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AN EXTENSION OF THE CONCEPT OF GRADIENT SYSTEMS WHICH IS STABLE UNDER PERTURBATION

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Resumo:

Um semigrupo não-linear é *gradiente* se tem uma função de Lyapunov. Esta é uma condição sufuciente para caraterizar um atrator *gradiente-semelhante* e suas perturbações (caso hiperbólico). Esta característica se perde quando estudamos a dinamica assintótica de sistemas dinâmicos não-lineares e não autônomos. Para estes a definção de sistema gradiente não é clara.

Neste trabalho definimos *semigrupos não lineares gradiente-semelhantes* como um conceito intermediário entre os semigrupos não-lineares *gradientes* e aqueles que possuem um *attrator gradiente-semelhante.*

Provamos que este conceito é estável sob perturbações autônomas ou não. Consequentemente, recuperamos e estendemos os principais resultados na teoria de sistemas dinâmicos autonômos ou não-autônomos relativamente a continuidade, caracterização e atração exponencial de atratores.

AN EXTENSION OF THE CONCEPT OF GRADIENT SYSTEMS WHICH IS STABLE UNDER PERTURBATION

ALEXANDRE N. CARVALHO† AND JOSÉ A. LANGA‡

ABSTRACT. A semigroup is said to be gradient if it has a Lyapunov function (see [13]). This is a sufficient condition to characterize a gradient-like attractor and its perturbations (see [6]). This approach is missed when we study nonlinear dynamics associated to non-autonomous dynamical systems, where a natural definition of a gradient system is unclear. We define in this work a *gradient-like semigroup* as an intermediate concept between a gradient system and a system possessing a gradient-like attractor, and we prove that the concept is robust under autonomous or non-autonomous perturbations. Thus, we will recover and extend the main results in the autonomous and non-autonomous theory of dynamical systems with respect to continuity, characterization and exponential attraction of their attractors.

1. INTRODUCTION

In the theory of infinite dimensional dynamical systems, one problem that has received a lot of attention in the last few decades is the description of the geometric structure of their associated invariant sets. Any characterization of the structures inside an invariant set of a given dynamical system helps to describe the asymptotic behavior (possible chaotic) of the modeled phenomena under study. Although a large literature has been developed in the finite dimensional framework related to this problem, in an infinite dimensional setting the study of the structure of attractors is a difficult task, and just in very particular examples (see [13, 2, 26, 22] a detailed description of is available. What it is essentially known is that, if a dynamical system with a finite number of equilibria is gradient; that is, it has an associated Lyapunov function, then its attractor can be characterized by the unstable set of the set of equilibria. Moreover, if all equilibria are hyperbolic, this structure is stable under autonomous perturbation of the system.

Recently, in [6] (see also [27]), the authors consider non-autonomous perturbations of autonomous gradient semigroups for which all equilibria are hyperbolic. In both articles the authors have started from the proof that the hyperbolic equilibria under perturbation give rise to the same number of hyperbolic global solutions for the perturbed problem, proving that the perturbed attractor *contains* the same structures as the limiting one. Then, using the properties of the Lyapunov function (backwards and forwards convergence to equilibria and non-existence of homoclinic structures) for the limiting problem they prove that the perturbed attractor does not contain anything else.

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In this paper, without using a Lyapunov function, we translate the concept of gradient semigroups into a concept that we call *gradient-like semigroups* which strictly include the gradient semigroups. In particular, it allows that the semigroups have periodic orbits which is not allowed for gradient semigroups and are not considered in [6, 27]. In addition, this concept can be extended to non-autonomous evolution processes.

The gradient-like semigroups are shown to be stable under perturbation (autonomous or non-autonomous). That is, without talking about hyperbolicity (but assuming the continuity of the targeted global solutions) we prove that the attractor of a gradient-like semigroup under perturbation (autonomous or not) maintains its characterization.

In the autonomous case, the attractor of a gradient-like semigroup has a Morse decomposition (see [11, 19, 25]), so that we are showing that Morse decomposition are stable under perturbations. The concept of Morse decomposition can be extended to evolution processes (non-autonomous dynamical systems) and if a semigroup has an attractor which has a Morse decomposition, a perturbation (autonomous or not) of this semigroup has also a pullback attractor with a Morse decomposition.

In order to state our main results we introduce some terminology. Let $\mathcal{Z}$ be a Banach space. A nonlinear evolution process is a two parameter family $\{S(t,\tau):\ t\geq\tau\in\mathbb{R}\}$ of continuous operators from $\mathcal{Z}$ into itself such that

1) $S(\tau,\tau)=I$,
2) $S(t,\sigma)S(\sigma,\tau)=S(t,\tau)$, for each $t\geq\sigma\geq\tau$, and
3) $(t,\tau)\mapsto S(t,\tau)z_0$ is continuous for $t\geq\tau,\ z_0\in\mathcal{Z}$.

A continuous function $z:\mathbb{R}\to\mathcal{Z}$ is called a global solution for the evolution process $\{S(t,\tau):t\geq\tau\}$ if it satisfies

$$S(t,\tau)z(\tau)=z(t),\ \text{for all } t\geq\tau\in\mathbb{R}.$$

A nonlinear semigroup (or autonomous evolution process) is a family $\{S(t):\ t\geq 0\}$ with the property that $\{S(t,\tau)=S(t-\tau):t\geq\tau\in\mathbb{R}\}$ is an evolution process.

Definition 1.1. *A set $\mathcal{A}\subseteq\mathcal{Z}$ is said to be the* global attractor *for $\{S(t):t\geq 0\}$ if it is*

(i) *compact,*
(ii) *invariant, i.e. $S(t)\mathcal{A}=\mathcal{A}$ for all $t\geq 0$, and*
(iii) *it attracts bounded subsets B of $\mathcal{Z}$,*

$$\operatorname{dist}(S(t)B,\mathcal{A})\to 0 \ \textit{as}\ t\to\infty.$$

Since any fixed set A will not, in general, be invariant in the above sense for a non-autonomous process, it is natural to define *invariance* in this context as follows:

- A family $\{\mathcal{A}(t)\subset\mathcal{Z}:t\in[\sigma,\infty)\}$ is invariant under $S(\cdot,\cdot)$ if $S(t,\tau)\mathcal{A}(\tau)=\mathcal{A}(t)$ for all $t\geq\tau\geq\sigma$.

In the autonomous case the attractor is exactly the union of all its global orbits see [26]),

$$\mathcal{A}=\{z:\ \text{there is a bounded global solution through } z\}. \tag{1.1}$$

In the non-autonomous case, the definition of an 'attractor' that has the same characterization as the union of all globally-defined bounded orbits,

$$\{\mathcal{A}(t) : t \in \mathbb{R}\} = \{\xi(t) : \xi(\cdot) : \mathbb{R} \to \mathcal{Z} \text{ is bounded and } S(t,\tau)\xi(\tau) = \xi(t)\} \tag{1.2}$$

is the pullback attractor (see [9], [10], [16], [24]) :

Definition 1.2. *A family of compact sets $\{\mathcal{A}(t) \subset \mathcal{Z} : t \in \mathbb{R}\}$ is a* pullback attractor *for $\{S(t,\tau) : t \geq \tau \in \mathbb{R}\}$ if it is invariant and attracts all bounded subsets of $\mathcal{Z}$ 'in the pullback sense', i.e.*

$$\lim_{\tau \to -\infty} \operatorname{dist}(S(t,\tau)B, \mathcal{A}(t)) = 0, \ \forall t \in \mathbb{R}.$$

For autonomous problems, it is clear that the concept of a pullback attractor coincides with the standard definition of the attractor, while the characterization in (1.2) shows that this notion is in some sense a 'natural' generalization (see [6] for a discussion). However, the pullback attractor will not necessarily enjoy any kind of forward attraction. Except in specific situations, the pullback behaviour and the forwards behaviour will not be related (see Theorems 4.1 and 4.2, and [8, 23, 18] for other specific cases).

Let $\xi : \mathbb{R} \to \mathcal{Z}$ be a global solution of a nonlinear evolution process $\{T(t,\tau) : t \geq \tau \in \mathbb{R}\}$. The set $\Gamma = \{\xi(t) : t \in \mathbb{R}\}$ is called trace of ξ. Let ξ^* be a global solution of $\{T(t,\tau) : t \geq \tau \in \mathbb{R}\}$ and let Γ^* its trace. We say that $\mathcal{S} = \{\xi_1^*, \cdots \xi_n^*\}$ is a set of isolated global solutions if there exists $\delta > 0$ such that $\mathcal{O}_\delta(\Gamma_i^*) \cap \mathcal{O}_\delta(\Gamma_j^*) = \emptyset$, $1 \leq i < j \leq n$, where Γ_i^* is the trace of ξ_i^* and $\mathcal{O}_\delta(\Gamma_i^*) := \{z \in \mathcal{Z} : \operatorname{dist}(z, \Gamma^*) < \delta\}$.

Let $\{T(t,\tau) : t \geq \tau \in \mathbb{R}\}$ be a nonlinear evolution process with a finite number of isolated global solutions $\mathcal{S} = \{\xi_1^*, \cdots \xi_n^*\}$ and pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$. Let Γ_i^* be the trace of ξ_i^*. We define (see [12, 21])

Definition 1.3. *For $\xi^* \in \mathcal{S}$, an $\epsilon-$chain from ξ^* to ξ^* is a sequence of real numbers $t_1, \tau_1, \cdots t_k, \tau_k$ with $t_i > \tau_i$, $1 \leq i \leq k$, $k \leq n$, and a sequence of vectors u_i, $1 \leq i \leq k$, such that $u_i \in \mathcal{O}_\epsilon(\Gamma_i^*)$ and $T(t_i, \tau_i)u_i \in \mathcal{O}_\epsilon(\Gamma_{i+1}^*)$, $1 \leq i \leq k$, with $\xi^* = \xi_{k+1}^* = \xi_1^*$. We say that $\xi^* \in \mathcal{S}$ is chain recurrent if there is an $\epsilon-$chain from ξ^* to ξ^* for each $\epsilon > 0$.*

Definition 1.4. *Let $\mathcal{Z}$ be a Banach space and $\{T(t,\tau) : t \geq \tau \in \mathbb{R}\}$ be a nonlinear process in $\mathcal{Z}$. Let $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ be the pullback attractor for $\{T(t,\tau) : t \geq \tau \in \mathbb{R}\}$. We say that $\{T(t,\tau) : t \geq \tau \in \mathbb{R}\}$ is gradient-like if the following two hypotheses are satisfied:*

(H1) *There exists a finite number of isolated global solutions $\{\xi_i^* : \mathbb{R} \to \mathcal{Z} : 1 \leq i \leq n\}$ such that, any global solution $\xi : \mathbb{R} \to \mathcal{Z}$ in $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ satisfies*

$$\lim_{t \to -\infty} \|\xi(t) - \xi_i^*(t)\|_{\mathcal{Z}} = 0 \text{ and } \lim_{t \to \infty} \|\xi(t) - \xi_j^*(t)\|_{\mathcal{Z}} = 0,$$

for some $1 \leq i, j \leq n$.

(H2) *$\mathcal{S} = \{\xi_1^*, \cdots, \xi_n^*\}$ does not contain any chain recurrent solution.*

Next we introduce the definitions of unstable and stable sets for which we will refer to as the unstable and stable manifolds, though they do not need to be a manifold (up to this point).

