^1(\Omega)$ if $N \geq 3$
- (b) $\mathbf{A}$ is bounded in $W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \times W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ with any $q \in (1, \infty)$ if $N \in \{1, 2\}$.

Proof: Note that $\mathbf{A}$ is invariant and, in particular, any point from $\mathbf{A}$ can be viewed as a $H_0^1(\Omega) \times H_0^1(\Omega)$ -limit of a sequence of the form

$$S(t_n)([\frac{u_{0n}}{v_{0n}}]) = e^{-A_\mu t_n} [\frac{u_{0n}}{v_{0n}}] + U(t_n, [\frac{u_{0n}}{v_{0n}}]), \text{ where } t_n \nearrow \infty \text{ and } \{[\frac{u_{0n}}{v_{0n}}]\} \subset \mathbf{A}. \quad (3.7)$$

Since $\mathbf{A}$ is bounded in $H_0^1(\Omega) \times H_0^1(\Omega)$, conditions (a), (b) follow from Lemmas 2.2, 3.2 and from the weak convergence property. $\square$

Lemma 3.4. *Under the assumptions of Lemma 3.3, if $N \geq 3$ then $\mathbf{A}$ is bounded in the norm of $H^2(\Omega) \cap H_0^1(\Omega) \times H^2(\Omega) \cap H_0^1(\Omega)$.*

Proof: If $\tilde{\sigma} = 1$, then the result coincides with Lemma 3.3 (i).

Suppose that $\tilde{\sigma} < 1$ and set $r_0 := \tilde{\sigma}$, $r_1 = \min\{1, \frac{N+2}{N-2}r_0\}$. Since $1 < \rho < \frac{N+2}{N-2} = \rho^*$, we infer from (1.7) that $|f(s)| \leq \tilde{c}(1 + |s|^{\rho^*})$ for $s \in \mathbb{R}$ and hence

$$\|f^e((I + A)\phi)\|_{E_2^{\frac{r_1-1}{2}}} \leq \tilde{c}\|f(\phi)\|_{E_2^{\frac{2N}{N+2(1-r_1)}}} \leq \hat{c}\left(1 + \|\phi\|_{E_2^{\frac{1+r_0}{2}}}^{\rho^*}\right), \quad \phi \in E_2^{\frac{1+r_0}{2}}. \quad (3.8)$$

As a consequence of Theorem 1.3 and Lemma 3.3 the set $\mathbf{A}$ is invariant and bounded in $E_2^{\frac{1+r_0}{2}} \times E_2^{\frac{1+r_0}{2}}$, so that (3.8) implies

$$\begin{aligned} \|\Phi_{-\frac{r_1}{2}}\mathcal{F}(\Phi_{-1}S(s)(\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}))\|_{E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}}} &= \|\mathcal{F}(\Phi_{-1}S(s)(\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}))\|_{E_2^{\frac{r_1-1}{2}} \times E_2^{\frac{r_1-1}{2}}} \\ &\leq \hat{c}\left(1 + \sup_{\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in \mathbf{A}} \|u_0\|_{E_2^{\frac{1+r_0}{2}}}^{\rho^*}\right), \quad \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in \mathbf{A}, \quad s \geq 0. \end{aligned} \quad (3.9)$$

Estimating in a similar manner as in Lemma 3.2 we get

$$\begin{aligned} \|U(t, \begin{bmatrix} u_0 \\ v_0 \end{bmatrix})\|_{E_2^{\frac{1+r_1}{2}} \times E_2^{\frac{1+r_1}{2}}} &= \left\| \Phi_{-1-\frac{r_1}{2}} \Phi_1 \int_0^t e^{-\mathcal{A}_\mu(t-s)} \mathcal{F}(\Phi_{-1}S(s)(\begin{bmatrix} u_0 \\ v_0 \end{bmatrix})) ds \right\|_{E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}}} \\ &\leq \int_0^t \|e^{-\mathcal{A}_\mu(t-s)}\|_{L(E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}})} \|\Phi_{-\frac{r_1}{2}}\mathcal{F}(\Phi_{-1}S(s)(\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}))\|_{E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}}} ds \end{aligned} \quad (3.10)$$

and hence (using (2.2) and (3.9))

$$\sup_{\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in \mathbf{A}} \sup_{t \geq 0} \|U(t, \begin{bmatrix} u_0 \\ v_0 \end{bmatrix})\|_{E_2^{\frac{1+r_1}{2}} \times E_2^{\frac{1+r_1}{2}}} < \infty. \quad (3.11)$$

