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NON-AUTONOMOUS SEMILINEAR EVOLUTION EQUATIONS WITH ALMOST SECTORIAL OPERATORS

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Resumo

Inspirados na teoria de semigrupos com crescimento α, construímos os *processos de evolução lineares com crescimento* α com a finalidade de lidar com problemas singularmente não autônomos que apresentam natureza singular no instante inicial. A teoria abstrata linear é aplicada ao estudo de problemas parabólicos singularmente não autônomos semilineares. Provamos que, sob condições naturais, um conceito razoável de solução pode ser dado para tais problemas parabólicos singularmente não autônomos com natureza singular no instante inicial. Consideramos aplicações a problemas parabólicos singularmente não autônomos em espaços de funções Hölder contínuas e a problemas parabólicos singularmente não autônomos em domínios $\Omega \subset \mathbb{R}^n$ com alças unidimensionais.

NON-AUTONOMOUS SEMILINEAR EVOLUTION EQUATIONS WITH ALMOST SECTORIAL OPERATORS

ALEXANDRE N. CARVALHO†, TOMASZ DLOTKO‡, AND MARCELO J. D. NASCIMENTO*

ABSTRACT. Inspired by the theory of semigroups of growth α, we construct *an evolution process of growth* α. The abstract theory is next applied to study semilinear singular non-autonomous parabolic problems. We prove that, under natural assumptions, a reasonable concept of solution can be given to such semilinear singularly non-autonomous problems. Applications are considered to non-autonomous parabolic problems in space of Hölder continuous functions and to a parabolic problem in a domain $\Omega \subset \mathbb{R}^n$ with a one dimensional handle.

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1. INTRODUCTION

In this paper we consider the Cauchy problem for singular non-autonomous evolution equation

$$\begin{cases} \dfrac{dx}{dt} + A(t)x = f(x), \\ x(\tau) = x_0 \in X, \end{cases}$$

in a Banach space X. In the usual theory (see [19, 18]) each operator $A(t)$ is sectorial and the family $\{A(t)\}$ is Hölder continuous with respect to time t (which will be precised later). Our interest is to consider operators $A(t)$ whose resolvents contain a sector, but for which the resolvent operators do not satisfy the estimate required to be sectorial. Instead we assume that for each t the operator $A(t)$ is *almost sectorial.*

To better explain the results we need to introduce some terminology. Recall first the definition of an almost sectorial operator (see [4, 17]). A closed linear operator $A(t)$: $D(A(t)) \subset X \to X$ is called *almost sectorial* if for some $\theta \in (0, \frac{\pi}{2})$,

$$\rho(-A(t)) \supset \Sigma_\theta := \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg \lambda| \leq \pi - \theta\} \cup \{0\} \tag{1.1}$$

and, for some $0 < \alpha < 1$,

$$\|(\lambda + A(t))^{-1}\|_{L(X)} \leq \frac{C}{|\lambda|^\alpha}, \quad \forall \lambda \in \Sigma_\theta. \tag{1.2}$$

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Consequently,

$$\|A(t)(\lambda + A(t))^{-1}\|_{L(X)} \leq 1 + C|\lambda|^{1-\alpha}, \quad \forall \lambda \in \Sigma_\theta. \tag{1.3}$$

It is impossible to conclude from the above estimates that $A(t)$ generates a strongly continuous semigroup. However we will show that $A(t)$ generates a semigroup with a singularity at $t = 0$ in a sense that will be specified later.

Theory of semigroups of growth $1-\alpha$ was investigated earlier by Da Prato [8], when $1-\alpha$ is a non negative integer. In [14], the author extend the notion for $1-\alpha > 0$ (see [20]). Recall de definition of semigroup of growth $1-\alpha$.

Definition 1.1 ([14]). *Let $1-\alpha > 0$. Then a family of operators $\{T(t) : t \geq 0\}$ on X that satisfies the conditions*

(*i*) *$T(t+s) = T(t)T(s)$ for every $t, s > 0$,*
(*ii*) *If $T(t)u = 0$ for all $t > 0$ then $u = 0$,*
(*iii*) *$\|t^{1-\alpha}T(t)\|$ is bounded as t tends to zero,*
(*iv*) *$X_0 = \bigcup_{t>0} T(t)[X]$ is dense in X,*

is called the semigroup of growth $1-\alpha$.

Note, that the condition (*i*) of the above definition need not be satisfied for $t = 0$ or $s = 0$. Examples of semigroups of growth $1-\alpha$ will be found in Kreĭn [11]. Inspired by the theory of semigroups of growth $1-\alpha$, we define an *evolution process of growth* $1-\alpha$ of the following manner.

Definition 1.2. *Let $1-\alpha > 0$. A family $\{U(t,\tau) :\ t > \tau \in \mathbb{R}\} \subset L(X)$ that satisfies*

(1) *$U(t,\sigma)U(\sigma,\tau) = U(t,\tau)$, for any $t > \sigma > \tau$, $\tau \in \mathbb{R}$,*
(2) *$\|(t-\tau)^{1-\alpha}U(t,\tau)\| \leq M$, $\forall t > \tau$,*
(3) *$(t,\tau) \mapsto U(t,\tau)v_0$ is continuous for $t > \tau$, $v_0 \in X$,*

is called a linear evolution process of growth $1-\alpha$.

Let Γ be the boundary of the sector of Σ_θ oriented in such a way that the imaginary part is increasing. If

$$T_{A(\tau)}(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + A(\tau))^{-1} d\lambda, \tag{1.4}$$

by (1.2) we have that $\{T_{A(\tau)}(t) : t > 0\} \subset L(X)$. Then, from (1.2), we obtain that $T_{A(\tau)}(t)$ is a semigroup of growth order $1-\alpha$. Theory of linear evolution process of growth α is presented in Section 2. The notion of *linear evolution process of growth* α, $\{U(t,\tau) : t > \tau \in \mathbb{R}\}$, is defined there allowing us to solve linear homogeneous problem

$$\frac{dx}{dt} + A(\tau)x = 0,\ x(0) = x_0.$$

The main result of that section, Theorem 2.4, asserts local solvability of the integral equation

$$U(t,\tau) = T_{A(\tau)}(t-\tau) + \int_\tau^t U(t,s)[A(\tau) - A(s)]T_{A(\tau)}(s-\tau)ds,$$

using Banach contraction principle in a complete metric space

$$E = \{U \in C([\tau, t_0), L(X)) : \sup_{t\in[\tau,\, t_0)} (t_0 - t)^{1-\alpha} \|U(t)\|_{L(X)} < \infty\}.$$

The corresponding semilinear parabolic problem is studied in Section 3. Assuming that the nonlinear term fulfills suitable growth restriction we prove there its local solvability (see Theorem 3.2). Applications of the abstract theory to semilinear parabolic problems in Hölder spaces $C_0^{\mu}(\bar{\Omega})$ and to parabolic problems in n dimensional domains with one dimensional handles are described next in Section 4.

The notion of an almost sectorial operator was studied by W. von Wahl in [22]. An interesting investigation of functional calculus for such operators were presented recently in [16, 17]. Our studies of the semigroups generated by almost sectorial operators have their origin in the paper [4]. Here we extend the existence results known in case of autonomous problems to time dependent equations and connected with them *processes of growth* α.

2. Abstract theory of evolution processes of growth α

Let X be a Banach space and $A(t) : D(A(t)) \subset X \to X, t \geq 0$, a family of closed operators. In this section we introduce the problem

$$\begin{cases} \dfrac{du}{dt} + A(t)u = f(t) \\ u(0) = u_0 \in X, \end{cases} \tag{2.1}$$

where the operators $A(t)$ have some special features that we describe below. In particular, any of the operators $A(t)$ need not generate an analytic semigroup nor a C^0–semigroup. Moreover, f is an appropriate nonlinearity defined on X.

Remark 2.1. *Throughout this paper the name "analytic semigroup" is reserved for C^0 (strongly continuous) analytic semigroup (as in* [15]*). The semigroups consider here, called "semigroups of growth α", need not be strongly continuous at $t = 0$, also their generator A need not have dense domain, but they are analytic in an open sector of the complex plane $\mathbb{C}$ (compare the definition in* [13]*, p. 34).*

Assumption. Assume that there exists another Banach space Z with $Z \to X$ such that

$$A_Z(t) : D(A_Z(t)) \subset Z \to Z \tag{2.2}$$

with $D(A_Z(t)) = \{x \in D(A(t)) \cap Z : A(t)x \in Z\}$, and $A_Z(t)$ satisfies

$$\|(\lambda + A_Z(t))^{-1}\|_{L(Z)} \leq \frac{C}{|\lambda|}, \quad \forall \lambda \in \Sigma_\theta.$$

Remark 2.2. *While at the first view this assumption looks strange, it is automatically satisfied for the natural and important examples described at the end of the paper.*

For $0 < \nu < 1$ we can define fractional powers $A_Z(t)^{-\nu} \in L(X^0)$ of $A_Z(t)$ as

$$A_Z(t)^{-\nu} = \frac{\sin \pi\nu}{\pi} \int_0^\infty s^{-\nu}(s + A_Z(t))^{-1} ds,$$

and observe that, for $\nu > 1-\alpha$, the integral operator

$$A(t)^{-\nu} := \frac{\sin \pi\nu}{\pi}\int_0^\infty s^{-\nu}(s+A(t))^{-1}ds$$

defines an operator in $L(X)$ that coincides with $A_Z(t)^{-\nu}$ on Z. We introduce $A(t)^{\nu}$ as the inverse of $A(t)^{-\nu}$ and set $X^\nu = D(A(t)^\nu)$ with the graph norm.

