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A Feynman-Kac solution to an impulsive equation of Schrödinger type

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Resumo

Quando uma força é aplicada em uma partícula que descreve um movimento browniano, o movimento resultante admite uma função de estado que satisfaz uma equação do tipo Schrödinger. Consideramos, neste trabalho, um processo em que um movimento browniano é substituído por um processo que possui transições brownianas em todo tempo, exceto nos momentos de tempo onde o processo é sujeito a uma ação impulsiva. Utilizando a formulação de Feynman-Kac, baseada em integração de Riemann generalizada, examinamos a equação do movimento resultante.

Abstract

If a force is applied to a particle undergoing Brownian motion, the resulting motion has a state function which satisfies a diffusion or Schrödinger-type equation. We consider a process in which Brownian motion is replaced by a process which has Brownian transitions at all times other than specified times at which the transitions have an additional "impulsive" displacement. Using a Feynman-Kac formulation based on generalized Riemann integration, we examine the resulting equation of motion.

1 Introduction

As an introduction, we give a broad outline of the underlying ideas and methodology of the paper.

1.1 Some Underlying Ideas

When some system parameter has a discontinuity, the term "impulse" or "jump" can be a vivid way of describing this characteristic of the system.

Sometimes the state of a system can be described by a differential equation. For instance, a diffusion can be described by a parabolic partial differential equation satisfied by some function of displacement and time.

The purpose of this paper is to examine the relationship between discontinuities or impulsive changes in the state function which characterize the diffusion, and discontinuities in the underlying diffusion itself. We use a Feynman-Kac formulation to show the connection between these two classes of discontinuities.

Our method of analysis is based on the generalized Riemann approach of Henstock. In effect, our Feynman-Kac formulation of the problem is a generalized Riemann (or Henstock) integral.

The generalized Riemann integral is an adaptation of the standard Riemann integral such that Riemann sums can be used to give results for which Lebesgue methods are usually required. The general idea of this is as follows. We have some domain which is partitioned by means of a finite collection $\{I\}$ of disjoint sets, which we can think of as "intervals", with $|I|$ denoting the measure of an interval I. By "shrinking" the partitions, we can estimate the Riemann integral of a function $f(x)$ of values x in the domain by forming the Riemann sums $\sum f(x)|I|$.

In the standard Riemann integral, in any term $f(x)|I|$ of the Riemann sum, the only restriction on the choice of the evaluation point x is that it should belong to the corresponding partitioning interval I. The generalized Riemann adaptation is to make the selection of each interval I in the partition depend on the choice of each corresponding evaluation point x in $\sum f(x)|I|$. What difference does this make? It means that we can form the Riemann sums in a way which is sensitive to, or responsive to, the local behavior of the integrand. For instance, if f is highly oscillatory in a particular neighborhood, taking very large positive and negative values there, we can force the local terms of the Riemann sum to correspond to the local behavior of f. So in a scenario where f has a positive value at x and a negative value at the nearby point x', the partitioning intervals I, I' can be chosen so that the Riemann sum $\ldots + f(x)|I| + f(x')|I'| + \ldots$ captures the variation of f; so that, in this scenario, we produce in the Riemann sum a cancelation effect from the neighboring x, x'.

In this way it is found that we can define an integral of f which is equal to the Lebesgue integral of f whenever the latter exists. We call this the generalized Riemann integral (also known as Henstock integral or Henstock-Kurzweil integral).

Instead of using the Lebesgue measure $|I|$, we can use arbitrary interval functions $\mu(I)$, and the resulting definition of an integral $\int f(x)\mu(I)$ by Riemann sums remains valid.

More generally, instead of integrating a product $f(x)\mu(I)$, we can integrate functions $h(x, I)$ by taking Riemann sum estimates $\sum h(x, I)$ over x-dependent partitions $\{I\}$ of the domain of integration.

The discussion above can be read in a way which assumes that the domain of integration is a bounded real interval $[a, b]$, so that each of the partitioning intervals I is itself a bounded real interval. But the points made in the discussion remain valid if the domain of integration is a more general, multi-dimensional space, such as $\mathbb{R}^n$, in which some of the partitioning intervals are not bounded or compact.