Definition 1.5. *The unstable manifold of a global hyperbolic solution ξ_η^* to $S(t,\tau)$ is the set*

$$W_\eta^{\mathrm{u}}(\xi_\eta^*) = \{\, (\tau,\zeta) \in \mathbb{R}\times\mathcal{Z} : \ \textit{there is a backwards solution } S(t,\tau)\zeta \\ \textit{such that } \lim_{t\to-\infty} \|S(t,\tau)\zeta - \xi_\eta^*(t)\|_{\mathcal{Z}} = 0\}.$$

The stable manifold of a hyperbolic solution ξ_η^* to $S(t,\tau)$ is the set

$$W_\eta^{\mathrm{s}}(\xi_\eta^*) = \{\, (\tau,\zeta) \in \mathbb{R}\times\mathcal{Z} : \ \textit{there is a forwards solution } S(t,\tau)\zeta \\ \textit{satisfying such that } \lim_{t\to+\infty} \|S(t,\tau)\zeta - \xi_\eta^*(t)\|_{\mathcal{Z}} = 0\}.$$

The intersection of the unstable (stable) manifold with a neighborhood of the curve $(\cdot,\xi(\cdot))$ in $\mathbb{R}\times\mathcal{Z}$ is called a local unstable (stable) manifold and is denoted by $W_{\eta,\mathrm{loc}}^{\mathrm{u}}$ ($W_{\eta,\mathrm{loc}}^{\mathrm{s}}$).

We are now ready to state the main results of this paper

Theorem 1.6. *Let $\mathcal{Z}$ be a Banach space, $\eta\in[0,1]$ and $\{T_\eta(t,\tau) : t\geq\tau\in\mathbb{R}\}$ be a nonlinear evolution process in $\mathcal{Z}$. Assume that there exists a uniform absorbing compact set B_0 (independent of η). Let $\{\mathcal{A}_\eta(t) : t\in\mathbb{R}\}$ be the pullback attractor for $\{T_\eta(t,\tau) : t\geq\tau\in\mathbb{R}\}$. Assume that $\{T_\eta(t,\tau) : t\geq\tau\in\mathbb{R}\}$, $\eta\in[0,1]$, has a finite number of isolated global solutions $\mathcal{S}=\{\xi_{1,\eta}^*,\cdots,\xi_{n,\eta}^*\}$ which behave continuously as $\eta\to 0$ ($\sup_{1\leq i\leq n}\sup_{t\in\mathbb{R}}\|\xi_{i,\eta}^*(t)-\xi_{i,0}^*(t)\|_{\mathcal{Z}}\stackrel{\eta\to 0}{\longrightarrow}0$). Assume also that $\|T_\eta(t+\tau,\tau)u-T_0(t+\tau,\tau)u\|_{\mathcal{Z}}\stackrel{\eta\to 0}{\longrightarrow}0$ uniformly for $\tau\in\mathbb{R}$, t in compact subsets of $[0,\infty)$ and for u in compact subsets of $\mathcal{Z}$.*

If $\{S(t) : t\geq 0\}$ is a gradient-like nonlinear semigroup, $T_0(t,\tau)=S(t-\tau)$, $t\geq\tau$; that is, (H1)-(H2) are satisfied for $\{T_0(t,\tau) : t\geq\tau\in\mathbb{R}\}$, then there exists $\eta_0>0$ such that, for all $\eta\leq\eta_0$, $\{T_\eta(t,\tau) : t\geq\tau\in\mathbb{R}\}$ is a gradient-like nonlinear process. Consequently, there exists $\eta_0>0$ such that

$$\mathcal{A}_\eta(t)=\cup_{i=1}^n W^u(\xi_{i,\eta}^*)(t),\ t\in\mathbb{R} \textit{ and } \forall\eta\in[0,\eta_0].$$

Theorem 1.6 generalizes the characterization result in [6] to perturbation of autonomous gradient-like nonlinear semigroups. Hence, the limit problem need not to have a Lyapunov function nor ξ_i^* has to be an equilibrium point. Of course it remains to prove the continuity of the isolated global solutions $\xi_{i,\eta}^*$ at $\eta=0$ which is known to hold; e.g, for normally hyperbolic global solutions (see [4]). That opens the possibility for considering non-autonomous perturbations of gradient-like nonlinear semigroups with periodic solutions. Indeed, for each $1\leq m\in\mathbb{N}$, the problem

$$\begin{aligned}\dot r &= \begin{cases} \pi^{-1}(1-\frac{1}{2m+1}-r)^3\sin\frac{\pi}{1-r}, & r<1-\frac{1}{2m+1}\\ -(1-\frac{1}{2m+1}-r)^2, & r\geq 1-\frac{1}{2m+1}\end{cases}\\ \dot\theta &= 1\end{aligned}\tag{1.3}$$

has an attractor $A_m=\{|r|\leq 1-\frac{1}{2m+1}\}$, which is the union of the unstable manifolds of ξ_j^*, $1\leq j\leq 2m+1$. Where ξ_j is the $2\pi-$periodic solution corresponding to $r=1-\frac{1}{j}$,

$1 \leq j \leq 2m+2$. These periodic solutions are normally hyperbolic solutions, if k is even the orbit is unstable and if k is odd the orbit is stable. In this case, it is easy to see that the attractor A_m is the union of the unstable manifolds of the of periodic solutions $\{\xi_j^* : 1 \leq j \leq 2m+1\}$. Theorem 1.6 implies that any small non-autonomous perturbation of this system will lead a gradient-like evolution process and the perturbed attractor is characterized.

Observe that Theorem 1.6 implies that the pullback attractors for $\{T_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ are characterized as the union of the unstable manifolds of the isolated global solutions. Hyperbolicity is not required up to this point. Nonetheless, the persistence of the isolated global solutions will require some kind of hyperbolicity (in general normal hyperbolicity, see [4], [25]).

Suppose $\{T_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$, $\eta \in [0,1]$, is a family of nonlinear evolution processes with the property that there exist $c = c(B_0)$ and $L > 0$ such that

$$\|T_\eta(t+\tau,\tau)u - T_\eta(t+\tau,\tau)v\| \leq ce^{Lt}\|u-v\|, \text{ for all } \tau \in \mathbb{R},\ u,v \in B_0. \tag{1.4}$$

Our next result proves that a non-autonomous perturbation of a gradient-like semigroup with all equilibria hyperbolic has an exponential forwards (and pullback) attractor.

Theorem 1.7. *Let $\mathcal{Z}$ be a Banach space, $\eta \in [0,1]$ and $\{T_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ be a nonlinear evolution process in $\mathcal{Z}$. Assume that there exists a uniform absorbing compact set B_0 (independent of η). Let $\{\mathcal{A}_\eta(t) : t \in \mathbb{R}\}$ be the pullback attractor for $\{T_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$. Assume also that $\|T_\eta(t+\tau,\tau)u - T_0(t+\tau,\tau)u\|_{\mathcal{Z}} \xrightarrow{\eta\to 0} 0$ uniformly for $\tau \in \mathbb{R}$, t in compact subsets of $[0,\infty)$ and for u in compact subsets of $\mathcal{Z}$. Assume that $\{T_0(t) : t \geq 0 \in \mathbb{R}\}$, has a finite number of isolated equilibria $\mathcal{S} = \{y_1^*, \cdots, y_n^*\}$ and that (3.4) is satisfied. Then there are hyperbolic global solutions $\{\xi_{1,\eta}^*, \cdots, \xi_{n,\eta}^*\}$ of (3.5) such that $\sup_{t\in\mathbb{R}} \|\xi_{i,\eta}^*(t) - y_i^*\|_{\mathcal{Z}} \xrightarrow{\eta\to 0} 0$.*

If $\{T_0(t) : t \geq 0\}$ is a gradient-like nonlinear semigroup, then there exists $\eta_0 > 0$ such that

$$\mathcal{A}_\eta(t) = \cup_{i=1}^n W^u(\xi_{i,\eta}^*)(t),\ \forall t \in \mathbb{R} \text{ and } \forall \eta \in [0,\eta_0]$$

where $\mathcal{A}_0(t) = \mathcal{A}$ for all $t \in \mathbb{R}$ is the global attractor for $\{T_0(t) : t \geq 0\}$.

Furthermore, there exists $\gamma > 0$ such that, for $B \subset \mathcal{Z}$ bounded,

$$\sup_{\tau\in\mathbb{R}} \operatorname{dist}(T_\eta(t+\tau,\tau)u_0, \mathcal{A}_\eta(t+\tau)) \leq c(B)e^{-\gamma t}, \text{ for all } u_0 \in B. \tag{1.5}$$

This paper is organized as follows. In Section 2 we present the results for the autonomous case emphasizing the generalization of previous known results. In Section 3 we extend the results to the non-autonomous case. In Section 4 we write the consequences of the results of Section 3 to asymptotically autonomous evolution processes. Finally, Section 5 focuses in the comparison of the concept of gradient-like semigroups with the concept of semigroups with attractors having a Morse decomposition and to point out several problems that remain open.

2. Gradient semigroups, gradient-like semigroups and semigroups with 'gradient-like' attractors

We firstly need to draw a distinction between gradient systems and systems with 'gradient-like' attractors.

Definition 2.1. *We say that a nonlinear semigroup* $\{S(t) : t \geq 0\}$ *is* gradient *if* $\{S(t)z : t \geq 0\}$ *is relatively compact for each* $z \in \mathcal{Z}$ *and there exists a continuous function* $V : \mathcal{Z} \to \mathbb{R}$ *such that*

- $t \mapsto V(S(t)z) : [0, \infty) \to \mathcal{Z}$ *is non-increasing for each* $z \in \mathcal{Z}$.
- *If* $z \in \mathcal{Z}$ *is such that there is a global solution* $\xi(\cdot) : \mathbb{R} \to \mathcal{Z}$ *through* $\xi(0) = z$ *and there exists a* $t^* \in \mathbb{R}$ *such that* $V(\xi(t)) = V(z)$ *for all* $t \geq t^*$ *or for all* $t \leq t^*$, *then* z *is a solution for* $\{S(t) : t \geq 0\}$ *(and so in fact* $V(\xi(t)) = V(z)$ *for all* $t \in \mathbb{R}$*).*

The function $V : \mathcal{Z} \to \mathbb{R}$ *is called a Lyapunov function for* $\{S(t) : t \geq 0\}$.

The stable and unstable manifolds of an equilibrium y_0^*, $W^{\mathrm{s}}(y_0^*)$ and $W^{\mathrm{u}}(y_0^*)$ respectively, are defined as follows:

$$W^{\mathrm{s}}(y_0^*) = \{\, z \in \mathcal{Z} : \lim_{t \to +\infty} \|S(t)z - y_0^*\|_{\mathcal{Z}} = 0\}.$$

$$W^{\mathrm{u}}(y_0^*) = \{\, z \in \mathcal{Z} : \text{ there is a backwards solution } y(t) \text{ for } \{S(t) : t \geq 0\}$$
$$\text{satisfying } y(\tau) = z \text{ and such that } \lim_{t \to -\infty} \|y(t) - y_0^*\|_{\mathcal{Z}} = 0\}.$$

The following result is classical and can be found, for instance, in [13]

Theorem 2.2. *If* $\{S(t) : t \geq 0\}$ *is a gradient system that has a global attractor* $\mathcal{A}$, $V : \mathcal{Z} \to \mathbb{R}$ *is its Lyapunov function and* $\{S(t) : t \geq 0\}$ *has a finite number of hyperbolic solutions* y_i^*, $1 \leq i \leq n$, *then* $\mathcal{A}$ *is given by*

$$\mathcal{A} = \bigcup_{i=1}^{n} W_0^{\mathrm{u}}(y_i^*), \tag{2.1}$$

i.e. the attractor is 'gradient-like'. Furthermore if we denote by $\{\mathfrak{n}_1, \cdots, \mathfrak{n}_p\}$ *the set of all distinct values of* $V(y_j^*)$, *ordered so that* $\mathfrak{n}_i < \mathfrak{n}_j$, $1 \leq i < j \leq p \leq n$, *and define* $\mathcal{E}_k = \{y_i^* \in \mathcal{E} : V(y_i^*) = \mathfrak{n}_k\}$, *then if* $y(\cdot) : \mathbb{R} \to \mathcal{Z}$ *is a global solution for* $\{S(t) : t \geq 0\}$, *there are* k_1, k_2 *with* $1 \leq k_1 < k_2 \leq p$, $y_i^* \in \mathcal{E}_{k_1}$ *and* $y_j^* \in \mathcal{E}_{k_2}$, *such that*

$$\lim_{t \to -\infty} y(t) = y_j^* \text{ and } \lim_{t \to +\infty} y(t) = y_i^*.$$

The above result shows that if $\{S(t) : t \geq 0\}$ is a gradient nonlinear semigroup the structure of $\mathcal{A}$ and its asymptotic dynamics are completely understood. This is essentially the class of nonlinear semigroups in Banach spaces for which a detailed knowledge of the structure of the attractor is available. An attractor of the form (2.1) we term *'gradient-like'*. Clearly the class of systems with gradient-like attractors is larger than those that are strictly gradient.

