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**APPLICATIONS OF THE NON-STANDARD VERSION OF THE
BORSUK-ULAM THEOREM**

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Applications of the non-standard version of the Borsuk-Ulam theorem

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Neste artigo, nós generalizamos o resultado provado por Pannwitz em [5] e apresentamos algumas aplicações da versão não padrão do teorema de Borsuk-Ulam provado em [1].

In this paper, we generalize the result showed by Pannwitz in [5] and we give some applications of the non-standard version of the Borsuk-Ulam theorem proved in [1].

1. INTRODUCTION

Given topological spaces X and Y , a continuous map $f : X \rightarrow Y$ is called a *free map* if there exists a continuous map $\varphi : X \rightarrow X$ which has the property that $f(x) \neq f(\varphi(x))$, for any $x \in X$. This definition was introduced by Hopf in 1937. In [3], among other results, he proved that for any $n \geq 3$ there exists a *free map* $f : S^n \rightarrow \mathbb{R}^n$. In the same paper, Hopf asked whether the following questions holds:

1. There exists a *free map* $f : S^2 \rightarrow \mathbb{R}^2$.
2. In the affirmative case, for which natural numbers $n > 2$ there exists a *free map* from n -sphere S^n into Euclidean 2-space $\mathbb{R}^2$.

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The Hopf's questions had been answered in 1952 by Erika Pannwitz¹. In [5], she showed the existence of a *free map* $f : S^n \rightarrow \mathbb{R}^2$, for any $n \geq 0$.

Let us consider a topological space X and a continuous map $\varphi : X \rightarrow X$. In [1] we proved that, under certain conditions on X and φ , for a given map $f : X \rightarrow \mathbb{R}^2$, although the existence of a point $x \in X$ such that x and $\varphi(x)$ are collapsed by f cannot always be assured, a very interesting phenomenon concerning the three points $u = f(\varphi(x))$, $v = f(x)$ and $w = f(\varphi^2(x))$ occurs when $f(x) \neq f(\varphi(x))$, for any $x \in X$: for at least a point x in a special subset of X , u belongs to the line segment determined by v and w and we obtained a non-standard version of the Borsuk-Ulam theorem.

Specifically, we proved the following

THEOREM 1.1. *Let X be Hausdorff space and A a compact, connected, and locally pathwise connected subset of X . Let $\varphi : X \rightarrow X$ be continuous map such that $\varphi(A) \subset A$. Suppose that*

$$Id_* - \varphi_*^1 : i_*(H_1(A, \mathbb{Q})) \rightarrow i_*(H_1(A, \mathbb{Q})) \quad (1)$$

is a surjective map. Then for every continuous map $f : X \rightarrow \mathbb{R}^2$, either there exists a point $x \in X$ such that $f(x) = f(\varphi(x))$, or there exists a point $x \in A$ such that $f(\varphi(x))$ belongs to the closed line segment in $\mathbb{R}^2$ determined by $f(x)$ and $f(\varphi^2(x))$, that is,

$$f(\varphi(x)) \in [f(x), f(\varphi^2(x))]. \quad (2)$$

Remark 1. 1. Let us observe that when $A = X$ in Theorem 1.1, one has $i_*(H_1(A, \mathbb{Q})) = H_1(X, \mathbb{Q})$.

The following consequence of Theorem 1.1 was proved in [1, Corollary 2.10].

COROLLARY 1.1. *Let X be a Hausdorff space and A a compact, connected, and locally pathwise connected subset of X . Let $\varphi : X \rightarrow X$ be a continuous map such that $\varphi(A) \subset A$ and $(\varphi|_A)^3 = Id_A$. Suppose that*

$$Id_* - \varphi_*^1 : i_*(H_1(A, \mathbb{Q})) \rightarrow i_*(H_1(A, \mathbb{Q}))$$

is a surjective map. Then for every continuous map $g : X \rightarrow \mathbb{R}$, there exists a point $x \in X$ such that $g(x) = g(\varphi(x)) = g(\varphi^2(x))$.