The scenario we tackle in this paper requires us to consider displacements x_t at various times t in some time interval $]\tau', \tau[$, and also to consider the possibility that, at arbitrary times $\tau' < t_1 < \cdots < t_{n-1} < \tau$, the displacements x_{t_j} satisfy $u_j \leq x_{t_j} \leq v_j$ for $1 \leq j \leq n-1$; or $x_j \in \mathrm{Cl}(I_j)$, where we write $I_j = [u_j, v_j[$ and $x_j = x_{t_j}$ for each j.

Writing

$$x = (x_t)_{t \in]\tau', \tau[} \quad \text{and} \quad I = \{x : x_j \in I_j,\ 1 \leq j \leq n-1\}$$

we are led to consider Riemann sums such as $\sum f(x)\mu(I)$. The corresponding integrals are $\int f(x)\mu(I)$. The domain of integration is the set $\{x\}$, where each x is a mapping of the form

$$x :]\tau', \tau[\mapsto \mathbb{R}, \quad \text{with } x_t = x(t) \in \mathbb{R} \quad \text{for } \tau' < t < \tau.$$

We denote this domain by $\mathbb{R}^{]\tau', \tau[}$, which can be viewed as a Cartesian product of $\mathbb{R}$ by itself uncountably many times. The partitioning intervals I are cylindrical subsets of $\mathbb{R}^{]\tau', \tau[}$.

The framework of generalized Riemann integration outlined above can be adapted to this scenario, and this is explained in more detail in the following sections.

Treating the elements x as sample paths in some version of the Brownian motion, we develop a Feynman-Kac representation

$$u(\xi, \tau) = \int_{\mathbb{R}^{]\tau', \tau[}} f(x)\mu(I),$$

with $x(\tau) := \xi$, of the solutions $u(\xi, \tau)$ of a partial differential equation

$$\frac{\partial u}{\partial \tau} - \frac{1}{2}\frac{\partial^2 u}{\partial \xi^2} + V(\xi)u = 0.$$

where V is a potential function.

With the aid of this theoretical framework, we can relate discontinuities in $u(\xi, \tau)$ to "impulses" in the sample paths x.

1.2 Outline of the Theory

To begin, we define the different kinds of cylindrical intervals

$$I = P_N^{-1}(I_1 \times \cdots \times I_{n-1})$$

which are used to partition the domain of integration $\mathbb{R}^{]\tau', \tau[}$. Given

$$N = \{t_1, \ldots, t_n\} \subset]\tau', \tau[, \quad \text{with } \tau' = t_0 < t_1 < \cdots < t_{n-1} < t_n = \tau,$$

P_N is the projection function which maps $\mathbb{R}^{]\tau',\tau[}$ to the n-dimensional product space $\mathbb{R}^N$ or $\mathbb{R}^n$.

In general, if we want to define an integral $\int_{\mathbb{R}^{]\tau',\tau[}} f(x)\mu(I)$, we must show how the approximating Riemann sums $\sum f(x)\mu(I)$ are constructed, and that is done in the next section.

We are especially interested in volume functions (or measures) μ on the sets I which are related to the Brownian motion function in which each difference $x(t_j) - x(t_{j-1})$ is normally distributed with mean zero and variance $t_j - t_{j-1}$, giving

$$\mu(I) = \int_{I_1} \cdots \int_{I_{n-1}} \prod_{j=1}^{n} \left(\frac{e^{-\frac{1}{2}\frac{(y_j - y_{j-1})^2}{t_j - t_{j-1}}}}{\sqrt{2\pi(t_j - t_{j-1})}} \right) dy_1 \dots dy_{n-1} \tag{1}$$

as the joint probability that the motion takes a value x_j in I_j at time t_j, for $1 \leq j \leq n-1$. (There is a technical, notational reason for having $1, \dots, n-1$, rather than $1, \dots, n$, as the range of j.) The finite-dimensional expression on the righthand side of (1) is used to give meaning to the function on the left which has an infinite-dimensional domain. The essential simplicity of the expression on the right helps to simplify the analysis, in comparison with other formulations of the theory which introduce measurable sets at this point, instead of the simpler cylindrical intervals I. Because if the integrand $f(x)$ takes the value 1 for all x, then $\int_{\mathbb{R}^{]\tau',\tau[}} f(x)\mu(I)$ is approximated by Riemann sums $\sum \mu(I)$ whose value, in every case, turns out to be

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{j=1}^{m} \left(\frac{e^{-\frac{1}{2}\frac{(y_j - y_{j-1})^2}{t_j - t_{j-1}}}}{\sqrt{2\pi(t_j - t_{j-1})}} \right) dy_1 \dots dy_{m-1},$$

where $\{t_1, \dots, t_{m-1}\}$ is the maximal of the sets of times $\{\cdots, t_j, \cdots\}$ which appear, as variable sets, in each of the terms $\mu(I)$ of the Riemann sum.