In general one does not expect that a perturbation of a gradient semigroup gives rise to a new gradient semigroup, the main difficulty being to prove that the perturbed problem has a Lyapunov function. In [6] we prove that a perturbation of a gradient semigroup which has a global attractor gives rise to a new semigroup which posseses a gradient-like attractor. However, we do not prove that a perturbation of a semigroup with a gradient-like attractor will have a gradient-like attractor. Actually, at this point we are only able to prove that the perturbed attractor contains the same structure as the limit attractor and therefore the attractors behave upper and lower semicontinuously. In our proofs the Lyapunov function plays an essential role. It is not difficult to see (see the example in [13] pages 2 and 3) that, in general, a gradient-like-attractor is not coming from a gradient system and a perturbation of the attractor may not be a gradient-like attractor.

This brings up the question of what (natural) dynamical properties of semigroups ensure that they have gradient-like attractors and such that (the properties) are stable under perturbation. Of course, such properties must be satisfied by gradient semigroups and its perturbations.

With this in mind we define the following concept of *gradient-like semigroups* generalizing the concept of gradient semigroups while maintaining its essential dynamics. We will prove that the properties defining gradient-like semigroups are stable under perturbation.

Definition 2.3. *Consider a nonlinear semigroup $\{S(t) : t \geq 0\}$ with a finite number of stationary solutions $\mathcal{E} = \{y_1^*, \cdots y_n^*\}$. Let $z^* \in \mathcal{E}$ and $\epsilon > 0$. An $\epsilon-$chain from z^* to z^* is a sequence $\{y_1, \cdots, y_{k+1}\}$ of points in $\mathcal{Z}$ and a sequence of positive numbers $\{t_1, \cdots, t_k\}$ such that $\|y_i - y_i^*\|_{\mathcal{Z}} < \epsilon$, $1 \leq i \leq k+1$, $z^* = y_1^* = y_{n+1}^*$ and $\|S(t_i)y_i - y_{i+1}^*\|_{\mathcal{Z}} < \epsilon$, $1 \leq i \leq k$. We say that $z^* \in \mathcal{E}$ is chain recurrent if there is an $\epsilon-$chain from z^* to z^* for each $\epsilon > 0$.*

Definition 2.4. *Let $\{S(t) : t \geq 0\}$ be a nonlinear semigroup with a finite number of stationary solutions $\mathcal{E} = \{y_1^*, \cdots y_n^*\}$ and assume that it has a global attractor $\mathcal{A}$. We say that $\{S(t) : t \geq 0\}$ is a*gradient-like semigroup *if the following two hypotheses are satisfied:*

(G1) *Given a global solution $\xi : \mathbb{R} \to \mathcal{Z}$ in $\mathcal{A}$, there exists $i, j \in \{1, \cdots, n\}$ such that*

$$\lim_{t \to -\infty} \|\xi(t) - y_i^*\| = 0 \text{ and } \lim_{t \to \infty} \|\xi(t) - y_j^*\| = 0.$$

(G2) *$\mathcal{E} = \{y_1^*, \cdots, y_n^*\}$ does not contain any chain recurrent point.*

The assumptions (G1) and (G2) carry the important dynamic properties of a system with a Lyapunov function (see also Lemmas 2.5 and 2.6 below). Clearly, from (G1), we have that $\mathcal{A} = \cup_{i=1}^n W^u(y_i^*)$. Also, the hypothesis (G2) says that no finite number of orbits may produce a closed contour. The great advantage of these assumptions over Lyapunov functions is its stability under perturbation. This fact will allow us to give characterization of the attractor which are perturbation of gradient systems, perturb it again and still be able to give characterization of the new perturbed system.

Lemma 2.5. *Let $\{S(t) : t \geq 0\}$ be a nonlinear semigroup with a finite number of stationary solutions $\mathcal{E} = \{y_1^*, \cdots y_n^*\}$ which has a global attractor $\mathcal{A}$ and satisfies (G1) and (G2). Given $\delta < \frac{1}{2} \min\{\|z_i^* - z_j^*\|_{\mathcal{Z}} : 1 \leq i, j \leq k, \ i \neq j\}$ and a bounded set $B \subset \mathcal{Z}$, there is a positive number $T = T(\delta, B)$ such that $\{S(t)u_0 : 0 \leq t \leq T\} \cap \cup_{i=1}^n B_\delta(z_i^*) \neq \emptyset$ for all $u_0 \in B$.*

Proof: We argue by contradiction. Assume that there is a sequence u_k in B and a sequence of positive numbers t_k such that $\{S(t)u_k : 0 \leq t \leq t_k\} \cap \cup_{i=1}^n B_\delta(z_i^*) = \emptyset$. Extracting subsequences we have that there is a global solution $\xi : \mathbb{R} \to \mathcal{Z}$ such that $S(t + \frac{t_k}{2})u_k \to \xi(t)$ uniformly in compact subsets of $\mathbb{R}$. Clearly $\xi(t) \in \mathcal{A}$, for all $t \in \mathbb{R}$. Hence, $\xi(t) \notin \cup_{i=1}^n B_\delta(z_i^*)$ for all $t \in \mathbb{R}$ and this contradicts (G1).$\square$

Lemma 2.6. *Let $\{S(t) : t \geq 0\}$ be a nonlinear semigroup with a finite number of stationary solutions $\mathcal{E} = \{y_1^*, \cdots y_n^*\}$ which has a global attractor $\mathcal{A}$ and satisfies (G1) and (G2). Given $0 < \delta < \frac{1}{2}\min\{\|z_i^* - z_j^*\|_\mathcal{Z} : 1 \leq i, j \leq k,\ i \neq j\}$, there is a $\delta' > 0$ such that, if for some $1 \leq i \leq n$, $\|u_0 - z_i^*\|_\mathcal{Z} < \delta'$ and, for some $t_1 > 0$, $\|S(t)u_0 - z_i^*\|_\mathcal{Z} \geq \delta$, then $\|S(t)u_0 - z_i^*\|_\mathcal{Z} > \delta'$ for all $t \geq t_1$.*

Proof: Assume that, for some $1 \leq i \leq n$, there is a sequence u_k in $\mathcal{Z}$ with $\|u_k - z_i^*\|_\mathcal{Z} < \frac{1}{k}$ and sequences $t_k < \tau_k$ of positive numbers such that $\|S(t_k)u_k - z_i^*\|_\mathcal{Z} \geq \delta$ and $\|S(\tau_k)u_k - z_i^*\|_\mathcal{Z} < \frac{1}{k}$. Clearly t_k is bounded from below and that contradicts (G2).$\square$

Before we proceed with the analysis of the attractors of *gradient-like semigroups* under perturbation let us establish the equivalence between condition (G2) and the absence of homoclinic structures.

Definition 2.7. *Let $\{S(t) : t \geq 0\}$ be a nonlinear semigroup with a finite number of stationary solutions $\mathcal{E} = \{y_1^*, \cdots y_n^*\}$ and assume that it has a global attractor $\mathcal{A}$. A* homoclinic structure *in $\mathcal{A}$ is a set $\{z_1^*, \cdots, z_k^*\} \subset \mathcal{E}$ and a set of global solutions $\{\xi_i : \mathbb{R} \to \mathcal{Z},\ 1 \leq i \leq k\}$ in $\mathcal{A}$ such that, making $z_{k+1}^* := z_1^*$,*

$$\lim_{t\to-\infty} \xi_i(t) = z_i^*, \quad \lim_{t\to+\infty} \xi_i(t) = z_{i+1}^*, \quad 1 \leq i \leq k.$$

Lemma 2.8. *If $\{S(t) : t \geq 0\}$ is a nonlinear semigroup with a finite number of stationary solutions $\mathcal{E} = \{y_1^*, \cdots y_n^*\}$ which has a global attractor $\mathcal{A}$ and satisfies (G1), then (G2) is satisfied if and only if $\mathcal{A}$ does not have any homoclinic structure.*

Proof: Assuming that $\mathcal{A}$ has a homoclinic structure it is easy to see that some of the equilibria in it is chain recurrent. On the other hand, if one of the equilibria in $z^* \in \mathcal{E}$ is chain recurrent, there is a subset $\{z_1^*, \cdots, z_{\ell+1}^*\}$ and, for each positive integer k, points $x_1^k, \cdots, x_{\ell+1}^k$ and positive numbers $t_1^k, \cdots, t_\ell^k$ such that

$$\|x_i^k - z_i^*\|_\mathcal{Z} < \frac{1}{k}, \quad 1 \leq i \leq \ell + 1, \quad \|S(t_i^k)x_i^k - z_{i+1}^*\|_\mathcal{Z} < \frac{1}{k}, \quad 1 \leq i \leq \ell.$$

Choose $0 < 2\delta < \min\{\|z_i^* - z_j^*\|_\mathcal{Z} : 1 \leq i, j \leq k,\ i \neq j\}$ and choose $\tau_i^k > 0$ such that $\|S(t)x_i^k - z_i^*\|_\mathcal{Z} < \delta$, for all $0 \leq t < \tau_i^k$. Note that $\tau_i^k \to +\infty$ as $k \to +\infty$. For $t \in [-\tau_i^k, t_i^k - \tau_i^k]$ let $\xi_i^k(t) = S(\tau_i^k + t)x_i^k$. Taking subsequences we define the global solutions $\xi_i : \mathbb{R} \to \mathcal{Z}$ by $\xi_i(t) = \lim_{k\to\infty} \xi_i^k(t)$. Since each $\xi_i(t)$ must converge to an equilibrium solution as $t \to +\infty$ and as $t \to -\infty$ and since $\xi_i(t) \in B_\delta(z_i^*)$ for all $t \leq 0$ we have that $\xi_i(t) \to z_i^*$ as $t \to -\infty$. Also, there is a time $\sigma_i^k > 0$ such that $\xi_i^k(t) \in B_\delta(z_{i+1}^*)$ for all $t \in [\sigma_i^k, t_i^k - \tau_i^k]$. It is easy to see that the sequence $\{\sigma_i^k\}_{k=1}^\infty$ is bounded and that $t_i^k - \tau_i^k - \sigma_i^k \to +\infty$ as $k \to \infty$. Consequently $\xi_i(t) \to z_{i+1}^*$ as $t \to +\infty$.

The set $\{z_1^*, \cdots, z_k^*\} \subset \mathcal{E}$ and the set of global solutions $\{\xi_i : \mathbb{R} \to \mathcal{Z},\ 1 \leq i \leq k\}$ are such that,

$$\lim_{t\to-\infty} \xi_i(t) = z_i^*, \quad \lim_{t\to+\infty} \xi_i(t) = z_{i+1}^*, \quad 1 \leq i \leq k,$$

with $z_{k+1}^* := z_1^*$. Hence $\mathcal{A}$ has a homoclinic structure.$\square$

We are now ready to prove that (G1) and (G2) are stable under perturbation, i.e. the concept of gradient-like system is robust under perturbation.

