UNIVERSIDADE DE SÃO PAULO

Instituto de Ciências Matemáticas e de Computação

ISSN 0103-2577

THE σ -ISOTYPIC DECOMPOSITION AND THE σ -INDEX OF REVERSIBLE-EQUIVARIANT SYSTEMS

PATRÍCIA H. BAPTISTELLI
MIRIAM MANOEL

Nº 290

NOTAS

Série Matemática

São Carlos – SP
Fev./2008

The σ−isotypic decomposition and the σ−index of reversible-equivariant systems

28th February 2008

PATRÍCIA H. BAPTISTELLI[1]

Departamento de Matemática, Centro de Ciências Exatas
Universidade Estadual de Maringá
Av. Colombo, 5790, 87020-900
Maringá, PR - Brazil

MIRIAM MANOEL[2]

Departamento de Matemática, ICMC
Universidade de São Paulo
13560-970 Caixa Postal 668,
São Carlos, SP - Brazil

Abstract

In this work we describe the complete construction process of subspaces that are left invariant by linear Γ-reversible-equivariant mappings, where Γ is a compact Lie group of all the symmetries and reversing symmetries of such mappings. These subspaces are the σ-isotypic components, first introduced by Lamb and Roberts in [14] and that correspond to the isotypic components for purely equivariant systems. In addition, by representation theory methods, two algebraic formulae are established for the computation of the σ-index of a closed subgroup of Γ. The results obtained here are to be applied to general reversible-equivariant systems, but are of particular interest for the more subtle of the two possible cases, namely the non self-dual case. A series of examples is presented.

Keywords: symmetry, reversing symmetry, invariant subspaces, Haar integral, character theory.

2000 AMS Classification: 35B32; 37G40

[1]Email address: phbaptistelli@uem.br
[2]Email address: miriam@icmc.usp.br

Resumo

Neste trabalho descrevemos o processo completo de construção dos subespaços que são deixados invariantes por aplicações lineares Γ-reversíveis-equivariantes, onde Γ é um grupo de Lie compacto de todas as simetrias (equivariâncias) e anti-simetrias (reversibilidades) de tais aplicações. Estes subespaços são as componentes σ-isotípicas, introduzidas inicialmente por Lamb e Roberts em [14] e que correspondem às componentes isotípicas para os sistemas puramente equivariantes. Além disso, por métodos da teoria de representação de grupos, duas expressões algébricas são estabelecidas para o cálculo do σ-índice de subgrupos fechados de Γ. Os resultados obtidos aqui se aplicam a sistemas reversíveis-equivariantes em geral, mas são de particular interesse ao caso mais complicado dentre os dois casos possíveis, a saber, o chamado caso não auto-dual. Apresentamos uma série de exemplos.

1 Introduction

Reversible-equivariant dynamical systems are characterized by the simultaneous occurrence of symmetries (equivariances) and reversing symmetries. In their mathematical formulation, the set Γ of these elements have a group structure, this group acting linearly on the space of the variables. We shall assume throughout hat Γ is a compact Lie group. In the dynamics point of view, trajectories of such systems are taken into trajectories of the same systems preserving direction in time by the symmetries and inverting direction in time by the reversing symmetries. The results of this paper were obtained motivated mainly by the interest in the analysis of reversible-equivariant bifurcation problems.

The effect of symmetries in dynamical systems has become an important area of research during the last thirty years. We mention the important contributions in the systematic study of symmetries in bifurcation theory given by Sattinger [15], Vanderbauwhede [17] and Golubitsky, Stewart and Schaeffer [10], among many others. The effect of reversing symmetries in local and global dynamics has been firstly investigated in the context of purely reversible dynamical systems by Arnol'd [2], Devaney [8] and Sevryuk [16]. After that, a lot of attention has been given in this direction. See Lamb and Roberts [13] for a historical survey and comprehensive bibliography.

The interest on reversible-equivariant systems is more recent and, as in the purely equivariant case, the main techniques applied to the description of the vector fields as well as to the study of global and local dynamics fall into group representation theory and singularities. The first impulse to the subject came from the classification of the Γ-reversible-equivariant linear systems by Lamb and Roberts [14], where they consider the two possible representations of Γ that can occur in the reversible-equivariant setting, namely the *self-dual* and *non self-dual* representations (Definition 2.3). Methods of singularity theory were developed in [4], in the same lines as Golubitsky *et al.* [10], for the analysis and classification of steady-state bifurcations in the self-dual case. The main point in that case is the existence of a reversible-equivariant isomorphism that establishes a one-to-one correspondence with an associated purely equivariant bifurcation. Antoneli *et al.* [1] present an algorithm for finding the general form of Γ-reversible-equivariant vector fields by employing algebraic techniques of invariant theory. Others recent results in this direction are also found in [3, 6, 7]. The starting point for the study of systems that are simultaneously equivariant and reversible is to introduce a group homomorphism $\sigma : \Gamma \to \mathbf{Z}_2 \cong \{\pm 1\}$ that defines the symmetries and the reversing symmetries of the problem in question, an element $\gamma \in \Gamma$ being a symmetry if $\sigma(\gamma) = 1$, and a reversing symmetry if $\sigma(\gamma) = -1$. The present paper uses representation theory of compact Lie groups for the investigation of two subjects for those systems: the σ-isotypic decomposition and the σ-index.

Regarding the first subject, we observe that the structure of the representation of Γ on a finite-dimensional vector space V allows the decomposition of this space in a direct sum of *irreducible* representations. This guarantees the existence of the *isotypic components*, each one combining all the irreducible representations that are in a fixed isomorphic class, that are invariant by any purely equivariant linear mapping. Correspondingly, the construction of components that are left invariant by any reversible-equivariant linear mapping is of great importance in the systematic study of reversible-equivariant systems. These are the *σ-isotypic components*, introduced by Lamb and Roberts in [14]. The first purpose of the present paper is to complete the construction of those subspaces given therein and to detail a comparison between them.