Recalling that any point from $\mathbf{A}$ can be viewed as a $H_0^1(\Omega) \times H_0^1(\Omega)$ -limit of a sequence of the form given in (3.7), via weak convergence we infer from (2.2), (3.7), (3.11) that

$$\sup_{\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in \mathbf{A}} \|\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\|_{E_2^{\frac{1+r_1}{2}} \times E_2^{\frac{1+r_1}{2}}} < \infty. \quad (3.12)$$

If $r_1 = \frac{N+2}{N-2}r_0 < 1$, then we repeat the above procedure and obtain that

$$\mathbf{A} \text{ is bounded in } E_2^{\frac{1+r_2}{2}} \times E_2^{\frac{1+r_2}{2}}, \text{ where } r_2 = \min\{1, (\frac{N+2}{N-2})^2 r_0\}. \quad (3.13)$$

In a finite number of k -steps we will then reach $r_k = 1$, which gives the result. $\square$

Lemma 3.5. *Under the assumptions and notation of Lemma 3.3, if $N \geq 3$ and a number $q \in (1, \infty)$ is such that $\mathbf{A}$ is bounded in $E_q^1 \times E_q^1$ then $\mathbf{A}$ is bounded in $E_{q^*}^1 \times E_{q^*}^1$, where*

$$\begin{aligned} q^* &= \frac{N}{\rho^*} \frac{q}{N - 2q} \quad \text{in the case when } N > 2q, \\ q^* &\text{ is an arbitrary number from } (1, \infty) \text{ when } N \leq 2q. \end{aligned} \quad (3.14)$$

Proof: For $1 < \rho < \frac{N+2}{N-2} = \rho^*$ we infer from (1.7) that

$$\|f^e((I + A)\phi)\|_{E_{q^*}} \leq \tilde{c}\|f(\phi)\|_{E_{q^*}} \leq \tilde{c}(1 + \|\phi\|_{E_q^1}^{\rho^*}), \quad \phi \in E_q^1. \quad (3.15)$$

Since $\mathbf{A}$ is invariant and bounded in $E_q^1 \times E_q^1$, (3.15) implies for $[\frac{u_0}{v_0}] \in \mathbf{A}$ and $s \geq 0$

$$\|\Phi_{-\frac{1}{2}} \mathcal{F}(\Phi_{-1} S(s)([\frac{u_0}{v_0}]))\|_{E_{q^*}^{-\frac{1}{2}} \times E_{q^*}^{-\frac{1}{2}}} = \|\mathcal{F}(\Phi_{-1} S(s)([\frac{u_0}{v_0}]))\|_{E_{q^*} \times E_{q^*}} \leq \hat{c} \left(1 + \sup_{[\frac{u_0}{v_0}] \in \mathbf{A}} \|u_0\|_{E_q^1}^{\rho^*}\right).$$

From this and (2.2) we thus have for certain constant $D > 0$ and all $[\frac{u_0}{v_0}] \in \mathbf{A}$, $t \geq 0$,

$$\begin{aligned} \|U(t, [\frac{u_0}{v_0}])\|_{E_{q^*}^1 \times E_{q^*}^1} &= \left\| \Phi_{-1-\frac{1}{2}} \Phi_1 \int_0^t e^{-A_\mu(t-s)} \mathcal{F}(\Phi_{-1} S(s)([\frac{u_0}{v_0}])) ds \right\|_{E_{q^*}^{-\frac{1}{2}} \times E_{q^*}^{-\frac{1}{2}}} \\ &\leq \int_0^t \|e^{-A_\mu(t-s)}\|_{L(E_{q^*}^{-\frac{1}{2}} \times E_{q^*}^{-\frac{1}{2}})} \left\| \Phi_{-\frac{1}{2}} \mathcal{F}(\Phi_{-1} S(s)([\frac{u_0}{v_0}])) \right\|_{E_{q^*}^{-\frac{1}{2}} \times E_{q^*}^{-\frac{1}{2}}} ds \leq D. \end{aligned}$$