Theorem 2.9. *Let $\mathcal{Z}$ be a Banach space, $\eta \in [0,1]$ and $\{S_\eta(t) : t \geq 0\}$ be a nonlinear semigroup in $\mathcal{Z}$. Assume that there exists a compact absorbing set B_0 (independent of η). Let $\mathcal{A}_\eta$ be the attractor for $\{S_\eta(t) : t \geq 0\}$. Assume that $\{S_\eta(t) : t \geq 0\}$, $\eta \in [0,1]$, has a finite number of isolated equilibria $\mathcal{S} = \{y_{1,\eta}^*, \cdots, y_{n,\eta}^*\}$ which behave continuously as $\eta \to 0$ ($\|y_{i,\eta} - y_{i,0}\|_\mathcal{Z} \xrightarrow{\eta\to 0} 0$, $1 \leq i \leq n$). Assume also that $\|S_\eta(t)u - S_0(t)u\|_\mathcal{Z} \xrightarrow{\eta\to 0} 0$ uniformly for t in compact subsets of $[0,\infty)$ and for u in compact subsets of $\mathcal{Z}$.*

Assume also that $\{S_0(t) : t \geq 0\}$ satisfies (G1) and (G2), then there exists $\eta_0 > 0$ such that $\{S_\eta(t) : t \geq 0\}$ satisfies (G1) and (G2) for all $\eta \leq \eta_0$.

Proof: Note $z_{i,\eta}^* \to z_{i,0}^* =: z_i^*$ and that there is a $\delta > 0$ such that, for suitably small η, if a solution ξ_η satisfies $\|\xi_\eta(t) - z_i^*\|_\mathcal{Z} \leq \delta$ for all $t \geq t_0$ and for some $t_0 > 0$, then $\xi_\eta(t) \to z_{i,\eta}^*$ as $t \to \infty$.

We argue by contradiction to prove that for all suitably small η, $\{S_\eta(t) : t \geq 0\}$ satisfies (G1). Assume that there is a sequence $\eta_k \xrightarrow{k\to\infty} 0$ and corresponding global solutions ξ_k in $\mathcal{A}_{\eta_k}$ such that

$$\liminf_{t\to\infty} \operatorname{dist}(\xi_k(t), \mathcal{E}) > \delta. \tag{2.2}$$

Since $\xi_k(t) \to \xi_0(t)$ uniformly in compact subsets of $\mathbb{R}$ and since $\xi_0(t) \xrightarrow{t\to\infty} z_i^*$, for some $1 \leq i \leq n$, we have that, given $\ell > 0$ there is a $t_\ell > 0$ and $k_\ell \in \mathbb{N}$ such that $\|\xi_k(t_\ell) - z_i^*\|_\mathcal{Z} < \frac{1}{\ell}$, for each $k \geq k_\ell$. From (2.2), there exists $t'_\ell > t_\ell$ such that $\|\xi_{k_\ell}(t) - z_i^*\|_\mathcal{Z} < \delta$ for all $t \in [t_\ell, t'_\ell)$ and $\|\xi_{k_\ell}(t'_\ell) - z_i^*\|_\mathcal{Z} = \delta$. Taking subsequences, if necessary, let $\xi_1(t) = \lim_{\ell\to\infty} \xi_{k_\ell}(t + t_\ell)$. Then, since $t'_\ell - t_\ell \to \infty$ and $t_\ell \to +\infty$ as $\ell \to \infty$, $\|\xi_1(t) - z_i^*\|_\mathcal{Z} \leq \delta$ for all $t \leq 0$ and consequently $\xi_1(t) \to z_i^*$ as $t \to -\infty$. And $\xi_1(t) \to z_j^*$ as $t \to \infty$ with $i \neq j$ by (G1) for $\eta = 0$. From the fact that $\xi_{k(\ell)} \to \xi_1(t)$ uniformly in compact subsets of $\mathbb{R}$ we have that, for each $m \in \mathbb{N}$, there is a time $t_m > 0$ and $k_m \in \mathbb{N}$ such that $\|\xi_k(t_m) - z_j^*\|_\mathcal{Z} < \frac{1}{m}$ for all $k \geq k_m$. Again, from (3.2), there exists $t'_m > t_m$ such that $\|\xi_{k_m}(t) - z_j^*\|_\mathcal{Z} < \delta$ for all $t \in [t_m, t'_m)$ and $\|\xi_{k_m}(t'_m) - z_j^*\|_\mathcal{Z} = \delta$. Proceeding exactly as before we obtain a global solution $\xi_2 : \mathbb{R} \to \mathcal{Z}$ such that $\xi_2(t) \to z_j^*$ as $t \to -\infty$ and $\xi_2(t) \to z_r^*$ as $t \to \infty$ with $r \notin \{i,j\}$. In a finite number of steps we arrive at a contradiction. This proves that there is a $\eta_0 > 0$ such that, for all global solution ξ_η in $\mathcal{A}_\eta$ with $\eta \leq \eta_0$, we have that

$$\lim_{t\to\infty} \|\xi_\eta(t) - z_i^*\| = 0.$$

To prove that there is a $\eta_1 > 0$ such that, for all global solution ξ_η in $\mathcal{A}_\eta$ with $\eta \leq \eta_1$, we have that

$$\lim_{t\to-\infty} \|\xi_\eta(t) - z_j^*\| = 0,$$

we proceed exactly in the same manner. This completes the proof that, for all suitably small η, $\{S_\eta(t) : t \geq 0\}$ satisfies (G1).

Let us prove that, for all suitably small η, $\{S_\eta(t) : t \geq 0\}$ satisfies (G2). Again we argue by contradiction. Assume that there is a sequence $z_1^*, \cdots, z_{p+1}^*$ in $\mathcal{E}$, a sequence $\eta_k \to 0$, global solutions $\xi_{k,i}$ in $\mathcal{A}_{\eta_k}$, and times $t_1^k, \cdots, t_p^k$ such that

$$\|\xi_{k,i}(t_i^k) - z_{i+1}^*\| < \frac{1}{k}, \quad 1 \leq i \leq p, \quad z_1^* = z_{p+1}^*.$$

Proceeding as in the proof of (G1) we construct a homoclinic structure for $\{S_0(t) : t \geq 0\}$ and arrive at a contradiction.$\square$

As a immediate consequence of this theorem we obtain the following generalization of the characterization result in [6] for autonomous perturbation of semigroups.

Theorem 2.10. *Let $\{S_\eta(t) : t \geq 0\}$ be a family of nonlinear semigroups with a finite number of stationary solutions $\mathcal{E}_\eta = \{y_{1,\eta}^*, \cdots y_{n,\eta}^*\}$ which have global attractor $\mathcal{A}_\eta$. Assume that $\{S_0(t) : t \geq 0\}$ is a gradient-like semigroup and that $\|S_\eta(t)u - S_0(t)u\|_{\mathcal{Z}} \to 0$ uniformly in compact subsets of $[0, \infty)$ and uniformly for u in compact subsets of $\mathcal{Z}$. Then, there exists $\eta_0 > 0$ such that*

$$\mathcal{A}_\eta = \cup_{i=1}^n W^u(y_{i,\eta}^*), \ \forall \eta \in [0, \eta_0].$$

Remark 2.11. *We observe that, up to this point, we have not explicitly used the hyperbolicity of the equilibria y_i^*, $1 \leq i \leq n$. Hence the results may hold in some cases for which hyperbolicity fails.*

Note also that we may replace the finite set of equilibria by a finite set of isolated global solutions, changing accordingly the definition of gradient-like semigroups. Since that naturally arises when dealing with evolution processes we leave the consideration of such generalizations to Section 3.

Remark 2.12. *Note that Theorem 2.2 proves that a gradient semigroup is also a gradient-like semigroup. Moreover, a perturbation of a gradient semigroup, in general, is not a gradient semigroup. Theorem 2.9 proves that a gradient-like semigroup $\{S(t) : t \geq 0\}$ is stable under perturbation. Since gradient semigroups are also gradient-like semigroups we have that a perturbation of a gradient semigroup is a gradient-like semigroup. Section 5 in [6] shows examples of regular perturbations of gradient semigroups for which the perturbed semigroups have gradient-like-attractors but are not gradient.*

On the other hand, observe that a gradient-like attractor is a global attractor with the structure shown in (2.1). But, for instance, an attractor can exhibit this structure and for $u_0 \in \mathcal{Z}$ its omega limit is not just one of the stationary points y_j^*, so that (G1) does not hold.

If we take the example shown in [13], page 2, and change the arrow directions we have an example of a semigroup with a gradient-like attractor. The perturbation of this semigroup will have an attractor which is not gradient-like (contains a periodic orbit) but behaves upper and lower semicontinuously. Note that both (G1) and (G2) fail in this case. This shows that the class of systems with a gradient-like attractor is larger that the one of gradient-like systems.

As a consequence, our concept of gradient-like system is actually an intermediate one between the ones of a gradient system and a system possessing a gradient-like attractor.

In Section 3 we show that the concept of gradient-like nonlinear semigroups can be extended in a natural way to nonlinear evolution processes.

Next, we introduce the linearization of a semigroup around an equilibrium point. The aim is to show lower semicontinuity of attractors for gradient-like semigroups for which all equilibria are hyperbolic under perturbation (which does not follow yet from the previous results). We also pursue a property that the attractor obtained is an exponential attractor (see [2]) with the aim to obtain also the same result in Section 3 under non-autonomous perturbation.

Assume that $f_\eta : \mathcal{Z} \to \mathcal{Z}$ is a continuously differentiable, globally Lipschitz and bounded function. Assume also that, for each $r > 0$,

$$\sup_{\|z\| \leq r} \|f_\eta(z) - f_0(z)\|_{\mathcal{Z}} + \|f'_\eta(z) - f'_0(z)\|_{\mathcal{L}(\mathcal{Z})} \overset{\eta\to 0}{\longrightarrow} 0. \tag{2.3}$$

Let $t \mapsto S_\eta(t-\tau)y_0$ be the solution of

$$S_\eta(t-\tau)y_0 = e^{\mathfrak{B}(t-\tau)}y_0 + \int_\tau^t e^{\mathfrak{B}(t-s)} f_\eta(S_\eta(s-\tau)y_0)\, ds. \tag{2.4}$$

A solution of (2.4) is an equilibrium solution if it satisfies

$$\mathfrak{B}y + f_\eta(y) = 0. \tag{2.5}$$

Suppose that y^*_η is solution of (2.5). It follows that, if $A_\eta = \mathfrak{B} + f'_\eta(y^*_\eta)$, then A_η generates a strongly continuous semigroup $\{e^{A_\eta t} : t \geqslant 0\}$ of bounded linear operators.

Definition 2.13. *An equilibrium solution y^*_η to (2.4) is said to be hyperbolic if the following items are satisfied:*

(1) *The spectrum of A_η does not intersect the imaginary axis and the set $\sigma^+_\eta = \{\lambda \in \sigma(A_\eta) : \mathrm{Re}\lambda > 0\}$ is compact.*
This allows us to choose a smooth closed simple curve γ in $\rho(A_\eta) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\}$ that is positively orientated and encloses σ^+; we can then define the projection

$$\mathcal{Q}_\eta = \mathcal{Q}_\eta(\sigma^+_\eta) = \frac{1}{2\pi \mathrm{i}} \int_\gamma (\lambda I - A_\eta)^{-1}\, d\lambda. \tag{2.6}$$

If $\mathcal{Z}^+_\eta = \mathcal{Q}_\eta(\mathcal{Z})$, $\mathcal{Z}^-_\eta = (I - \mathcal{Q}_\eta)(\mathcal{Z})$, and $A^\pm_\eta = (A_\eta)|_{\mathcal{Z}^\pm}$, then $\mathcal{Z} = \mathcal{Z}^+_\eta \oplus \mathcal{Z}^-_\eta$, A^-_η generates a strongly continuous semigroup on $\mathcal{Z}^-_\eta$ and $A^+_\eta \in L(\mathcal{Z}^+_\eta)$.