Regarding our second subject, we observe that a Γ-reversible-equivariant mapping on V can be viewed, with appropriate actions of Γ on the source and target, as an equivariant mapping. More specifically, the action of Γ on the target, denoted by V_σ, is the dual action of Γ on the source V (Definition 3.1). Hence, for $\Sigma \subseteq \Gamma$ a subgroup, every Γ-reversible-equivariant mapping maps the fixed-point subspace of Σ in V, $\mathrm{Fix}_V(\Sigma)$, to the fixed-point subspace of Σ in its dual V_σ, $\mathrm{Fix}_{V_\sigma}(\Sigma)$. The *$\sigma-$index* of Σ in V, denoted by $s_V(\Sigma)$, is then defined as the difference between the dimensions of $\mathrm{Fix}_V(\Sigma)$ and $\mathrm{Fix}_{V_\sigma}(\Sigma)$ (Definition 2.7). The knowledge of this number through the isotropy lattice of Γ has its interest in the analysis of the bifurcation diagrams in Γ-reversible-equivariant problems. In fact, according to Buono *et al.* [6], the structure of the set of equilibria in an open neighborhood of a generic equilibrium with isotropy subgroup Σ of a reversible-equivariant vector field must be decribed to obtain its bifurcation diagrams. They establish the dimension of families of reversible-equivariant equilibria with a given isotropy subgroup Σ in terms of $s_V(\Sigma)$ and obtain results about the bifurcation structure of some families. The second purpose of the present work is to establish two alternative formulae for the direct computation of $s_V(\Sigma)$, without knowing the values of the dimensions of $\mathrm{Fix}_V(\Sigma)$ and $\mathrm{Fix}_{V_\sigma}(\Sigma)$. One of these expressions is given by using the character of the representation of the normal subgroup Σ_+ formed by all symmetries of Σ, in terms of a Haar integral over Σ_+. Now, the symbolic computation packages GAP [9] and SINGULAR [12] have the character function implemented in their libraries. Hence, for examples where one uses those programmes, the usage of our expression for $s_V(\Sigma)$ make its computation much faster and simpler. Although the formulae are general, they turn out to be particularly useful for the class of non self-dual representations of Γ. In fact, when V is a self-dual vector space, $s_V(\Sigma)$ is zero, for all $\Sigma \subseteq \Gamma$. However, $s_V(\Sigma)$ can be either zero, positive or negative if V is non self-dual, so its value must be investigated.

The paper is organized as follows. In Section 2 we recall some concepts and results from group representation theory and equivariant theory of compact Lie groups. In Section 3 we describe the complete process of construction of the σ-isotypic components of a vector space V that represents the space of variables of a reversible-equivariant system. In Section 4 we present an algebraic expression for $s_V(\Sigma)$, where $\Sigma \subseteq \Gamma$ is a closed Lie subgroup. This shall be given in terms of a Haar integration over Σ_+. We end with Subsection 4.1, where we use the structure of the isotypic decomposition of V to obtain another expression for $s_V(\Sigma)$ for a special type of representations.

2 Preliminaries

We start this section by presenting the definitions and some basic results on representation theory of compact Lie groups and of reversible-equivariant theory.

2.1 Group representation

Let Γ be a compact Lie group and V a finite-dimensional vector space. We start by recalling that to a linear action of Γ on V,

$$\begin{array}{rcl} \Gamma \times V & \to & V \\ (\gamma, x) & \mapsto & \gamma x, \end{array}$$

there corresponds a group homomorphism from Γ to the group $\mathbf{GL}(V)$ of invertible linear transformations,

$$\begin{array}{rcl} \rho \,:\, \Gamma & \to & \mathbf{GL}(V) \\ \gamma & \mapsto & \rho(\gamma) \end{array},$$

$\rho(\gamma)(x) = \gamma x$, called a *representation* of Γ on V. We shall denote by (ρ, V) the vector space V under the representation ρ.

Being compact and a Lie group, Γ admits an invariant measure, denoted by $\mathrm{d}\gamma$, which is unique up to a constant multiple. Then, for $f : \Gamma \to \mathbf{R}$ continuous, we consider the *Haar integral* of f over Γ, which is denoted by $\int_{\gamma\in\Gamma} f(\gamma)\mathrm{d}\gamma$. We assume that the integral has been normalized, so that $\int_\Gamma 1\mathrm{d}\gamma = 1$. We refer to Bröcker and tom Dieck [5] for the definition and a proof of existence of the Haar measure on a compact Lie group. Using the Haar integral, we construct a $\Gamma-$invariant inner product on V (see Golubitsky *et al.* [10, Proposition XII 1.3]). As a consequence, we can identify Γ with a closed subgroup of the orthogonal group $\mathbf{O}(n)$. Hence, with no loss of generality, we assume Γ acting orthogonally on V.

We now fix a homomorphism of Lie groups

$$\sigma : \Gamma \to \mathbf{Z}_2 \cong \{\pm 1\}. \tag{1}$$

Notice that σ defines a unidimensional representation of Γ, with corresponding action on $\mathbf{R}$ given by $(\gamma, x) \mapsto \sigma(\gamma)x$.

Let us denote by $\Gamma_+ = \ker\sigma$ and Γ_- the complement of Γ_+ in Γ. Motivated by the study of reversible-equivariant dynamics, the following definition is given:

Definition 2.1 *Consider the homomorphism σ given in (1). An element in $\Gamma_+ = \ker\sigma$ is called a* symmetry *of Γ and an element in Γ_- is called a* reversing symmetry *of Γ. When Γ_- is non-empty, $\Gamma = \Gamma_+ \dot\cup\, \Gamma_-$ is called a* reversing symmetry group.

