Instituto de Ciências Matemáticas de São Carlos

ISSN - 0103-2577

A generalization of Alexander's torus theorem to higher dimensions

LAÉRCIO APARECIDO LUCAS
OZIRIDE MANZOLI NETO
OSAMU SAEKI

Nº 28

NOTAS DO ICMSC
Série Matemática

São Carlos
Abr./1995

A generalization of Alexander's torus theorem to higher dimensions

Laércio Aparecido Lucas
Academia da Força Aérea,
Rua D-3 Casa 9 - AFA,
13630-000, Pirassununga, SP, Brazil

Oziride Manzoli Neto
Departamento de Matemática,
ICMSC-USP,
Caixa postal 668,
13560-970, São Carlos, SP, Brazil

Osamu Saeki*
Department of Mathematics,
Faculty of Science,
Hiroshima University,
Higashi-Hiroshima 739, Japan

Abstract. Let $f : S^p \times S^q \to S^{p+q+1}$ be a smooth embedding. We show that the closure of one of the components of $S^{p+q+1} - f(S^p \times S^q)$ is diffeomorphic to $S^p \times D^{q+1}$ if $p \geq q \geq 1$ and $p + q \neq 3$, and is homeomorphic to $S^2 \times D^2$ if $p = 2$ and $q = 1$. This is a generalization of Alexander's torus theorem [**A**] for the case $p = q = 1$. This result for the case $p + q \neq 3$ has been obtained by Kosinski [**Ko**], Wall [**Wa2**] and Goldstein [**G**]. Our proof is different from theirs and gives a new approach to the problem.

1991 *Mathematics Subject Classification.* Primary 57R40; Secondary 57R80.
*The third author has been partially supported by CNPq, Brazil.

1. Introduction

Alexander [**A**] has shown that a piecewise linearly embedded torus in the three sphere S^3 bounds a solid torus in S^3, which is known as Alexander's torus theorem (see also [**R**]). This theorem holds also for smooth embeddings. After Alexander, Kosinski [**Ko**] considered a smooth embedding of a product of spheres $S^p \times S^q$ into S^{p+q+1} and showed that the image bounds an embedded $D^{p+1} \times S^q$ in S^{p+q+1} under the dimension assumptions that $p > q > 1$ and p is odd when $q = 2$. He used the Smale theory, or the h-cobordism theorem for higher dimensions, extensively to prove the result and he needed the dimension assumptions for technical reasons. After that, Wall [**Wa2**] considered the same problem and he obtained the same conclusion under the assumption that $p+q \neq 3$, using similar techniques (note that he assumes a conjecture concerning h-cobordisms of $S^1 \times S^3$ to itself for the case $p + q = 4$; however, this conjecture has been solved by Shaneson [**Sh**] and the results of Wall hold also in this case). Goldstein [**G**] considered the same problem in the piecewise linear category and solved it under the assumption that $p \geq q > 1$ and $p + q > 4$, using a different method.

Our purpose of this paper is to prove rigorously the same result in the differentiable category (topological category if $p+q = 3$) without any dimension assumptions. More precisely, our main result is as follows.

THEOREM 1.1. *Let $f : S^p \times S^q \longrightarrow S^{p+q+1}$ be a smooth embedding. We suppose that $p \geq q \geq 1$ and that $p+q \neq 3$. Then the closure of one of the components of $S^{p+q+1} - f(S^p \times S^q)$ is diffeomorphic to $S^p \times D^{q+1}$. Furthermore, when $p+q = 3$ (i.e., when $p = 2$ and $q = 1$), it is homeomorphic to $S^2 \times D^2$.*

Our technique is basically the same as that of Kosinski; namely we prove the theorem through an extensive use of the h-cobordism theorem. The main difficulties lie in the cases where $q = 1$ or $p = q$, which Kosinski did not consider. In the former case, it is not trivial that one of the components of the complement $S^{p+q+1} - f(S^p \times S^q)$ is simply connected, which we show by using van Kampen's theorem. Furthermore, the technique of Kosinski does not work for this dimension. In the latter case, a technical difficulty occurs when showing a certain map is a homotopy equivalence. We avoid this difficulty, using Whitney's theorem on cancelling self-intersections of immersed manifolds.

In the course of the proof, we have obtained some results concerning the diffeomorphism type of a compact simply connected manifold with spherical homology and with boundary $S^p \times S^q$ (see Propositions 2.6, 3.1 and 4.3 and Remark 4.8), which seem interesting in themselves.

As corollaries to Theorem 1.1 we also obtain the following theorems, which have been obtained by Kosinski [**Ko**] under his dimension assumptions.

THEOREM 1.2. *Let $f_i : S^p \times S^q \to S^{p+q+1}$ $(i = 1, 2)$ be smooth embeddings, where $p, q \geq 2$. Then there exists a diffeomorphism $h : S^{p+q+1} \to S^{p+q+1}$ such that $h(f_1(S^p \times S^q)) = f_2(S^p \times S^q)$.*

THEOREM 1.3. *Let $f_i : S^p \to S^n$ $(i = 1, 2)$ be two smooth embeddings, both with trivial normal bundles, where $n - p \geq 3$ and $p \geq 1$. Then $S^n - f_1(S^p)$ and $S^n - f_2(S^p)$ are diffeomorphic.*

Furthermore, we have the following.

THEOREM 1.4. *Let $f : S^p \times S^1 \to S^{p+2}$ be a smooth embedding $(p \geq 1)$. Then f is smoothly unknotted (topologically unknotted if $p = 2$) if and only if one of the components of $S^{p+2} - f(S^p \times S^1)$ is homotopy equivalent to S^1.*

Here a smooth embedding $f : S^p \times S^q \to S^{p+q+1}$ is smoothly (or topologically) unknotted if there exists a diffeomorphism (respectively a homeomorphism) $h : S^{p+q+1} \to S^{p+q+1}$ such that $h(f(S^p \times S^q))$ is the image of a standard embedding. For example, Theorem 1.2 states that if p or q is greater than or equal to 2, then every embedding $f : S^p \times S^q \to S^{p+q+1}$ is unknotted.

THEOREM 1.5. *Let $f : S^p \times S^q \to S^{p+q+1}$ be a smooth embedding $(p, q \geq 1)$. Then one of the closures of the complements of $f(S^p \times S^q)$, denoted by C_1, has the homology of S^p. Furthermore, if C_1 is a homotopy S^p, then it is diffeomorphic (homeomorphic if $p + q = 3$) to $S^p \times D^{q+1}$.*

Throughout the paper, all manifolds and maps are assumed to be differentiable of class C^∞ and all homology and cohomology groups are with integral coefficients unless otherwise specified. The symbol "$\cong$" will denote a diffeomorphism between manifolds or an appropriate isomorphism between algebraic objects.