The latter integral can be evaluated by iterated integration, and this is demonstrated in a later section. Alternatively, with $x_0 = x(t_0) = x(\tau')$ and $x_n = x(t_n) = x(\tau)$, basic probability theory tells us that this integral gives the probability density function of a normal random variable $x(\tau) - x(\tau')$ with mean zero and variance $\tau - \tau'$, so the value of the integral is

$$\frac{e^{-\frac{(x(\tau)-x(\tau'))^2}{2(\tau-\tau')}}}{\sqrt{2\pi(\tau - \tau')}}.$$

Some of these ideas occur in the analysis of Brownian motion, but usually expressed somewhat differently.

1.3 Impulsive Processes with Drift

When the underlying Brownian process undergoes impulsive changes of amounts J_k at specified times τ_k, then we get a process

$$z = \{z_t\} = \{z(t) : t \in]\tau', \tau[\}, \quad \text{where } z(t) = x(t) + \sum_{\tau_k \leq t} J_k.$$

From this we are led to consider a measure

$$\mu(I) = \int_{I_1} \cdots \int_{I_{n-1}} \prod_{j=1}^{n} \left(\frac{e^{-\frac{1}{2}\frac{(z_j - y_{j-1})^2}{t_j - t_{j-1}}}}{\sqrt{2\pi(t_j - t_{j-1})}} \right) dy_1 \ldots dy_{n-1}.$$

where $z_j = y_j + J_j$, if t_j is one of the instants τ_k, and $z_j = y_j$ otherwise.

Our purpose is to investigate systems in which the impulsive process is subject to some external force which produces a further manifestation of drifting motion in the process, represented by a potential function V which, at any time t, depends on the displacement $z(t)$. The study of systems of this kind leads to examination of the function

$$\begin{aligned} f(x) &= \exp\left(-\sum_{j=1}^{n} V(z(t_j))(t_j - t_{j-1}) \right) \\ &= \exp\left(-\sum_{j=1}^{n} V\left(x(t_j) + \sum_{\tau_i \leq t_j} J_i \right) (t_j - t_{j-1}) \right) \end{aligned}$$

defined for $x \in \mathbb{R}^{]\tau', \tau[}$. (Indeed, f depends also on the choice of the set $\{t_1, \ldots, t_n\}$ which, like x, is a variable in successive terms of a Riemann sum. Suitable notation for this dependence is presented in a later section.)

The state function $u(\xi, \tau)$ describing the evolution of this system, with $\xi = x(\tau)$, is often obtained as a solution to an appropriate parabolic diffusion equation, and sometimes this has a Feynman-Kac representation

$$u(\xi, \tau) = \int_{\mathbb{R}^{]\tau', \tau[}} f(x)\mu(I).$$

We investigate each of these methods of determining $u(\xi, \tau)$ and, by examining Riemann sum estimates of u, we show how discontinuities in u are related to impulsive phenomena in the underlying process z. The latter can be regarded as initial conditions or boundary conditions for the constitutive partial differential equation.

Our investigation is restricted to impulses J which are deterministic functions of the displacement $x(t)$, at pre-determined times $t = \tau_i$ for $i = 1, 2, \ldots, p$. In the concluding section of the paper, we illustrate the theory by explicit evaluation of u when each of the impulses is a constant.

2 Preliminaries

In this section we present the basic definitions and notation of the theory of Henstock Integral in Function Spaces. We also include some fundamental results which are necessary for understanding the basis of the theory.

2.1 The Henstock Integral in Function Space

Let $\mathbb{R}$ denote the set of real numbers and let $\mathbb{R}_+$ denote the set of positive real numbers. Let I be a real interval of the form:

$$]-\infty,\ v[, \qquad [u,\ v[\quad \text{or} \quad [u,\ +\infty[. \tag{2}$$

A partition of $\mathbb{R}$ is a finite collection of disjoint intervals I whose union is $\mathbb{R}$. We say that I is attached to x (or associated with x) if

$$x = -\infty, \qquad x = u \text{ or } v, \qquad x = +\infty,$$

respectively.