The product of two symmetries or two reversing symmetries is a symmetry, and the product of a symmetry and a reversing symmetry is a reversing symmetry. Also, if γ is a symmetry (reversing symmetry), then γ^{-1} is a symmetry (reversing symmetry). When σ is non-trivial, $\Gamma_+ \lhd \Gamma$ is a normal subgroup of Γ. In this case, we choose an arbitrary $\delta \in \Gamma_-$ and write $\Gamma_- = \delta\, \Gamma_+$, so Γ is decomposed as a disjoint union $\Gamma = \Gamma_+ \,\dot\cup\, \delta\, \Gamma_+$.

Definition 2.2 *Let ρ be a representation of Γ on V. Given the homomorphism σ in (1), we define the* dual representation *of ρ as*

$$\begin{array}{rcl} \rho_\sigma \, : \, \Gamma & \rightarrow & \mathbf{GL}(V) \\ \gamma & \mapsto & \sigma(\gamma)\rho(\gamma). \end{array} \tag{2}$$

Notice that $(\rho_\sigma)_\sigma = \rho$. In what follows, the representation (ρ_σ, V) shall also be denoted by V_σ.

Let us now recall the notion of equivalence between two representations of Γ. If ρ and η are two representations of Γ on V and W, respectively, then the two representations are *equivalent*, or said to be *isomorphic*, if there exists a linear isomorphism $T : (\rho, V) \rightarrow (\eta, W)$ such that $T(\rho(\gamma)x) = \eta(\gamma)T(x)$, for all $\gamma \in \Gamma,\ x \in V$. As mentioned in Section 1, the recognition of those representations that are isomorphic to its dual is the crucial point for the systematic study of a class of reversible-equivariant vector fields. These are the representations defined as follows:

Definition 2.3 *A representation ρ of Γ is called* self-dual *if it is isomorphic to ρ_σ. In this case, we say that (ρ, V) is a* self-dual *space.*

We end this section presenting a result, Corollary 2.5, that shall be used in Section 4 to establish a formula for the σ-index of a subgroup $\Sigma \subseteq \Gamma$. This result is an immediate consequence of the theorem below, which is concerned with the integration over a compact Lie group given by an iteration of integrals.

Theorem 2.4 ("Fubini") *Let Γ be a compact Lie group, Δ a closed subgroup. Let $\mathrm{d}(\gamma\Delta)$ denote the normalized Haar measure on the quotient Γ/Δ. For any continuous real-valued function f on Γ,*

$$\int_\Gamma f(\gamma)\,\mathrm{d}\gamma = \int_{\Gamma/\Delta} \left(\int_\Delta f(\gamma\eta)\,\mathrm{d}\eta \right) \mathrm{d}(\gamma\Delta) \,.$$

Proof. See Bröcker and tom Dieck [5, Proposition I 5.16]. □

If Γ admits a reversing symmetry δ and if $\Sigma \subset \Gamma$ is a subgroup with $\delta \in \Sigma$, then $\Sigma = \Sigma_+ \dot\cup \Sigma_-$, where $\Sigma_- = \Sigma \cap \Gamma_-$ and $\Sigma_+ = \Sigma \cap \Gamma_+$. In this case, $\Sigma/\Sigma_+ \cong \mathbf{Z}_2$, so the theorem above reduces to the following:

Corollary 2.5 *Let Γ be a reversing symmetry group and let $\Sigma \subseteq \Gamma$ be a closed Lie subgroup with $\delta \in \Sigma$ a reversing symmetry. If $f : \Sigma \rightarrow \mathbf{R}$ is a continuous function, then*

$$\int_\Sigma f(\gamma)\,\mathrm{d}\gamma = \frac{1}{2}\left[\int_{\Sigma_+} f(\gamma)\,\mathrm{d}\gamma + \int_{\Sigma_+} f(\delta\gamma)\,\mathrm{d}\gamma \right].$$

2.2 Character of a representation

Character theory is an essencial tool in invariant theory and sustain important properties of group representations. We refer to Bröcker and tom Dieck [5] or James and Liebeck [11] for details.

Given a representation ρ of Γ on V, the *character* of ρ is the function $\chi_V : \Gamma \to \mathbf{R}$,

$$\chi_V(\gamma) = \operatorname{tr}(\rho(\gamma)),$$

where $\operatorname{tr}(\rho(\gamma))$ denotes the trace of the matrix of $\rho(\gamma)$. Notice that if $\gamma, \delta \in \Gamma$, then

$$\chi_V(\gamma\delta\gamma^{-1}) = \chi_V(\delta),$$

so χ_V is constant on the conjugacy classes of Γ. In addition, isomorphic representations have the same character. We also have:

Lemma 2.6 *Let Γ be a reversing symmetry group and V a self-dual vector space. Then, $\chi_V(\gamma) = 0$, for any reversing symmetry γ .*

Proof. See Antoneli *et al.* [1]. □

Let $\Sigma \subset \Gamma$ be a subgroup. The *fixed-point subspace of* Σ is the subspace of V given by

$$\operatorname{Fix}_V(\Sigma) = \{x \in V \ : \ \rho(\gamma)x = x, \ \forall \gamma \in \Sigma\}.$$

We now define the object of Section 4, which gives the difference between the dimensions of the fixed-point subspaces of Σ in V and in its dual V_σ.

Definition 2.7 *Let Σ be a subgroup of Γ. We define the $\sigma-$index of Σ in V as*

$$s_V(\Sigma) = \dim \operatorname{Fix}_V(\Sigma) - \dim \operatorname{Fix}_{V_\sigma}(\Sigma).$$

The following lemma provides an algebraic expression for the dimension of $\operatorname{Fix}_V(\Sigma)$ by means of the Haar integration over Σ and the character function.