This work has been done during the third author's stay in ICMSC, Universidade de São Paulo, Brazil. He would like to thank the people there for their help and hospitality.

2. Basic lemmas and the case where $p > q \geq 2$

Let $f : S^p \times S^q \longrightarrow S^{p+q+1}$ be a smooth embedding, where $p \geq q \geq 1$. Note that $S^{p+q+1} - f(S^p \times S^q)$ always consists of two connected components. We denote the two components by C_1' and C_2', and the closure of C_i' in S^{p+q+1} by C_i $(i = 1, 2)$.

LEMMA 2.1. *One of C_1, C_2 has the homology of S^p and the other has the homology of S^q.*

Proof. Set $E = S^{p+q+1} - f(S^p \times S^q)$. By Alexander's duality, we have immediately that

$$\tilde{H}_i(E) \cong \begin{cases} \mathbf{Z} & (\text{if } i = p, q \text{ or } 0) \\ 0 & (\text{otherwise}) \end{cases}$$

for $p \neq q$ and

$$\tilde{H}_i(E) \cong \begin{cases} \mathbf{Z} \oplus \mathbf{Z} & (\text{if } i = p = q) \\ \mathbf{Z} & (\text{if } i = 0) \\ 0 & (\text{otherwise}) \end{cases}$$

for $p = q$. Thus, for the proof of Lemma 2.1, it suffices to show that, for one of the components, say C_1', we cannot have $\tilde{H}_i(C_1') = 0$ for all i. Suppose that $\tilde{H}_i(C_1') = 0$ for all i. Then we have $\tilde{H}_i(C_1) = 0$ for all i, since C_1' and C_1 are homotopy equivalent. Note that C_1 is a compact codimension-0 submanifold of S^{p+q+1} and that $\partial C_1 = f(S^p \times S^q)$. Set $\Sigma_1 = f(S^p \times \{*\})$ and $\Sigma_2 = f(\{*\} \times S^q)$, which are embedded submanifolds in $\partial C_1 = f(S^p \times S^q)$ and which intersect each other transversely in one point. Since $H_p(C_1) = 0$, there exists a $(p+1)$-cycle $(D, \partial D)$ in $(C_1, \partial C_1)$ whose boundary is homologous to Σ_1 in ∂C_1. Then we see easily that the intersection number of $[D, \partial D] \in H_{p+1}(C_1, \partial C_1)$ and $[\Sigma_2] \in H_q(C_1)$ is equal to ± 1, where $[D, \partial D]$ and $[\Sigma_2]$ are the homology classes represented by $(D, \partial D)$ and Σ_2 respectively. This implies that $H_q(C_1) \neq 0$, which contradicts our assumption. Thus we cannot have $\tilde{H}_i(C_1') = 0$ for all i. This completes the proof of Lemma 2.1. ||

In the following, we assume that C_1 has the homology of S^p and C_2 has the homology of S^q.

Set $\Sigma_1 = f(S^p \times \{*\})(\subset \partial C_1)$. We push Σ_1 into the interior of C_1, using a normal vector field on $f(S^p \times S^q)$ in S^{p+q+1} pointing toward $\mathrm{Int}C_1$. Let G denote a sufficiently small tubular neighborhood of this pushed submanifold in $\mathrm{Int}C_1$.

LEMMA 2.2. *The manifold G is always diffeomorphic to $S^p \times D^{q+1}$.*

Proof. First note that the normal bundle ν of Σ_1 in $f(S^p \times S^q)$ is isomorphic to the normal bundle of $S^p \times \{*\}$ in $S^p \times S^q$, which is clearly trivial. Then the normal bundle of Σ_1 in S^{p+q+1} is isomorphic to the Whitney sum of ν and the normal bundle ν_1 of $f(S^p \times S^q)$ in S^{p+q+1} restricted to Σ_1. Note that ν_1 is trivial, since $f(S^p \times S^q)$ is an orientable codimension-1 embedded submanifold of an orientable manifold S^{p+q+1}. Thus $\nu_1|\Sigma_1$ is also trivial and hence the normal bundle of Σ_1 in S^{p+q+1} is trivial. Since the pushed submanifold is isotopic to Σ_1 in S^{p+q+1}, its normal bundle is isomorphic to that of Σ_1. Hence the tubular neighborhood G of the pushed submanifold is diffeomorphic to $S^p \times D^{q+1}$. This completes the proof. ||

Let $i : \partial C_1 \to C_1$ denote the inclusion map.

LEMMA 2.3. *When $p > q$, $H_p(C_1)$ is generated by $i_*[\Sigma_1]$, where $[\Sigma_1] \in H_p(\partial C_1)$ is the homology class represented by Σ_1.*

Proof. Since $H_q(C_1) = 0$, there exists a $(q+1)$-cycle $(D, \partial D)$ in $(C_1, \partial C_1)$ whose boundary is homologous to Σ_2 in ∂C_1. Then the intersection number of $(D, \partial D)$ and the cycle obtained by pushing Σ_1 into C_1' is obviously equal to the intersection number of ∂D and Σ_1 in ∂C_1 up to sign, which is equal to ± 1. Hence $i_*[\Sigma_1] \in H_p(C_1) \cong \mathbf{Z}$ is nonzero and primitive. This completes the proof. ||

Set $V = \overline{C_1 - G}$.

LEMMA 2.4. *When $p > q \geq 2$, $\partial C_1, \partial G$ and V are simply connected and V is an h-cobordism between ∂G and $f(S^p \times S^q)(= \partial C_1)$.*

Proof. Since the codimension of the submanifold Σ_1' of C_1 obtained by pushing Σ_1 into the interior is equal to $q+1 \geq 3$, we see that $\pi_1(C_1 - \Sigma_1') \cong \pi_1(C_1)$.

On the other hand, since $C_1 \cap C_2 = f(S^p \times S^q)$ and S^{p+q+1} are simply connected, we see that C_1 and C_2 are simply connected by van Kampen's theorem. Hence we have $\pi_1(V) \cong \pi_1(C_1 - \Sigma_1') \cong \pi_1(C_1) = \{1\}$. The simply connectedness of ∂C_1 and ∂G follows from the fact that they are diffeomorphic to $S^p \times S^q$. Finally, we have $H_*(V, \partial G) \cong H_*(C_1, G)$ by excision and the latter group is trivial by Lemma 2.3. Hence V is an h-cobordism between ∂G and ∂C_1. ||

Since $\dim V = p+q+1 \geq 6$, we see that V is diffeomorphic to the product $\partial G \times [0,1]$ by the h-cobordism theorem ([Sm1, Mi]). Hence we have $C_1 = G \cup V \cong G \cup \partial G \times [0,1] \cong G$, which is diffeomorphic to $S^p \times D^{q+1}$ by Lemma 2.2. This completes the proof of Theorem 1.1 for the case $p > q \geq 2$.