It will be goal in this paper to give some applications of Theorem 1.1. In Section 2, we generalize the Pannwitz's construction, replacing the sphere S^n by more general topological space. As a consequence, using Corollary 1.1, we obtain a result about coincidence

¹Erika Pannwitz (1904-1975) was a PhD student of Heinz Hopf in Berlin in the 1930s. She was the first person to show that knotted polygonal curves have quadriseccants.

points. In Section 3, we describe an useful method to verify the hypotheses of Theorem 1.1. Further, we construct examples of topological spaces of dimension one for which Theorem 1.1 can be applied. In particular, in Section 4, when X is a connected graph, we obtain an interesting physical interpretation, by using the condition of alignment of three points, given by (2).

Throughout the paper, we assume that homology groups will always have rational coefficients. Thus the p th homology group $H_p(X)$ of a topological space X is considered a vector space over the field of rational numbers; the symbol $\tilde{H}_*(X)$ denotes the reduced homology groups.

2. A GENERALIZATION OF THE PANNWITZ'S CONSTRUCTION

The main result of this section was motivated upon ideas of Pannwitz [5]. For each $n \geq 0$, she constructed maps $\varphi_n : S^n \rightarrow S^n$ and $f_n : S^n \rightarrow \mathbb{R}^2$ such that $f_n(x) \neq f_n(\varphi_n(x))$, for any $x \in S^n$. Such construction was made by induction on n . Just observing that the suspension of S^n is S^{n+1} , a similar construction can be made if we replace the sphere S^n by more general topological space X and S^{n+1} by its suspension SX . This is a nice way to generalize this result.

For a given topological space X , we will denote by SX the suspension of X , that is, the quotient of $X \times I$ obtained by collapsing $X \times \{1\}$ to one point and $X \times \{-1\}$ to another point. We prove the following

THEOREM 2.1. *Suppose that there exists a free map $g : X \rightarrow \mathbb{R}^2$, that is, there is a map $\rho : X \rightarrow X$ such that $g(p) \neq g(\rho(p))$, for any $p \in X$. For any $p \in X$, let us denote $g(p) = (u(p), v(p))$ and $g(\rho(p)) = (u(\rho(p)), v(\rho(p)))$. If $v(p) \leq v(\rho(p))$ for any $x \in X$ then there is a free map $f : SX \rightarrow \mathbb{R}^2$.*

Proof. Denote by O_1 and O_2 the points $(1, p)$ and $(-1, p)$ in SX , respectively. Thus, we can define maps $\varphi : SX \rightarrow SX$ and $f : SX \rightarrow \mathbb{R}^2$ by

$$\begin{aligned}\varphi(t, p) &= (-t|t|, \rho(p)) \\ f(t, p) &= tf(O_1) + (1 - |t|)g(p), \quad \forall (t, p) \in SX,\end{aligned}\tag{3}$$

where $f(O_1) = (1, 0)$, $f(O_2) = (-1, 0)$ and $-1 \leq t \leq 1$. We write

$$f(t, p) = (x(t, p), y(t, p)) \quad \text{and} \quad f(\varphi(t, p)) = (x(\varphi(t, p)), y(\varphi(t, p))).$$

Without loss of generality, we can assume that the image $g(X)$ is entirely contained in the upper half-plane $v > 0$. Now we prove that f is a free map. For any $(t, p) \in SX$, with $0 < |t| < 1$, one has that

$$\begin{aligned}y(t, p) &= (1 - |t|)v(p); \\ y(\varphi(t, p)) &= y(-t|t|, \rho(p)) = (1 - t^2)v(\rho(p)).\end{aligned}$$