Let $\overline{\mathbb{R}}$ denote the union of the domain of integration $\mathbb{R}$ with the set of associated points x of the intervals I of $\mathbb{R}$, so that $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty,\ +\infty\}$.

In generalized Riemann integration, the convention is that the domain of integration is the space which is partitioned by intervals. A point x is not always an element of the associated interval I to which it is attached. Thus the set of associated points x may constitute a set which differs from the domain of integration. In our case, the domain of integration is $\mathbb{R}$ and the set of associated points is $\overline{\mathbb{R}}$.

Let $\delta : \overline{\mathbb{R}} \to \mathbb{R}_+$ be a positive function defined for $x \in \overline{\mathbb{R}}$. If I is attached to x, we say that $(x,\ I)$ is δ-fine if

$$v < -\frac{1}{\delta(x)}, \qquad v - u < \delta(x), \qquad \text{or} \qquad u > \frac{1}{\delta(x)}, \tag{3}$$

respectively.

In this version of the integral, the attached or associated points x of an interval I are its vertices. In another version (see [19]), they are chosen from the union of I with the vertices: that is, the closure of I in the open-interval topology. These two versions are equivalent whenever the integrator (measure or interval function) is finitely additive, because if x is an interior point of $[u,\ v[$, then $f(x)m([u,\ v[) = f(x)m([u,\ x[) + f(x)m([x,\ v[)$, see [19]. In yet another version (see [6]), an equivalent of the Lebesgue integral is produced if the associated or attached intervals of a point x are the intervals I satisfying $I \subseteq\ ]x - \delta(x),\ x + \delta(x)[$. In this case, the attached points of an interval may be outside the closure of I in the open-interval topology. In all cases, the domain of integration is the space which is partitioned by the intervals.

If $N = \{t_1, ..., t_n\}$ is a finite set, with $\mathbb{R}_{t_j} = \mathbb{R}$ and $\overline{\mathbb{R}}_{t_j} = \overline{\mathbb{R}}$, let $x = (x(t_1), ..., x(t_n))$ denote any element of

$$\prod\{\overline{\mathbb{R}}_{t_j} : t_j \in N\} = \overline{\mathbb{R}}^N.$$

Denote $x(t_j)$ by x_j, $1 \le j \le n$. For each $t_j \in N$, let $I_j = I(t_j)$ denote an interval of form (2). Then $I = I_1 \times ... \times I_n$ is an interval of $\prod\{\mathbb{R}_{t_j} : t_j \in N\} = \mathbb{R}^N$. A pair (x, I) is associated, or attached, in $\mathbb{R}^N$ if each (x_j, I_j) is associated in $\mathbb{R}$, $1 \le j \le n$, that is, x is a vertex of I in $\overline{\mathbb{R}}^N$. Given a function $\delta : \overline{\mathbb{R}}^N \to \mathbb{R}_+$, an associated pair (x, I) of the domain $\mathbb{R}^N$ is δ-fine if each (x_j, I_j) satisfies one of the inequalities in (3), depending on the kind of interval I_j (see (2)). A finite collection $\mathcal{E} = \{(x_j, I_j)\}$ of associated pairs (x_j, I_j), where each (x_j, I_j) is associated in $\mathbb{R}^N$, is a *division* of $\mathbb{R}^N$ if the intervals I_j are disjoint with union $\mathbb{R}^N$, and the division is *δ-fine* if each of the pairs (x_j, I_j), $1 \le j \le n$, is δ-fine. A proof of the existence of such a δ-fine division is given in [19], Theorem 4.1.

Let B denote any infinite set, and let $\mathcal{F}(B)$ denote the family of finite subsets of B. In what follows, we consider the product space $\prod_{t \in B} \mathbb{R}_t$ with $\mathbb{R}_t = \mathbb{R}$ for each $t \in B$, that is, the set of all functions on B to $\mathbb{R}$. We prefer to use, for this product, the notation $\mathbb{R}^B$ which is usual in the theory of stochastic processes.