Moreover, for $1 > \alpha > \nu > 1-\alpha$ (consequently; $\alpha > \frac{1}{2}$),

$$\begin{aligned}
&\|A(t)^{-1+\nu}x\|_X \\
&\le \frac{\sin \pi\nu}{\pi}\Big(\int_0^\tau s^{-1+\nu}\left\|(s+A(t))^{-1}x\right\|_X ds + \int_\tau^\infty s^{-1+\nu}\left\|(s+A(t))^{-1}x\right\|_X ds\Big)\\
&\le \frac{\sin \pi\nu}{\pi}\Big(\int_0^\tau s^{-1+\nu}\left\|A(t)(s+A(t))^{-1}A(t)^{-1}x\right\|_X ds + \int_\tau^\infty s^{-1+\nu}\left\|(s+A(t))^{-1}x\right\|_X ds\Big)\\
&\le \frac{\sin \pi\nu}{\pi}\Big(M_1\int_0^\tau s^{-1+\nu}ds\left\|A(t)^{-1}x\right\|_X + M_2\int_\tau^\infty s^{-1-\alpha+\nu}ds\left\|x\right\|_X\Big)\\
&\le \frac{\sin \pi\nu}{\pi}\Big(\frac{M_1}{\nu}\tau^{\nu}\left\|A(t)^{-1}x\right\|_X + \frac{M_2}{\alpha-\nu}\tau^{\nu-\alpha}\left\|x\right\|_X\Big)\\
&\le \frac{\sin \pi\nu}{\pi}M_1^{1-\frac{\nu}{\alpha}}M_2^{\frac{\nu}{\alpha}}\Big(\frac{1}{\nu}+\frac{1}{\alpha-\nu}\Big)\left\|A(t)^{-1}x\right\|_X^{1-\frac{\nu}{\alpha}}\left\|x\right\|_X^{\frac{\nu}{\alpha}},
\end{aligned}$$

where, in the last inequality, we minimize the right hand side for $\tau \in (0,\infty)$. Thus, for $x \in D(A(t))$ we have

$$\|A(t)^\nu x\|_X \le \frac{\sin \pi\nu}{\pi}M_1^{1-\frac{\nu}{\alpha}}M_2^{\frac{\nu}{\alpha}}\Big(\frac{1}{\nu}+\frac{1}{\alpha-\nu}\Big)\|x\|_X^{1-\frac{\nu}{\alpha}}\|A(t)x\|_X^{\frac{\nu}{\alpha}}. \tag{2.3}$$

Using the fact that $\frac{\pi}{\sin \pi r} = \Gamma(r)\Gamma(1-r)$ and the expression

$$(\lambda + A(t))^{-1} = \int_0^\infty e^{-\lambda s}T_{A(t)}(s)ds, \quad \forall\lambda > 0,$$

we have for $\nu > 1-\alpha$,

$$A(t)^{-\nu} = \frac{1}{\Gamma(\nu)}\int_0^\infty s^{\nu-1}T_{A(t)}(s)ds. \tag{2.4}$$

Let Γ be the boundary of the sector of Σ_θ oriented in such a way that the imaginary part is increasing. Observe that

$$\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + A(\tau))^{-1}d\lambda,$$

converge in the uniform operator topology of $L(X)$ for all $t > 0$; in fact,

$$\begin{aligned}
\Big\|\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + A(\tau))^{-1}d\lambda\Big\|_{L(X)} &\le \frac{1}{2\pi}\int_\Gamma e^{-\cos\theta|\lambda|t}\|(\lambda + A(\tau))^{-1}\|_{L(X)}d|\lambda|\\
&\le \frac{1}{2\pi}\int_\Gamma e^{-\cos\theta|\lambda|t}\frac{C}{|\lambda|^\alpha}d|\lambda|\\
&\le \frac{t^{\alpha-1}}{2\pi}\int_\Gamma e^{-\cos\theta|\mu|}\frac{C}{|\mu|^\alpha}d|\mu|.
\end{aligned}$$

Hence, if

$$T_{A(\tau)}(t)=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda+A(\tau))^{-1}d\lambda, \tag{2.5}$$

we have that $\{T_{A(\tau)}(t):t>0\}\subset L(X)$ and there is a constant $C>0$ such that

$$\|T_{A(\tau)}(t)\|_{L(X)}\le Ct^{-1+\alpha}. \tag{2.6}$$

Note that $\{T_{A(\tau)}(t):t\ge 0\}$ decays exponentially.

In the same way as when $-A(\tau)$ generates a usual analytic semigroup, one can show that the semigroup property, $T_{A(\tau)}(t+s)=T_{A(\tau)}(t)T_{A(\tau)}(s)$, $t,s>0$, is satisfied. Also, since the operator $A(\tau)$ is closed and since, from (1.3), the integral

$$\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda+A(\tau))^{-1}d\lambda,$$

is convergent we have that $A(\tau)T_{A(\tau)}(t)\in L(X)$ for all $t>0$ and

$$\begin{aligned}
\|A(\tau)T_{A(\tau)}(t)\|_{L(X)}&\le\left\|\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}A(\tau)(\lambda+A(\tau))^{-1}d\lambda\right\|_{L(X)}\\
&\le\frac{1}{2\pi}\int_\Gamma e^{-\cos\theta|\lambda|t}\|A(\tau)(\lambda+A(\tau))^{-1}\|_{L(X)}d|\lambda|\\
&\le\frac{1}{2\pi}\int_\Gamma e^{-\cos\theta|\lambda|t}C(1+|\lambda|^{1-\alpha})d|\lambda|\\
&=\frac{1}{2\pi}\int_\Gamma e^{-\cos\theta|\mu|}C(1+t^{\alpha-1}|\mu|^{1-\alpha})t^{-1}d|\mu|\\
&\le\max\{t^{-1},t^{-2+\alpha}\}\frac{1}{2\pi}\int_\Gamma e^{-\cos\theta|\mu|}C(1+|\mu|^{1-\alpha})d|\mu|\\
&\le M\max\{t^{-1},t^{-2+\alpha}\}.
\end{aligned} \tag{2.7}$$

Note that, for $0<t<1$, we have

$$\|A(\tau)T_{A(\tau)}(t)\|_{L(X)}\le Mt^{-2+\alpha}. \tag{2.8}$$

In particular, $T_{A(\tau)}(t)x_0\in D(A)$ for all $t>0$. It follows from of the moment inequality (2.3) and from (2.6) and (2.8) that, for $\alpha>\nu$

$$\left\|A^\nu T_{A(\tau)}(t)\right\|_{L(X)}\le Ct^{-1+\alpha-\frac{\nu}{\alpha}}.$$

Note that, if $0<1-\alpha<\nu<\alpha^2$ (consequently, $1-\nu>1-\alpha$ and $1>\alpha>\frac{\sqrt{5}-1}{2}$), then $-1<\alpha-1-\frac{\nu}{\alpha}$.

The next lemma (for its proof see [4]) says that, in some sense, the semigroup $\{T_{A(\tau)}(t):t\ge 0\}$ defines the solution operator for linear homogeneous problem

$$\begin{cases}\dfrac{dx}{dt}+A(\tau)x=0,\\ x(0)=x_0\in X.\end{cases}$$

Lemma 2.3. *The semigroup $T_{A(\tau)}(t) : (0,\infty) \to L(X)$ is differentiable and*

$$\frac{d}{dt}T_{A(\tau)}(t) = \frac{1}{2\pi i}\int_0^\infty \lambda e^{\lambda t}(\lambda + A(\tau))^{-1}d\lambda.$$

In addition, for each $x_0 \in X$, we have that

$$\frac{d}{dt}T_{A(\tau)}(t)x_0 + A(\tau)T_{A(\tau)}(t)x_0 = 0, \ t > 0.$$

Consider the problem

$$\frac{dx}{dt} + A(t)x = 0 \tag{2.9}$$

in a Banach space X. We shall denote by $U(t,\tau)$ the solution satisfying initial condition

$$v(\tau) = I \quad (I \text{ is the identity operator on } X). \tag{2.10}$$

Note that, for fixed $\tau \in \mathbb{R}$, the family $\{T_{A(\tau)}(t-\tau) : t > \tau\}$ is the solution operator family associated to

$$\frac{dx}{dt} = A(\tau)x. \tag{2.11}$$

Therefore, if $\{U(t,\tau) \in L(X) : t > \tau \in \mathbb{R}\}$ is the linear evolution process of growth α associated to (2.9), the difference $\{U(t,\tau) - T_{A(\tau)}(t-\tau) : t > \tau\}$ is the family of the solution operators associated to

$$\frac{dx}{dt} + A(t)x = -[A(t) - A(\tau)]T_{A(\tau)}(t-\tau) \tag{2.12}$$

with trivial initial condition. Hence we have

$$U(t,\tau) = T_{A(\tau)}(t-\tau) + \int_\tau^t U(t,s)[A(\tau) - A(s)]T_{A(\tau)}(s-\tau)ds. \tag{2.13}$$

To obtain the evolution process of growth α, $\{U(t,\tau) \in L(X) : t > \tau \in \mathbb{R}\}$, we will use (2.13); that is, we show that, for each $\tau \in \mathbb{R}$, equation (2.13) has an unique continuous solution $\{U(t,\tau) \in L(X) : t > \tau\}$.

Assume that

(a) The operator $A(t) : D \subset X \to X$ is a closed linear operator (its domain D is independent of t) and for some $0 < \alpha < 1$, it satisfies the equation (1.2). To express this fact we will say that the family $A(t)$ is *uniformly almost sectorial.*

(b) Moreover, there exists $C > 0$ and $\epsilon \in (0,1]$ such that

$$\|[A(t) - A(\tau)]A^{-1}(s)\|_{L(X)} \le C(t-\tau)^\epsilon, \ \forall t,\tau,s \in \mathbb{R}. \tag{2.14}$$

To express this fact we will say that the function $A(t)$ is *uniformly Hölder continuous.*

Fix $t_0 > \tau$ and denote by $U(t)$ the operator $U(t_0,t)$. For $\alpha + \epsilon > 1$ consider the complete metric space

$$E = \{U \in C([\tau,t_0), L(X)) : \sup_{t\in[\tau,\,t_0)} (t_0 - t)^{1-\alpha}\|U(t)\|_{L(X)} < \infty\}$$

equipped with the metric

$$\|U\|_E = \sup_{t\in[\tau,\, t_0)} (t_0 - t)^{1-\alpha}\|U(t)\|_{L(X)}.$$

Let S be the map defined by

$$(SU)(t) = T_{A(t)}(t_0 - t) + \int_t^{t_0} U(s)[A(t) - A(s)]T_{A(t)}(s - t)ds. \tag{2.15}$$

If $U \in E$, then the integral in the above equality is convergent and

$$t \mapsto \int_t^{t_0} U(s)[A(t) - A(s)]T_{A(t)}(s - t)ds$$

is in $C([\tau, t_0), L(X))$.