REMARK 2.5. In this case, a similar argument shows that C_2 is also diffeomorphic to the product $D^{p+1} \times S^q$. We also note that our technique of the proof for this case is essentially the same as Kosinski's [Ko].

As an easy consequence of our proof, we obtain the following.

PROPOSITION 2.6. *Let M be a compact simply connected $(p+q+1)$-dimensional manifold $(p > q \geq 2)$ with the following properties:*
(1) ∂M is diffeomorphic to $S^p \times S^q$, and
(2a) $H_*(M) \cong H_*(S^p)$ or
(2b) $H_*(M) \cong H_*(S^q)$.
Then M is diffeomorphic to $S^p \times D^{q+1}$ in the case (2a) or $D^{p+1} \times S^q$ in the case (2b).

3. The case where $p = q \geq 2$

In this case, by Lemma 2.1, both C_1 and C_2 have the homology of S^p. Furthermore, it is easy to see that they are simply connected. Hence it suffices to prove the following.

PROPOSITION 3.1. *Let M be a compact simply connected $(2p+1)$-dimensional manifold $(p \geq 2)$ with the following properties:*
(1) ∂M is diffeomorphic to $S^p \times S^p$, and
(2) $H_*(M) \cong H_*(S^p)$.
Then M is diffeomorphic to $S^p \times D^{p+1}$.

For the proof of Proposition 3.1, first we prepare some lemmas. Let $i : \partial M \to M$ denote the inclusion map.

LEMMA 3.2. *There exists a primitive homology class $\gamma \in H_p(\partial M)$ such that $i_*(\gamma) = 0$.*

Proof. Consider the exact sequence of homology as follows:

$$H_p(\partial M) \xrightarrow{i_*} H_p(M) \longrightarrow H_p(M, \partial M).$$

Since M has the same homology as S^p, we have that $H_p(M, \partial M) \cong H^{p+1}(M) = 0$. Hence we see that

$$H_p(\partial M) \xrightarrow{i_*} H_p(M) \longrightarrow 0$$

is exact. Note that this exact sequence splits, since $H_p(M)$ is free. Hence $\ker(i_*)$ is a direct summand of $H_p(\partial M) \cong \mathbf{Z} \oplus \mathbf{Z}$ of rank one. This completes the proof. ||

LEMMA 3.3. *When p is even, the self-intersection number $\gamma \cdot \gamma$ is equal to 0.*

Proof. First note that $\gamma \cdot \gamma = < \gamma^*, \gamma >$, where $\gamma^* \in H^p(\partial M)$ is the Poincaré dual of γ. Consider the following commutative diagram with exact rows, where the vertical isomorphisms are induced by the Poincaré-Lefschetz duality:

$$\begin{array}{ccccc}
H^p(M) & \xrightarrow{i^*} & H^p(\partial M) & \longrightarrow & H^{p+1}(M, \partial M) \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
H_{p+1}(M, \partial M) & \longrightarrow & H_p(\partial M) & \xrightarrow{i_*} & H_p(M).
\end{array}$$

Using the above diagram, we see that there exists some $\beta \in H^p(M)$ such that $i^*(\beta) = \gamma^*$, since $i_*(\gamma) = 0$. Then we have $\gamma \cdot \gamma = <\gamma^*, \gamma> = <i^*(\beta), \gamma> = <\beta, i_*(\gamma)> = 0$. This completes the proof. $||$.

LEMMA 3.4. *There exists a homology class $\delta \in H_p(\partial M)$ such that $\gamma \cdot \delta = 1$. Furthermore, when p is even, we can choose δ so that $\delta \cdot \delta = 0$.*

Proof. Since γ is a primitive homology class, there exists a homology class $\delta' \in H_p(\partial M)$ such that $\gamma \cdot \delta' = \pm 1$. Changing the orientation of δ' if necessary, we may assume that $\gamma \cdot \delta' = 1$. When p is even, since the intersection form of ∂M is of even type, we have $\delta' \cdot \delta' = 2m$ for some integer m. Then setting $\delta = \delta' - m\gamma$, we have

$$\begin{aligned} \gamma \cdot \delta &= \gamma \cdot (\delta' - m\gamma) \\ &= \gamma \cdot \delta' - m(\gamma \cdot \gamma) \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \delta \cdot \delta &= (\delta' - m\gamma) \cdot (\delta' - m\gamma) \\ &= (\delta' \cdot \delta') - 2m(\delta' \cdot \gamma) + m^2(\gamma \cdot \gamma) \\ &= 2m - 2m \\ &= 0. \end{aligned}$$

This completes the proof. $||$

Now we consider the case where p is even. Since ∂M is diffeomorphic to $S^p \times S^p$, in the following we identify M with $S^p \times S^p$. Set $\xi_1 = [S^p \times \{*\}]$ and $\xi_2 = [\{*\} \times S^p]$, which generate $H_p(\partial M) = H_p(S^p \times S^p)$. We may assume that $\xi_i \cdot \xi_i = 0$ $(i = 1, 2)$ and $\xi_1 \cdot \xi_2 = 1$, choosing suitable orientations of $S^p \times \{*\}$ and $\{*\} \times S^p$. Since $\delta \cdot \delta = \gamma \cdot \gamma = 0$ and $\delta \cdot \gamma = 1$, we see that the endomorphism $\eta : H_p(S^p \times S^p) \to H_p(S^p \times S^p)$ defined by $\eta(\xi_1) = \delta$ and $\eta(\xi_2) = \gamma$ is an automorphism of $(H_p(S^p \times S^p), \ \cdot \ )$.