Lemma 2.8 *Let Γ be a compact Lie group acting on V and let $\Sigma \subset \Gamma$ be a Lie subgroup. Then,*

$$\dim \operatorname{Fix}_V(\Sigma) = \int_\Sigma \chi_V(\gamma)\, \mathrm{d}\gamma \ . \tag{3}$$

Proof. See [10, Theorem XIII 2.3]. □

If Σ is finite, then (3) becomes

$$\dim \operatorname{Fix}_V(\Sigma) = \frac{1}{|\Sigma|} \sum_{\gamma \in \Sigma} \chi_V(\gamma).$$

2.3 Equivariant mappings and isotypic components

Let Γ be a compact Lie group acting linearly on two finite-dimensional vector spaces V and W, with representations (ρ, V) e (η, W).

A mapping $g : (\rho, V) \to (\eta, W)$ is $\Gamma-$*equivariant* if

$$g(\rho(\gamma)x) = \eta(\gamma)g(x), \quad \forall\gamma \in \Gamma,\ x \in V. \tag{4}$$

When $(\eta, W) = (\rho, V)$, we say that g in (4) is *purely equivariant* (or simply equivariant) and, in this case, $\Gamma = \Gamma_+$ is the *symmetry group* of g. One important aspect of Γ-equivariant mappings is that they send fixed-point subspaces into fixed-point subspaces, i.e.,

$$g(\mathrm{Fix}_V(\Sigma)) \subseteq \mathrm{Fix}_W(\Sigma),$$

for any subgroup $\Sigma \subseteq \Gamma$.

We say that a subspace $U \subseteq V$ is Γ-*invariant* if $\gamma u \in U$ for all $u \in U$, $\gamma \in \Gamma$. If, in addition, the only Γ-invariant subspaces of U are $\{0\}$ and U, then the representation of Γ on U is called *irreducible* and U is called a Γ-*irreducible* subspace of V. A $\Gamma-$invariant subspace admits a complement in V which is also $\Gamma-$invariant (see Golubitsky *et al.* [10, Proposition XII 2.1]). As a consequence, we have the following result:

Theorem 2.9 *Let Γ be a compact Lie group acting on V.*

(a) Up to isomorphism, there exists a finite number of Γ-irreducible subspaces $U_k \subset V$, $k = 1, \dots, m$, U_k not isomorphic to U_j if $k \neq j$.

(b) Let V_k be the sum of all $\Gamma-$irreducible subspaces of V that are isomorphic to U_k. Then,

$$V = V_1 \oplus \cdots \oplus V_m. \tag{5}$$

(c) If $T : V \to V$ is a Γ-equivariant linear mapping, then $T(V_k) \subseteq V_k$, for all $k = 1, \dots, m$.

Proof. See Golubitsky *et al.* [10, Theorems XII 2.5 and 3.5]. □

Definition 2.10 *The subspaces V_k given in Theorem 2.9 are called* isotypic components *of V. The decomposition (5) is called* isotypic decomposition.

By definition, the isotypic components are unique and can be decomposed as a direct sum of isomorphic Γ-irreducible subspaces. Hence, if U is a $\Gamma-$irreducible subspace of V, then $U \subset V_k$ for a unique $k \in \{1, \dots, m\}$.

3 Reversible-equivariant mappings and the σ−isotypic decomposition

We start this section with the definition of mappings that model systems in the presence of symmetries and reversing symmetries, for which the results of this paper are devoted for.

Definition 3.1 *Let Γ be a compact Lie group and assume that its one-dimensional representation σ given on (1) is non-trivial. We say that a mapping $g : V \to V$ is* Γ−reversible-equivariant *if*

$$g(\rho(\gamma)x) = \sigma(\gamma)\rho(\gamma)g(x), \quad \forall \gamma \in \Gamma, \ x \in V. \tag{6}$$

The reversibility-equivariance condition (6) can be written as

$$g(\rho(\gamma)x) = \rho_\sigma(\gamma)g(x), \tag{7}$$

where ρ_σ is the dual representation of Γ on V defined in (2). It follows that a Γ−reversible-equivariant mapping is Γ−equivariant from (ρ, V) to (ρ_σ, V), that is, when the action of Γ on the target is the dual of its action on the source. If σ is trivial, then g in (7) is purely equivariant. When $\Gamma \simeq \mathbb{Z}_2$ and the identity is the unique symmetry of Γ, g in (7) is called *purely reversible.*

The purpose of this section is to describe the construction of subspaces that are left invariant by any Γ−reversible-equivariant linear mapping. This process corresponds to the construction of the isotypic components of V in the purely equivariant case (Definition 2.10).

Let ρ be a representation of Γ on V and let $\{(\rho_1, U_1), \ldots, (\rho_m, U_m)\}$ be the set of irreducible representations of Γ, with $\rho_k = \rho_{|U_k}$, such that each isomorphism class of irreducible representations contains precisely one of the U_k's, $k = 1, \ldots, m$. In what follows, we write U_k to denote (ρ_k, U_k) and $(U_k)_\sigma$ to denote the dual $((\rho_k)_\sigma, U_k)$, where $(\rho_k)_\sigma = \rho_{\sigma|U_k}$.

Firstly, we note that the decomposition of V into isotypic components under the representation ρ coincides (up to isomorphism) with the decomposition of V into isotypic components under its dual representation ρ_σ. This is directly verified by observing that :

(i) a subspace U_k is irreducible under ρ if, and only if, it is irreducible under ρ_σ;

(ii) two subspaces U_k and U_j are isomorphic if, and ony if, their duals $(U_k)_\sigma$ and $(U_j)_\sigma$ are isomorphic (by the same isomorphism).

Based on that, we generalize to the reversible-equivariant context two useful results [10, Lemma XII 3.4 and Theorem XII 3.5]) in representation theory of compact Lie groups. These are Lemma 3.2 and Theorem 3.5 below.