We are now able to formulate

Theorem 2.4. *There exists a unique solution of an integral equation* (2.15) *in the metric space* E.

The proof of this result will be given in a number of lemmas presented below.

Lemma 2.5. *For* $t, s, \xi \in \mathbb{R}$, *the following inequalities hold*

$$\|T_{A(t)}(\tau) - T_{A(s)}(\tau)\|_{L(X)} \le c\tau^{-2+2\alpha}(t - s)^\epsilon, \quad \textit{for } \tau > 0, \tag{2.16}$$

$$\|A(\xi)[T_{A(t)}(\tau) - T_{A(s)}(\tau)]\|_{L(X)} \le c\tau^{-3+2\alpha}(t - s)^\epsilon, \quad \textit{for } \tau > 0. \tag{2.17}$$

Proof: Note that

$$(\lambda + A(t))^{-1} - (\lambda + A(s))^{-1} = (\lambda + A(t))^{-1}[A(s) - A(t)](\lambda + A(s))^{-1}.$$

Thus, due to (2.5), we get

$$[T_{A(t)}(\tau) - T_{A(s)}(\tau)] = -\frac{1}{2\pi i}\int_\Gamma e^{\lambda\tau}(\lambda + A(t))^{-1}[A(t) - A(s)](\lambda + A(s))^{-1}d\lambda. \tag{2.18}$$

To prove (2.16) note that

$$\begin{aligned}
&\|T_{A(t)}(\tau) - T_{A(s)}(\tau)\| \\
&= \| - \frac{1}{2\pi i}\int_\Gamma e^{\lambda\tau}(\lambda + A(t))^{-1}[A(t) - A(s)]A(s)^{-1}A(s)(\lambda + A(s))^{-1}d\lambda\| \\
&\le \frac{1}{2\pi}\int_\Gamma e^{-\cos\theta|\lambda|\tau}\frac{c_1}{|\lambda|^\alpha}|\lambda|^{1-\alpha}d|\lambda|\ C(t - s)^\epsilon \\
&\le \frac{Cc_1}{2\pi}\int_\Gamma e^{-\cos\theta|u|}\tau^{-1+2\alpha}|u|^{1-2\alpha}\frac{d|u|}{\tau}\ (t - s)^\epsilon \\
&\le \frac{Cc_1}{2\pi}\tau^{-2+2\alpha}\int_\Gamma e^{-\cos\theta|u|}|u|^{1-2\alpha}d|u|\ (t - s)^\epsilon \le c\tau^{-2+2\alpha}(t - s)^\epsilon.
\end{aligned}$$

This concludes the proof of (2.16). To prove (2.17), note that

$$A(t)\Big[T_{A(t)}(\tau) - T_{A(s)}(\tau)\Big] = -\frac{1}{2\pi i}\int_\Gamma e^{\lambda\tau}A(t)(\lambda + A(t))^{-1}[A(t) - A(s)](\lambda + A(s))^{-1}d\lambda.$$

Then, it follows that

$$\begin{aligned}
\|A(\xi)[T_{A(t)}(\tau) - T_{A(s)}(\tau)]\|_{L(X)} &\le c_1\|A(t)[T_{A(t)}(\tau) - T_{A(s)}(\tau)]\|_{L(X)} \\
&\le c_1\| -\frac{1}{2\pi i}\int_\Gamma e^{\lambda\tau}A(t)(\lambda + A(t))^{-1}[A(t) - A(s)](\lambda + A(s))^{-1}d\lambda\| \\
&\le \frac{c_2}{2\pi}\int_\Gamma e^{-\cos\theta|\lambda|\tau}|\lambda|^{2-2\alpha}d|\lambda|\ (t-s)^\epsilon \\
&\le \frac{c_2}{2\pi}\int_\Gamma e^{-\cos\theta|u|}\frac{|u|^{2-2\alpha}}{\tau^{2-2\alpha}}\frac{d|u|}{\tau}\ (t-s)^\epsilon \\
&\le c\tau^{-3+2\alpha}(t-s)^\epsilon,
\end{aligned}$$

which shows (2.17). □

Lemma 2.6. *The functions* $(0,\infty)\times\mathbb{R}\ni(\tau,s)\mapsto T_{A(s)}(\tau)\in L(X)$ *and* $(0,\infty)\times\mathbb{R}\times\mathbb{R}\ni(\tau,t,s)\mapsto A(t)T_{A(s)}(\tau)\in L(X)$ *are continuous in the uniform operator topology of* $L(X)$.

Corollary 2.7. *The functions*

$$(t,\tau)\mapsto T_{A(\tau)}(t-\tau),\quad (t,\tau)\mapsto T_{A(t)}(t-\tau),$$

$$(t,\tau)\mapsto[A(\tau)-A(t)]T_{A(\tau)}(t-\tau)\ \text{and}\ (t,\tau)\mapsto[A(\tau)-A(t)]T_{A(t)}(t-\tau)$$

are continuous from $\{(t,\tau)\in\mathbb{R}^2 : t>\tau\}$ *into* $L(X)$ *in the uniform operator topology.*

It follows from the Corollary 2.7 that $t\mapsto T_{A(t)}(t_0-t)$ is in E. In what follows, we prove that S is a contraction from E into itself and therefore has a unique fixed point in E.

Recall that the *beta function* $\mathbf{B}(\cdot,\cdot):(0,\infty)\times(0,\infty)\longrightarrow(0,\infty)$ is defined by

$$\mathbf{B}(a,b)=\int_0^1 s^{a-1}(1-s)^{b-1}ds.$$

Firstly, note that if $U\in E$, then thanks to (2.6), (2.14) and (2.8),

$$\begin{aligned}
&(t_0-t)^{1-\alpha}\|(SU)(t)\|_{L(X)}\le(t_0-t)^{1-\alpha}\|T_{A(t)}(t_0-t)\|_{L(X)} \\
&+(t_0-t)^{1-\alpha}\int_t^{t_0}\|U(s)\|_{L(X)}\|[A(t)-A(s)]A^{-1}(t)\|_{L(X)}\|A(t)T_{A(t)}(s-t)\|_{L(X)}ds \\
&\le C+(t_0-t)^{1-\alpha}\int_t^{t_0}M(t_0-s)^{\alpha-1}C(s-t)^\epsilon M(s-t)^{-2+\alpha}ds \\
&\le C+C_1\mathbf{B}(\alpha,\alpha+\epsilon-1)(t_0-t)^{\alpha+\epsilon-1}.
\end{aligned}$$

Now, given $U,V\in E$, it follows that

$$(SU)(t)-(SV)(t)=\int_t^{t_0}[U(s)-V(s)][A(t)-A(s)]T_{A(t)}(s-t)ds,$$

and thus

$$
\begin{aligned}
&(t_0-t)^{1-\alpha}\left\|(SU)(t)-(SV)(t)\right\|_{L(X)}\\
&\le (t_0-t)^{1-\alpha}\int_t^{t_0}\|U(s)-V(s)\|_{L(X)}\|A(s)-A(t)\|_{L(X^1,X)}\|T_{A(t)}(s-t)\|_{L(X,X^1)}ds\\
&\le (t_0-t)^{1-\alpha}\int_t^{t_0}C(t_0-s)^{\alpha-1}(s-t)^{\epsilon}C(s-t)^{-2+\alpha}ds\sup_{t\le s\le t_0}\left\{\|U(s)-V(s)\|_{L(X)}\right\}\\
&\le C_1(t_0-t)^{1-\alpha}\int_t^{t_0}(t_0-s)^{\alpha-1}(s-t)^{\epsilon+\alpha-2}ds\,\|U-V\|_E\\
&\le C_1\mathbf{B}(\alpha,\alpha+\epsilon-1)(t_0-t)^{\alpha+\epsilon-1}\|U-V\|_E.
\end{aligned}
$$

Also, similarly,

$$
\begin{aligned}
&(t_0-t)^{1-\alpha}\left\|(S^2U)(t)-(S^2V)(t)\right\|_{L(X)}=(t_0-t)^{1-\alpha}\|S(SU)(t)-S(SV)(t)\|_{L(X)}\\
&\le (t_0-t)^{1-\alpha}\int_t^{t_0}\|(SU)(s)-(SV)(s)\|_{L(X)}\|A(s)-A(t)\|_{L(X^1,X)}\|T_{A(t)}(s-t)\|_{L(X,X^1)}ds\\
&\le (t_0-t)^{1-\alpha}\int_t^{t_0}C(t_0-s)^{2\alpha+\epsilon-2}(s-t)^{\epsilon}C(s-t)^{-2+\alpha}ds\sup_{t\le s\le t_0}\left\{\|U(s)-V(s)\|_{L(X)}\right\}\\
&\le C_1(t_0-t)^{1-\alpha}\int_t^{t_0}(t_0-s)^{2\alpha+\epsilon-2}(s-t)^{\epsilon+\alpha-2}ds\,\|U-V\|_E\\
&\le C_1\mathbf{B}(2\alpha+\epsilon-1,\alpha+\epsilon-1)(t_0-t)^{2(\alpha+\epsilon-1)}\|U-V\|_E.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&(t_0-t)^{1-\alpha}\left\|(S^3U)(t)-(S^3V)(t)\right\|_{L(X)}=(t_0-t)^{1-\alpha}\left\|S(S^2U)(t)-S(S^2V)(t)\right\|_{L(X)}\\
&\le (t_0-t)^{1-\alpha}\int_t^{t_0}\left\|(S^2U)(s)-(S^2V)(s)\right\|_{L(X)}\|A(s)-A(t)\|_{L(X^1,X)}\|T_{A(t)}(s-t)\|_{L(X,X^1)}ds\\
&\le (t_0-t)^{1-\alpha}\int_t^{t_0}C(t_0-s)^{3\alpha+2\epsilon-3}(s-t)^{\epsilon}C(s-t)^{-2+\alpha}ds\sup_{t\le s\le t_0}\left\{\|U(s)-V(s)\|_{L(X)}\right\}\\
&\le C_1(t_0-t)^{1-\alpha}\int_t^{t_0}(t_0-s)^{3\alpha+2\epsilon-3}(s-t)^{\epsilon+\alpha-2}ds\,\|U-V\|_E\\
&\le C_1\mathbf{B}(3\alpha+2\epsilon-2,\alpha+\epsilon-1)(t_0-t)^{3(\alpha+\epsilon-1)}\|U-V\|_E.
\end{aligned}
$$