From our preceding assumption, $g(\rho(X))$ is contained in the half-plane $v > 0$ and by hypotheses we have that $0 < v(p) \leq v(\rho(p))$. Furthermore, it follows from $(1 - |t|) < (1 - t^2)$ that for any $(t, p) \in SX$, with $0 < |t| < 1$

$$y(t, p) = (1 - |t|)v(p) < (1 - t^2)v(\rho(p)) = y(\varphi(t, p)),$$

which implies $f(t, p) \neq f(\varphi(t, p))$, for any $(t, p) \in SX$, with $0 < |t| < 1$. Moreover, $f(O_1) \neq f(O_2)$ and since f coincides with the free map g in points of kind $(0, p)$, we conclude that $f(t, p) \neq f(\varphi(t, p))$, for any $(t, p) \in SX$ and therefore f is a free map. $\blacksquare$

The following corollary is an important application of Corollary 1.1 and it is based in the definition of the map φ , given in the proof of Theorem 2.1.

A $\mathbb{Z}_p$ -space is a pair (X, ρ) , where X is a topological space equipped with a free action of the cyclic group $\mathbb{Z}_p$, generated by a periodic homeomorphism $\rho : X \rightarrow X$ of period p .

COROLLARY 2.1. *Let (X, ρ) be a $\mathbb{Z}_3$ -space. Suppose that X is a compact, connected, locally pathwise connected and Hausdorff space. Then there exists $\varphi : SX \rightarrow SX$ satisfying:*
(i) $\varphi|_X = \rho$ and $\varphi(x) \neq x$, for any $x \in SX$;
(ii) for every continuous map $g : SX \rightarrow \mathbb{R}$ there exists a point $x \in SX$ such that $g(x) = g(\varphi(x)) = g(\varphi^2(x))$.

Proof. It follows from step (3) of the proof of Theorem 2.1 that there exists a map $\varphi : SX \rightarrow SX$ such that $\varphi|_X = \rho$ and $\varphi(x) \neq x$, for any $x \in SX$. On the other hand, since (X, ρ) is a $\mathbb{Z}_3$ -space, one has $\varphi(X) \subset X$ and $(\varphi|_X)^3 = Id_X$. Moreover, the inclusion map $i : X \rightarrow SX$ induces trivial homomorphism in homology, which implies $i_*(H_1(X; \mathbb{Q})) = 0$, consequently $Id_* - \varphi_*$ is a surjective map. It follows from Corollary 1.1, that for every continuous map $g : SX \rightarrow \mathbb{R}$, there exists $x \in X$ such that $g(x) = g(\varphi(x)) = g(\varphi^2(x))$. $\blacksquare$

EXAMPLE 2.1. Let $\rho : S^1 \rightarrow S^1$ be a rotation through an angle $2\pi/3$. We have that (S^1, ρ) is a $\mathbb{Z}_3$ -space. It follows from Corollary 2.1 that there exists $\varphi : S^2 \rightarrow S^2$ such that for every continuous map $g : S^2 \rightarrow \mathbb{R}$, there exists $x \in X$ such that $g(x) = g(\varphi(x)) = g(\varphi^2(x))$.

It is important to emphasize that in Corollary 2.1, is not necessary that the cyclic group $\mathbb{Z}_3$ acts freely on SX , we only need to assume that it acts freely on X . Let us note that, in this conditions, the classical Borsuk-Ulam theorems (see, for example [4]) cannot be applied to prove that every continuous map $g : SX \rightarrow \mathbb{R}$ sends three points of some orbit of φ to a single point, that is, $g(x) = g(\varphi(x)) = g(\varphi^2(x))$, for some $x \in SX$. Therefore, Corollary 2.1 is a special result about coincidence points in which the group only acts freely on some subspace of a certain topological space. This situation is illustrated by Example 2.1, since the cyclic group $\mathbb{Z}_3$ acts freely on the sphere S^1 , but does not acts freely on sphere S^2 .