Viewing now points from $\mathbf{A}$ as $H_0^1(\Omega) \times H_0^1(\Omega)$ -limits of sequences of the form (3.7) we obtain via (2.2), (3.7) and the weak convergence property that $\sup_{[\frac{u_0}{v_0}] \in \mathbf{A}} \|[\frac{u_0}{v_0}]\|_{E_{q^*}^1 \times E_{q^*}^1} < \infty$. $\square$

The next simple lemma establishes, together with Lemma 3.5 and Lemma 3.4 the asymptotic bootstrapping procedure.

Lemma 3.6. *If $N \geq 3$ and $\rho^* = \frac{N+2}{N-2}$, then the sequence $\{q_k\}$ defined as*

$$q_0 = 2, \quad q_{k+1} = \frac{N}{\rho^*} \frac{q_k}{N - 2q_k} \quad \text{if } \frac{N}{2} > q_k, \quad q_{k+1} = N \quad \text{if } \frac{N}{2} \leq q_k, \quad (3.16)$$

has the property that $q_k = N$ for all k sufficiently large.

Proof: Note that $q_1 \geq q_0$ and hence $q_{k+1} \geq q_k$ for all $k \in \mathbb{N}$. If $q_k < \frac{N}{2}$ for each $k \in \mathbb{N}$, then we would have $q_0 \leq \lim q_n = \frac{N(\rho^*-1)}{2\rho^*} = 2\frac{N}{N+2} < q_0$ which is impossible. $\square$

With these, all the conclusions of Theorem 1.3 connected with the subcritical case (1.8) are proved.

4. ASYMPTOTIC SMOOTHNESS OF $\{S(t)\}$ AND THE ATTRACTOR IN THE CRITICAL CASE

This section is devoted to the proof of existence of attractor for the semigroup $\{S(t)\}$ associated in $H_0^1(\Omega) \times H_0^1(\Omega)$ to the evolution problem (1.1)-(1.2) in the critical case (1.9), completing the proof of Theorem 1.3. It only remains to prove the asymptotic compactness of $\{S(t)\}$. To that end we will employ Alekseev's Nonlinear Variation of Constants Formula (see [2, 4]). We first need to prove differentiability of the map $f^e : H^{-1}(\Omega) \rightarrow H^{-1}(\Omega)$.

Lemma 4.1. *Suppose that $f \in C^1(\mathbb{R}, \mathbb{R})$, $N > p$, and $|f'(s)| \leq c(1 + |s|^{\frac{2p}{N-p}})$, $s \in \mathbb{R}$, for some $c > 0$. Then the map $f^e = f \circ (I + A)^{-1}$ is of the class $C^1(W^{-1,p}(\Omega), W^{-1,p}(\Omega))$.*

Proof: Since $(I + A)^{-1}$ is a linear isomorphism from $W^{-1,p}(\Omega)$ into $W^{1,p}(\Omega)$ it is Frechet differentiable between these spaces with the derivative $(I + A)^{-1}$. Therefore, it remains to ensure that f is continuously Frechet differentiable from $W^{1,p}(\Omega)$ into $W^{-1,p}(\Omega)$.

Under our assumptions a sufficient condition for this is

$$\|f'(\theta(u + h) + (1 - \theta)u) - f'(u)\|_{L^{\frac{N}{2}}(\Omega)} \rightarrow 0 \quad \text{when } \|h\|_{W^{1,p}(\Omega)} \rightarrow 0, \quad (4.1)$$

where θ varies in the interval $[0, 1]$.