Following with the iterations, we get

$$
\begin{aligned}
&\|(S^nU)(t)-(S^nV)(t)\|_{L(X)}\\
&\le C\mathbf{B}(\alpha,\alpha+\epsilon-1)\mathbf{B}(2\alpha+\epsilon-1,\alpha+\epsilon-1)\dots\mathbf{B}(n\alpha+(n-1)(\epsilon-1),\alpha+\epsilon-1)\\
&\times(t_0-t)^{n(\alpha+\epsilon-1)}\|U-V\|_E.
\end{aligned}
$$

Recalling that $\mathbf{B}(s,t)=\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}$, where Γ is the *Gamma function*, we have

$$
\mathbf{B}(\alpha,\alpha+\epsilon-1)\dots\mathbf{B}(n\alpha+(n-1)(\epsilon-1),\alpha+\epsilon-1)=\frac{\Gamma(\alpha)[\Gamma(\alpha+\epsilon-1)]^n}{\Gamma((n+1)\alpha+n(\epsilon-1))}.
$$

That gives,

$$\|(S^nU)(t)-(S^nV)(t)\|_{L(X)} \leq \frac{C\Gamma(\alpha)[\Gamma(\alpha+\epsilon-1)(t_0-t)^{\alpha+\epsilon-1}]^n}{\Gamma((n+1)\alpha+n(\epsilon-1))}\|U-V\|_E.$$

Consider the power series

$$\sum_{n=1}^{\infty}\frac{C\Gamma(\alpha)}{\Gamma((n+1)\alpha+n(\epsilon-1))}s^n.$$

If its radius of convergence is infinite, for each $s\in\mathbb{R}$ there is $n\in\mathbb{N}$, such that

$$\frac{C\Gamma(\alpha)[\Gamma(\alpha+\epsilon-1)(t_0-t)^{\alpha+\epsilon-1}]^n}{\Gamma((n+1)\alpha+n(\epsilon-1))}<1,$$

and then by the Banach contraction principle S^n has an unique fixed point in E and, consequently, S has an unique fixed point in E.

To verify that the radius of convergence is infinite, we apply the ratio test. Dividing the $(n+1)^{\text{th}}$ coefficient of the series by its n^{th} coefficient we obtain

$$\Gamma(\alpha+\epsilon-1)^{-1}\mathbf{B}((n+1)\alpha+n(\epsilon-1),\alpha+\epsilon-1).$$

Since

$$\mathbf{B}((n+1)\alpha+n(\epsilon-1),\alpha+\epsilon-1)=\int_0^1 s^{(n+1)\alpha+n(\epsilon-1)-1}(1-s)^{\alpha+\epsilon-2}ds,$$

and for each $0<s\leq 1$ the integrand is bounded by $(1-s)^{\alpha+\epsilon-1}$ and pointwise converges to zero as $n\to\infty$ it follows from the dominated convergence theorem that $\mathbf{B}((n+1)\alpha+n(\epsilon-1),\alpha+\epsilon-1)\overset{n\to\infty}{\longrightarrow}0$. The proof of existence of a unique fixed point of S in E is completed.

As t_0 was arbitrary in $[0,T]$, (2.13) has an unique solution $U(t,\tau)$ which is continuous in the uniform operator topology for all $t>\tau$.

Proposition 2.8. *Let $0<\alpha<1$ and $\beta<\alpha$, then*

$$\|A(t)^\beta T_{A(t)}(\tau)\|_{L(X)}\leq C\max\{\tau^{-1+\alpha-\frac{\beta}{\alpha}},\tau^{-1+\alpha-\beta}\}.$$

Proof: It follows from (2.3), (2.6) and (2.7) that

$$\begin{aligned}\|A(t)^\beta T_{A(t)}(\tau)x\| &\leq C(\alpha,\beta)\|T_{A(t)}(\tau)x\|^{1-\frac{\beta}{\alpha}}\|A(t)T_{A(t)}(\tau)x\|^{\frac{\beta}{\alpha}}\\ &\leq C(\alpha,\beta)\left(\tau^{-1+\alpha}\right)^{1-\frac{\beta}{\alpha}}\left(\max\{\tau^{-1},\tau^{-2+\alpha}\}\right)^{\frac{\beta}{\alpha}}\\ &\leq C(\alpha,\beta)\max\{\tau^{-1+\alpha-\frac{\beta}{\alpha}},\tau^{-1+\alpha-\beta}\}.\end{aligned}$$

□

The proof of the two next results is due to Sobolevskiĭ [19], Theorem 2, we only go to emphasize the necessary adjustments to ours case.

Proposition 2.9. *Let $0<\alpha<1$, $\gamma-\beta<\alpha$, and $\beta<-2\alpha+\frac{\epsilon\alpha}{1-\alpha}$, then*

$$\|A(t)^\gamma T_{A(t)}(t-\tau)A(\tau)^{-\beta}\|_{L(X)}\leq C(t-\tau)^{-1+\alpha-\frac{\gamma-\beta}{\alpha}}. \tag{2.19}$$

Proof: We have

$$A(t)^{\gamma}T_{A(t)}(t-\tau)A(\tau)^{-\beta} = A^{\gamma-\beta}(t)T_{A(t)}(t-\tau) + A(t)^{\gamma}T_{A(t)}(t-\tau)[A(\tau)^{-\beta} - A(t)^{-\beta}].$$

The norm of the first term does not exceed

$$C(\gamma,\beta)(t-\tau)^{-1+\alpha-\frac{\gamma-\beta}{\alpha}}.$$

For the second term, it follows from (2.4) that

$$A(\tau)^{-\beta} - A(t)^{-\beta} = \frac{1}{\Gamma(\beta)}\int_0^{\infty}[T_{A(\tau)}(s) - T_{A(t)}(s)]s^{\beta-1}ds. \tag{2.20}$$

Now, for any $\eta \in [0,1]$, using (2.16) and (2.17), we get from (2.3) that for $\eta < \alpha$

$$\|A(t)^{\eta}[T_{A(\tau)}(s) - T_{A(t)}(s)]\| \le Cs^{-2+2\alpha-\frac{\eta}{\alpha}}(t-\tau)^{\epsilon}. \tag{2.21}$$

Let $0 < \beta < 1$. From (2.20), (2.21) and closedness of the operators it follow that for any η, with $2\alpha + \beta > 2 + \frac{\eta}{\alpha}$,

$$\begin{aligned}\|A(t)^{\eta}[A(t)^{-\beta} - A(\tau)^{-\beta}]\| &\le \frac{1}{\Gamma(\beta)}\int_0^{\infty} Ce^{-\delta s}s^{-2+2\alpha-\frac{\eta}{\alpha}}(t-\tau)^{\epsilon}s^{\beta-1}ds \\ &\le \frac{C(t-\tau)^{\epsilon}}{\Gamma(\beta)}\int_0^{\infty} e^{-\delta s}s^{-2+2\alpha-\frac{\eta}{\alpha}}s^{\beta-1}ds \\ &\le C(\alpha,\beta,\eta)(t-\tau)^{\epsilon}.\end{aligned} \tag{2.22}$$

If $2 > \beta \ge 1$, then (2.22) holds for $\eta = 1$. In fact,

$$A(t)[A(t)^{-\beta} - A(\tau)^{-\beta}] = [A(\tau) - A(t)]A^{-1}(\tau)A^{-(\beta-1)}(\tau) + [A^{-(\beta-1)}(t) - A^{-(\beta-1)}(\tau)],$$

and the result follows by (2.14) and through the bound already obtained for the case $\beta < 1$.

Using (2.22), we have

$$\begin{aligned}&\|A(t)^{\gamma}T_{A(t)}(t-\tau)[A(\tau)^{-\beta} - A(t)^{-\beta}]\| \le \|A^{\gamma-\eta}(t)T_{A(t)}(t-\tau)\|\,\|A(t)^{\eta}[A(t)^{-\beta} - A(\tau)^{-\beta}]\| \\ &\le C(\alpha,\gamma)e^{-\delta(t-\tau)}(t-\tau)^{-1+\alpha-\frac{\gamma-\eta}{\alpha}}C(\gamma,\beta)(t-\tau)^{\epsilon} \le C(\alpha,\beta,\gamma)e^{-\delta(t-\tau)}(t-\tau)^{-1+\alpha-\frac{\gamma-\eta}{\alpha}+\epsilon},\end{aligned}$$

where $\frac{\beta}{\alpha} - \epsilon < \frac{\eta}{\alpha} < -2 + 2\alpha + \beta$ and, consequently $\beta < -2\alpha + \frac{\epsilon\alpha}{1-\alpha}$, which concludes the proof. □

Proposition 2.10. *Let $0 < \alpha < 1$, $\alpha + \epsilon > \max\{1, \frac{\gamma}{\alpha}\}$, $\gamma - \beta < \alpha$ and $\beta < -2\alpha + \frac{\epsilon\alpha}{1-\alpha}$, then*

$$\|A(t)^{\gamma}U(t,\tau)A(\tau)^{-\beta}\|_{L(X)} \le C(t-\tau)^{-1+\alpha-\frac{\gamma-\beta}{\alpha}}. \tag{2.23}$$