3. APPLICATIONS OF THEOREM 1.1 TO GRAPHS

In order to apply Theorem 1.1, we need to know the homomorphism $Id_* - \varphi_*^1$. The Propositions 3.1 and 3.2 establish efficient methods which allows to verify when this map is surjective.

PROPOSITION 3.1. *Let X be a topological space, $\rho : X \rightarrow X$ a continuous map and $\varphi : S(X) \rightarrow S(X)$ defined as in (3). Let us denote by $[\varphi_*^1]$ and $[\rho_*^0]$ the matrices of the linear transformations $\varphi_*^1 : H_1(SX; \mathbb{Q}) \rightarrow H_1(SX; \mathbb{Q})$ and $\rho_*^0 : H_0(X; \mathbb{Q}) \rightarrow H_0(X; \mathbb{Q})$, respectively and let I be the identity matrix. Then*

$$\det(I - [\varphi_*^1]) = (1/2) \cdot \det(I + [\rho_*^0]). \quad (4)$$

Proof. Let us observe that a basic property of the induced homomorphism $\varphi_*^p : H_p(S(X); \mathbb{Q}) \rightarrow H_p(S(X); \mathbb{Q})$ is the following

$$\varphi_*^p = \begin{cases} (-1)^p \rho_*^{p-1}, & \text{if } p > 1 \\ (-1)^p \tilde{\rho}_*^{p-1}, & \text{if } p = 1. \end{cases} \quad (5)$$

where $\tilde{\rho}_*$ denotes the induced homomorphism by ρ between the reduced homology groups. In particular, if $p = 1$, one has

$$\varphi_*^1 = -\tilde{\rho}_*^0. \quad (6)$$

Now, let us describe how the matrices of the induced homomorphisms

$$\begin{aligned} \rho_*^0 : H_0(X; \mathbb{Q}) &\rightarrow H_0(X; \mathbb{Q}) \\ \tilde{\rho}_*^0 : \tilde{H}_0(X; \mathbb{Q}) &\rightarrow \tilde{H}_0(X; \mathbb{Q}) \end{aligned}$$

are related: if $x_1, x_2, \dots, x_k$ denote points in each connected component $X_1, X_2, \dots, X_k$ of X , we may choose a basis $\{u_1, u_2, \dots, u_k\}$ for the finite-dimensional rational vector space $H_0(X, \mathbb{Q})$, with $u_i = [x_i]$ a generator of $H_0(X_i, \mathbb{Q})$, for $i = 1, 2, \dots, k$. From $\{u_1, u_2, \dots, u_k\}$, we can produce a basis $\beta = \{v_1, v_2, \dots, v_k\}$ for $H_0(X, \mathbb{Q})$ by defining

$$v_i = \begin{cases} u_i, & \text{if } i = 1 \\ u_i - u_1, & \text{if } i = 2, \dots, k. \end{cases} \quad (7)$$

Thus, the matrix of ρ_*^0 with respect to the basis β is the $k \times k$ block matrix

$$[\rho_*^0]_\beta = \begin{bmatrix} I_1 & O_{1 \times k-1} \\ C_{k-1 \times 1} & [\tilde{\rho}_*^0]_{\tilde{\beta}} \end{bmatrix} \quad (8)$$

where I_1 denotes the 1×1 identity matrix, $O_{1 \times k-1}$ denotes the $1 \times k-1$ zero matrix and $[\tilde{\rho}_*^0]_{\tilde{\beta}}$ denotes the $k-1 \times k-1$ matrix of $\tilde{\rho}_*^0$ with respect to the basis $\tilde{\beta} = \{v_2, v_3, \dots, v_k\}$

of $\tilde{H}_0(X; \mathbb{Q})$. From (6), one has $[\varphi_*^1] = -[\tilde{\rho}_*^0]_{\tilde{\beta}}$, which implies that

$$\det(I - [\varphi_*^1]) = \det(I + [\tilde{\rho}_*^0]_{\tilde{\beta}}). \quad (9)$$