Since we know that $|f'(\theta(u+h_n) + (1-\theta)u) - f'(u)|$ tends to zero a.e. in Ω as $\|h_n\|_{W^{1,p}(\Omega)} \rightarrow 0$, we will apply in what follows the Inverse Dominated Convergence Theorem. As a consequence of this theorem we infer from the fact that $\|h_n\|_{L^{\frac{Np}{N-p}}(\Omega)} \rightarrow 0$ that there exists a subsequence $\{n_k\}$, for which $h_{n_k} \rightarrow 0$ a.e. in Ω and also

$$|h_{n_k}(x)|^{\frac{Np}{N-p}} \leq |g(x)|^{\frac{Np}{N-p}} \quad \text{for a.e. } x \in \Omega,$$

with g being a certain element of $L^{\frac{Np}{N-p}}(\Omega)$.

Fixing u and recalling the growth of f' we have a.e. in Ω

$$|f'(\theta(u+h_{n_k}) + (1-\theta)u) - f'(u)|^{\frac{N}{2}} \leq c(1 + |h_n|^{\frac{Np}{N-p}} + |u|^{\frac{Np}{N-p}}) \leq c(1 + |g|^{\frac{Np}{N-p}} + |u|^{\frac{Np}{N-p}}),$$

where $u, g \in L^{\frac{Np}{N-p}}(\Omega)$. Condition (4.1) now follows easily. $\square$

Next we introduce a decomposition of the nonlinearity f which plays an essential role in the proof of condition (1.6).

Lemma 4.2. *Suppose that $f \in C^1(\mathbb{R}, \mathbb{R})$ and $\limsup_{|s| \rightarrow \infty} f(s)s^{-1} = \mu \in \mathbb{R}$. Then there are functions $f_1 \in C^1(\mathbb{R}, \mathbb{R})$ and $f_2 : \mathbb{R} \rightarrow \mathbb{R}$ linear such that $f(s) = f_1(s) + f_2(s)$ and $sf_1(s) \leq 0$ for all $s \in \mathbb{R}$,*

Proof: It is sufficient to define $f_2(s) = f(0) + Ks$, where $K \geq \max\{\mu, f'(0)\}$ is a constant for which $s^{-1}(f(s) - f(0)) < K$ for all $s \neq 0$. $\square$

A decomposition of the nonlinear term $\mathcal{F}$ and of the semigroup of global solutions of (1.13) can now be established in accordance with the decomposition of f .

Lemma 4.3. *Suppose that $f \in C^1(\mathbb{R}, \mathbb{R})$, $\limsup_{|s| \rightarrow \infty} f(s)s^{-1} = \mu \in \mathbb{R}$, $N > p$, and $|f'(s)| \leq c(1 + |s|^{\frac{2p}{N-p}})$, $s \in \mathbb{R}$, for some $c > 0$.*

Under these assumptions the following conditions hold.

(C1) *There are functions $f_1 \in C^1(\mathbb{R}, \mathbb{R})$ and $f_2 : \mathbb{R} \rightarrow \mathbb{R}$ linear such that $f(s) = f_1(s) + f_2(s)$, $sf_1(s) \leq 0$ for $s \in \mathbb{R}$, and*

$$\mathcal{F}\left(\begin{bmatrix} v \\ w \end{bmatrix}\right) = \left[f_1((I+A)^{-1}v) \right] + \left[f_2((I+A)^{-1}v) \right] =: \mathcal{F}_1\left(\begin{bmatrix} v \\ w \end{bmatrix}\right) + \mathcal{F}_2\left(\begin{bmatrix} v \\ w \end{bmatrix}\right) \quad (4.2)$$

for $\begin{bmatrix} v \\ w \end{bmatrix} \in W^{-1,p}(\Omega) \times W^{-1,p}(\Omega)$, where

$$\begin{aligned} \mathcal{F}_1 &\in C^1(W^{-1,p}(\Omega) \times W^{-1,p}(\Omega), W^{-1,p}(\Omega) \times W^{-1,p}(\Omega)) \\ \mathcal{F}_2 &\in C^1(W^{-1,p}(\Omega) \times W^{-1,p}(\Omega), W^{1,p}(\Omega) \times W^{1,p}(\Omega)). \end{aligned} \quad (4.3)$$