Lemma 3.2 *Let Γ be a reversing symmetry group acting on V. Let $L : V \to V$ be a Γ-reversible-equivariant linear mapping and let $W \subset V$ be a $\Gamma-$irreducible subspace. So $L(W)$ is $\Gamma-$invariant and either $L(W) = \{0\}$ or the representations of Γ on W and on $L(W)$ are isomorphic.*

Proof. It is a direct adaptation of the proof of [10, Lemma XII 3.4] to reversible-equivariant linear mappings. □

Next we present two results that derive from Lemma 3.2 above.

Proposition 3.3 *Let U_k e U_j be non-trivial $\Gamma-$irreducible subspaces of V and let $T : U_k \to U_j$ be a $\Gamma-$reversible-equivariant linear mapping. Then, T is non-zero if, and only if, it is an isomorphism.*

Proof. The sufficiency is obvious. For the necessity, we take a Γ-reversible-equivariant extension of the non-zero $T : U_k \mapsto U_j$ to V and apply Lemma 3.2. □

Remark 3.4 *From Proposition 3.3, it follows that for a given Γ-irreducible $U_k \subseteq V$, $k \in \{1, \ldots, m\}$, a necessary and sufficient condition for the existence of an irreducible $U_j \subseteq V$ isomorphic to the dual of U_k is the existence of a non-zero Γ-reversible-equivariant $T : V \to V$ such that $T(U_k) \neq \{0\}$. In other words, for any Γ-reversible-equivariant linear $L : V \to V$, we have that $L(U_k) = \{0\}$ if, and only if, there is no $j \in \{1, \ldots, m\}$ such that U_k is isomorphic to $(U_j)_\sigma$.*

Theorem 3.5 *Let Γ be a reversing symmetry group and let $L : V \to V$ be a Γ-reversible-equivariant linear mapping. Decompose V into isotypic components V_k's as in (5), each corresponding to the Γ-irreducible U_k, $k = 1, \ldots, m$. Then,*

(a) *If U_k is self-dual, then $L(V_k) \subseteq V_k$.*

(b) *If U_k is non self-dual and there exists a Γ-irreducible U_j which is isomorphic to $(U_k)_\sigma$, then $L(V_k) \subseteq V_j$, for $j \neq k$.*

(c) *If U_k is non self-dual and there is no Γ-irreducible U_j which is isomorphic to $(U_k)_\sigma$, then $L(V_k) = \{0\}$.*

Proof. For $k \in \{1, \ldots, m\}$, write V_k as a direct sum of $\Gamma-$irreducible subspaces that are isomorphic to U_k,

$$V_k = W_{k1} \oplus \cdots \oplus W_{k\alpha(k)}. \tag{8}$$

Let $\ell \in \{1, \ldots, \alpha(k)\}$. If there is U_j isomorphic to $(U_k)_\sigma$, then $W_{k\ell}$ is isomorphic to $(U_j)_\sigma$. By Lemma 3.2, either $L(W_{k\ell}) = \{0\}$ or $L(W_{k\ell})$ is isomorphic a U_j. In both cases, $L(W_{k\ell}) \subset V_j$ and, by linearity, it follows that $L(V_k) \subseteq V_j$. By the same argument, $L(V_j) \subseteq V_k$. Hence, we conclude parts (a) and (b), with $j = k$ for the first case and $j \neq k$ for the second.

If there is no U_j isomorphic to $(U_k)_\sigma$, then $L(W_{k\ell}) = \{0\}$ (see Remark 3.4). By linearity, $L(V_k) = \{0\}$. □

Remark 3.6 *As claimed in [14], each isotypic block V_k is self-dual if, and only if, the correspondent $\Gamma-$irreducible U_k is self-dual. Here we ratify this claim, taking now into account the three possible cases* (a)-(c) *of Theorem 3.5 that can occur: write V_k as in (8). If U_k is self-dual, then $W_{k\ell}$ is self-dual, for all $\ell \in \{1, \ldots, \alpha(k)\}$. Therefore V_k is self-dual. On the other hand, suppose that U_k is non self-dual. Then, any Γ-reversible-equivariant linear mapping $L : V \to V$ satisfies either $L(V_k) \subseteq V_j$, $j \neq k$, or $L(V_k) = \{0\}$ (cases* (b) *and* (c) *of Theorem 3.5, respectively). In both cases, the isotypic block V_k is non self-dual.*

The following result establishes the desired construction of the invariant subspaces and it is now an immediate consequence of Theorem 3.5.

Corollary 3.7 *Let V be decomposed into isotypic components V_k's as in (5), $k = 1, \ldots, m$. Then, there exists an order-2 permutation π of $\{1, \ldots, m\}$ such that the subspace*

$$\widehat{V}_k = V_k + V_{\pi(k)} \tag{9}$$

is left invariant by any $\Gamma-$reversible-equivariant linear mapping, where $\pi(k) = k$ for cases (a) *and* (c) *of the Theorem 3.5 and $\pi(k) = j$ for case* (b), for some *$j \in \{1, \ldots, m\}$, $j \neq k$.*

Definition 3.8 *The subspaces $\widehat{V}_k$'s given in (9) are called* σ-isotypic components *of V. The decomposition*

$$V = \widehat{V}_1 \oplus \ldots \oplus \widehat{V}_q$$

is called σ-isotypic decomposition *of V.*

We notice that the number q of subspaces $\widehat{V}_k$'s is at most m.