Proof: For any $x \in D(A^{1-\beta}(\tau))$ $(0 \le \beta \le 1)$, it follows from closedness of $A(\tau)^{-\beta}$ and (2.15) that

$$\begin{aligned}U(t,\tau)A(\tau)^{-\beta}x &= T_{A(t)}(t-\tau)A(\tau)^{-\beta}x \\ &\quad + \int_{\tau}^{t} T_{A(t)}(t-s)[A(s) - A(t)]U(s,\tau)A(\tau)^{-\beta}xds.\end{aligned} \tag{2.24}$$

Using (2.19) with $\gamma = 1$ for $1-\beta < \alpha$, we have next

$$\begin{aligned}
\|A(t)U(t,\tau)A^{-\beta}(\tau)x\| &\le \|A(t)T_{A(t)}(t-\tau)A(\tau)^{-\beta}x\| \\
&+ \int_\tau^t \|A(t)T_{A(t)}(t-s)\| \, \|[A(s)-A(t)]A^{-1}(s)\| \, \|A(s)U(s,\tau)A(\tau)^{-\beta}x\| ds \\
&\le C(1,\beta)\,(t-\tau)^{-1+\alpha-\frac{\beta-1}{\alpha}}\|x\| \\
&+ C^2 \int_\tau^t (t-s)^{-2+\alpha+\epsilon}\|A(s)U(s,\tau)A(\tau)^{-\beta}x\| ds.
\end{aligned}$$

From this and the Generalized Gronwall Lemma ([10], Exercise 4, p. 190) it follows that

$$\|A(t)U(t,\tau)A^{-\beta}(\tau)x\| \le C_1(1,\beta)(t-\tau)^{-1+\alpha-\frac{1-\beta}{\alpha}}\|x\|.$$

As $D(A^{1-\beta}(\tau))$ is dense in X, the inequality (2.23) holds for $1-\beta < \alpha$ and $\gamma = 1$.

For $\gamma < \alpha$, and $\beta < -2\alpha + \frac{\epsilon\alpha}{1-\alpha}$, it follows from (2.13) and (2.23) that

$$\begin{aligned}
\|A(t)^\gamma U(t,\tau)A(\tau)^{-\beta}x\| &\le \|A(t)^\gamma T_{A(t)}(t-\tau)A(\tau)^{-\beta}x\| \\
&+ \int_\tau^t \|A(t)^\gamma T_{A(t)}(t-s)\| \, \|[A(s)-A(t)]A^{-1}(s)\| \, \|A(s)U(s,\tau)A(\tau)^{-\beta}x\| ds \\
&\le Ce^{-\delta(t-\tau)}(t-\tau)^{-1+\alpha-\frac{\gamma-\beta}{\alpha}}\|x\| \\
&+ C\int_\tau^t e^{-\delta(t-s)}(t-s)^{-1+\alpha-\frac{\gamma}{\alpha}+\epsilon}(s-\tau)^{\gamma-1}ds\|x\|
\end{aligned}$$

where $\gamma = \min\{\beta, 1\}$ and $x \in D(A^{1-\gamma}(\tau))$. This show (2.23). □

Let

$$\varphi_1(t,\tau) = [A(\tau) - A(t)]T_{A(\tau)}(t-\tau), \tag{2.25}$$

and assume that the integral equation

$$\Phi(t,\tau) = \varphi_1(t,\tau) + \int_\tau^t \Phi(t,s)\varphi_1(s,\tau)ds \tag{2.26}$$

has an unique solution $\Phi(t,\tau)$, $t > \tau$. Setting

$$\bar{U}(t,\tau) = T_{A(\tau)}(t-\tau) + \int_\tau^t T_{A(s)}(t-s)\Phi(s,\tau)ds,$$

and noting that

$$\begin{aligned}
\int_\tau^t \bar{U}(t,s)\varphi_1(s,\tau)ds &= \int_\tau^t [T_{A(s)}(t-s) + \int_s^t T_{A(\theta)}(t-\theta)\Phi(\theta,s)d\theta]\varphi_1(s,\tau)ds \\
&= \int_\tau^t T_{A(s)}(t-s)\varphi_1(s,\tau)ds + \int_\tau^t T_{A(s)}(t-s)\int_\tau^s \Phi(s,\theta)\varphi_1(\theta,\tau)d\theta ds \\
&= \int_\tau^t T_{A(s)}(t-s)\Phi(s,\tau)ds,
\end{aligned}$$

we have that $\bar{U}(t,\tau) = U(t,\tau)$ for all $t \geq \tau$, and

$$U(t,\tau) = T_{A(\tau)}(t-\tau) + \int_\tau^t T_{A(s)}(t-s)\Phi(s,\tau)ds. \tag{2.27}$$

It remains to show that (2.26) has a unique solution to ensure that (2.27) holds. The expression (2.27) will be used to study regularity properties of $t \mapsto U(t,\tau)x_0$.

Note that $\varphi_1(s,\tau) = [A(\tau) - A(s)]T_{A(\tau)}(s-\tau)$ is continuous in the uniform operator topology in all variables for $s > \tau$ (see Corollary 2.7). From (2.14) and (2.8) we have that

$$\|\varphi_1(s,\tau)\|_{L(X)} \leq C(s-\tau)^{\alpha+\epsilon-2}. \tag{2.28}$$

With this in mind we show that (2.26) has an unique solution $\Phi(t,\tau)$ which is continuous in the uniform operator topology for all (t,τ) with $t > \tau$ and which satisfies

$$\|\Phi(t,\tau)\|_{L(X)} \leq C(t-\tau)^{\alpha+\epsilon-2}. \tag{2.29}$$

The procedure is analogous to that used to obtain $U(t,\tau)$. For $\alpha+\epsilon > 1$ and fixed $t_0 \in \mathbb{R}$, with $t_0 > \tau$ consider the complete metric space

$$Y = \{\Phi \in C([\tau,t_0), L(X)) : \sup_{t\in[\tau,\, t_0)} (t_0-t)^{2-\alpha-\epsilon}\, \|\Phi(t)\|_{L(X)} < \infty\},$$

with the metric

$$\|\Phi\|_Y = \sup_{t\in[\tau,\, t_0)} (t_0-t)^{2-\alpha-\epsilon}\, \|\Phi(t)\|_{L(X)}\,.$$

Define the map G by the formula

$$(G\Phi)(t) = \varphi_1(t_0,t) + \int_t^{t_0} \Phi(s)\varphi_1(s,t)ds.$$

Clearly G takes Y into $C([\tau,t_0), L(X))$. We prove next that G takes Y into itself and that, for some $n \in \mathbb{N}$, G^n is a contraction. Indeed,

$$(t_0-t)^{2-\alpha-\epsilon}\|(G\Phi)(t)\|_{L(X)} \leq C\left[1 + (t_0-t)^{\alpha+\epsilon-1}\|\Phi\|_Y \int_0^1 (1-s)^{\alpha+\epsilon-2}s^{\alpha+\epsilon-2}ds\right]$$

which proves that $G : Y \longrightarrow Y$. Now, for $\Phi, \Psi \in Y$,

$$(G\Phi)(t) - (G\Psi)(t) = \int_t^{t_0} [\Phi(s) - \Psi(s)]\varphi_1(s,t)ds$$

and

$$(t_0-t)^{2-\alpha-\epsilon}\|(G\Phi)(t) - (G\Psi)(t)\|_{L(X)} \leq C\mathbf{B}(\alpha+\epsilon-1, \alpha+\epsilon-1)(t_0-t)^{\alpha+\epsilon-1}\,\|\Phi-\Psi\|_Y\,.$$

Using this estimate and iterating we obtain

$$\begin{aligned}
&\|(G^n\Phi)(t) - (G^n\Psi)(t)\|_Y \\
&\leq C^n\mathbf{B}(\alpha+\epsilon-1, \alpha+\epsilon-1)\mathbf{B}(2(\alpha+\epsilon-1), \alpha+\epsilon-1)\ldots\mathbf{B}(n(\alpha+\epsilon-1), \alpha+\epsilon-1) \\
&\quad \times (t_0-t)^{n(\alpha+\epsilon-1)}\,\|\Phi-\Psi\|_Y\,,
\end{aligned}$$

which means

$$\|(G^n\Phi)(t)-(G^n\Psi)(t)\|_{L(X)} \leq \frac{C^n\Gamma(\alpha+\epsilon-1)[\Gamma(\alpha+\epsilon-1)(t_0-t)^{\alpha+\epsilon-1}]^n\,\|\Phi-\Psi\|_Y}{\Gamma((n+1)(\alpha+\epsilon-1))}.$$

From Banach contraction principle G has an unique fixed point in $C([\tau,\ t_0), L(X))$, which concludes the proof that (2.26) has a unique solution. Also the proof of Theorem 2.4 is thus finished.

3. Local solvability of semilinear parabolic problems

For $\gamma-\beta<1$ and $\alpha-\frac{\gamma-\beta}{\alpha}>1-\alpha$, we consider an abstract semilinear parabolic problem of the form

$$\begin{cases} \dfrac{du}{dt}+A(t)u=f(u),\ t>\tau, \\ u(\tau)=u_0\in X^\gamma, \end{cases} \tag{3.1}$$

where $\{A(t): t\in\mathbb{R}\}$ is a family of linear operators satisfying conditions (a) and (b) of Section 2. Assume that $f: X^\gamma\to X^\beta$ and that there are constants $c>0$ and $q>1$ such that

$$\|f(v)-f(w)\|_{X^\beta}\leq c\|v-w\|_{X^\gamma}\left(1+\|v\|_{X^\gamma}^{q-1}+\|w\|_{X^\gamma}^{q-1}\right),\quad \text{for all } v,w\in X^\gamma \tag{3.2}$$

with

$$q<\frac{\alpha-\frac{\gamma-\beta}{\alpha}}{1-\alpha}. \tag{3.3}$$

We will show that for each $\tau\in\mathbb{R}$ and arbitrary $u_0\in X^\gamma$ the problem (3.1) has a unique local mild solution defined on certain interval $(\tau,\tau+\tau_0)$ with τ_0 dependent on u_0.