Let $x = x_B$ denote any element of $\overline{\mathbb{R}}^B$. With

$$N = N_B = \{t_1, ..., t_n\} \in \mathcal{F}(B),$$

let $x(N) = x(N_B)$ denote a point $(x_1, ..., x_n) = (x(t_1), ..., x(t_n))$ of $\overline{\mathbb{R}}^N$. Consider the projection

$$P_N : \mathbb{R}^B \to \mathbb{R}^N, \qquad P_N(x) = (x(t_1), ..., x(t_n)),$$

and similarly we define the projection $\overline{P}_N : \overline{\mathbb{R}}^B \to \overline{\mathbb{R}}^N$. Then, to each interval $I_1 \times ... \times I_n$ of $\mathbb{R}^N$ there corresponds a cylindrical interval $I[N] := P_N^{-1}(I_1 \times ... \times I_n)$, which is a subset of $\mathbb{R}^B$. It is often convenient to denote $I_1 \times ... \times I_n$ by $I(t_1) \times ... \times I(t_n)$ or $I(N)$, so $I[N] = I(N) \times \mathbb{R}^{B \setminus N}$. Similarly,

$$\overline{P}_N(x_B) = x(N) \in \overline{\mathbb{R}}^N, \quad \text{for} \quad x = x_B \in \overline{\mathbb{R}}^B.$$

Given $x \in \overline{\mathbb{R}}^B$ and $I[N] \subset \mathbb{R}^B$, we say that $(x, I[N])$ is associated in $\mathbb{R}^B$, if $(x(N), I(N))$ is an associated pair in $\mathbb{R}^N$. Our domain of integration is $\mathbb{R}^B$ and the set of associated points is $\overline{\mathbb{R}}^B$.

Definition 2.1 *A finite collection $\mathcal{E} = \{(x, I[N]) : x \in \overline{\mathbb{R}}^B$ and $N \in \mathcal{F}(\mathcal{B})\}$ of associated pairs is said to be a division of $\mathbb{R}^B$, if the intervals $I[N]$ are disjoint and have union $\mathbb{R}^B$.*

Divisions of cylindrical intervals in $\mathbb{R}^B$ are defined similarly.

We now address the question of a gauge for $\mathbb{R}^B$, that is, a rule which determines which associated point-interval pairs $(x, I[N])$ are admissible, as elements of a division, in forming a Riemann sum approximation of an integral in the infinite dimensional space $\mathbb{R}^B$. To do this, we define mappings L_B on the sets of associated points $\overline{\mathbb{R}}^B$ of the domain of integration $\mathbb{R}^B$, and mappings δ_B on $\overline{\mathbb{R}}^B \times \mathcal{F}(B)$, which give us an effective class of gauges. Let

$$L_B : \overline{\mathbb{R}}^B \to \mathcal{F}(B), \qquad L_B(x) \in \mathcal{F}(B);$$
$$\delta_B : \overline{\mathbb{R}}^B \times \mathcal{F}(B) \to \mathbb{R}_+, \qquad 0 < \delta_B(x, N) < +\infty.$$

A choice of L_B and δ_B gives us a representative member of this class of gauges:

$$\gamma_B := (L_B,\, \delta_B). \tag{4}$$

We say that an associated point-interval pair $(x,\, I[N])$ is γ_B-fine if

$$N \supseteq L_B(x), \quad \text{and} \quad (x(N),\, I(N)) \text{ is } \delta_B\text{-fine in } \mathbb{R}^N.$$

Definition 2.2 *Let $\gamma = (L,\, \delta)$ and $\gamma_1 = (L_1,\, \delta_1)$ be two gauges. We say that $\gamma \prec \gamma_1$ if $L(x) \supseteq L_1(x)$ for each $x \in \overline{\mathbb{R}}^B$, and if $\delta(x,\, N) \leq \delta_1(x,\, N)$ for each $(x,\, N) \in \overline{\mathbb{R}}^B \times \mathcal{F}(B)$.*

Definition 2.3 *A division $\mathcal{E} = \{(x,\, I[N]) : x \in \overline{\mathbb{R}}^B$ and $N \in \mathcal{F}(\mathcal{B})\}$ of the domain of integration is γ_B-fine, or is a γ_B-division, if each of the pairs $(x,\, I[N])$ is γ_B-fine. In this case, we denote $\mathcal{E}$ by $\mathcal{E}_{\gamma_B}$.*

The space $\mathbb{R}^B$ admits a γ_B-division. This result is stated next and a proof of it can be found in [18], Theorem 1.