As mentioned before, the σ-isotypic components have first appeared in [14]. The irreducibles considered in that paper for the construction of those components correspond to cases (a) and (b) of Theorem 3.5 (see Subsection 3.2 therein). The construction described above shows that, in addition to these two types, there is one more case, namely (c), which appears in an extensive number of examples of reversible-equivariant systems. A typical and very simple example is as follows:

Example 3.9 *Consider the orthogonal group $\Gamma = \mathbf{O}(2)$ acting on $\mathbb{R}^3$, with the rotations $\theta \in \mathbf{SO}(2)$ acting by rotating the (x, y)-plane and leaving the z-axis fixed, and the flip $\kappa \in \mathbf{O}(2)$ acting by the reflection with respect to the x-axis. Take $\Gamma_+ = \mathbf{SO}(2)$. The two isotypic components are*

$$V_1 = U_1 = \{(x, y, 0)\}, \quad V_2 = U_2 = \{(0, 0, z)\}.$$

We have that V_1 is self-dual, so $\widehat{V_1} = V_1$. Now, V_2 is non self-dual, with $L(V_2) = \{0\}$ for all $\mathrm{O}(2)$-reversible-equivariant linear mapping L (which falls in case (c)*), so $\widehat{V_2} = V_2$. Hence, $\widehat{V_2}$ is a σ-isotypic component that is not given by the sum of two distinct isotypic blocks, which is the non self-dual case considered in [14].*

4 The $\sigma-$index

The purpose of this section is to establish formulae for the computation of the σ-index of Σ on V (Definition 2.7)

$$s_V(\Sigma) = \dim \mathrm{Fix}_V(\Sigma) - \dim \mathrm{Fix}_{V_\sigma}(\Sigma), \tag{10}$$

reducing the problem of finding $\dim \mathrm{Fix}_V(\Sigma)$ and $\dim \mathrm{Fix}_{V_\sigma}(\Sigma)$ to the direct computation of the dimensional difference. Notice that the σ-index is constant on the conjugacy classes of Γ. Moreover, as mentioned in the introductory section, if V is self-dual, then $s_V(\Sigma) = 0$, for all $\Sigma \subseteq \Gamma$. Hence, the results of this section are useful for the non self-dual cases, for which the σ-index may assume any value (zero, positive or negative).

In the theorem below we give an algebraic expression for $s_V(\Sigma)$ in terms of a Haar integral over Σ_+, depending only on the character of its representation on V. In Subsection 4.1, we obtain another explicit expression for $s_V(\Sigma)$ for a class of representations in terms of the σ-indices of Σ on the non self-dual $\Gamma-$irreducible subspaces of V.

Theorem 4.1 *Let Γ be a reversign symmetry group and ρ a representation of Γ on V. If $\Sigma \subset \Gamma$ is a closed Lie subgroup with Σ_- non-empty, then for fixed (but arbitrary) $\delta \in \Sigma_-$*

$$s_V(\Sigma) = \int_{\Sigma_+} \chi_V(\delta\gamma)\, \mathrm{d}\gamma\ , \tag{11}$$

where χ_V is the character of (ρ, V).

Proof. It follows directly from Lemma 2.8 and Corollary 2.5. □

If Σ_- is empty, it is direct from (10) that $s_V(\Sigma) = 0$. If Σ_- is non-empty and Σ is finite, then (11) becomes

$$s_V(\Sigma) = \frac{1}{\mid \Sigma_+ \mid} \sum_{\gamma \in \Sigma_+} \chi_V(\delta\gamma),$$

for a $\delta \in \Sigma_-$ fixed. In this case, $\delta\gamma$ runs over Σ_- when γ runs over Σ_+. In addition, $\mid \Sigma_+ \mid = \frac{|\Sigma|}{2}$, so we rewrite the expression above as

$$s_V(\Sigma) = \frac{2}{\mid \Sigma \mid} \sum_{\gamma \in \Sigma_-} \chi_V(\gamma). \tag{12}$$

We now illustrate the usage of Theorem 4.1. With the aim of applying our results to the analysis of branching of equilibria in reversible-equivariant bifurcations, we compute $s_V(\Sigma)$ for Σ an isotropy subgroup and Σ_- non-empty.

Example 4.2 *Consider the standard action of the dihedral group* D_n *on* $\mathbf{C}$ *generated by the complex conjugation and by the rotation* ξ *of angle* $\frac{2\pi}{n}$*:*

$$\kappa z = \bar{z} \quad e \quad \xi z = e^{i\frac{2\pi}{n}} z.$$

Take κ *as a symmetry and* ξ *as a reversing symmetry. This implies that* n *is even and* $\Gamma_+ = \mathrm{D}_{\frac{n}{2}}$ *(see Baptistelli and Manoel [3]). We have that* $\mathrm{tr}(\xi) = 2\cos(\frac{2\pi}{n})$ *is non-zero if, and only if,* $n \neq 4$*. Hence, such representation of* D_n *on* $\mathbf{C}$ *is non self-dual for* $n \neq 4$ *(Lemma 2.6). For* $n = 4$*, this representation is self dual, since*

$$L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is the matrix of a D_4*-reversible-equivariant linear map.*

Up to conjugacy, Σ *is* $\mathbf{Z}_2(\xi\kappa)$ *or* D_n*. For* $n = 4$*,* $s_{\mathbf{C}}(\Sigma) = 0$*. For* $n \neq 4$*, the isotropy* $\mathbf{Z}_2(\xi\kappa)$ *is generated by a reversing symmetry which has null character, so from (12)* $s_{\mathbf{C}}(\mathbf{Z}_2(\xi\kappa)) = 0$*. Reversing symmetries of* D_n *which are reflections have the form* $\kappa\xi^k$*, with* $1 \leq k \leq n-1$ *odd, and have null trace. On the other hand, reversing symmetries that are rotations are of the form* ξ^k*, with* $1 \leq k \leq n-1$ *odd, and have trace equal to* $2\cos[\frac{2k\pi}{n}]$*. From (12),*

$$s_{\mathbf{C}}(\mathrm{D}_n) = \frac{2}{n}\sum_{m=0}^{\frac{n}{2}-1} \cos\left[\frac{(4m+2)\pi}{n}\right].$$