Definition 3.1. *A function $u:(\tau,\tau+\tau_0)\to X^\gamma$ is said to be local mild solution to* (3.1), *if $(\tau,\tau+\tau_0)\ni t\mapsto w(t):=u(t)-U(t,\tau)u_0\in X^\gamma$ is continuous, $\lim_{t\to\tau^+}\|w(t)\|_{X^\gamma}=0$ and*

$$u(t)=U(t,\tau)u_0+\int_\tau^t U(t,s)f(u(s))ds,$$

holds for all $t\in(\tau,\tau+\tau_0)$, where $\{U(t,\tau): t>\tau\in\mathbb{R}\}$ is the process of growth α corresponding to the linear part of (3.1).

According to the above definition, since the process $\{U(t,\tau): t>\tau\in\mathbb{R}\}$ is singular at $t=\tau$, solutions to (3.1) are assumed to have the same kind of singularity at $t=\tau$ as the process U.

For τ, u_0 as in (3.1), $\mu>0$ (to be choosen later) introduce the metric space

$$K(\tau_0,u_0):=\{v\in C((\tau,\tau+\tau_0),X^\gamma): \sup_{t\in(\tau,\tau+\tau_0)}\|v(t)-U(t,\tau)u_0\|_{X^\gamma}\leq\mu\}, \tag{3.4}$$

with metric in $K(\tau_0,u_0)$ given by

$$\|\phi-\psi\|_{K(\tau_0,u_0)}=\sup_{t\in(\tau,\tau+\tau_0)}\|\phi(t)-\psi(t)\|_{X^\gamma},\quad \phi,\psi\in K(\tau_0,u_0). \tag{3.5}$$

The following result holds

Theorem 3.2. *Under the condition (3.2) with parameters satisfying (3.3) and for suitably small $\tau_0 > 0$, there exists a unique local mild solution to (3.1) in $K(\tau_0, u_0)$.*

Banach fixed point theorem in $K(\tau_0, u_0)$ will be used to prove existence of this solution to (3.1). We introduce an operator $T : K(\tau_0, u_0) \to C((\tau, \tau + \tau_0), X^\gamma)$ as

$$Tv(t) := U(t,\tau)u_0 + \int_\tau^t U(t,s)f(v(s))ds. \tag{3.6}$$

We need to show that, for a sufficiently small τ_0,
(*i*) $T : K(\tau_0, u_0) \to K(\tau_0, u_0)$,
(*ii*) T is a contraction in $K(\tau_0, u_0)$.
To prove the first claim let $v \in K(\tau_0, u_0)$ (with a sufficiently small τ_0, fixed for a moment). Note that

$$\begin{aligned}(t-\tau)^{1-\alpha}\|v(t)\|_{X^\gamma} &\le (t-\tau)^{1-\alpha}\|v(t) - U(t,\tau)u_0\|_{X^\gamma} + (t-\tau)^{1-\alpha}\|U(t,\tau)u_0\|_{X^\gamma} \\ &\le (t-\tau)^{1-\alpha}\mu + c\|u_0\|_{X^\gamma}.\end{aligned}$$

Hence, if B is a bounded subset of X^γ,

$$k = \tau_0^{1-\alpha}\mu + c\sup_{u_0\in B}\|u_0\|_{X^\gamma}$$

we have that

$$\begin{aligned}
\|Tv(t) - U(t,\tau)u_0\|_{X^\gamma} &\le \int_\tau^t \|U(t,s)f(v(s))\|_{X^\gamma} ds \\
&\le c\int_\tau^t \|U(t,s)\|_{\mathcal{L}(X^\beta,X^\gamma)}\|f(v(s))\|_{X^\beta} ds \\
&\le c\int_\tau^t (t-s)^{-1+\alpha-\frac{\gamma-\beta}{\alpha}}\left(1 + \|v(s)\|^q_{X^\gamma}\right) ds \le \frac{c}{\alpha - \frac{\gamma-\beta}{\alpha}}(t-\tau)^{\alpha-\frac{\gamma-\beta}{\alpha}} \\
&+ c\int_\tau^t (t-s)^{-1+\alpha-\frac{\gamma-\beta}{\alpha}}(s-\tau)^{-(1-\alpha)q}((s-\tau)^{(1-\alpha)q}\|v(s)\|_{X^\gamma})^q ds \\
&\le \frac{c}{\alpha - \frac{\gamma-\beta}{\alpha}}(t-\tau)^{\alpha-\frac{\gamma-\beta}{\alpha}} + ck^q\int_\tau^t (t-s)^{-1+\alpha-\frac{\gamma-\beta}{\alpha}}(s-\tau)^{-(1-\alpha)q} ds \\
&\le \frac{c}{\alpha - \frac{\gamma-\beta}{\alpha}}(t-\tau)^{\alpha-\frac{\gamma-\beta}{\alpha}} + ck^q(t-\tau)^{\alpha-\frac{\gamma-\beta}{\alpha}-(1-\alpha)q}\mathbf{B}\Big(1-(1-\alpha)q, \alpha - \frac{\gamma-\beta}{\alpha}\Big) \\
&\le \frac{c}{\alpha - \frac{\gamma-\beta}{\alpha}}\tau_0^{\alpha-\frac{\gamma-\beta}{\alpha}} + ck^q\tau_0^{\alpha-\frac{\gamma-\beta}{\alpha}-(1-\alpha)q}\mathbf{B}\Big(1-(1-\alpha)q, \alpha - \frac{\gamma-\beta}{\alpha}\Big) \\
&\le \mu,
\end{aligned} \tag{3.7}$$

for suitable small τ_0 and for all $u_0 \in B$, provided that $\alpha - \frac{\gamma-\beta}{\alpha} - (1-\alpha)q > 0$, or equivalently that

$$q < \frac{\alpha - \frac{\gamma-\beta}{\alpha}}{1-\alpha}$$

but this is the condition (3.3).

In particular, for each $v \in K(\tau_0, u_0)$ we have that

$$\|Tv(t) - U(t,\tau)u_0\|_{X^\gamma} \to 0, \quad \text{as } t \to 0. \tag{3.8}$$

Hence, with this choice of τ_0, we have that $T : K(\tau_0, u_0) \to K(\tau_0, u_0)$. Note that this procedure can be carried out uniformly for u_0 in bounded subsets of X^γ.

The proof of the second claim is similar. We have that for $v, w \in K(\tau_0, u_0)$,

$$\begin{aligned}\|Tv(t) - Tw(t)\|_{X^\gamma} &\leq \int_\tau^t \|U(t,s)(f(v(s) - f(w(s)))\|_{X^\gamma} ds \\ &\leq c \int_\tau^t \|U(t,s)\|_{\mathcal{L}(X^\beta, X^\gamma)} \|f(v(s)) - f(w(s))\|_{X^\beta} ds \\ &\leq c \int_\tau^t (t-s)^{-1+\alpha-\frac{\gamma-\beta}{\alpha}} \|v(s) - w(s)\|_{X^\gamma} \left(1 + \|v(s)\|_{X^\gamma}^{q-1} + \|w(s)\|_{X^\gamma}^{q-1}\right) ds.\end{aligned}$$

Concluding the above estimate we get

$$\begin{aligned}\|Tv - Tw\|_{X^\gamma} &\leq c \int_\tau^t (t-s)^{-1+\alpha-\frac{\gamma-\beta}{\alpha}} \|v - w\|_{K(\tau_0,u_0)} ds \\ &+ 2c\, k^{q-1} \int_\tau^t (t-s)^{-1+\alpha-\frac{\gamma-\beta}{\alpha}} \|v - w\|_{K(\tau_0,u_0)} (s-\tau)^{-(1-\alpha)(q-1)} ds \\ &\leq c\|v - w\|_{K(\tau_0,u_0)} \left[\frac{\tau_0^{\alpha-\frac{\gamma-\beta}{\alpha}}}{\alpha - \frac{\gamma-\beta}{\alpha}} + 2k^{q-1}\tau_0^{\alpha-\frac{\gamma-\beta}{\alpha}-(1-\alpha)(q-1)}\, \mathbf{B}\Big(1-(1-\alpha)(q-1), \alpha - \frac{\gamma-\beta}{\alpha}\Big)\right].\end{aligned}$$

Note that the expression within parenthesis tends to zero as τ_0 tends to zero provided that $\alpha - \frac{\gamma-\beta}{\alpha} - (1-\alpha)(q-1) > 0$, or equivalently that

$$q < \frac{1 - \frac{\gamma-\beta}{\alpha}}{1-\alpha}. \tag{3.9}$$

Clearly, if the exponent q in (3.2) satisfies (3.3) then it also satisfies (3.9). Hence, we obtain that, for suitably small τ_0, T is a contraction in $K(\tau_0, u_0)$ and this concludes the proof of Theorem 3.2.

4. Applications.

We will show two applications of the described above abstract theory. We start with the standard parabolic equation with time dependent coefficients settled in the space of Hölder continuous functions.

4.1. **Semilinear parabolic problems in Hölder spaces.**

Recall that (see e.g. [9], Chapter 3) the Hölder space $C^\mu(\bar{\Omega})$, $\mu \in (0,1)$, is a Banach space of uniformly continuous functions $v : \bar{\Omega} \to \mathbb{R}$, equipped with the norm

$$\|v\|_\mu = \sup_{x\in\bar{\Omega}} |v(x)| + \sup_{x,y\in\bar{\Omega}, 0<|x-y|\leq 1} \frac{|v(x) - v(y)|}{|x-y|^\mu} =: \|v\|_{C(\bar{\Omega})} + H^\mu(v).$$

Example of an almost sectorial operator which is not sectorial has been first introduced by W. von Wahl in [22] (see also [17]). We recall it here for completeness of the presentation.