(C2) *If $\tilde{T}(t, \begin{bmatrix} v_0 \\ w_0 \end{bmatrix})$ solves*

$$\frac{d}{dt} \begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix} + \mathcal{A}_\mu \begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix} = \mathcal{F}_1\left(\begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix}\right), \quad t > 0, \quad \begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix}_{t=0} = \begin{bmatrix} v_0 \\ w_0 \end{bmatrix} \quad (4.4)$$

and $T(t, \begin{bmatrix} v_0 \\ w_0 \end{bmatrix})$ solves

$$\frac{d}{dt} \begin{bmatrix} v \\ w \end{bmatrix} + \mathcal{A}_\mu \begin{bmatrix} v \\ w \end{bmatrix} = \mathcal{F}_1\left(\begin{bmatrix} v \\ w \end{bmatrix}\right) + \mathcal{F}_2\left(\begin{bmatrix} v \\ w \end{bmatrix}\right), \quad t > 0, \quad \begin{bmatrix} v \\ w \end{bmatrix}_{t=0} = \begin{bmatrix} v_0 \\ w_0 \end{bmatrix}, \quad (4.5)$$

then

$$T(t, [\frac{v_0}{w_0}]) = \tilde{T}(t, [\frac{v_0}{w_0}]) + \int_0^t (\partial \tilde{T})(t-s, T(s, [\frac{v_0}{w_0}])) \mathcal{F}_2(T(s, [\frac{v_0}{w_0}])) ds, \quad t > 0, \quad (4.6)$$

where $\partial \tilde{T}$ denotes the partial derivative of the map

$$(0, \infty) \times W^{-1,p}(\Omega) \times W^{-1,p}(\Omega) \ni (t, [\frac{v_0}{w_0}]) \rightarrow \tilde{T}(t, [\frac{v_0}{w_0}]) \in W^{-1,p}(\Omega) \times W^{-1,p}(\Omega)$$

with respect to $[\frac{v_0}{w_0}]$ and

$$\mathcal{F}_2([\frac{v}{w}]) = [K(I+A)^{-1}v + f(0)], \quad [\frac{v}{w}] \in W^{-1,p}(\Omega) \times W^{-1,p}(\Omega).$$

(C3) Function $(\partial \tilde{T})(\cdot, [\frac{v_0}{w_0}]) \mathcal{F}_2([\frac{v_0}{w_0}])$ is a mild solution of

$$\frac{d}{dt} \mathcal{W} + \mathcal{A}_\mu \mathcal{W} = \mathcal{F}'_1(\tilde{T}(t, [\frac{v_0}{w_0}])) \mathcal{W}, \quad t > 0, \quad \mathcal{W}(0) = \mathcal{F}_2([\frac{v_0}{w_0}]). \quad (4.7)$$

Proof: Condition (C1) follows from Lemmas 4.1 and 4.2.

Since $\mathcal{A}_\mu$ can be viewed as a sectorial operator in $W^{-1,p}(\Omega) \times W^{-1,p}(\Omega)$ and (4.3) holds, then (C2) follows from [4, Lemma 7]. For the proof of (C3) we refer to [10, Theorem 3.4.4]. $\square$

In Lemma 4.4 below we will obtain some suitable estimates for $\tilde{T}(\cdot, \cdot)$, $T(\cdot, \cdot)$ and $\mathcal{W}(\cdot)$.

Lemma 4.4. *If (1.4) and (1.9) are satisfied then Lemma 4.3 applies with $p = 2$ and the following properties hold.*