For $n = 2$*,* $s_{\mathbf{C}}(\mathrm{D}_2) = -1$*. For* $n \geq 6$*,*

$$s_{\mathbf{C}}(\mathrm{D}_n) = \frac{2}{n}\left(\cos\frac{2\pi}{n} + \cos\frac{2.3\pi}{n} + \cdots + \cos\frac{2(n-3)\pi}{n} + \cos\frac{2(n-1)\pi}{n}\right) = 0.$$

Example 4.3 *Consider the standard action of* $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ *on the plane generated by the flips*

$$\kappa_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \kappa_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Take κ_1 *a reversing symmetry and* κ_2 *a symmetry, so this representation is non self-dual.*

Here, Σ *is* $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ *or* $\mathbb{Z}_2(\kappa_1)$*. The isotropy* $\mathbb{Z}_2(\kappa_1)$ *is generated by a reversing symmetry with null trace. From (12), we get* $s(\mathbb{Z}_2(\kappa_1)) = 0$ *and* $s(\mathbb{Z}_2 \oplus \mathbb{Z}_2) = -1$*.*

Example 4.4 *Consider the action of* $\mathbf{O}(2)$ *on* $\mathbf{R}^3$ *of Example 3.9, which is non self-dual. The isotropy subgroups are* $O(2)$, $\mathbf{SO}(2)$, $\mathbf{Z}_2(\kappa)$ *and* 1. *Here,* Σ *is* $O(2)$ *or* $\mathbf{Z}_2(\kappa)$. *From (12),* $s_{\mathbf{R}^3}(\mathbf{Z}_2(\kappa)) = -1$. *From (11),*

$$s_{\mathbf{R}^3}(\mathbf{O}(2)) = \int_{\mathbf{SO}(2)} \chi_{\mathbf{R}^3}(\kappa\theta)\, \mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \chi_{\mathbf{R}^3}(\kappa\theta)\, \mathrm{d}\theta = -1.$$

4.1 The σ−index for a class of representations

Let Γ be a reversing symmetry group acting on V. Consider V decomposed into isotypic blocks V_k, $k = 1, \ldots, m$. If V admits at least one Γ-irreducible subspace U_k self-dual, assume with no loss of generality that U_k is self-dual for $k \in I = \{1, \ldots, p\}$ and it is non self-dual for $k \in J = \{p+1, \ldots, m\}$. Otherwise, I is empty.

In Section 3 we have classified the Γ-irreducible subspaces U_k's into three types, as given in Theorem 3.5. In this subsection, we give a formula for the σ-index for cases where the irreducibles satisfy (a) or (b) of that theorem, which - as mentioned before - are the types considered by Lamb and Roberts [14]. More specifically, the representations that we investigate here shall satisfy the following assumptions:

- $J \neq \emptyset$
- for all $k \in J$, U_k is of type (b) of Theorem 3.5.

Let $\Sigma \subseteq \Gamma$ be a subgroup such that Σ_- is non-empty. Our aim is deduce an expression for $s_V(\Sigma)$ when V is a representation under the assumption above. We obtain $s_V(\Sigma)$ in terms of $s_{U_k}(\Sigma)$, for $k \in J$. Notice that $s_{U_k}(\Sigma) = 0$ for $k \in I$. More precisely, we have the following:

Theorem 4.5 *Under all the conditions above,*

$$s_V(\Sigma) = \sum_{k=p+1}^{\frac{m-p}{2}} \Big(\alpha(k) - \alpha(\pi(k))\Big) s_{U_k}(\Sigma), \tag{13}$$

where π *is the permutation of Corollary 3.7 and* $\alpha(k)$ *is the number of* Γ−*irreducible subspaces that sum direct to give the isotypic block* V_k.

Proof. Since $\mathrm{Fix}_V(\Sigma) = \bigoplus_{k=1}^m \mathrm{Fix}_{V_k}(\Sigma)$, then

$$\dim \mathrm{Fix}_V(\Sigma) = \sum_{k=1}^m \dim \mathrm{Fix}_{V_k}(\Sigma),$$

so the σ−index of $\Sigma \subseteq \Gamma$ in V can be write as

$$s_V(\Sigma) = \sum_{k=1}^m s_{V_k}(\Sigma) = \sum_{k=p+1}^m s_{V_k}(\Sigma). \tag{14}$$

By the existence of the permutation π of Corollary 3.7, $m-p$ must be even, and the $\sigma-$isotypic decomposition of V is given by

$$V = \widehat{V}_1 \oplus \ldots \oplus \widehat{V}_p \oplus \widehat{V}_{p+1} \oplus \ldots \oplus \widehat{V}_{\frac{m-p}{2}}, \tag{15}$$

where $\widehat{V}_k = V_k$, for $k \in I$, and $\widehat{V}_k = V_k \oplus V_{\pi(k)}$, for $k \in \{p+1, \ldots, \frac{m-p}{2}\} \subseteq J$. From (14) and (15),

$$s_V(\Sigma) = \sum_{k=p+1}^{\frac{m-p}{2}} \Big(s_{V_k}(\Sigma) + s_{V_{\pi(k)}}(\Sigma) \Big). \tag{16}$$

Now, from (8), we have that $\dim \mathrm{Fix}_{V_k}(\Sigma) = \alpha(k) \dim \mathrm{Fix}_{U_k}(\Sigma)$. Also,

$$\dim \mathrm{Fix}_{U_{\pi(k)}}(\Sigma) = \dim \mathrm{Fix}_{(U_k)_\sigma}(\Sigma),$$

for all $k \in \{p+1, \ldots, \frac{m-p}{2}\}$. Then,

$$s_{V_{\pi(k)}}(\Sigma) = \alpha(\pi(k)) \Big(\dim \mathrm{Fix}_{(U_k)_\sigma}(\Sigma) - \dim \mathrm{Fix}_{U_k}(\Sigma) \Big).$$