Theorem 2.1 *For any infinite set B and for any given gauge γ_B, there exists a γ_B-fine division of $\mathbb{R}^B$.*

Suppose h is a function of attached pairs $(x,\, I[N])$. Sometimes $h(x,\, I[N])$ is undefined for certain $x \in \overline{\mathbb{R}}_+^B$ (or $\overline{\mathbb{R}}^B$), such as those x for which $x(t) = 0$ or ∞ for $t \in N$. In such cases, we may take $h(x,\, I[N])$ to be zero, and these terms are omitted from the Riemann sum. If E is an elementary set (an interval or a finite union of intervals), the variation of h in E is given by

$$\inf_{\gamma_B} \left\{ \sup_{\mathcal{E}_{\gamma_B}} \{(\mathcal{E}_{\gamma_B})|h(x,\, I[N])|\} \right\},$$

where $\mathcal{E}_{\gamma_B}$ is any γ_B-division of E.

More generally, if X is any subset of $\mathbb{R}^B$, the variation of h in $\mathbb{R}^B$ relative to X is

$$\inf_{\gamma_B} \left\{ \sup_{\mathcal{E}_{\gamma_B}} \{(\mathcal{E}_{\gamma_B})|h(x,\, I[N])|1_X(x)\} \right\},$$

where $1_X(x)$ is the characteristic function or indicator function of X and $\mathcal{E}_{\gamma_B}$ is any γ_B-division of $\mathbb{R}^B$. We say that h is of bounded variation in X, if its variation in X is finite. We say that h is VBG* (or h is of generalized bounded variation in $\mathbb{R}^B$), if $\mathbb{R}^B$ is the union of disjoint sets X_j with h being of bounded variation in X_j, $j = 1, 2, \ldots$.

The generalized Riemann integral of a function h of an associated pair $(x,\, I[N])$ is defined as follows (see [11]).

Definition 2.4 *The function h is generalized Riemann integrable over $\mathbb{R}^B$, with integral $\alpha = \int_{\mathbb{R}^B} h$, if, given $\epsilon > 0$, there exists a gauge γ_B such that*

$$\left| \sum_{(x,\, I[N]) \in \mathcal{E}_{\gamma_B}} h(x,\, I[N]) - \alpha \right| < \epsilon$$

for every γ_B-division $\mathcal{E}_{\gamma_B}$ of $\mathbb{R}^B$.

Sometimes we integrate functions $h(I[N])$ which do not depend on the associated point x of the variable $I[N]$. In generalized Riemann integration, this must be handled carefully. We should think of the integrand as $h(x, I[N]) = h(I[N])$ for every x associated with $I[N]$. Thus, even though the variable x does not appear explicitly in the integrand, the terms $\sum h(I[N])$ of the Riemann sum still depend on the x's of the division $\{(x, I[N])\}$ which determines the Riemann sum.

Now we define variational equivalence of two functions of $(x,\ I[N])$. Two functions $h_1(x,\ I[N])$ and $h_2(x,\ I[N])$ are variationally equivalent in $\mathbb{R}^B$, if $h_1 - h_2$ has variation zero in $\mathbb{R}^B$([11], page 32). It is easy to show that h_1 is variationally equivalent to h_2 if, given $\epsilon > 0$, there exists γ_B such that, for all divisions $\mathcal{E}_{\gamma_B}$,

$$\sum_{(x,\ I[N]) \in \mathcal{E}_{\gamma_B}} |h_1(x,\ I[N]) - h_2(x,\ I[N])| < \epsilon.$$

If h_1 is integrable in $X \subseteq \mathbb{R}^B$ and if h_2 is variationally equivalent to h_1, then h_2 is integrable in X and $\int_X h_1 = \int_X h_2$ (see [11], Proposition 18, page 32 for a proof). This result is important because sometimes, when we want to establish a property of $\int_X h_1$, it is easier to demonstrate it first for an integral $\int_X h_2$, where h_2 is "equivalent", in the variational sense, to h_1.

2.2 Limit theorem in function space integration

The following result corresponds to Lebesgue's dominated convergence theorem. A proof of it can be found in [11], Proposition 33.