Example 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^{4m} regular boundary. In the space $C^\mu(\bar{\Omega})$ of Hölder continuous functions ($\mu \in (0,1)$) consider elliptic differential operator of order $2m$*

$$Av(x) = \sum_{|\delta| \leq 2m} a_\delta(x) D^\delta v(x),$$

with the domain $D(A) = \{v \in C^{2m+\mu}(\bar{\Omega}) : D^\delta u = 0 \text{ on } \partial\Omega, |\delta| \leq m-1\}$, where $D^\delta = \prod_{j=1}^n (-i\frac{\partial}{\partial x_j})^{\delta_j}$, δ is a multi-index. The coefficients $a_\delta : \bar{\Omega} \to \mathbb{C}$ are assumed to satisfy the following conditions

$$\begin{aligned} &(i)\ a_\delta \in C^\mu(\bar{\Omega}) \text{ for } |\delta| \leq 2m, \\ &(ii)\ a_\delta(x) \in \mathbb{R} \text{ for } x \in \bar{\Omega} \text{ and } |\delta| = 2m, \\ &(iii) \text{ There exists } M > 0 \text{ such that } M^{-1}|\xi|^{2m} \leq \sum_{|\delta|=2m} a_\delta(x)\xi^\delta \leq M|\xi|^{2m}, \forall \xi \in \mathbb{R}^n,\ \forall x \in \bar{\Omega}. \end{aligned} \tag{4.1}$$

For certain $\nu > 0$ the operator $A + \nu$ will be almost sectorial with parameter $\alpha = 1 - \frac{\mu}{2m}$. Note that in particular the minus Laplace operator fulfills the above conditions.

Following von Wahl's example in case of time dependent processes consider the second order parabolic problem

$$\begin{cases} u_t = L(t,x)u + g(u), \quad t > 0,\ x \in \Omega, \\ u(0,x) = u_0(x) \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.2}$$

where

$$L(t,x)u := \sum_{i,j=1}^n a_{ij}(t,x) \frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Assume that there is a constant $L > 0$ such that the coefficients $a_{ij} : \mathbb{R} \times \bar{\Omega} \to \mathbb{R}$ are uniformly continuous together with its derivatives and satisfy the conditions

$$\begin{aligned} &\sup_{x \in \Omega} |a_{ij}(t,x) - a_{ij}(s,x)| \leq L|t-\tau|, \quad \text{for all } t, \tau \in \mathbb{R},\ |t-\tau| < 1 \\ &\sup_{1 \leq k \leq n} \sup_{x \in \Omega} |\frac{\partial a_{ij}}{\partial x_k}(t,x) - \frac{\partial a_{ij}}{\partial x_k}(s,x)| \leq L|t-\tau| \text{ for all } t, \tau \in \mathbb{R},\ |t-\tau| < 1 \end{aligned} \tag{4.3}$$

and, uniformly with respect to t, ellipticity condition (4.1) (iii).

We need to check that assumptions (a) and (b) of Section 2 are satisfied for this example. Let $A(t) : D(A(t)) \subset C_0^\mu(\bar{\Omega}) \to C_0^\mu(\bar{\Omega})$ (subscript 0 means the subspace consisting of all functions vanishing at $\partial\Omega$) be an abstract operator defined by $L(t,x)$

$$A(t)v = L(t,\cdot)v, \quad v \in D(A(t)).$$

First we will show that condition (a) is satisfied, which means that there are $\alpha \in (0,1]$ and $M > 0$ such that

$$\|(\lambda + A(t))^{-1}\|_{L(C_0^\mu(\Omega))} \leq \frac{M}{|\lambda|^\alpha}, \quad \text{for all } \lambda \in \Sigma_\theta \text{ and } \forall t \in \mathbb{R}. \tag{4.4}$$

For clarity of further calculations let us discuss now the connections between various constants appearing below. Fix first exponent $\mu \in (0,1)$ and choose the parameter

$$s \in [\mu + \frac{n}{p}, 1 + \frac{n}{p}). \tag{4.5}$$

Next, let $\epsilon, \epsilon' \in (0,1)$ be such that $0 < s - 2\epsilon < s - 2\epsilon' < \mu$. Note that the last condition together with (4.5) require that

$$\frac{n}{2p} \leq \epsilon' < \epsilon < \frac{\mu}{2} + \frac{n}{2p}. \tag{4.6}$$

In the proof of (4.4) we will use known properties of the operators $A(t)$ in the scale $\tilde{X}^r := H^{2r+s}_{p,\{B_j\}}(\Omega)$ of Bessel potentials spaces (see [21], p. 321), that is Calderon-Zygmund type estimate

$$\|(\lambda + A(t))^{-1}\|_{\mathcal{L}(\tilde{X}^0, \tilde{X}^1)} \leq c, \tag{4.7}$$

and sectoriality condition for this operators

$$\|(\lambda + A(t))^{-1}\|_{\mathcal{L}(\tilde{X}^0)} \leq \frac{c}{|\lambda|}. \tag{4.8}$$

Thanks to the Sobolev type embedding (see [21], p. 328), $H^s_{p,\{B_j\}}(\Omega) \subset C_0^\mu(\bar{\Omega})$, valid for μ, s satisfying (4.5), we have

$$\begin{aligned} \|(\lambda + A(t))^{-1}\phi\|_{C_0^\mu(\bar{\Omega})} &\leq c\|(\lambda + A(t))^{-1}\phi\|_{\tilde{X}^0} \\ &\leq \|(\lambda + A(t))^{-1}\|_{\mathcal{L}(H^{s-2\epsilon}_{p,\{B_j\}}(\Omega), H^s_{p,\{B_j\}}(\Omega))}\|\phi\|_{H^{s-2\epsilon}_{p,\{B_j\}}(\Omega)} \leq \frac{c}{|\lambda|^{1-\epsilon}}\|\phi\|_{H^{s-2\epsilon}_{p,\{B_j\}}(\Omega)}, \end{aligned} \tag{4.9}$$

where the following from (4.7), (4.8) and the *moment inequality* estimate

$$\|(\lambda + A(t))^{-1}\|_{\mathcal{L}(\tilde{X}^{-\epsilon}, \tilde{X}^0)} \leq \frac{c}{|\lambda|^{1-\epsilon}},$$

has been used. We will extend next (4.9) based on the estimate

$$\|\phi\|_{H^{s-2\epsilon}_{p,\{B_j\}}(\Omega)} \leq c\|\phi\|_{W_p^{s-2\epsilon'}(\Omega)} \leq c\|\phi\|_\mu, \tag{4.10}$$

in which the first inequality follows from the embedding in [21], p. 180. The second inequality will be obtained considering the, so called, *intrinsic norm* of the space $W_p^r(\Omega)$ (see [1], p. 208, [21], p. 323)

$$\|\phi\|_{W_p^r(\Omega)} = \left(\|\phi\|^p_{W_p^m(\Omega)} + \sum_{|\delta|=m} \int_\Omega \int_\Omega \frac{|D^\delta\phi(x) - D^\delta\phi(y)|^p}{|x-y|^{n+\sigma p}} dxdy \right)^{\frac{1}{p}}, \tag{4.11}$$

where $r = m + \sigma$, with $\sigma \in (0,1), m \in \mathbb{N}$. Note that letting $p \to +\infty$ in (4.11) with $r = s - 2\epsilon', m = 1$, we obtain at the right hand side

$$\max\left(\|\phi\|_{L^\infty(\Omega)}, \text{ ess} \sup_{x,y\in\Omega, x\neq y} \frac{|\phi(x) - \phi(y)|}{|x-y|^{s-2\epsilon'}} \right),$$

which is an equivalent norm of $C^{s-2\epsilon'}(\bar{\Omega})$. Further, $C^\mu(\bar{\Omega}) \subset C^{s-2\epsilon'}(\bar{\Omega})$ when $s - 2\epsilon' \leq \mu$. Since we can allow arbitrary large value of p (the role of Sobolev type spaces in this proof is only auxiliary), we obtain an estimate

$$\|(\lambda + A(t))^{-1}\phi\|_{C_0^\mu(\bar{\Omega})} \leq \frac{c}{|\lambda|^{1-\epsilon}} \|\phi\|_{C_0^\mu(\bar{\Omega})}, \; Re\lambda > 0, \tag{4.12}$$

verifying condition (4.4) with $\alpha = 1 - \epsilon$ (the value of ϵ was evaluated in (4.6)). Note that for large p we obtained von Wahl's type estimate

$$\alpha = 1 - \epsilon > 1 - \frac{\mu}{2}.$$

Next, we need to check condition (2.14); that is, there exists $\epsilon \in (0,1)$ such that

$$\|[A(t) - A(\tau)]A^{-1}(s)]\phi\|_\mu \leq c(t-\tau)^\epsilon \|\phi\|_\mu, \; \forall t, \tau \in \mathbb{R}, \; |t - \tau| < 1.$$

Since the operator $A^{-1}(s)$ is bounded linear from $C_0^\mu(\bar{\Omega})$ into $D(A(t))$, it suffices to show that for $\psi := A^{-1}(s)\phi$,

$$\|[A(t) - A(\tau)]\psi\|_\mu \leq c(t-\tau)^\epsilon \|\phi\|_\mu, \quad \tau \leq t \in \mathbb{R},$$

which, in an explicit form, means the estimate

$$\left\| \sum_{i,j=1}^n [a_{ij}(t,\cdot) - a_{ij}(\tau,\cdot)] \frac{\partial^2 \psi}{\partial x_i \partial x_j} \right\|_\mu \leq c(t-\tau)^\epsilon \|\phi\|_\mu. \tag{4.13}$$

It follows from our assumption (4.3) on the coefficients that

$$\|a_{ij}(t,\cdot) - a_{ij}(\tau,\cdot)\|_\mu \leq c\|a_{ij}(t,\cdot) - a_{ij}(\tau,\cdot)\|_{C^1(\Omega)} \leq L|t-\tau| \leq L|t-\tau|^\epsilon, \tag{4.14}$$

for $|t - \tau| < 1, \epsilon \in (0,1)$, which justifies condition (2.14) since $C^\mu(\bar{\Omega})$ is a Banach algebra and

$$\left\| \frac{\partial^2 \psi}{\partial x_i \partial x_j} \right\|_\mu \leq c\|\phi\|_\mu.$$

Finally we need to impose condition on the nonlinear term g in (4.2) assuring that (3.2) holds and, consequently, the problem is locally solvable. Since in our example, for $1 > \mu^+ > \mu$, for all $s > \mu^+ - \mu$ and for all $p \geq 1$, $L^{p_0}(\Omega) \subset H^{-s}_{p,\{B_j\}}(\Omega) \subset X^{\gamma - \frac{\mu^+}{2}}$ with $p_0 \geq \frac{np}{n+sp}$ and $X^\gamma = C^\mu(\bar{\Omega}) \subset L^\infty(\Omega)$, then it suffices to assume that $g : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function. Indeed, when $B \subset C^\mu(\bar{\Omega})$ is bounded and $v, w \in B$, we have

$$\|g(v) - g(w)\|_{X^{\gamma - \frac{\mu^+}{2}}} \leq c\|g(v) - g(w)\|_{L^{p_0}(\Omega)} \leq cL_B\|v - w\|_{L^{p_0}(\Omega)} \leq cL_B\|v - w\|_{X^\gamma}, \tag{4.15}$$

where L_B is the Lipschitz constant for g in a large interval containing the image of every function in B. This proves the required condition and concludes the example.