LEMMA 3.5. *Every automorphism of $(H_p(S^p \times S^p), \ \cdot \ )$ is realized by an orientation preserving diffeomorphism of $S^p \times S^p$, provided that p is even.*

Proof. Let $\eta : H_p(S^p \times S^p) \to H_p(S^p \times S^p)$ be an automorphism. Let the representation matrix of η with respect to the base $\{\xi_1, \xi_2\}$ be denoted by A. Then we have

$$ {}^tA \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, $$

since the intersection matrix of $S^p \times S^p$ with respect to the base $\{\xi_1, \xi_2\}$ is equal to

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This shows that there are exactly 4 automorphisms, which correspond to

$$\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

These automorphisms are easily realized by orientation preserving diffeomorphisms of $S^p \times S^p$. This completes the proof. ||

Now we consider the case where $p = 2$. Let $\psi : S^2 \times S^2 \to S^2 \times S^2$ be an orientation preserving diffeomorphism such that $\psi_* = \eta$, whose existence is guaranteed by the previous lemma (see also [Wa1]). Set $X = M \cup_\psi S^2 \times D^3$, where we attach ∂M and $\partial(S^2 \times D^3)$ by ψ. In the following, we identify ∂M and $S^2 \times S^2$ by the diffeomorphism ψ. Let $j_1 : S^2 \times S^2 \to M$ and $j_2 : S^2 \times S^2 \to S^2 \times D^3$ be the inclusion maps. Note that $j_1 = i \circ \psi$.

LEMMA 3.6. *$(j_1)_*(\xi_1)$ is a generator of $H_2(M)(\cong H_2(S^2) \cong \mathbf{Z})$.*

Proof. Since $(j_1)_*(\xi_2) = (i \circ \psi)_*(\xi_2) = i_*(\gamma) = 0$, there exists a 3-cycle $(D, \partial D)$ in $(M, \partial M)$ such that ∂D is homologous to $\{*\} \times S^2$ in $S^2 \times S^2$. Then we see that the intersection number $[D, \partial D] \cdot (j_1)_*(\xi_1)$ in M is equal to the intersection number $\xi_2 \cdot \xi_1 = 1$ in ∂M up to sign, where $[D, \partial D] \in H_3(M, \partial M)$ is the homology class represented by $(D, \partial D)$. Hence $(j_1)_*(\xi_1) \in H_2(M)$ is nonzero and is a primitive homology class. This implies that it is a generator of $H_2(M) \cong \mathbf{Z}$. This completes the proof. ||

LEMMA 3.7. *X is a simply connected closed 5-dimensional manifold with $H_2(X) \cong \mathbf{Z}$.*

Proof. Simply connectedness of X follows easily from the fact that M and $S^2 \times D^3$ are simply connected and that $M \cap S^2 \times D^3$ is connected. Consider the following Mayer-Vietoris exact sequence of homology:

$$H_2(S^2 \times S^2) \xrightarrow{\theta} H_2(M) \oplus H_2(S^2 \times D^3) \longrightarrow H_2(X) \longrightarrow H_1(S^2 \times S^2),$$

where $\theta = ((j_1)_*, -(j_2)_*)$ and $H_1(S^2 \times S^2) = 0$. Note that $(j_1)_*(\xi_1)$ is a generator of $H_2(M)$ by the previous lemma, that $(j_2)_*(\xi_1)$ is also a generator of $H_2(S^2 \times$

D^3) and that $(j_1)_*(\xi_2) = 0$ and $(j_2)_*(\xi_2) = 0$. Thus the image of θ is equal to $\{(m(j_1)_*(\xi_1), -m(j_2)_*(\xi_1)) : m \in \mathbf{Z}\}$ and $H_2(X)$ is isomorphic to $\mathbf{Z}$. This completes the proof. ||

Note that, by the above proof, $(j_3)_* : H_2(M) \to H_2(X)$ and $(j_4)_* : H_2(S^2 \times D^3) \to H_2(X)$ are isomorphisms, where $j_3 : M \to X$ and $j_4 : S^2 \times D^3 \to X$ are the inclusion maps.

LEMMA 3.8. *The second Stiefel-Whitney class $w_2(X) \in H^2(X; \mathbf{Z}_2)$ vanishes.*

Proof. First observe that $\partial M \cong S^2 \times S^2$ is a spin manifold. We fix its spin structure (in fact, a spin structure of $S^2 \times S^2$ is unique, since $H^1(S^2 \times S^2; \mathbf{Z}_2) = 0$). Furthermore, we have $H^2(M, \partial M; \mathbf{Z}_2) \cong H_3(M; \mathbf{Z}_2) = 0$ and $H^2(S^2 \times D^3, \partial(S^2 \times D^3); \mathbf{Z}_2) \cong H_3(S^2 \times D^3; \mathbf{Z}_2) = 0$. Hence the relative Stiefel-Whitney classes $w_2(M, \partial M)$ and $w_2(S^2 \times D^3, \partial(S^2 \times D^3))$ vanish. Hence we have $w_2(X) = 0$. (For details, see the proof of (4.3) of [**Bo**].) This completes the proof. ||

Thus X is a simply connected closed 5-dimensional manifold with vanishing second Stiefel-Whitney class and with $H_2(X) \cong \mathbf{Z}$. Hence by [**Sm2**], we see that X is diffeomorphic to $S^2 \times S^3$. Then there exists a compact 6-dimensional manifold W such that $\partial W = X$ and that W is diffeomorphic to $S^2 \times D^4$. Let $k : X \to W$ denote the inclusion map.

LEMMA 3.9. *The inclusions $k \circ j_3 : M \to W$ and $k \circ j_4 : S^2 \times D^3 \to W$ are homotopy equivalences.*

Proof. By the remark just after the proof of Lemma 3.7, we know that $(j_3)_* : H_2(M) \to H_2(X)$ and $(j_4)_* : H_2(S^2 \times D^3) \to H_2(X)$ are isomorphisms. Since $k_* : H_2(X) \to H_2(W)$ is clearly an isomorphism, we see that $(k \circ j_3)_* : H_2(M) \to H_2(W)$ and $(k \circ j_4)_* : H_2(S^2 \times D^3) \to H_2(W)$ are isomorphisms. Since $H_*(M) \cong H_*(S^2 \times D^3) \cong H_*(W)$, we see that $k \circ j_3$ and $k \circ j_4$ induce isomorphisms on homology. Since $M, S^2 \times D^3$ and W are simply connected, they are actually homotopy equivalences by Whitehead's theorem. This completes the proof. ||

Let $c : \partial(S^2 \times D^3) \times [0, 1] \to S^2 \times D^3$ be a collar neighborhood of $\partial(S^2 \times D^3)$ in $S^2 \times D^3$ with $\partial(S^2 \times D^3) \times \{0\}$ corresponding to $\partial(S^2 \times D^3)$. Then we see that the triad $(W; M, S^2 \times D^3 - Y)$ is an h-cobordism relative to the boundary, where $Y = c^{-1}[0, 1/2)$. Since W is simply connected and $\dim W = 6$, we see that M is diffeomorphic to $S^2 \times D^3$ by the relative h-cobordism theorem. This completes the proof of Proposition 3.1 for $p = 2$.