Hence, (16) becomes

$$s_V(\Sigma) = \sum_{k=p+1}^{\frac{m-p}{2}} \Big(\alpha(k) - \alpha(\pi(k)) \Big) s_{U_k}(\Sigma),$$

as desired. □

Example 4.6 *Consider the action of the group $\mathbb{Z}_2$ on $\mathbf{R}^3$ generated by the reflection*

$$\kappa = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

which we take to be a reversing symmetry. Such a representation of $\mathbb{Z}_2$ is non self-dual. The two isotypic components are

$$V_1 = \{(x, y, 0)\}, \quad V_2 = \{(0, 0, z)\},$$

where $U_1 = \{(x, 0, 0) : x \in \mathbb{R}\}$, $U_2 = \{(0, 0, z) : z \in \mathbb{R}\}$ are distinct $\mathbb{Z}_2-$irreducible subspaces and $\alpha(1) = 2\alpha(2) = 2$. We have that U_1 and U_2 are subspaces of type (b) *of Theorem 3.5, with $\pi(1) = 2$. Hence, we can use (13) to get that*

$$s_{\mathbb{R}^3}(\mathbb{Z}_2) = s_{U_1}(\mathbb{Z}_2) = 1.$$

References

[1] F. Antoneli, P.H. Baptistelli, A.P.S. Dias and M. Manoel, Invariant theory and reversible-equivariant vector fields. Submitted.

[2] V.I. Arnol'd, Reversible Systems. *Nonlinear and Turbulent Processes in Physics* **3** (1984), 1161-1174.

[3] P.H. Baptistelli and M. Manoel, Some Results on Reversible-Equivariant Vector Fields. *Cadernos de Matemática* **6** (2005), 237-263.

[4] P.H. Baptistelli and M. Manoel, The classification of reversible-equivariant steady-state bifurcations on self-dual spaces. *Math. Proc. Camb. Phil. Soc.*, To appear.

[5] T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups.* Graduate Texts in Mathematics **98**, Springer-Verlag, New York, 1995.

[6] P.L. Buono, J.S.W. Lamb and R.M. Roberts, Bifurcation and Branching of Equilibria of Reversible Equivariant Vector Fields. In preparation.

[7] C.A. Buzzi and J.S.W. Lamb, Reversible equivariant Hopf bifurcation. *Arch. Ration. Mech. Anal.* **175** (2005), 39-84.

[8] R.L Devaney, Reversible Diffeomorphisms and Flows. *Trans. Amer. Math. Soc.* **218** (1976), 89-113.

[9] The GAP Group: GAP - Groups, Algorithms, and Programming. Version 4.4.9, 2006. (`http://www.gap-system.org`)

[10] M. Golubitsky, I. Stewart and D. Schaeffer, *Singularities e Groups in Bifurcation Theory, Vol. II.* Appl. Math. Sci. **69**, Springer-Verlag, New York, 1985.

[11] G. James and M. Liebeck, *Representations and Characters of Groups. 2nd ed.* Cambridge University Press, New York, 2001.

[12] G.M. Greuel, G. Pfister and H. Schönemann, SINGULAR 3.0. A Computer Algebra System for Polynomial Computations. Centre for Computer Algebra, University of Kaiserslautern (2005). (`http://www.singular.uni-kl.de`)

[13] J.S.W. Lamb and J.A.G. Roberts, Time-reversal Symmetry in Dynamical Systems: A Survey. *Physica* D **112** (1998), 1-39.

[14] J.S.W Lamb and R.M. Roberts, Reversible Equivariant Linear Systems. *J. Diff. Eq.* **159** (1999), 239-279.

[15] D.H. Sattinger, *Group Theoretic Methods in Bifurcation Theory.* Lecture Notes in Mathematics **762**, Springer-Verlag, New York, 1979.

[16] M.B. Sevryuk, *Reversible Systems.* Lecture Notes in Mathematics **1211**, Springer-Verlag, New York, 1986.

[17] A. Vanderbauwhede, *Local bifurcation and symmetry.* Research Notes in Mathematics **75**, Pitman, London, 1982.

NOTAS DO ICMC

SÉRIE MATEMÁTICA

289/2007 CARVALHO, A.N.; DLOTKO, T.; NASCIMENTO, M.J.D. – Non –autonomous semilinear evolution equations with almost sectorial operators.

288/2007 CARVALHO, A.N.; DLOTKO, T. – Dynamics of the viscous Cahn-Hilliard equations.

287/2007 BIASI, C.; MATTOS, D.; SANTOS, E.L. – Applications of the non-standard Borsuk-Ulam theorem.

286/2007 NABARRO, A.C.; TARI, F. – Families of surfaces and conjugate curve congruences.

285/2007 FEDERSON, M. – Converse Lyapunov theorems for retarded functional differential equations.

284/2007 ANTONELI, F.; DIAS, A. P.; BAPTISTELLI, P. H.; MANOEL, M. G. – Invariant theory and reversible-equivariant vector fields.

283/2007 TAHZIBI, A. – Ergodic properties of dynamical systems beyond uniform hiperbolicity (38th Iranian Mathematical Conference) Zanjan-Iran.

282/2007 LLIBRE, J.; OLIVEIRA, R.D.S. – Phase portraits of quadratic polynomial vector fields having a rational first integral of degree3.

281/2007 BONOTTO, E.M.; GRULHA JR., N.G. – Lyapunov stability of closed sets in impulsive semidynamical systems.

280/2007 CARBINATTO, M.C.; RYBAKOWSKI, K.P. – On the suspension isomorphism for index braids in a singular perturbation problem.