4.2. **Domains with a handle.** Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Consider the problem

$$\begin{cases} u_t = a(t,x)\Delta u - u + f(u), & x \in \Omega,\ t > 0, \\ u(t,x) = 0, & x \in \partial\Omega,\ t \geq 0, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{4.16}$$

This problem can be written abstractly as

$$\frac{du}{dt} + A(t)u = f(u), \quad u(0) = u_0, \tag{4.17}$$

where $X = L^p(\Omega)$, $A(t) : D(A(t)) \subset X \to X$, $D(A(t)) = H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$, and

$$A(t)u = -a(t,x)\Delta u + u.$$

Assume that $a(t,x)$ is continuously differentiable in x and that there are positive constants c_0 and c_1 such that $0 < c_0 \leq a(t,x) \leq c_1$.

Assume further that $b(t,x) := \nabla_x a(t,x) \in L^\infty(\Omega^n)$ for all $t \in \mathbb{R}$ and that $t \mapsto a(t,\cdot) \in L^\infty(\Omega)$, $t \mapsto b(t,\cdot) \in L^\infty(\Omega^n)$ are Hölder continuous functions with exponent $\epsilon > 0$ and constant C.

As for the nonlinearities f, we assume that $f \in C^1(\mathbb{R},\mathbb{R})$ and satisfies the growth condition

$$|f'(u)| \leq c. \tag{4.18}$$

Operator $-A(t)$ is the infinitesimal generator of an analytic semigroup which satisfies

$$\|(A(t) + \lambda I)^{-1}\|_{L(X)} \leq \frac{M}{|\lambda|}, \text{ for } t \in \mathbb{R} \text{ and } \lambda \in \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\},$$

and

$$\|[A(t) - A(\tau)]A^{-1}(s)\|_{L(X)} \leq C(t-\tau)^\epsilon, \text{ for some } \epsilon \in (0,1] \text{ and for all } t,\tau,s \in \mathbb{R}.$$

Lemma 4.2 ([6]). *The operator $A(t) : D(A(t)) \subset X \to X$ is the infinitesimal generator of an analytic semigroup and satisfies conditions* (1.2) *(with $\alpha = 1$) and* (2.14).

Inspired by [4] we consider the following example. For $g \in C^1([0,1],(0,\infty))$, let $U^p = L^p(\Omega) \oplus L_g^p(0,1)$, where $L_g^p(0,1)$ is the space $L^p(0,1)$ with the norm

$$\|\phi\|_{L_g^p(0,1)} = \left(\int_0^1 g(s)|\phi(s)|^p ds\right)^{\frac{1}{p}}.$$

For $P_0, P_1 \in \partial\Omega$ and $p > n/2$ let $\mathcal{A}(t) : \mathcal{D}(\mathcal{A}(t)) \subset U^p \to U^p$ be the operator defined by

$$\begin{aligned} &D(\mathcal{A}(t)) = \big\{(w,v) \in U^p : w \in D(\Delta_N^\Omega),\ (gv_x)_x \in L^p(0,1),\ v(0) = w(P_0),\ v(1) = w(P_1)\big\} \\ &\mathcal{A}(t)(w,v) = \Big(-a(t,x)\Delta w + w, -\frac{1}{g}(gv_x)_x + v\Big), \quad (w,v) \in D(\mathcal{A}(t)), \end{aligned} \tag{4.19}$$

where Δ_N^Ω is the Laplacian with homogeneous Neumann boundary conditions in $L^p(\Omega)$ and the domain $D(\Delta_N^\Omega) = \{u \in H^{2,p}(\Omega) : \frac{\partial u}{\partial n} = 0 \text{ in } \partial\Omega\}$.

For $p > n/2$ we have that $D(\Delta_N^\Omega)$ is continuously embedded in $C(\bar{\Omega})$. This tells us that the functions in $D(\Delta_N^\Omega)$ have trace at P_0 and P_1.

Proposition 4.3 ([4]). *For each $t \in \mathbb{R}$, the operator $\mathcal{A}(t)$ defined by* (4.19) *has the following properties*

(i) $D(\mathcal{A}(t))$ is dense in U^p,

(ii) $\mathcal{A}(t)$ is a closed operator,

(iii) $\mathcal{A}(t)$ has compact resolvent, and

(iv) $\rho(\mathcal{A}(t)) \supset \Sigma_\theta$ and the estimates hold

$$\|(\mathcal{A}(t)+\lambda)^{-1}\|_{L(U^p)} \leqslant \frac{C}{|\lambda|^\alpha}, \tag{4.20}$$

$$\|\mathcal{A}(t)(\mathcal{A}(t)+\lambda)^{-1}\|_{L(U^p)} \leqslant C(1+|\lambda|^{1-\alpha})$$

for some $0 < \alpha < 1 - \frac{n}{2p} < 1$ and there are constants $C > 0$ and $\epsilon \in (0,1]$ such that

$$\|[\mathcal{A}(t) - \mathcal{A}(\tau)]\mathcal{A}^{-1}(s)\|_{L(U^p)} \leq C(t-\tau)^\epsilon, \ \forall t, \tau, s \in \mathbb{R}. \tag{4.21}$$

If $f : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous, it defines a Nemitskiĭ operator from U^p into itself by $f^e(u,v) = (f_\Omega(u), f_I(v))$, with $f_\Omega(u)(x) = f(u(x))$, $x \in \Omega$, and $f_I(v)(s) = f(v(s))$, $s \in (0,1)$.

Consider the semilinear evolution equation

$$\begin{cases} \dfrac{du}{dt} = \mathcal{A}(t)u + f^e(u) \\ u(0) = u_0 \in U^p, \end{cases} \tag{4.22}$$

where u vary in the Banach space U^p and $\mathcal{A}(t) : D(\mathcal{A}(t)) \subset U^p \to U^p$ is the linear operator defined by (4.19) and where $f^e : U^p \to U^p$ is such that

$$\|f^e(u) - f^e(v)\|_{U^p} \leq c\|u-v\|_{U^p}.$$

It follows from the result in Section 3 with $\gamma = \beta = 0$ that (4.22) has a unique solution for each $u_0 \in U^p$.

References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press Inc., New York, 1975.

[2] H. Amann, *Linear and Quasilinear Parabolic Problems. Volume I: Abstract Linear Theory*, Birkhäuser Verlag, Basel, 1995.

[3] J. Appell, P.P. Zabrejko, *Nonlinear Superposition Operators*, Cambridge University Press, Cambridge, 1990.

[4] J. Arrieta, A. N. Carvalho, G. Lozada-Cruz, *Dynamics in dumbbell domains II: The limiting problem*, submitted.

[5] J. Arrieta, A. N. Carvalho, Abstract parabolic problems with critical nonlinearities and applications to Navier-Stokes and heat equations, *Trans. Amer. Math. Soc.*, **352** (2000), 285-310.

[6] A. N. Carvalho, M. J. D. Nascimento, *Singularly non-autonomous semilinear parabolic problems with critical exponents and applications*, preprint.

[7] J. W. Cholewa, T. Dlotko, *Global Attractors in Abstract Parabolic Problems*, Cambridge University Press, Cambridge, 2000.

[8] G. Da Prato, *Semigruppi di crescenza n*, Ann. Scuola Norm. Sup. Piza, **20** (1966), 753-782.

[9] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice Hall, Inc., Englewood Cliffs, N.J., 1964.
[10] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer-Verlag, Berlin, 1981.
[11] S. G. Kreĭn, *Linear Differential Equations in Banach Space*, AMS, Providence, 1971.
[12] R. deLaubenfels, *Existence Families, Functional Calculi and Evolution Equations*, Springer-Verlag, Berlin, 1994.
[13] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser, Basel, 1995.
[14] N. Okazawa, *A generation theorem for semigroups of growth order* α, Tôhoku Math. J., **26** (1974), 39-51.
[15] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.
[16] F. Periago, *Global existence, uniqueness, and continuous dependence for a semilinear initial value problem*, J. Math. Anal. Appl., **280** (2003), 413-423.
[17] F. Periago, B. Straub, *A functional calculus for almost sectorial operators and applications to abstract evolution equations*, J. Evol. Equations, **2** (2002), 41-68.
[18] H. Tanabe, *Equations of Evolution*, Pitman, London, 1979.
[19] P. E. Sobolevskiĭ, *Equations of parabolic type in a Banach space*, Amer. Math. Soc. Transl., **49** (1965), 1-62.
[20] S. P. Toropova, *Relation of semigroups with singularity to integrated semigroups*, Russian Mathematics, **(3) 47** (2003), 66-74.
[21] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.
[22] W. von Wahl, *Gebrochen Potenzen eines elliptischen Operators und parabolische Differentialgleichungen in Räumen hölderstetinger Funktionen*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl., **11** (1972), 231-258.

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