Next we consider the case $p > 2$. Let $\delta \in H_p(\partial M)$ be the homology class as in Lemma 3.4. Since $\pi_1(\partial M) = 1$ and $H_i(\partial M) = 0$ for $0 < i < p$, we see that $\pi_p(\partial M)$ is isomorphic to $H_p(\partial M)$ by the Hurewicz theorem. In particular, there exists a continuous map $\varphi : S^p \to \partial M$ such that $\varphi_*[S^p] = \delta$, where $[S^p] \in H_p(S^p)$ is the fundamental class. Then by [**Wh**] or [**H1**], we see that φ is homotopic to an embedding $\varphi' : S^p \to \partial M$. Let S denote the submanifold of M which is obtained by pushing $\varphi'(S^p)$ into the interior of M, using the normal vector field of ∂M pointing toward $\mathrm{Int}M$.

LEMMA 3.10. *The normal bundle of S in M is trivial.*

Proof. Let $c' : \partial M \times [0, 1] \to M$ be a collar neighborhood of ∂M in M. Then we may assume that S is contained in the interior of $c'(\partial M \times [0, 1])$. Furthermore, $\partial M \cong S^p \times S^p$ can be embedded in $\mathbf{R}^{2p+1}$. For example, it is embedded as the boundary of a closed tubular neighborhood of a standardly embedded p-sphere in $\mathbf{R}^{2p+1}$. Hence $\partial M \times [0, 1]$ can also be embedded in $\mathbf{R}^{2p+1}$. This implies that the normal bundle of S in M is equivalent to that of an embedded p-sphere in $\mathbf{R}^{2p+1}$. On the other hand, we see easily that the normal bundle ν of an embedded q-sphere in $\mathbf{R}^r$ is trivial, provided that $r - q > q$. This is because $\nu \oplus \varepsilon^{q+1} \cong \nu \oplus TS^q \oplus \varepsilon^1 \cong \varepsilon^{r+1}$, which implies that ν is always stably trivial, where ε^m denotes the trivial m-plane bundle over S^q. Hence the normal bundle of S in M is trivial. This completes the proof. ||

Let G denote the closed tubular neighborhood of S in $\mathrm{Int}M$ and let $l : G \to M$ denote the inclusion map. Note that G is diffeomorphic to $S^p \times D^{p+1}$ by Lemma 3.10.

LEMMA 3.11. *$l_* : H_i(G) \to H_i(M)$ is an isomorphism for every i.*

Proof. We have only to show the lemma for $i = p$. Since $H_p(G)$ is generated by the class $[S]$ represented by S, which is isotopic to $\varphi'(S^p)$ in M, we see that the image of l_* is generated by $i_*(\delta)$. On the other hand, since $i_*(\gamma) = 0$ in $H_p(M)$, there exists a $(p+1)$-cycle $(D, \partial D)$ in $(M, \partial M)$ such that ∂D is homologous to γ in ∂M. Then we see that the intersection number $[D, \partial D] \cdot i_*(\delta)$ in M is equal to the intersection number $\gamma \cdot \delta$ in ∂M up to sign, which is equal to ± 1. Hence we see that $i_*(\delta)$ is a non-zero primitive class of $H_p(M) \cong \mathbf{Z}$. Thus l_* is surjective and hence it is an isomorphism. This completes the proof. ||

Set $V = M - \mathrm{Int}G$. Then we see that $\pi_1(V) = 1$, since $\pi_1(V) \cong \pi_1(M - S')$ and S' has codimension $p+1 \geq 4$. Furthermore, we have $H_*(V, \partial G) \cong H_*(M, G) = 0$ by excision and Lemma 3.11. Thus $(V; \partial M, \partial G)$ is an h-cobordism and hence V is diffeomorphic to $\partial G \times [0,1]$, which implies that $M = G \cup V$ is diffeomorphic to $S^p \times D^{p+1} \cup \partial(S^p \times D^{p+1}) \times [0,1] \cong S^p \times D^{p+1}$. This completes the proof of Proposition 3.1 and hence the proof of Theorem 1.1 for the case $p = q \geq 2$.

4. The case where $q = 1$

In this case, the techniques used in §2 do not work. This is because, in general, $\pi_1(V)$ is not necessarily isomorphic to $\pi_1(\partial G) \cong \mathbf{Z}$ and V is not an h-cobordism nor an s-cobordism. We will prove the theorem for this case using a technique similar to that used in §3 for the case $p = q = 2$.

Recall that C_1 has the homology of S^p. The following lemma is proved in [**Wa2**]. Here we give another proof.

LEMMA 4.1. *When $q = 1$ and $p \geq 2$, C_1 is simply connected.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}[column sep=small]
 & \pi_1(f(S^p \times S^1)) \arrow[dl, "(i_1)_*"'] \arrow[dr, "(i_2)_*"] \arrow[dd, "\bar{i}_*"] & \\
\pi_1(C_1) \arrow[dr, "(\bar{i}_1)_*"'] & & \pi_1(C_2) \arrow[dl, "(\bar{i}_2)_*"] \\
 & \pi_1(S^{p+2}), &
\end{tikzcd}$$

where $i_1, i_2, \bar{i}_1, \bar{i}_2$ and $\bar{i}$ are the obvious inclusion maps. Note that $(i_2)_* : H_1(f(S^p \times S^1)) \to H_1(C_2)$ is an isomorphism, which is proved using an argument similar to that of Lemma 2.3. Hence $(i_2)_* : \pi_1(f(S^p \times S^1)) \to \pi_1(C_2)$ is injective.

Now suppose that $(i_1)_*$ is injective. Then we see that $\pi_1(S^{p+2})$ is the amalgamated free product of $\pi_1(C_1)$ and $\pi_1(C_2)$ with $(i_1)_*(\pi_1(f(S^p \times S^1)))$

and $(i_2)_*(\pi_1(f(S^p \times S^1)))$ identified by van Kampen's theorem. Since, $(i_1)_*$ and $(i_2)_*$ are injective, we see that $(\bar{i}_1)_*$ and $(\bar{i}_2)_*$ are also injective (see, for example, [MKS, Theorem 4.3]). This is a contradiction, since $\pi_1(f(S^p \times S^1)) \cong \mathbf{Z}$ and $\pi_1(S^{p+2}) = 1$.