- (P1) $H^{-1}(\Omega) \times H^{-1}(\Omega) \ni [\frac{v_0}{w_0}] \rightarrow \tilde{T}(t, [\frac{v_0}{w_0}]) \in H^{-1}(\Omega) \times H^{-1}(\Omega)$ (where $t \geq 0$) is a C^0 semigroup of global solutions to (4.4) with bounded (positive) orbits of bounded sets.
- (P2) $H^{-1}(\Omega) \times H^{-1}(\Omega) \ni [\frac{v_0}{w_0}] \rightarrow T(t, [\frac{v_0}{w_0}]) \in H^{-1}(\Omega) \times H^{-1}(\Omega)$ (where $t \geq 0$) is a C^0 semigroup of global solutions to (4.5) with bounded (positive) orbits of bounded sets.
- (P3) If B is a bounded subset of $E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}}$ and $[\frac{\chi_1}{\chi_2}] \in E_2^{\frac{\sigma-1}{2}}$ for some $\sigma \in [0, 1]$ then there is a constant M_B such that

$$\sup_{s \geq 0} \sup_{[\frac{v_0}{w_0}] \in B} \|\mathcal{F}'_1(\tilde{T}(s, [\frac{v_0}{w_0}])) [\frac{\chi_1}{\chi_2}]\|_{E_2^{\frac{\sigma-1}{2}} \times E_2^{\frac{\sigma-1}{2}}} \leq M_B \|\frac{\chi_1}{\chi_2}\|_{E_2^{\frac{\sigma-1}{2}} \times E_2^{\frac{\sigma-1}{2}}}. \quad (4.8)$$

- (P4) If B is a bounded subset of $E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}}$ then for a certain constant $N_B > 0$

$$\sup_{s \geq 0} \sup_{[\frac{v_0}{w_0}] \in B} \|\mathcal{F}_2(T(s, [\frac{v_0}{w_0}]))\|_{E_2 \times E_2} \leq N_B. \quad (4.9)$$

- (P5) If $\tau > 0$ and B is bounded in $E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}}$, then there is a constant $\mathcal{R}(\tau, B)$ such that for each $[\frac{v_0}{w_0}] \in \{T(s, [\frac{v_0}{w_0}]), s \geq 0, [\frac{v_0}{w_0}] \in B\} =: \hat{B}$ the mild solution $\mathcal{W}$ of (4.7) fulfils the estimate

$$\sup_{t \in [0, \tau]} \|\mathcal{W}(t)\|_{E_2 \times E_2} \leq \mathcal{R}(\tau, B); \quad (4.10)$$

in particular

$$\begin{aligned} & \text{for all } \begin{bmatrix} \hat{v}_0 \\ \hat{w}_0 \end{bmatrix} \in \hat{B} \text{ the corresponding values of } \mathcal{W}(t) \text{ until} \\ & \text{the time } \tau \text{ lie in a certain compact subset of } E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}}. \end{aligned} \quad (4.11)$$

Proof: Properties (P1), (P2) follows from Theorem 2.1 and properties of the functional $\mathcal{L}$ in Lemma 2.5, which lemma is applicable since both f and f_1 satisfy (1.4).

For the proof of (P3) we first note that $\mathcal{F}'_1\left(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right)\begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \left[f'_1((I+A)^{-1}z_1)(I+A)^{-1}\chi_1 \right]$ and hence

$$\begin{aligned} \|\mathcal{F}'_1\left(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right)\begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix}\|_{E_2^{\frac{\sigma-1}{2}} \times E_2^{\frac{\sigma-1}{2}}} &\leq d\|f'_1((I+A)^{-1}z_1)(I+A)^{-1}\chi_1\|_{E^{\frac{2N}{N+2(1+\sigma)}}} \\ &\leq d\|f'_1((I+A)^{-1}z_1)\|_{E^{\frac{N}{2}}}\|(I+A)^{-1}\chi_1\|_{E^{\frac{2N}{N-2(1+\sigma)}}} \\ &\leq d\|f'_1((I+A)^{-1}z_1)\|_{E^{\frac{N}{2}}}\|(I+A)^{-1}\chi_1\|_{E^{\frac{1+\sigma}{2}}} \\ &= d\|f'_1((I+A)^{-1}z_1)\|_{E^{\frac{N}{2}}}\|\chi_1\|_{E^{\frac{\sigma-1}{2}}}. \end{aligned}$$