(2) *There are constants $\bar{M} \geq 1$ and $\beta > 0$ such that*

$$\begin{aligned} \|e^{A^+_\eta t}\|_{L(\mathcal{Z}^+)} &\leqslant \bar{M}e^{\beta t}, \quad t \leqslant 0, \\ \|e^{A^-_\eta t}\|_{L(\mathcal{Z}^-)} &\leqslant \bar{M}e^{-\beta t}, \quad t \geqslant 0. \end{aligned} \tag{2.7}$$

We consider an underlying semigroup $\{S_0(t) : t \geq 0\}$, and a family $\{S_\eta : t \geq 0\}$ of non-autonomous processes that converge to $\{S_0(t) : t \geq 0\}$ as $\eta \to 0$. We assume that the equilibria of $\{S_0(t) : t \geq 0\}$ become hyperbolic global solutions for $\{S_\eta(t) : t \geq 0\}$ for η sufficiently small, and that the corresponding stable and unstable manifolds change continuously.

The intersection of the unstable manifold with a neighborhood of y_η^* is termed the local unstable manifold, which we write $W^{\mathrm{u}}_{\mathrm{loc}}(y_\eta^*)$. The existence of local unstable manifolds as graphs near a hyperbolic equilibrium is well-known (see [5] for a proof).

Lemma 2.14. *Let y_η^* be stationary solution of* (2.4); *that is, a solution of* (2.5). *Assume* (2.3) *and that y_η^* is hyperbolic in the sense of Definition 2.13. If Q_η is the projection in Definition 2.13, there is a neighborhood V of y_η^* (which may be chosen independently of all suitably small η) and a function $\Sigma_\eta : R(Q_\eta) \to Ker(Q_\eta)$ such that*

$$W^u(y_\eta^*) \cap V = \{y_\eta^* + Q_\eta(u) + \Sigma_\eta(Q_\eta u) : u \in \mathcal{Z}\} \cap V,$$

$$\sup_{u \in V} \|Q_\eta(u) - Q_0 u\| + \|\Sigma_\eta(Q_\eta u) - \Sigma_0(Q_0)u\|_{\mathcal{Z}} \stackrel{\eta \to 0}{\longrightarrow} 0$$

and there exists $\gamma > 0$ such that, for any $u_0 \in V$ and as long as $S(t)u_0 \in V$,

$$\|(I - Q_\eta)(S_\eta(t)u_0) - \Sigma_\eta^u(Q_\eta(S_\eta(t)u_0))\|_{\mathcal{Z}} \leq Me^{-\gamma t}.$$

In this case, if one only assumes that $\{S_0(t) : t \geq 0\}$ has a gradient-like attractor $\mathcal{A}_0$, all its stationary solutions are hyperbolic and $\{S_\eta(t) : t \geq 0\}$ has an attractor $\mathcal{A}_\eta$, then it is shown (see for example [5]) that the attractors $\mathcal{A}_\eta$ of $\{S_\eta(t) : t \geq 0\}$ behave continuously as $\eta \to 0$; that is,

$$\mathrm{dist}_H(\mathcal{A}_\eta, \mathcal{A}_0) \to 0 \quad \text{as} \quad \eta \to 0,$$

where $\mathrm{dist}_H(X, Y) = \max[\mathrm{dist}(X, Y), \mathrm{dist}(Y, X)]$. In fact this continuity result is proved by showing that the attractors for the perturbed problem *contain* (possibly strictly) the union of the unstable manifolds of the hyperbolic equilibria, while the remaining part of the attractor for the perturbed problem (if it exists) is small.

The main result in Carvalho et al. [6] is that under the additional assumption that the unperturbed problem is *gradient* (has a Lyapunov function), then there is no 'remainder', and the attractor has the same structure as the autonomous attractor; that is

$$\mathcal{A}_\eta = \bigcup_{j=1}^{n} W^{\mathrm{u}}(y_{i,\eta}^*),$$

where the $y_{i,\eta}^*$ are the hyperbolic equilibria corresponding to the hyperbolic equilibria y_j^* in the original problem. We also show in [6] that every solution converges to one of the $y_{i,\eta}^*$ as $t \to \pm\infty$.

The above remarks together with Theorem 2.9 generalizes the results in [6] to the class of gradient-like dynamical systems.

Next we show, following [2] (see also [27]), that the global attractor of a gradient-like semigroup is also an exponential attractor. This proof will be extended in Section 3 to pullback attractors of gradient-like non-autonomous evolution processes.

Suppose $\{S(t) : t \geq 0\}$ is Lipschitz continuous. In particular, there exist $c = c(B_0)$ and $L > 0$ such that

$$||S(t)u - S(t)v|| \leq ce^{Lt}||u - v||, \text{ for all } u, v \in B_0. \tag{2.8}$$

Theorem 2.15. *Suppose $\{S(t) : t \geq 0\}$ is a gradient-like semigroup,* (2.8) *holds and $\mathcal{A}$ its associated attractor. Then*

$$\mathcal{A} = \bigcup_{j=1}^{n} W^{u}(y_j^*) \tag{2.9}$$

and there exists $\gamma > 0$ such that, for all $B \subset \mathcal{Z}$ bounded

$$\operatorname{dist}(S(t)u_0, \mathcal{A}) \leq c(B)e^{-\gamma t}, \text{ for all } u_0 \in B. \tag{2.10}$$

Proof: Clearly (2.9) holds.

To prove (2.10) we first choose $\delta < \frac{1}{2}\min\{\|y_i^* - y_j^*\| : 1 \leq i \leq j,\ i \neq j\}$ such that $B_\delta(y_i^*) \subset V_i$ and V_i is the neighborhood given in Lemma 2.14 for y_i^*. From Lemma 2.6, for all suitably small δ, there exists $\delta' = \delta'(\delta) < \delta$ such that, if $u_0 \in B_{\delta'}(y_i^*)$ and for some $t_1 > 0$

$$S(t_1)u_0 \notin B_\delta(y_i^*),$$

then

$$S(t)u_0 \notin B_\delta(y_i^*), \text{ for all } t \geq t_1.$$

Now, let B be a bounded subset of $\mathcal{Z}$ and B_0 be a closed ball centered at $z = 0$ that contains B and $\cup_{z\in\mathcal{A}} B_\delta(z)$. From Lemma 2.5, there exists $T = T(\delta', B_0)$ such that, for all $u_0 \in B_0$

$$S(t)u_0 \in \mathcal{O}_{\delta'} = \bigcup_{i=1}^{n} B_{\delta'}(y_i^*) \text{ for some } t \leq T.$$

Thus, given $u_0 \in B_0$, there are sequences $\{t_-^i\}_{i=0}^M$ and $\{t_+^i\}_{i=0}^M$, $M \leq n$ and $\{y_i^*\}_{i=0}^M$ such that

$$t_-^0 = \tau, \qquad t_+^i - t_-^{i-1} \leq T, \qquad t_-^M = +\infty$$

for which

$$S(t)u_0 \in \mathcal{O}_\delta(y_i^*),$$

for all $t \in [t_+^i, t_-^i]$ and $i \in \{1, \ldots, M\}$. Then, by Lemma 2.14

$$\operatorname{dist}(S(t)u_0, \mathcal{A}) \leq c_0(B_0)e^{-\gamma t}, \text{ for all } t \in [t_+^i, t_-^i].$$

On the other hand, for $t \in [t_-^{i-1}, t_+^i]$, $t = s + t_-^{i-1}$, for some $s \leq T$, and using (2.8) we have that

$$\begin{aligned}\mathrm{dist}(S(t)u_0, \mathcal{A}) &= \mathrm{dist}(S(s+t_-^{i-1})u_0, \mathcal{A})\\ &= \mathrm{dist}(S(s)S(t_-^{i-1})u_0, S(s)\mathcal{A})\\ &\leq c_1(B_0)e^{kT}\mathrm{dist}(S(t_-^{i-1})u_0, \mathcal{A})\\ &\leq c_1(B_0)e^{kT}c_0(B_0)e^{-\gamma t_-^{i-1}} = c(T, B_0)e^{-\gamma t}. \square\end{aligned}$$

3. Non-autonomous gradient-like dynamical systems

The definition of a gradient-like semigroup given in Section 2 can be extended to an evolution process. In this section we prove that all properties observed for gradient-like semigroups can be extended also to gradient-like evolution processes.

Note that, if $T(t,\tau) = S(t-\tau)$, $t \geq \tau$, with $\{S(t) : t \geq 0\}$ being a nonlinear semigroup, Definition 1.3 and Definition 1.4 generalizes Definitions 2.3 and 2.4, by also including the possibility of ξ_i^* are global isolated solutions.

Definition 3.1. *Let $\{S(t) : t \geq \tau \in \mathbb{R}\}$ be a nonlinear semigroup with a finite number of isolated global solutions $\mathcal{S} = \{\xi_1^*, \cdots \xi_n^*\}$ and assume that it has a global attractor $\mathcal{A}$. A* homoclinic structure *in $\mathcal{A}$ is a set $\{\xi_1^*, \cdots, \xi_k^*\} \subset \mathcal{S}$ and a set of global solutions $\{\xi_i : \mathbb{R} \to \mathcal{Z},\ 1 \leq i \leq k\}$ in $\mathcal{A}$ such that, making $\xi_{k+1}^* := \xi_1^*$,*

$$\lim_{t\to-\infty} \xi_i(t) = \xi_i^*(t), \quad \lim_{t\to+\infty} \xi_i(t) = \xi_{i+1}^*(t), \quad 1 \leq i \leq k.$$

The proof of the following lemma is similar to the proof of Lemma 2.8.

Lemma 3.2. *Let $\{S(t) : t \geq \tau \in \mathbb{R}\}$ be a gradient-like nonlinear semigroup in the sense of Definition 1.4. Then the attractor $\mathcal{A}$ for $\{S(t) : t \geq 0\}$ does not contain any homoclinic structures.*

As in the case of semigroups we prove that a non-autonomous perturbation of a gradient-like nonlinear semigroup is a gradient-like nonlinear evolution process.

Theorem 3.3. *Let $\mathcal{Z}$ be a Banach space, $\eta \in [0,1]$ and $\{T_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ be a nonlinear evolution process in $\mathcal{Z}$. Assume that there exists a uniform absorbing compact set B_0 (independent of η). Let $\{A_\eta(t) : t \in \mathbb{R}\}$ be the pullback attractor for $\{T_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$. Assume that $\{T_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$, $\eta \in [0,1]$, has a finite number of isolated global solutions $\mathcal{S} = \{\xi_{1,\eta}^*, \cdots, \xi_{n,\eta}^*\}$ which behave continuously as $\eta \to 0$ ($\sup_{1\leq i\leq n}\sup_{t\in\mathbb{R}} \|\xi_{i,\eta}(t) - \xi_{i,0}(t)\|_{\mathcal{Z}} \xrightarrow{\eta\to 0} 0$). Assume also that $\|T_\eta(t+\tau,\tau)u - T_0(t+\tau,\tau)u\|_{\mathcal{Z}} \xrightarrow{\eta\to 0} 0$ uniformly for $\tau \in \mathbb{R}$, t in compact subsets of $[0,\infty)$ and for u in compact subsets of $\mathcal{Z}$.*

If $\{T_0(t) : t \geq 0\}$ is a gradient-like nonlinear semigroup, $T_0(t,\tau) = T_0(t-\tau)$, $t \geq \tau$; that is, (H1)-(H2) are satisfied for $\{T_0(t,\tau) : t \geq \tau \in \mathbb{R}\}$, then there exists $\eta_0 > 0$ such that, for all $\eta \leq \eta_0$, $\{T_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ is a gradient-like nonlinear process.

As a immediate consequence of this theorem we obtain the following generalization of the characterization result which generalizes the main one in [6] for perturbations of evolution processes.