Let t be a generator of $\pi_1(f(S^p \times S^1))$ represented by $f(\{*\} \times S^1)$. Since $(i_1)_*$ is not injective, $(i_1)_*(t^{r'}) = 1$ for some positive integer r'. Let r denote the minimum positive integer satisfying this property. We will show that $r = 1$ as follows. Consider the following commutative diagram:

$$\begin{tikzcd}
 & \pi_1(f(S^p \times S^1))/T \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] \arrow[dd, "\omega"] & \\
\pi_1(C_1) \arrow[dr, "\bar{\alpha}"'] & & \pi_1(C_2)/N \arrow[dl, "\bar{\beta}"] \\
 & \pi_1(S^{p+2}), &
\end{tikzcd}$$

where T is the (normal) subgroup of $\pi_1(f(S^p \times S^1))$ generated by t^r, N is the normal subgroup generated by $(i_2)_*(t^r)$ and $\alpha, \beta, \bar{\alpha}, \bar{\beta}$ and ω are the obvious maps induced by $(i_1)_*, (i_2)_*, (\bar{i}_1)_*, (\bar{i}_2)_*$ and $\bar{i}_*$ respectively. Note that, by the definition of r, α is injective. Now suppose that β is not injective. Then we see that there exists an integer s with $0 < s < r$ such that $(i_2)_*(t^s) \in N$. Thus, by the definition of N, we have

$$(i_2)_*(t^s) = a_1((i_2)_*(t^r))^{\pm 1} a_1^{-1} \cdots a_m((i_2)(t^r))^{\pm 1} a_m^{-1}$$

for some $a_1, \cdots, a_m \in \pi_1(C_2)$. Then, in the abelianized group H of $\pi_1(C_2)$, we have

$$(i_2)_*(t^s) = ((i_2)_*(t^r))^e \tag{4.2}$$

for some integer e. Here we note that H is isomorphic to $H_1(C_2) \cong \mathbf{Z}$. Now it is easy to show that $H_1(C_2)$ is generated by the class of $(i_2)_*(t)$, using the same argument as in the proof of Lemma 2.3. Then the equation (4.2) implies that $s - re = 0$, since $((i_2)_*(t))^{s-re} = 1$ in H. This contradicts the assumption that $0 < s < r$. Hence β must be injective.

Then by the well-known theorem on amalgamated free products and van Kampen's theorem, we see that ω is injective. Then r must be equal to ± 1, since the order of $\pi_1(f(S^p \times S^1))/T$ is equal to r and that of $\pi_1(S^{p+2})$ is equal to 1. Then by van Kampen's theorem again, we see that $\pi_1(S^{p+2})$ is isomorphic to the free product of $\pi_1(C_1)$ and $\pi_1(C_2)/N$. This implies that $\pi_1(C_1)$ is trivial. This completes the proof of Lemma 4.1. ||

Note that the above proof also shows that $\pi_1(C_2)$ is normally generated by $(i_2)_*(t)$.

By Lemma 4.1, C_1 is a simply connected smooth $(p+2)$-dimensional manifold whose homology is the same as S^p and whose boundary is diffeomorphic to $S^p \times S^1$. Thus, for the proof of Theorem 1.1 for the case $p > q = 1$, it suffices to show the following.

PROPOSITION 4.3. *Let M be a compact simply connected $(p+2)$-dimensional manifold $(p \geq 2)$ with the following properties:*
(1) ∂M is diffeomorphic to $S^p \times S^1$, and
(2) $H_*(M) \cong H_*(S^p)$.
Then M is diffeomorphic to $S^p \times D^2$ if $p > 2$ and M is homeomorphic to $S^2 \times D^2$ if $p = 2$.

Proof. First consider the case $p = 2$. Then the proposition follows from [**Bo**, Remark (5.3) (i)]. Note that, since $H_1(\partial M)$ has no torsion, the intersection form of M is trivial and is even. Furthermore, it is known that $\mathcal{H}_+(S^1 \times S^2)$ ([**Bo**]) acts transitively on $\mathrm{Spin}(S^1 \times S^2)$ (see also [**St**]).

Now suppose that $p \geq 3$. Let $\psi : S^p \times S^1 \to \partial M$ be a diffeomorphism. We see easily that the inclusion $\psi(S^p \times \{*\}) \to M$ induces a homotopy equivalence by the same argument as in the proof of Lemma 2.3. Denote by X the union of M and $S^p \times D^2$ attached by the diffeomorphism ψ along their boundaries. Note that X is a simply connected closed $(p+2)$-dimensional manifold.

LEMMA 4.4.
$$H_*(X) \cong H_*(S^p \times S^2).$$

The above lemma follows easily from the fact that the inclusion $\psi(S^p \times \{*\}) \to M$ is a homotopy equivalence and the Mayer-Vietoris exact sequence as follows:

$$\cdots \to H_i(S^p \times S^1) \to H_i(M) \oplus H_i(S^p \times D^2) \to H_i(X) \to H_{i-1}(S^p \times S^1) \to \cdots.$$

Now we continue the proof of Proposition 4.3. Set $S_1 = \Sigma_1 = S^p \times \{*\}$ and $S' = \Sigma_2 = \{*'\} \times S^1$, where $S_1, S' \subset S^p \times S^1 = \partial(S^p \times D^2)$. Note that S' bounds a smoothly embedded 2-disk $D_1 = \{*'\} \times D^2$ in $S^p \times D^2$. On the other hand, since M is simply connected, $\psi(S')(\subset \partial M)$ bounds an immersed 2-disk in M. Since M is of dimension greater than or equal to 5, we can change the immersed 2-disk so that it is actually an embedded 2-disk by transversality (for example, see [**H1**]). Denote by D_2 such an embedded 2-disk in M bounded by $\psi(S')$. We may assume that $D_2 \cap \partial M = \partial D_2$ and that D_2 is transverse to ∂M. Then we see that $S_2 = D_2 \cup D_1$ forms an embedded 2-sphere in X, which intersects S_1 transversely in one point $*' \times * \in \partial(S^p \times D^2)$.

LEMMA 4.5. *The induced map*

$$j_* : H_i(S_1 \cup S_2) \to H_i(X)$$

is an isomorphism for $i \neq p+2$, where $j : S_1 \cup S_2 \to X$ is the inclusion.

The above lemma follows from the proof of Lemma 4.4.