Theorem 2.2 *Suppose $h_j(x,\ N,\ I)$ is integrable in $\mathbb{R}^B$, $j = 1, 2, 3, \ldots$, and for each associated pair $(x,\ I[N])$, the sequence $\{h_j(x,\ I,\ N)\}_{j \geq 1}$ converges to a value $h(x,\ N,\ I)$. Suppose, if $\epsilon > 0$ is given, there exists a gauge γ_1 such that, whenever $(x,\ I[N])$ is γ_1-fine, there exists $j_0 = j_0(x,\ I[N]) > 0$ such that $j > j_0$ implies*

$$|h(x,\ N,\ I) - h_j(x,\ N,\ I)| < \epsilon g_0(x,\ N,\ I),$$

where g_0 is a positive function, integrable in $\mathbb{R}^B$. Suppose there exist functions $g_1(x,\ N,\ I)$ and $g_2(x,\ N,\ I)$, integrable in $\mathbb{R}^B$, and a gauge γ_2 satisfying

$$g_1(x,\ N,\ I) \leq h_j(x,\ N,\ I) \leq g_2(x,\ N,\ I)$$

for each j and each γ_2-fine $(x,\ I[N])$. Then h is generalized Riemann integrable in $\mathbb{R}^B$ and

$$\lim_{j \to +\infty} \int_{\mathbb{R}^B} h_j(x,\ N,\ I) = \int_{\mathbb{R}^B} h(x,\ N,\ I).$$

2.3 The Wiener Integral

An irregular movement of pollen suspended in water was observed by the botanist Robert Brown in 1828. This movement is called Brownian motion or Wiener process. In the beginning of the 20th century, important applications of the Brownian motion were discovered. The first was a theory of stock price fluctuations due to Bachelier (1900) [1]. The second was an investigation of the properties of the density of Brownian particles at given position and time due to Einstein [4]. Details about Brownian motion can be found in [2], [5], [8] and [9].

We now present a basic definition.

Definition 2.5 *A Wiener process or a Brownian motion, is a stochastic process* $\{W(t) = W_t : t \in \mathbb{T}\}$, $\mathbb{T} = [0, +\infty)$ *or* $\mathbb{T} = [0, T]$, $T \in \mathbb{R}_+$, *defined in a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and satisfying the following properties:*

a) $W(0) = 0$;

b) for each $s \geq 0$ *and each* $t > 0$, *the random variable* $W_{t+s} - W_s$ *has normal distribution with mean zero and variance* $\sigma^2 t$;

c) for each $n \geq 1$ *and times* $0 \leq t_0 \leq t_1 \leq ... \leq t_n$, *the random variables* $\{W_{t_r} - W_{t_{r-1}}\}_{1 \leq r \leq n}$ *are independent.*

The parameter σ^2 is known as the variance parameter. When $\sigma^2 = 1$, the process is called a standard Brownian motion. The existence of Brownian motion can be demonstrated by various arguments. See, for instance, [8]. Since the generalized Riemann integral is an extension of the Lebesgue integral [17], these arguments are valid in our discussion here. Our realization of Brownian motion uses $\Omega = \mathbb{R}^{[0,T]}$, and $W(t) = x(t) = x_t$.

If $0 = t_0 < t_1 < ... < t_n = \tau$, the probability that a Brownian particle, with initial displacement zero, will be found in $u_j \leq x_j < v_j$ at time t_j, $1 \leq j \leq n-1$, and at ξ at time τ, is

$$w(I) = \int_{u_1}^{v_1} ... \int_{u_{n-1}}^{v_{n-1}} \rho(x_1, t_1)\rho(x_2 - x_1, t_2 - t_1)...\rho(x_n - x_{n-1}, t_n - t_{n-1})dx_1...dx_{n-1},$$

where

$$\rho(x_j - x_{j-1}, t_j - t_{j-1}) = \sqrt{\frac{1}{4\pi D(t_j - t_{j-1})}} \exp\left(-\frac{(x_j - x_{j-1})^2}{4D(t_j - t_{j-1})}\right)$$

for every $1 \leq j \leq n$, where $x_0 = 0$, $t_0 = 0$ and D is the diffusion coefficient depending on medium viscosity and particle dimensions and containing Avogadro's number.

N. Wiener [20] showed that this interval function, $w(I)$, produces a measure in the space C of continuous functions x defined in $]0, \tau[$, with $x(0) = 0$ and $x(\tau) = \xi$.

In [7], M. Kac showed that if U is a continuous positive function of displacement and $u(x) = \exp\left(-\int U(x(t))dt\right)$, $x \in C$, then $\int_C u(x)dw$ exists and $\phi(\xi, \tau) = \int_C u(x)dw$ satisfies an equation analogous to Schrödinger's equation and the diffusion equation. This subject was treated in [11] using integrals in function spaces. Next, we present the treatment of this subject in [11].