Corollary 3.4. *Under the assumptions of Theorem 3.3, there exists $\eta_0 > 0$ such that*

$$A_\eta(t) = \cup_{i=1}^n W^u(\xi_{i,\eta}^*)(t), \ \ t \in \mathbb{R} \text{ and } \forall \eta \in [0, \eta_0].$$

We will need the following important lemma (Lemma 3.1 in [6]):

Lemma 3.5. *Let η_k be a sequence of positive numbers such that $\eta_k \to 0$ as $k \to \infty$. Assume that there is a sequence $\eta_k \stackrel{k\to\infty}{\longrightarrow} 0$, and global solutions $\xi_{\eta_k} : \mathbb{R} \to \mathcal{Z}$ in $\{A_{\eta_k}(t) : t \in \mathbb{R}\}$, $k \in \mathbb{Z}^+$. Then, for any sequence $\{t_k\}$ in $\mathbb{R}$, there is a subsequence which we again denote by ξ_{η_k} and a global solution $\xi : \mathbb{R} \to \mathcal{Z}$ in the attractor $\mathcal{A}$ for $\{T_0(t) : t \geq 0\}$ such that*

$$\lim_{k\to\infty} \xi_{\eta_k}(t + t_k) \to \xi(t) \tag{3.1}$$

uniformly for t in compact subsets of $\mathbb{R}$.

Proof of Theorem 3.3. Note $\xi_{i,\eta}^* \to \xi_{i,0}^* =: \xi_i^*$, uniformly in $\mathbb{R}$, and that there is a $\delta > 0$ such that, for suitably small η, if a solution ξ_η satisfies $\|\xi_\eta(t) - \xi_i^*(t)\|_{\mathcal{Z}} \leq \delta$ for all $t \geq t_0$ and for some $t_0 > 0$, then $\xi_\eta(t) \to \xi_{i,\eta}^*(t)$ as $t \to \infty$.

We argue by contradiction to prove that for all suitably small η, $\{T_\eta(t,\tau) : t \geq 0\}$ satisfies (H1). Assume that there exists a sequence $\eta_k \stackrel{k\to\infty}{\longrightarrow} 0$ and corresponding global solutions $\xi_k : \mathbb{R} \to \mathcal{Z}$ in $\{A_{\eta_k}(t) : t \in \mathbb{R}\}$ such that

$$\liminf_{t\to\infty} \operatorname{dist}(\xi_k(t), \xi_i^*(t)) > \delta, \quad 1 \leq i \leq n. \tag{3.2}$$

Since $\xi_k(t) \to \xi_0(t)$ uniformly in compact subsets of $\mathbb{R}$ and since $\|\xi_0(t) - \xi_i^*(t)\|_{\mathcal{Z}} \to 0$ as $t \to \infty$, for some $1 \leq i \leq n$, we have that, given $\ell > 0$ there is a $t_\ell > 0$ and $k_\ell \in \mathbb{N}$ such that $\|\xi_k(t_\ell) - \xi_i^*(t_\ell)\|_{\mathcal{Z}} < \frac{1}{\ell}$, for each $k \geq k_\ell$. From (3.2), there exists $t'_\ell > t_\ell$ such that $\|\xi_{k_\ell}(t) - \xi_i^*(t)\|_{\mathcal{Z}} < \delta$ for all $t \in [t_\ell, t'_\ell)$ and $\|\xi_{k_\ell}(t'_\ell) - \xi_i^*(t'_\ell)\|_{\mathcal{Z}} = \delta$. Taking subsequences, if necessary, let $\xi_1(t) = \lim_{\ell\to\infty} \xi_{k_\ell}(t+t_\ell)$. Then, since $t'_\ell - t_\ell \to \infty$ as $\ell \to \infty$, $\|\xi_1(t) - \xi_i^*(t)\|_{\mathcal{Z}} \leq \delta$ for all $t \leq 0$ and consequently $\xi_1(t) \to \xi_i^*(t)$ as $t \to -\infty$. And $\xi_1(t) \to \xi_j^*(t)$ as $t \to \infty$ with $i \neq j$. From the fact that $\xi_{k(\ell)}(t + t_\ell) \to \xi_1(t)$ uniformly in compact subsets of $\mathbb{R}$ we have that, for each $m \in \mathbb{N}$, there is a time $t_m > 0$ and $k_m \in \mathbb{N}$ such that $\|\xi_k(t_m) - \xi_j^*(t_m)\|_{\mathcal{Z}} < \frac{1}{m}$ for all $k \geq k_m$. Again, from (3.2), there exists $t'_m > t_m$ such that $\|\xi_{k_m}(t) - \xi_j^*(t)\|_{\mathcal{Z}} < \delta$ for all $t \in [t_m, t'_m)$ and $\|\xi_{k_m}(t'_m) - \xi_j^*(t'_m)\|_{\mathcal{Z}} = \delta$. Proceeding exactly as before we obtain a global solution $\xi_2 : \mathbb{R} \to \mathcal{Z}$ such that $\xi_2(t) \to \xi_j^*(t)$ as $t \to -\infty$ and $\xi_2(t) \to \xi_r^*(t)$ as $t \to \infty$ with $r \notin \{i, j\}$. In a finite number of steps we arrive at a contradiction. This proves that there is a $\eta_0 > 0$ such that, for all global solution ξ_η in $\{A_\eta(t) : t \in \mathbb{R}\}$ with $\eta \leq \eta_0$, we have that

$$\lim_{t\to\infty} \|\xi_\eta(t) - \xi_i^*(t)\|_{\mathcal{Z}} = 0.$$

To prove that there is a $\eta_1 > 0$ such that, for all global solution ξ_η in $\{A_\eta(t) : t \in \mathbb{R}\}$ with $\eta \leq \eta_1$, we have that

$$\lim_{t\to-\infty} ||\xi_\eta(t) - \xi_j^*(t)|| = 0,$$

we proceed exactly in the same manner. This completes the proof that, for all suitably small η, $\{T_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ satisfies (H1).

Let us prove that, for all suitably small η, $\{T_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ satisfies (H2). Again we argue by contradiction. Assume that there is a sequence $\xi_1^*, \cdots, \xi_{p+1}^*$ in $\mathcal{S}$, a sequence $\eta_k \to 0$, global solutions ξ_k in $\{A_{\eta_k} : t \in \mathbb{R}\}$, and times $t_1^k, \tau_1^k, \cdots, t_p^k, \tau_p^k$ with $t_i^k > \tau_i^k$, $1 \leq i \leq p$, such that

$$||\xi_k(\tau_i^k) - \xi_i^*(\tau_i^k)|| < \frac{1}{k}, \quad ||\xi_k(t_i^k) - \xi_{i+1}^*(t_i^k)|| < \frac{1}{k}, \quad 1 \leq i \leq p \text{ and } \xi_1^* = \xi_{p+1}^*$$

Proceeding as in the proof of (H1) we construct a homoclinic structure for $\{T_0(t,\tau) : t \geq \tau \in \mathbb{R}\}$ and arrive at a contradiction.$\square$

Next, we introduce the linearization of a nonlinear process around a global solution. The aim is to show lower semicontinuity of attractors for gradient-like processes for which all global isolated solutions are hyperbolic and which are perturbation of a nonlinear semigroup (which does not follow yet from the previous result). We also pursue a property that the pullback attractor obtained is an exponential pullback attractor and exponential forward attractor.

Definition 3.6. *We say that a linear process $\{U(t,\tau) : t \geq \tau \in \mathbb{R}\}$ has an exponential dichotomy with exponent ω and constant M if there exists a family of projections $\{Q(t) : t \in \mathbb{R}\} \subset L(\mathcal{Z})$ such that*

(1) *$Q(t)U(t,s) = U(t,s)Q(s)$, for all $t \geq s$.*
(2) *The restriction $U(t,s)_{|R(Q(s))}$, $t \geq s$ is an isomorphism from $R(Q(s))$ into $R(Q(t))$; we denote its inverse by $U(s,t) : R(Q(t)) \to R(Q(s))$.*
(3) *There are constants $\omega > 0$ and $M \geq 1$ such that*

$$\begin{aligned} &||U(t,s)(I - Q(s))||_{L(\mathcal{Z})} \leq Me^{-\omega(t-s)} \quad t \geq s \\ &||U(t,s)Q(s)||_{L(\mathcal{Z})} \leq Me^{\omega(t-s)}, \quad t \leq s. \end{aligned} \tag{3.3}$$

Assume that $f_\eta : \mathbb{R} \times \mathcal{Z} \to \mathcal{Z}$ is continuously differentiable, globally Lipschitz in the second variable and bounded. Assume also that, for each $r > 0$,

$$\sup_{||z||\leq r} \sup_{t\in\mathbb{R}} \{||f_\eta(t,z) - f_0(z)||_{\mathcal{Z}} + ||f_\eta'(t,z) - f_0'(z)||_{\mathcal{L}(\mathcal{Z})}\} \overset{\eta\to 0}{\longrightarrow} 0, \tag{3.4}$$

where $f'(t,z) \in \mathcal{L}(\mathcal{Z})$ denotes the derivative of f_η with respect to the second variable in (t,z). Let $t \mapsto T_\eta(t,\tau)y_0$ be the solution of

$$T_\eta(t,\tau)y_0 = e^{\mathfrak{B}(t-\tau)}y_0 + \int_\tau^t e^{\mathfrak{B}(t-s)} f_\eta(s, T_\eta(s,\tau)y_0)\, ds. \tag{3.5}$$

Suppose that ξ_η^* is a global solution of (3.5). If $B_\eta(t) = f_\eta'(t, \xi_\eta^*(t))$, consider the linear evolution process $\{U_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ defined by

$$U_\eta(t,\tau)z_0 = e^{A(t-\tau)}z_0 + \int_\tau^t e^{A(t-s)}B_\eta(s)U_\eta(s,\tau)z_0\, ds. \tag{3.6}$$

Definition 3.7. *Let $\xi_\eta^* : \mathbb{R} \to \mathcal{Z}$ be a global solution of (3.5). We say that ξ_η^* is hyperbolic if the linear evolution process $\{U_\eta(t,\tau) : t \geq \tau \in \mathbb{R}\}$ defined by (3.6) has exponential dichotomy. A global solution that has an exponential dichotomy will be called a* global hyperbolic solution.

Definition 3.8. *The unstable manifold of a global hyperbolic solution ξ_η^* to (3.5) is the set*

$$W_\eta^{\mathrm{u}}(\xi_\eta^*) = \{\, (\tau,\zeta) \in \mathbb{R}\times\mathcal{Z} : \text{ there is a backwards solution } z(t,\tau,\zeta) \text{ of } (3.5)$$
$$\text{satisfying } z(\tau,\tau,\zeta) = \zeta \text{ and such that } \lim_{t\to-\infty} \|z(t,\tau,\zeta) - \xi_\eta^*(t)\|_{\mathcal{Z}} = 0\}.$$

The stable manifold of a hyperbolic solution ξ_η^* to (3.5) is the set

$$W_\eta^{\mathrm{s}}(\xi_\eta^*) = \{\, (\tau,\zeta) \in \mathbb{R}\times\mathcal{Z} : \text{ there is a forwards solution } z(t,\tau,\zeta) \text{ of } (3.5)$$
$$\text{satisfying } z(\tau,\tau,\zeta) = \zeta \text{ and such that } \lim_{t\to+\infty} \|z(t,\tau,\zeta) - \xi_\eta^*(t)\|_{\mathcal{Z}} = 0\}.$$

The intersection of the unstable (stable) manifold with a neighborhood of the curve $(\cdot,\xi(\cdot))$ in $\mathbb{R}\times\mathcal{Z}$ is called a local unstable (stable) manifold and is denoted by $W_{\eta,\mathrm{loc}}^{\mathrm{u}}$ ($W_{\eta,\mathrm{loc}}^{\mathrm{s}}$).

The proof of the following result can be adapted from [5].