Let ν_k be the normal bundle of S_k in X $(k = 1, 2)$. We see easily that ν_1 is trivial (see the proof of Lemma 2.2). On the other hand, ν_2 may not be trivial. If it is not trivial, we consider the diffeomorphism $\tau : S^p \times S^1 \to S^p \times S^1$ which corresponds to the generator of $\pi_1(SO(p+1)) \cong \mathbf{Z}_2$ such that $\tau(\{*'\} \times S^1) = \{*'\} \times S^1$, and put $\psi' = \psi \circ \tau$. Then ψ' satisfies the same hypothesis as ψ and if we use ψ' when constructing X, then the normal bundle ν_2 of S_2 is trivial. Hence, we may assume that ν_2 is trivial.

Let N_k be a small tubular neighborhood of S_k in X $(k = 1, 2)$. By the argument in the previous paragraph, we see that $N_1 \cong S^p \times D^2$ and $N_2 \cong D^p \times S^2$. Hence we see that $N_1 \cup N_2 \cong S^p \times S^2 - \mathrm{Int} D^{p+2}$.

LEMMA 4.6. *Set $\Delta = X - \mathrm{Int}(N_1 \cup N_2)$. Then Δ is diffeomorphic to the $(p+2)$-dimensional disk.*

Proof. First note that $S^p \times D^2 - S_1$ and $M - S_1$ are simply connected, since S_1 lies on the boundaries of $S^p \times D^2$ and M, which are simply connected. Consequently, $S^p \times D^2 - (S_1 \cup D_1)$ and $M - (S_1 \cup D_2)$ are simply connected, since D_1 and D_2 have codimension greater than or equal to 3. Hence $M - \mathrm{Int}(N_1 \cup N_2)$ and $S^p \times D^2 - \mathrm{Int}(N_1 \cup N_2)$ are simply connected. This implies that Δ is also simply connected, since $(M \cap S^p \times D^2) - \mathrm{Int}(N_1 \cup N_2)$ is connected. Furthermore, by Poincaré-Lefschetz duality and excision, we have

$$H_i(\Delta) \cong H^{p+2-i}(\Delta, \partial\Delta) \cong H^{p+2-i}(X, N_1 \cup N_2) \cong H^{p+2-i}(X, S_1 \cup S_2).$$

Then by Lemma 4.5, we see that Δ is acyclic. Since Δ is simply connected, this implies that Δ is contractible. Furthermore, $\partial\Delta$ is diffeomorphic to the $(p+1)$-dimensional sphere. Hence by the generalized Poincaré conjecture proved by Smale ([**Sm1, Mi**]), we see that Δ is diffeomorphic to the $(p+2)$-dimensional disk. This completes the proof of Lemma 4.6. ||

Recall that $N_1 \cup N_2 \cong S^p \times S^2 - \mathrm{Int} D^{p+2}$. Lemma 4.6 shows that X is diffeomorphic to the connected sum $S^p \times S^2 \sharp \Sigma$ for some homotopy $(p+2)$-sphere Σ. Then changing ψ slightly on a $(p+1)$-disk in $\partial(S^p \times D^2)$, we may assume that X is diffeomorphic to $S^p \times S^2$.

Then X bounds a compact $(p+3)$-dimensional manifold W diffeomorphic to $S^p \times D^3 \cong S^p \times D^2 \times I$.

LEMMA 4.7. *The inclusions $M \to W$ and $S^p \times D^2 \to W$ are homotopy equivalences.*

The above lemma can be proved using homological arguments as in the proof of Lemma 3.9.

By Lemma 4.7, we see that W is a simply connected h-cobordism relative to the boundary between M and $S^p \times D^2 - Y$, where Y is a small open collar neighborhood. Hence, by the h-cobordism theorem, we see that M is diffeomorphic to $S^p \times D^2$. This completes the proof of Proposition 4.3 and hence Theorem 1.1 for the case $p > q = 1$. ||

Finally we note that when $p = q = 1$, Theorem 1.1 has been solved by Alexander [**A**]. This completes the proof of Theorem 1.1.

REMARK 4.8. In fact, examining the proof of Alexander's torus theorem in [**R**], we obtain the following. *Let M be a compact connected orientable irreducible 3-dimensional manifold such that ∂M is diffeomorphic to $S^1 \times S^1$ and $\pi_1(M) \cong \mathbf{Z}$. Then M is diffeomorphic to $S^1 \times D^2$.*

REMARK 4.9. When $q = 1$, C_2 is diffeomorphic (homeomorphic if $p = 2$) to a knot exterior. Consequently, the conclusion of Theorem 1.2 does not hold when $q = 1$.

REMARK 4.10. When $q = 1$, there are some inessential gaps in Wall's argument [**Wa2**]. For example, in the last paragraph of page 660, one cannot use a result of Smale if $q = 1$, since in this case $\pi_1(\partial C_1) \cong \pi_1(S^p \times S^1) \neq 1$. However his results remain valid.

Examining our proof of Theorem 1.1 carefully, we see that the same argument works for the proof of the following more general result.

THEOREM 4.11. *Let $f : S^p \times S^q \to \Sigma^{p+q+1}$ be a smooth embedding, where Σ^{p+q+1} is a homotopy $(p+q+1)$-sphere. We suppose that $p \geq q \geq 1$ and $p+q+1 \geq 5$. Then the closure of one of the components of $\Sigma^{p+q+1} - f(S^p \times S^q)$ is diffeomorphic to $S^p \times D^{q+1}$. Furthermore, when $p+q+1 = 4$ (i.e., when $p = 2$ and $q = 1$), it is homeomorphic to $S^2 \times D^2$.*

5. Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2. We have only to show the theorem in the case where f_2 is the standard embedding of $S^p \times S^q$. In other words, $f_2(S^p \times S^q)$ is the boundary of the tubular neighborhood of the standardly embedded q-sphere in S^{p+q+1}. By the proof of Theorem 1.1 together with our dimension assumptions, $f_1(S^p \times S^q)$ bounds an embedded manifold A diffeomorphic to $D^{p+1} \times S^q$. Note that A is a tubular neighborhood of S, where S is the embedded q-sphere corresponding to $\{0\} \times S^q \subset D^{p+1} \times S^q$. Then by [**H1**], there exists a diffeomorphism h of S^{p+q+1} such that $h(S)$ is the standardly embedded q-sphere. Then the result follows from the uniqueness of tubular neighborhoods. This completes the proof. ||

Proof of Theorem 1.3. By our assumption, the tubular neighborhood N_i of $f_i(S^p)$ is diffeomorphic to $S^p \times D^{n-p}$. Hence, when $p \geq 2$, by Theorem 1.2, $S^n - N_1$ and $S^n - N_2$ are diffeomorphic to each other. Since $S^n - N_i$ is diffeomorphic to $S^n - f_i(S^p)$, we have the desired conclusion. When $p = 1$, the result is obvious, since the isotopy class of an embedded circle in S^n $(n \geq 4)$ is unique ([**H1**]). This completes the proof. ||

Proof of Theorem 1.4. By the results of §4, one of the closures of the complements of the image is diffeomorphic (homeomorphic if $p = 2$) to a tubular neighborhood of an embedded codimension-2 sphere. Then the conclusion follows from the results of Levine [**L2**], Shaneson [**Sh**], and Freedman [**F**]. ||

Theorem 1.5 follows from Theorems 1.2 and 1.4.