On the other hand,

$$I + [\rho_*^0]_{\beta} = \begin{bmatrix} 2I_1 & O_{1 \times k-1} \\ C_{k-1 \times 1} & I + [\tilde{\rho}_*^0]_{\tilde{\beta}} \end{bmatrix} \quad (10)$$

From (9) and (10), we can compute the determinant

$$\det(I + [\rho_*^0]_{\beta}) = 2 \cdot \det(I + [\tilde{\rho}_*^0]_{\tilde{\beta}}) = 2 \cdot \det(I - [\varphi_*^1]), \quad (11)$$

and we obtain the desired formula

$$\begin{aligned} \det(I - [\varphi_*^1]) &= (1/2) \cdot \det(I + [\rho_*^0]_{\beta}) \\ &= (1/2) \cdot \det(I + [\rho_*^0]). \end{aligned} \quad (12)$$

■

In next example, we construct a *free map* from an 1-dimensional topological space into Euclidean 2-space $\mathbb{R}^2$ and we show that Theorem 1.1 is true in this case.

EXAMPLE 3.1. Let $X = \{x_1, x_2, x_3\}$ and define a map $\rho : X \rightarrow X$ by

$$\rho(x_1) = x_2, \rho(x_2) = x_3 \text{ and } \rho(x_3) = x_1. \quad (13)$$

Then $\rho(x) \neq x$, for any $x \in X$. It follows from (3), that there exist $\varphi : SX \rightarrow SX$ such that $\varphi(x) \neq x$, for any $x \in SX$. Let $f : X \rightarrow \mathbb{R}^2$ be a topological embedding and observe that $f(x) \neq f(\varphi(x))$, for any $x \in SX$, that is, f is a *free map*. We affirm that, there exists a point $x \in SX$ such that

$$f(\varphi(x)) \in [f(x), f(\varphi^2(x))], \quad (14)$$

where $[f(x), f(\varphi^2(x))]$ denotes the closed line segment in $\mathbb{R}^2$ joining the points $f(x)$ and $f(\varphi^2(x))$. To verify this assertion, it suffices to show that

$$Id_* - \varphi_*^1 : H_1(SX; \mathbb{Q}) \rightarrow H_1(SX; \mathbb{Q}) \quad (15)$$

is a surjective map and in this case, the result follows from Theorem 1.1. Since that $Id_* - \varphi_*^1$ is a linear transformation between vector spaces, let us check that $\det(Id_* - [\varphi_*^1]) \neq 0$. Let u_1, u_2, u_3 be generators of $H_0(X; \mathbb{Q})$, then $\rho_*^0(u_1) = u_2$, $\rho_*^0(u_2) = u_3$ and $\rho_*^0(u_3) = u_1$. The matrix of ρ_*^0 has the form

$$[\rho_*^0] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (16)$$

Moreover, one has that

$$I + [\rho_*^0] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad (17)$$

From Proposition 3.1,

$$(1/2) \cdot \det(I + [\rho_*^0]) = 1 = \det(I - [\varphi_*^1]) \neq 0, \quad (18)$$

and the result follows.

PROPOSITION 3.2. *Let X be a connected graph and let $\varphi : X \rightarrow X$ be a continuous map such that φ sends each vertex of X to a vertex of X . Then*

$$\det(Id - \varphi_{\#}^1) = \det(Id - \varphi_*^1) \cdot \det(Id - \varphi_{\#}^0)$$

Proof. Let us consider the homomorphisms

$$\begin{aligned} \varphi_{\#}^i &: C_i(X) \rightarrow C_i(X) \\ \varphi_*^i &: H_i(X) \rightarrow H_i(X), \text{ for } i = 0, 1. \end{aligned}$$