Lemma 3.9. *Assume (3.4) and let ξ_0^* be hyperbolic global solution of (3.5) with $\eta = 0$. Then, there is η_0 such that (3.5) has a unique hyperbolic global solution ξ_η^* in a neighborhood of ξ_0^* and $\sup_{1\leq i\leq n}\sup_{t\in\mathbb{R}} \|\xi_{i,\eta}(t) - \xi_{i,0}(t)\|_{\mathcal{Z}} \overset{\eta\to 0}{\longrightarrow} 0$. If $\{Q_\eta(t) : t\in\mathbb{R}\}$ is the family of projections in Definition 3.6 there is a neighborhood V of $(\cdot,\xi_\eta^*(\cdot))$ (which may be chosen independently of all suitably small η) and a family of function $\Sigma_\eta(t,\cdot) : R(Q_\eta(t)) \to Ker(Q_\eta(t))$ such that*

$$W^u(\xi_\eta^*)\cap V = \{(t, \xi_\eta^*(t) + Q_\eta(t)u + \Sigma_\eta(Q_\eta(t)u) : t\in\mathbb{R},\ u\in\mathcal{Z}\}\cap V,$$

$$\sup_{(t,u)\in V} \|Q_\eta(t)u - Q_0(t)u\| + \|\Sigma_\eta(t, Q_\eta(t)u) - \Sigma_0(Q_0(t))u\|_{\mathcal{Z}} \overset{\eta\to 0}{\longrightarrow} 0$$

and there exists $\gamma > 0$ such that, for any $u_0 \in V$ and as long as $T_\eta(t)u_0 \in V$,

$$\sup_{\tau\in\mathbb{R}} \|(I - Q_\eta(t))(T_\eta(t+\tau,\tau)u_0) - \Sigma_\eta^u(t, Q_\eta(t)(T_\eta(t+\tau,\tau)u_0))\|_{\mathcal{Z}} \leq Me^{-\gamma t}$$

In this case, if one only assumes that the attractor of $\{T_0(t,\tau) : t\geq\tau\in\mathbb{R}\}$ has a gradient-like pullback attractor $\{\mathcal{A}(t) : t\in\mathbb{R}\}$, all its isolated global solutions are hyperbolic and $\{T_\eta(t,\tau) : t\geq\tau\in\mathbb{R}\}$ has an attractor $\{\mathcal{A}_\eta(t) : t\in\mathbb{R}\}$, then it is shown (see [7]) that the attractors $\{\mathcal{A}_\eta(t) : t\in\mathbb{R}\}$ of $\{T_\eta(t,\tau) : t\geq\tau\in\mathbb{R}\}$ behave continuously as $\eta\to 0$; that is,

$$\sup_{t\in\mathbb{R}} \mathrm{dist}_H(\mathcal{A}_\eta(t), \mathcal{A}_0(t)) \to 0 \quad \text{as} \quad \eta\to 0.$$

In fact this continuity result is proved by showing that the attractors for the perturbed problem *contain* (possibly strictly) the union of the unstable manifolds of the global hyperbolic solutions. The main result in [6] is that under the additional assumption that the unperturbed problem is *gradient*, then there is no 'remainder', and the attractor has the same structure as the autonomous attractor. We also show in [6] that every solution converges to one of the $y^*_{i,\eta}$ as $t \to \pm\infty$.

Next we generalize the exponential rate of attraction in Theorem 2.15 to gradient-like nonlinear evolution process proving Theorem 1.7 (cf. [27]).

Proof of Theorem 1.7: To prove the exponential decay we proceed in three steps:

Step 1. Let η_0 be such that $\{T_\eta(t,\tau) : t \geq \tau\}$ satisfies (H1) and (H2) for all $\eta \leq \eta_0$. It holds that, for all δ small enough, there exists $\delta' = \delta'(\delta) < \delta$ and $\eta_1 > 0$ such that, for each $\eta \leq \eta_0$, if $u_0 \in B_{\delta'}(\xi^*_i(\tau))$, for some $\tau \in \mathbb{R}$ and for some $t_1 > 0$

$$T_\eta(t_1 + \tau, \tau)u_0 \notin B_\delta(\xi^*_{i,\eta}(t_1 + \tau)),$$

then

$$T_\eta(t + \tau, \tau)u_0 \notin B_{\delta'}(\xi^*_{i,\eta}(t + \tau)), \text{ for all } t \geq t_1.$$

We follow the argument in Lemma 2.6. Indeed, if not, there exist $\delta_0 > 0$ and sequences $\eta_k \to 0$, $\tau_k \in \mathbb{R}$, $t_k < t'_k$, and u_k such that, for some of the hyperbolic global solutions ξ^*_{i,η_k},

- $u_k \in B_{1/k}(\xi^*_{i,\eta_k}(\tau_k))$
- $T_{\eta_k}(t_k + \tau_k, \tau_k)u_k \notin B_{\delta_0}(\xi^*_{i,\eta_k}(t_k + \tau_k))$
- $T_{\eta_k}(t'_k + \tau_k, \tau_k)u_k \in B_{1/k}(\xi^*_{i,\eta_k}(t'_k + \tau_k))$.

Taking subsequences, it follows from Lemma 3.5 that

$$\xi_k(t) := T_{\eta_k}(t + t_k + \tau_k, t_k + \tau_k)(t_k + \tau_k, \tau_k)u_k \to \xi_0(t)$$

uniformly in compact subsets of $\mathbb{R}$ where $\xi_0 : \mathbb{R} \to \mathcal{Z}$ is a solution in $\mathcal{A}$. Consequently, $\xi_0(t) \to y^*_i$ as $t \to -\infty$ and $\xi_0(t) \to y^*_j$ as $t \to +\infty$ with $j \neq i$. Proceeding as in the proof of Theorem 3.3 we arrive at a contradiction.

That is clearly in contradiction with (H2).

Step 2. For every bounded set $B \subset \mathcal{Z}$ and every $\delta > 0$ there exists $\eta_2 > 0$ and $T = T(\delta, B)$ such that, for all $u_0 \in B$ and $\tau \in \mathbb{R}$

$$T_\eta(t + \tau, \tau)u_0 \in O_\delta := \bigcup_{i=1}^{n} B_\delta(y^*_\eta) \text{ for some } t \leq T.$$

Indeed, if not there exist $\varepsilon > 0$ $u_k \in B$, $\eta_k \to 0$, $\tau_k \in \mathbb{R}$ and $t_k \to +\infty$ such that

$$\operatorname{dist}(T_{\eta_k}(t_k + \tau_k, \tau_k)u_k, O_\delta) \geq \varepsilon,$$

for all $t \in [0, t_k]$. Now, if we define

$$\xi_k(t) := T_{\eta_k}(t + \tau + \frac{t_k}{2}, \tau + \frac{t_k}{2})T_{\eta_k}(\tau + \frac{t_k}{2}, \tau)u_k,$$

then

$$\operatorname{dist}(\xi_k(t), O_\delta) \geq \varepsilon,$$

for all $t \in [-t_k/2, t_k/2]$.

Again, we have that there exists $\zeta(t)$ global solution in $\mathcal{A}$ such that

$$\lim_{k\to+\infty} \xi_k(t) = \zeta(t),$$

for all $t \in \mathbb{R}$, uniformly on bounded subsets of $\mathbb{R}$. Thus,

$$\mathrm{dist}(\zeta(t), O_\delta) \geq \varepsilon,$$

for all $t \in \mathbb{R}$, which contradicts (H1).

Step 3. Now, by the two previous steps, there is $\bar{\eta} > 0$ such that, given $u_0 \in B_0$ and $\eta \leq \bar{\eta}$, one can find sequences $\{t_-^i\}_{i=0}^M$ and $\{t_+^i\}_{i=0}^M$, $M \leq n$ and $\{y_i^*\}_{i=0}^M$ such that

$$t_-^0 = \tau, \qquad t_+^i - t_-^{i-1} \leq T, \qquad t_-^M = +\infty$$

for which

$$T_{\eta_k}(t+\tau,\tau)u_0 \in \mathcal{O}_\delta(y_i^*),$$

for all $t \in [t_+^i, t_-^i]$ and $i \in \{1, \dots, M\}$. Then, by Lemma 3.9

$$\mathrm{dist}(T_{\eta_k}(t+\tau,\tau)u_0, A_{\eta_k}(t+\tau)) \leq c_0(B_0)e^{-\gamma t}, \text{ for all } t \in [t_+^i, t_-^i].$$

On the other hand, for $t \in [t_-^{i-1}, t_+^i]$, $t = s + t_-^{i-1}$, for some $s \leq T$, and using (1.4) we have that

$$\begin{aligned}
&\mathrm{dist}(T_{\eta_k}(t+\tau,\tau)u_0, A_{\eta_k}(t+\tau)) = \mathrm{dist}(T_{\eta_k}(s+t_-^{i-1}+\tau,\tau)u_0, A_{\eta_k}(t+\tau))\\
&= \mathrm{dist}(T_{\eta_k}(s+t_-^{i-1}+\tau, t_-^{i-1}+\tau)T_{\eta_k}(t_-^{i-1}+\tau,\tau)u_0, T_{\eta_k}(s+t_-^{i-1}+\tau, t_-^{i-1}+\tau)A_{\eta_k}(t_-^{i-1}+\tau))\\
&\qquad \leq c_1(B_0)e^{kT}\mathrm{dist}(T_{\eta_k}(t_-^{i-1}+\tau,\tau)u_0, A_{\eta_k}(t_-^{i-1}+\tau))\\
&\qquad \leq c_1(B_0)e^{kT}c_0(B_0)e^{-\gamma t_-^{i-1}} = c(T,B_0)e^{-\gamma t}. \square
\end{aligned}$$

4. Asymptotically autonomous dynamical systems

An asymptotically autonomous (backwards and/or forwards) dynamical system is one of the important examples which can be interpreted as a non-autonomous regular perturbation of an autonomous system (see [3, 21, 15, 20]). In this section we apply our result to these cases, generalizing the results in Section 4 of [6]. In particular, if we suppose that the long-time behaviour in the past and future are unrelated (i.e., there is no uniform regular perturbation of the limit system) we are still able to give a characterization of the associated pullback and forwards attractors. However, the uniform (with respect to the initial time) exponential convergence to these sets will be lost, so that only an exponential pullback attractor or an exponential forwards attractor will exist.

Consider a Banach space $\mathcal{Z}$ and the semilinear problem

$$\begin{aligned} \dot{y} &= \mathfrak{B}y + f(t,y)\\ y(\tau) &= y_0, \end{aligned} \tag{4.1}$$

where $\mathfrak{B} : D(\mathfrak{B}) \subset \mathcal{Z} \to \mathcal{Z}$ is the generator of a C^0-semigroup of bounded linear operators and $f(t,\cdot)$ is a differentiable function that is Lipschitz continuous in bounded subsets of $\mathcal{Z}$ with Lipschitz constant independent of t. If we denote by $t \mapsto T(t,\tau)y_0$ the solution for (4.1), then $\{T(t,\tau) : t \geq \tau \in \mathbb{R}\}$ defines a nonlinear process. We will assume that the problem (4.1) has a pullback attractor $\{\mathcal{A}(t) : t \in \mathbb{R}\}$.

Consider also the semilinear autonomous problem

$$\begin{aligned} \dot{y} &= \mathfrak{B}y + f_0(y) \\ y(\tau) &= y_0 \in \mathcal{Z}. \end{aligned} \tag{4.2}$$

Assume that (4.2) gives rise to a nonlinear gradient-like semigroup $\{S(t) : t \geq 0\}$ which has a global attractor $\mathcal{A}$.

4.1. **Asymptotically Autonomous Problems at $-\infty$.** Assume that

$$\lim_{t\to-\infty} \sup_{z\in B(0,r)} \{\|f(t,z) - f_0(z)\|_{\mathcal{Z}} + \|f_y(t,z) - f_0'(z)\|_{L(\mathcal{Z})}\} = 0, \qquad \text{for each } r > 0, \tag{4.3}$$

and that (4.2) has an autonomous attractor $\mathcal{A}_0$.