There have been a lot of results concerning embedded spheres in spheres and their normal bundles. For example, see [**H1, H2, L1, Ke1, Ke2, Ke3, Ke4, Ma, HLS**].

References

[**A**] J. W. Alexander, *On the subdivision of a 3-space by a polyhedron*, Proc. Nat. Acad. Sci. U. S. A. **10** (1924), 6–8.

[**Bo**] S. Boyer, *Simply-connected 4-manifolds with a given boundary*, Trans. Amer. Math. Soc. **298** (1986), 331–357.

[**F**] M. Freedman, *The topology of four-dimensional manifolds*, J. Diff. Geom. **17** (1982), 357–453.

[**G**] R. Z. Goldstein, *Piecewise linear unknotting of $S^p \times S^q$ in S^{p+q+1}*, Michigan Math. J. **14** (1967), 405–415.

[**H1**] A. Haefliger, *Differentiable imbeddings*, Bull. Amer. Math. Soc. **67** (1961), 109–112.

[**H2**] A. Haefliger, *Knotted $(4k-1)$-spheres in $6k$-space*, Ann. of Math. **75** (1962), 452–466.

[**HLS**] W. C. Hsiang, J. Levine and R. H. Szczarba, *On the normal bundle of a homotopy sphere embedded in Euclidean space*, Topology **3** (1965), 173–181.

[**Ke1**] M. Kervaire, *Normal bundle to a sphere in euclidean space*, Bull. Amer. Math. Soc. **63** (1957), 147.

[**Ke2**] M. Kervaire, *An interpretation of G. Whitehead's generalization of H. Hopf's invariant*, Ann. of Math. **69** (1959), 345–364.

[**Ke3**] M. Kervaire, *Sur le fibré normal à une sphère immergée dans un espace euclidean*, Comment. Math. Helv. **33** (1959), 121–131.

[**Ke4**] M. Kervaire, *Sur le fibré normal à une variété plongée dans l'espace euclidean*, Bull. Soc. Math. France **87** (1959), 397–401.

[**Ko**] A. Kosinski, *On Alexander's theorem and knotted spheres*, in "Topology of 3-manifolds and related topics," ed. by M. K. Fort Jr., Prentice-Hall, 1961, pp. 55–57.

[**L1**] J. Levine, *A classification of differentiable knots*, Ann. of Math. **82** (1965), 15–50.

[**L2**] J. Levine, *Unknotting spheres in codimension two*, Topology 4 (1965), 9–16.

[**MKS**] W. Magnus, A. Karrass and D. Solitar, "Combinatorial group theory: presentations of groups in terms of generators and relations," second revised edition, Dover Publications Inc., New York, 1976.

[Ma] W. S. Massey, *On the normal bundle of a sphere imbedded in euclidean space*, Proc. Amer. Math. Soc. **10** (1959), 959–964.

[Mi] J. Milnor, "Lectures on the h-cobordism theorem," Math. Notes, Princeton Univ. Press, Princeton, New Jersey, 1965.

[R] D. Rolfsen, "Knots and links," Math. Lect. Series 7, Publish or Perish Inc., Berkeley, 1976.

[Sh] J. L. Shaneson, *Embeddings with codimension two of spheres in spheres and h-cobordisms of $S^1 \times S^3$*, Bull. Amer. Math. Soc. **74** (1968), 972–974.

[Sm1] S. Smale, *On the structure of manifolds*, Amer. J. Math. **84** (1962), 387–399.

[Sm2] S. Smale, *On the structure of 5-manifolds*, Ann. of Math. **75** (1962), 38–46.

[St] R. Stong, *Simply-connected 4-manifolds with a given boundary*, Topology Appl. **52** (1993), 161–167.

[Wa1] C. T. C. Wall, *Diffeomorphisms of 4-manifolds*, J. London Math. Soc. **39** (1964), 131–140.

[Wa2] C. T. C. Wall, *Unknotting tori in codimension one and spheres in codimension two*, Proc. Camb. Phil. Soc. **61** (1965), 659–664.

[Wh] H. Whitney, *The self-intersections of a smooth n-manifold in $2n$-space*, Ann. of Math. **45** (1944), 220–246.

NOTAS DO ICMSC

SÉRIE MATEMÁTICA

027/95 BIASI, C. and GONÇALVES, D. L., *Equivalent conditions for n-equivalences module a class $\mathcal{C}$ of groups*

026/95 CARVALHO, A. N. and HINES, G., *Lower semicontinuity of unstable manifolds for gradient systems*

025/95 GARCIA, R.A.; MOCHIDA, D.; FUSTER, C.R. and RUAS, M. A. S., *Inflection points and topology of surfaces in 4-space*

024/95 CASSAGO JR., H.; GODOY, S. M.; and TÁBOAS, P.Z., *Periodic solutions of planar differential equations with two delays*

023/94 ZANI, S. L., *Two-weight norm inequalities for maximal functions on homogeneous spaces and boundary estimates*

022/94 LABOURIAU, I. S. and RUAS, M. A. S., *Singularities of equations of Hodgkin-Huxley type*

021/94 BRUCE, J.W. and MARAR, W.L., *Images and varieties*

020/94 LADEIRA, L.A.C., *Differentiability with respect to delays for a neutral differential-difference equation*

019/94 CARVALHO, A.N. and CUMINATO, J.A., *Reaction-diffusion problems in cell tissues*

018/94 CARVALHO, A.N. and RODRIGUEZ-BERNAL, A., *Global attractors for parabolic problems with nonlinear boundary conditions in fractional power spaces*

017/94 RODRIGUEZ-BERNAL, A., *Localized spatial homogenization and large diffusion*

016/94 GIONGO, M.A.P.A.P. and TÁBOAS, P.Z., *Effect of a vaccination on an epidemic model*

015/94 OLIVA, W.M., *Realidade Matemática: a controvérsia dos computadores, extraindo ordem do caos, medindo simetria e outros ensaios*