Since X is a connected 1-dimensional CW complex, we have that $C_2(X) = 0$ and therefore the boundary operator $\partial_2 : C_2(X) \rightarrow C_1(X)$ is trivial, which implies that

$$H_1(X) = \frac{Z_1(X)}{\text{Im}(\partial_2)} = Z_1(X)$$

One can write $C_1(X) = H_1(X) \oplus L_1(X)$, where $\partial_1|_{L_1(X)} : L_1(X) \rightarrow B_0(X)$ is an isomorphism. Let us consider the following commutative diagram

$$\begin{array}{ccc} C_1(X) & \xrightarrow{\partial_1} & B_0(X) \\ \varphi_{\#}^1 \downarrow & & \downarrow \varphi_{\#}^0 \\ C_1(X) & \xrightarrow{\partial_1} & B_0(X) \end{array} \quad (19)$$

Choose $B = B_1 \cup B_2$ as the basis for $C_1(X)$, where $B_1 = \{u_1, \dots, u_n\}$ is a basis for $H_1(X)$ and $B_2 = \{v_1, \dots, v_m\}$ is a basis for $L_1(X)$. Thus, the linear transformation $\varphi_{\#}^1$ is given

by formulas:

$$\begin{aligned}\varphi_{\#}^1(u_j) &= \sum_{i=1}^n a_{ij}u_i + \sum_{i=1}^m b_{ij}v_i, \quad j = 1, \dots, n \\ \varphi_{\#}^1(v_j) &= \sum_{i=1}^n c_{ij}u_i + \sum_{i=1}^m d_{ij}v_i, \quad j = 1, \dots, m.\end{aligned}$$

If we write $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ and $D = (d_{ij})$, then the matrix of $\varphi_{\#}^1$ is the block matrix

$$[\varphi_{\#}^1] = \begin{bmatrix} A & C \\ B & D \end{bmatrix} \quad (20)$$

One has that $A = (a_{ij})$ is the matrix of $\varphi_*^1 : H_1(X) \rightarrow H_1(X)$ and since that $\partial_1|_{L_1(X)}$ is an isomorphism between $L_1(X)$ and $B_0(X)$, it follows from diagram 19 that C is the zero matrix and $D = (d_{ij}) = [\tilde{\varphi}_{\#}^0]$. Therefore

$$[\varphi_{\#}^1] = \begin{bmatrix} [\varphi_*^1] & O \\ B & [\tilde{\varphi}_{\#}^0] \end{bmatrix} \quad (21)$$

Thus, the determinant of the matrix $I - [\varphi_{\#}^1]$ is equal to

$$\det(I - [\varphi_{\#}^1]) = \det(I - [\varphi_*^1]) \cdot \det(I - [\tilde{\varphi}_{\#}^0]),$$

which completes the proof of the proposition. $\blacksquare$

EXAMPLE 3.2. Consider X an oriented connected graph as indicated in Figure 1 and let us denote by $A = \{a_1, a_2, a_3\}$ the set of vertices of X and by $L = \{l_1, l_2, l_3, l_4\}$ the set of edges of X .

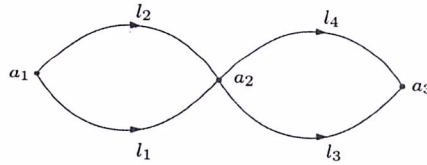


FIG. 1.

Let $\varphi : X \rightarrow X$ be a continuous map defined on vertices of X as follows

$$\varphi(a_1) = a_2, \quad \varphi(a_2) = a_3, \quad \varphi(a_3) = a_1$$

and let $\varphi_{\#}^1 : C_1(X) \rightarrow C_1(X)$ be the chain map induced by φ given by

$$\varphi_{\#}^1(l_1) = l_4, \quad \varphi_{\#}^1(l_2) = l_3, \quad \varphi_{\#}^1(l_3) = -l_1 - l_4, \quad \varphi_{\#}^1(l_4) = -l_2 - l_3. \quad (22)$$

Then φ sends vertex to vertex and $\varphi(x) \neq x$, for any $x \in X$. Let us check that $\det(I - \varphi_{\#}^1) \neq 0$. From (22), the matrix of $\varphi_{\#}^1$ is given by

$$[\varphi_{\#}^1] = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \quad \text{and thus} \quad I - [\varphi_{\#}^1] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{bmatrix}$$

which implies that $\det(I - [\varphi_{\#}^1]) = -3 \neq 0$ and it follows from Proposition 3.2 that $\det(I - [\varphi_{\#}^1]) \neq 0$.