With the aid of (1.9) we next obtain

$$\begin{aligned} \|\mathcal{F}'_1\left(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right)\begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix}\|_{E_2^{\frac{\sigma-1}{2}} \times E_2^{\frac{\sigma-1}{2}}} &\leq d\|c(1 + |(I+A)^{-1}z_1|^{\frac{4}{N-2}})\|_{E^{\frac{N}{2}}}\|\chi_1\|_{E^{\frac{\sigma-1}{2}}} \\ &\leq cd\left(\|1\|_{E^{\frac{N}{2}}} + \|(I+A)^{-1}z_1\|_{E^{\frac{2N}{N-2}}}\right)\|\chi_1\|_{E^{\frac{\sigma-1}{2}}} \leq \tilde{c}\left(1 + \|z_1\|_{E_2^{-\frac{1}{2}}}\right)\|\chi_1\|_{E^{\frac{\sigma-1}{2}}}. \end{aligned} \quad (4.12)$$

Property (P3) now follows from (P1) and (4.12).

In the proof of (P4) we note that $f_2(s) = f(0) + Ks$ is a linear function and therefore

$$\|\mathcal{F}_2\left(\begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix}\right)\|_{E_2 \times E_2} = \|f(0) + K(I+A)^{-1}\chi_1\|_{E_2} \leq \tilde{d}(1 + \|\chi_1\|_{E_2^{-\frac{1}{2}}}). \quad (4.13)$$

Thus (P4) follows from (P2) and (4.13).

To complete the proof we write the variation of constants formula for $\mathcal{W}$

$$\mathcal{W}(t) = e^{-\mathcal{A}_\mu t} \mathcal{F}_2\left(\begin{bmatrix} \hat{v}_0 \\ \hat{w}_0 \end{bmatrix}\right) + \int_0^t e^{-\mathcal{A}_\mu(t-s)} \mathcal{F}'_1(\tilde{T}(s, \begin{bmatrix} v_0 \\ w_0 \end{bmatrix})) \mathcal{W}(s) ds, \quad (4.14)$$

and apply $\Phi_{-\frac{1}{2}}$ to get with the aid of (P3), (P4) and (2.2)

$$\begin{aligned} \|\mathcal{W}(t)\|_{E_2 \times E_2} &= \|\Phi_{-\frac{1}{2}} \mathcal{W}(t)\|_{E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}}} = \|e^{-\mathcal{A}_\mu t} \Phi_{-\frac{1}{2}} \mathcal{F}_2\left(\begin{bmatrix} \hat{v}_0 \\ \hat{w}_0 \end{bmatrix}\right)\|_{E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}}} \\ &\quad + \int_0^t \|e^{-\mathcal{A}_\mu(t-s)}\|_{E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}}} \|\Phi_{-\frac{1}{2}} \mathcal{F}'_1(\tilde{T}(s, \begin{bmatrix} v_0 \\ w_0 \end{bmatrix})) \mathcal{W}(s)\|_{E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}}} ds \\ &\leq M_2 e^{-\omega t} N_B + \int_0^t M_2 e^{-\omega(t-s)} \|\mathcal{F}'_1(\tilde{T}(s, \begin{bmatrix} v_0 \\ w_0 \end{bmatrix})) \mathcal{W}(s)\|_{E_2 \times E_2} ds \\ &\leq M_2 N_B + \int_0^t M_2 M_B \|\mathcal{W}(s)\|_{E_2 \times E_2} ds. \end{aligned} \quad (4.15)$$

Relation (4.10) is now a consequence of the Gronwall's inequality and (4.11) follows via compact embedding $E_2 \hookrightarrow E_2^{-\frac{1}{2}}$. $\square$

To conclude the proof of asymptotic compactness for $\{S(t)\}$ it only remains to prove that $\{\tilde{T}(t)\}$ decays exponentially, uniformly in bounded sets.