Theorem 4.1. *Let $f : \mathbb{R} \times \mathcal{Z} \to \mathcal{Z}$ be a differentiable function that satisfies* (4.3). *Consider the initial value problem* (4.1). *Assume that* (4.2) *is gradient-like and $\mathcal{A}_0$ is its global attractor,*

(1) *Then the attractor $\{\mathcal{A}(\tau) : \tau \in \mathbb{R}\}$ for* (4.1) *is given by*

$$\mathcal{A}(\tau) = \cup_{i=1}^{n} W^{\mathrm{u}}(\xi_i^*)(\tau).$$

(2) *For each globally defined bounded solution $\xi(\cdot)$ of* (4.1) *there is i_- with $1 \leq i_- \leq n^-$ such that*

$$\lim_{t\to-\infty} \|\xi(t) - y_{i_-}^*\|_{\mathcal{Z}} = 0. \tag{4.4}$$

(3) *For every $B \subset \mathcal{Z}$ bounded*

$$\operatorname{dist}(T(t,t-\tau)B, \mathcal{A}(t)) \leq c(B)e^{-\gamma\tau}. \tag{4.5}$$

Proof: The proof of (1) is a consequence of Corollary 3.4 if we analyze (4.1) by considering the small non-autonomous perturbations of (4.2) obtained by replacing $f(t,y)$ by

$$f_\nu(t,y) = \begin{cases} f(t,y), & \text{if } t \leq -\nu \\ f(\nu,y), & \text{if } t > -\nu. \end{cases}$$

Indeed, for suitably large ν, there exists a pullback attractor $\{\mathcal{A}_\nu(s) : s \in \mathbb{R}\}$ for

$$\begin{aligned} \dot{y} &= \mathfrak{B}y + f_\nu(t,y) \\ y(\tau) &= y_0 \end{aligned} \tag{4.6}$$

given by $\mathcal{A}_\nu(s) = \cup_{i=1}^{n} W_\nu^{\mathrm{u}}(\xi_{i,\nu}^*)(s)$. To obtain the pullback attractor for (4.1) we first note that (4.6) and (4.1) coincide for $t \leq \tau \leq -\nu$. Hence $\mathcal{A}(t) = \mathcal{A}_\nu(t)$ for $t \leq -\nu$. To recover $\mathcal{A}(t)$ for $t \geq -\nu$ we only have to define $\mathcal{A}(t) = T(t,\tau)\mathcal{A}(\nu)$, for all $\tau \leq -\nu \leq t$.

Now, (2) is also essentially proved since, by (H1), every global solution approaches one of the equilibria $y_{i_-}^*$ as $t \to -\infty$, so that, in particular, (2) holds.

Finally, Theorem 1.7 gives (3).□

4.2. **Asymptotically Autonomous Problems at $+\infty$.**

Assume that

$$\lim_{t\to+\infty} \sup_{z\in B(0,r)} \{\|f(t,z)-f_0(z)\|_{\mathcal{Z}} + \|f_y(t,z)-f_0'(z)\|_{L(\mathcal{Z})}\} = 0, \qquad \text{for each } r>0, \tag{4.7}$$

and that (4.2) has an autonomous attractor $\mathcal{A}_0$. We note that the nonlinearity f_0 in this subsection may be different from that in the previous subsection and consequently the attractor $\mathcal{A}_0$ in this subsection may be different from that in the previous one. We assume in addition that is gradient-like; it follows from Theorem 1.7 that $\mathcal{A}_0$ is given by (2.1).

Consider $f_k(t,z)$ the function which coincides with f in $[k,\infty)\times\mathcal{Z}$ and which is equal to $f(k,z)$ for all $t<k$ and $z\in\mathcal{Z}$. Then

$$\lim_{k\to+\infty} \sup_{t\in\mathbb{R}} \sup_{z\in B(0,r_0)} \{\|f_k(t,z)-f_0(z)\|_{\mathcal{Z}} + \|(f_k)_y(t,z)-f_0'(z)\|_{L(\mathcal{Z})}\} = 0. \tag{4.8}$$

It has been proved in [6] that the family of attractors for

$$\begin{aligned} \dot{y} &= \mathfrak{B}y + f_k(t,y) \\ y(\tau) &= y_0 \end{aligned} \tag{4.9}$$

behaves upper and lower semicontinuously as $k\to\infty$ with the limit attractor being the attractor for (4.2), i.e.

$$\sup_{t\in\mathbb{R}} \operatorname{dist}(\mathcal{A}_k(t),\mathcal{A}_0)\to 0 \qquad \text{as } k\to\infty,$$

where $\operatorname{dist}(A,B)$ is the symmetric Hausdorff distance defined in Section 2.

Let k_0 be such that for $k\geq k_0$ the pullback attractor of (4.9) coincides with the union of the unstable manifolds of all those $\{\xi^*_{i,k}\}$ with $\sup_{t\in\mathbb{R}}\|\xi^*_{i,k}(t)-y^*_i\|_{\mathcal{Z}}\to 0$ as $k\to\infty$. Define $\mathcal{A}^+(t)=\mathcal{A}_{k_0}(t)$ for $t\geq k_0$. Note that $\mathcal{A}^+(t)$ is in fact the forwards image of the global attractor of the autonomous system $\dot{y}=\mathfrak{B}y+f(k_0,y)$ under the non-autonomous process $T(t,\tau)$.

Then we have the following result:

Theorem 4.2. *Assume* (4.8). *Then, there is a $t_0\in\mathbb{R}$ and a time dependent forwards attractor $\{\mathcal{A}^+(t): t\geq t_0\}$ for* (4.1). *Moreover, if $T(t,\tau)$ is gradient-like, it holds that, for τ big enough and $B\subset\mathcal{Z}$ bounded*

$$\operatorname{dist}(T(t+\tau,\tau)u_0,\mathcal{A}^+(t+\tau))\leq c(B)e^{-\gamma t}, \quad \textit{for all } u_0\in B.$$

5. Further Comments and Open Problems

We have proved in Section 3 that a non-autonomous perturbation of a gradient-like semigroup is a gradient-like evolution process. It remains open if the perturbation of a gradient-like evolution process is again a gradient-like evolution process. Lemma 3.5 is the crucial fact to the extension of the results in Section 2 to non-autonomous perturbation of gradient-like semigroups. The fact that an equivalent lemma does not immediately holds for perturbations of gradient-like nonlinear processes is what impair us to extend the results of Section 2 to perturbations of gradient-like nonlinear processes.

It is also true that (we will pursue that in a future work) if the attractor of a nonlinear semigroup is the union of the unstable manifold of a finite number of *normally hyperbolic* global solutions, then it is an exponential attractor. If, in addition, the nonlinear semigroup is gradient-like, a non-autonomous perturbation of it will have an exponential pullback attractor given as union of unstable manifolds of hyperbolic global solutions. If the perturbation is autonomous, that result together with the lower semicontinuity of attractors will follow from the results of [4] whereas if the perturbation is non-autonomous is yet to be proved.

Note also that our notion of gradient-like evolution process given in Definition 1.4 resembles the concept of *Morse decomposition* ([11, 19, 25]).

Let that $\{S(t) : t \geq 0\}$ be a nonlinear semigroup. In this case, a more general notion of gradient-like nonlinear semigroup can be written as

Definition 5.1. *Let $\mathcal{Z}$ be a Banach space and $\{S(t) : t \geq 0\}$ be a nonlinear semigroup in $\mathcal{Z}$. Assume that there exists an absorbing compact set B_0. Let $\mathcal{A}$ be the global attractor for $\{S(t) : t \geq 0\}$. We say that $\{S(t) : t \geq 0\}$ is a gradient-like nonlinear semigroup if the following two hypotheses are satisfied:*

(G1') *There exists a finite number of isolated global solutions $\{\xi_i^* : \mathbb{R} \to \mathcal{Z} : 1 \leq i \leq n\}$ such that, any global solution $\xi : \mathbb{R} \to \mathcal{Z}$ in $\{A(t) : t \in \mathbb{R}\}$ satisfies*

$$\lim_{t\to-\infty} \operatorname{dist}(\xi(t), \Gamma_i) = 0 \text{ and } \lim_{t\to\infty} \operatorname{dist}(\xi(t), \Gamma_j) = 0,$$

for some $1 \leq i, j \leq n$, where $\Gamma_i = \{\xi_i^*(t) : t \in \mathbb{R}\}$, $1 \leq i \leq n$.

(G2') *$\mathcal{S} = \{\xi_1^*, \cdots, \xi_n^*\}$ does not contain any chain recurrent solution (see Definition 1.3).*

On the other hand, a Morse decomposition of an attractor is defined as follows

Definition 5.2. *Let $\{S(t) : t \geq 0\}$ be a nonlinear semigroup with a global attractor $\mathcal{A}$. We say that $\mathcal{A}$ has a* Morse decomposition *if the following holds:*

(M1) *There is a finite number of non-empty, compact and disjoint invariant sets $\Gamma_1, \cdots, \Gamma_n$ in $\mathcal{A}$. The sets Γ_i, $1 \leq i \leq n$, are called the Morse sets.*

(M2) *The $\omega-$limit of each $z \in \mathcal{Z}$ is contained in one of the invariant sets Γ_i, $1 \leq i \leq n$.*

(M3) *For each $z \in \mathcal{A}$, either $z \in \cup_{i=1}^n \Gamma_i$ or, for each global solution ϕ through z there are $i = i(z)$ and $j = j(z)$ with $1 \leq i < j \leq n$ such that*

$$\lim_{t\to-\infty} \operatorname{dist}(\phi(t), \Gamma_i) = 0 \text{ and } \lim_{t\to\infty} \operatorname{dist}(\phi(t), \Gamma_j) = 0.$$

It is not difficult to see that a gradient-like semigroup in the sense of Definition 5.1 has a global attractor which has a Morse decomposition in the sense of Definition 5.2.

We can also define

Definition 5.3. *Let $\{S(t) : t \geq 0\}$ be a nonlinear semigroup with a finite number of isolated global solutions $\mathcal{S} = \{\xi_1^*, \cdots \xi_n^*\}$ and assume that it has a global attractor $\mathcal{A}$. A* homoclinic structure *in $\mathcal{A}$ is a sequence $\{n_1, \cdots, n_k\} \subset \{1, \cdots, n\}$, $\{\xi_{n_1}^*, \cdots, \xi_{n_k}^*\} \subset \mathcal{S}$ and a set of global solutions $\{\xi_i : \mathbb{R} \to \mathcal{Z},\ 1 \leq i \leq k\}$ in $\mathcal{A}$ such that, making $\xi_{k+1}^* := \xi_1^*$,*

$$\lim_{t\to-\infty} \operatorname{dist}(\xi_i(t), \Gamma_i), \quad \lim_{t\to+\infty} \operatorname{dist}(\xi_i(t), \Gamma_{i+1}), \quad 1 \leq i \leq k.$$

If in Definition 5.1 we replace (G1') by

(G2") The attractor $\mathcal{A}$ does not have any homoclinic structure,

then the new definition gives the same class of gradient-like nonlinear semigroups. Note that our results could have been written in the more general context of Morse decomposition of attractors for semigroups and processes. However, we have chosen to write the results by replacing the Morse sets by global solutions, which we think makes the arguments more clear.

On the other hand, in the non-autonomous context we have given a definition of a gradient-like nonlinear process. We have proved that non-autonomous perturbations of autonomous gradient-like nonlinear semigroups are gradient-like nonlinear process. That gives a *Morse decomposition for nonlinear processes.* This is, to our knowledge, the first example for which a Morse decomposition for pullback attractor has been given. Finally, for semigroups, we have also proved that Morse decomposition is stable under perturbations. For processes, we conjecture that it is also the case.

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