Remark 3. 1. Let X be a connected graph and $\varphi : X \rightarrow X$ such that the determinant of $I - [\varphi_{\#}^1]$ is not zero. It follows from Proposition 3.2 that the determinant of $I - [\varphi_{\#}^1]$ is not zero. Then, if $f : X \rightarrow \mathbb{R}^2$ is a topological embedding or an inclusion map, from Theorem 1.1 there exists $x \in X$ such that the three points

$$f(x), \quad f(\varphi(x)) \quad \text{and} \quad f(\varphi^2(x))$$

are lined up, with $f(\varphi(x))$ between $f(x)$ and $f(\varphi^2(x))$.

Remark 3. 2. Consider $\varphi : S^1 \rightarrow S^1$ a rotation through an angle $2\pi/3$ and let us denote by $f : S^1 \rightarrow \mathbb{R}^2$ the natural inclusion. Then, there is not a point $x \in S^1$ with the property that the three points $f(x)$, $f(\varphi(x))$ and $f(\varphi^2(x))$ are lined up. In this case, it is possible to verify that the hypothesis of Theorem 1.1 is not satisfied, since $\det(I - [\varphi_{\#}^1]) = 0$.

4. A PHYSICAL INTERPRETATION

It has been known the following physical interpretation of the classical Borsuk-Ulam theorem [2]: *there exists at least one pair of antipodal points on the surface of the earth having identical atmospheric pressures and temperatures.*

Next, we give an interesting physical interpretation of the non-standard version of the Borsuk-Ulam theorem.

Imagine three particles P_1 , P_2 and P_3 moving in 2-space in such a way that its positions at time t relative to some coordinate system are given by

$$x(t), \varphi(x(t)) \quad \text{and} \quad \varphi^2(x(t)),$$

respectively, where φ is a continuous map and when t varies through a time interval, the path traced out by the particles is a connected graph X .

We only assume that φ and X satisfy nice properties as in Remark 3.1. The results previously obtained guarantee that *there exists an instant t_0 such that the particles P_1 , P_2 and P_3 are lined up. Moreover, P_2 is between P_1 and P_3 .*

REFERENCES

1. Biasi, C., de Mattos, A *non-standard version of the Borsuk-Ulam Theorem*, Bull. Pol. Ac. Sc. Math., (53) (1) 111-119, 2005.
2. Borsuk, K., *Drei Stze ber die n -dimensionale euklidische Sphre*, Fund. Math., (20) 177-190, 1933.
3. Hopf, H. *Freie berdeckungen und freie Abbildungen*. Fund, Math. (28) 33-57, 1937.
4. Munkholm, H.J. *Borsuk-Ulam Theorems for proper $\mathbb{Z}_p$ -actions on (mod p homology) n -spheres*. Math. Scan. (24) 167-185, 1969.
5. Pannwitz, E. *Eine freie Abbildung der n -dimensionalen Sphre in die Ebene*, Math. Nachr. (7) 183-185, 1952.

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SÉRIE MATEMÁTICA

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- 278/2007 CARVALHO, A.N.; CHOLEWA, J.W. – Local well posedness, asymptotic behavior and asymptotic bootstrapping for a class of semilinear evolution equations of the second order in time.
- 277/2007 MATTOS, D.; SANTOS, E.L. – A parametrized Borsuk-Ulam theorem for a product of spheres with free Z_p – action and free S^1 - action.