Lemma 4.5. *Given $r > 0$ there are constants $c(r) > 0$ and $\beta(r) > 0$ such that*

$$\sup \left\{ \left\| \tilde{T}(t, \begin{bmatrix} v_0 \\ w_0 \end{bmatrix}) \right\|_{E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}}} : \left\| \begin{bmatrix} v_0 \\ w_0 \end{bmatrix} \right\|_{E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}}} \leq r \right\} \leq c(r) e^{-\beta(r)t}$$

Proof: The proof follows as in Proposition 3 of [4] using the functional

$$\mathcal{L}_\delta(v, v_t) = \mathcal{L}(v, v_t) + \delta \langle v, v_t \rangle_{E_2^{-\frac{1}{2}}}.$$

In fact, let

$$\begin{bmatrix} \tilde{v} \\ \tilde{v}_t \end{bmatrix} =: \tilde{T}(t, \begin{bmatrix} v_0 \\ w_0 \end{bmatrix}), \quad t \geq 0 \quad \text{and} \quad \left\| \begin{bmatrix} v_0 \\ w_0 \end{bmatrix} \right\|_{E_2^{-\frac{1}{2}} \times E_2^{-\frac{1}{2}}} \leq r.$$

Choosing $\delta > 0$ suitably small we have that

$$\begin{aligned} \frac{d}{dt} (\mathcal{L}_\delta(v, v_t)) &\leq - \left(\frac{\mu\lambda_1}{1 + \lambda_1} + \delta \right) \|v_t\|_{E_2^{-\frac{1}{2}}}^2 + \delta \langle v, -\mu\Lambda v_t - \Lambda v + f^e(v) \rangle_{E_2^{-\frac{1}{2}}} \\ &\leq - \left(\frac{\mu\lambda_1}{1 + \lambda_1} + \delta \right) \|v_t\|_{E_2^{-\frac{1}{2}}}^2 - \delta \langle v, \mu\Lambda v_t \rangle_{E_2^{-\frac{1}{2}}} - \delta \langle v, \Lambda v \rangle_{E_2^{-\frac{1}{2}}} \\ &\leq - \frac{\mu\lambda_1}{2 + 2\lambda_1} \|v_t\|_{E_2^{-\frac{1}{2}}}^2 - \frac{\delta}{2} \|v\|_{E_2^{-\frac{1}{2}}}^2. \end{aligned}$$

On the other hand using (2.11) we obtain, for suitably small δ , that

$$\begin{aligned} \mathcal{L}_\delta(v, v_t) &= \frac{1}{2} \|v_t\|_{E_2^{-\frac{1}{2}}}^2 + \frac{1}{2} \|v\|_{E_2^{-\frac{1}{2}}}^2 - \frac{1}{2} \|(I + A)^{-\frac{1}{2}} v\|_{E_2^{-\frac{1}{2}}}^2 + \delta \langle v, v_t \rangle_{E_2^{-\frac{1}{2}}} - \int_{\Omega} F((I + A)^{-1} v) dx \\ &\leq \frac{1}{2} \|v_t\|_{E_2^{-\frac{1}{2}}}^2 + \frac{1}{2} \|v\|_{E_2^{-\frac{1}{2}}}^2 + \delta \langle v, v_t \rangle_{E_2^{-\frac{1}{2}}} + \hat{c}(1 + \|v\|_{E_2^{-\frac{1}{2}}}^{\rho^* - 1}) \|v\|_{E_2^{-\frac{1}{2}}}^2 \\ &\leq \bar{c}(r) \left(\|v_t\|_{E_2^{-\frac{1}{2}}}^2 + \|v\|_{E_2^{-\frac{1}{2}}}^2 \right). \end{aligned} \tag{4.16}$$

The proof of the result now follows easily. $\square$

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(A. N. Carvalho) DEPARTAMENTO DE MATEMÁTICA, INSTITUTO DE CIÊNCIAS MATEMÁTICAS E DE COMPUTAÇÃO, UNIVERSIDADE DE SÃO PAULO-CAMPUS DE SÃO CARLOS, CAIXA POSTAL 668, 13560-970 SÃO CARLOS SP, BRAZIL

E-mail address: andcarva@icmc.usp.br

(J. W. Cholewa) INSTITUTE OF MATHEMATICS, SILESIAN UNIVERSITY, 40-007 KATOWICE, POLAND

E-mail address: jcholewa@ux2.math.us.edu.pl

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