Consider $\mathbb{R}^{]0,\,\tau[}$ to be the space of real valued functions x defined in $]0,\,\tau[$, with $x(0)=0$ and $x(\tau)=\xi$, where $\xi\in\mathbb{R}$. Let $0=t_0<t_1<...<t_n=\tau$, $N=\{t_1,t_2,...,t_{n-1}\}$ and $x(t_j)=x_j$, $0\leq j\leq n$, be such that $x_0=0$ and $x_n=\xi$. Define $I(t_j)=I_j=[u_j,\,v_j[$ and $\Delta x_j=v_j-u_j$, $1\leq j\leq n-1$. Set $y=(y_1,...,y_{n-1})\in I_1\times...\times I_{n-1}$. Define

$$w(x,\,N)=\prod_{j=1}^{n}\exp\left(-\frac{1}{2}\frac{(x_j-x_{j-1})^2}{t_j-t_{j-1}}\right)\prod_{j=1}^{n}[2\pi(t_j-t_{j-1})]^{-1/2},$$

$$w(I,\,x,\,N)=w(x,\,N)\prod_{j=1}^{n}\Delta x_j,$$

$$w(I,\,N)=\int_{I_{n-1}}...\int_{I_1}w(y,\,N)dy_1...dy_{n-1}. \tag{5}$$

The function $w(I,\,N)$ is generalized Riemann integrable in every elementary set $E\subseteq\mathbb{R}^{]0,\,\tau[}$ and, in particular,

$$\int_{\mathbb{R}^{]0,\,\tau[}}w(I,\,N)=\frac{1}{\sqrt{2\pi\tau}}\exp\left(-\frac{1}{2}\frac{\xi^2}{\tau}\right),$$

(see [11], Proposition 36 for a proof of this fact). As a consequence of Proposition 37 in [11], the functions $w(I,\,x,\,N)$ and $w(I,\,N)$ are variationally equivalent.

Suppose U is a real-valued function defined on $\mathbb{R}$. If $N=\{t_1,...,t_{n-1}\}\subseteq]0,\,\tau[$ and $x\in\mathbb{R}^{]0,\,\tau[}$, let

$$U_j=U(x_j)=U(x(t_j)),\qquad 0\leq j\leq n-1.$$

Let

$$u(x,\,N)=\exp\left(-\sum_{j=1}^{n}U_{j-1}(t_j-t_{j-1})\right)$$

and

$$u(I,\,N)=\int_{I(N)}u(x,\,N)w(x,\,N)dx(N).$$

For $\eta>0$, $0\leq\sigma-\tau<\eta$, $0\leq|\zeta-\xi|<\eta$, consider

$$h(\zeta,\,\sigma)=\frac{1}{\sqrt{2\pi(\sigma-t_{n-1})}}\exp\left(-\frac{1}{2}\frac{(\zeta-x_{n-1})^2}{\sigma-t_{n-1}}-U_{n-1}(\sigma-t_{n-1})\right),$$

$$W_1(I,\,x,\,N)=\prod_{j=1}^{n-1}\frac{\exp\left(-\left[\frac{1}{2}\frac{(x_j-x_{j-1})^2}{t_j-t_{j-1}}+U_{j-1}(t_j-t_{j-1})\right]\right)}{\sqrt{2\pi(t_j-t_{j-1})}}\Delta x(N),$$

$$W_2(I,\,x,\,N;\,\zeta,\,\sigma)=h(\zeta,\,\sigma)W_1(I,\,x,\,N).$$

In [11], Proposition 57, P. Muldowney proved that $\phi(\xi,\,\tau)=\int_{\mathbb{R}^{]0,\,\tau[}}u(I,\,N)$ is a Feynman-Kac representation of the solution of the equation

$$\frac{\partial\phi}{\partial\tau}=\frac{1}{2}\frac{\partial^2\phi}{\partial\xi^2}-U(\xi)\phi,$$

provided $U \geq 0$ is continuous at ξ, $W_2(I, x, N; \zeta, \sigma)$ is integrable in $\mathbb{R}^{]0, \tau[}$ for $0 \leq \sigma - \tau < \eta$, $0 \leq |\zeta - \xi| < \eta$ and the partial derivative $\dfrac{\partial \phi}{\partial \tau}$ exists.

In the present paper, we extend this result to processes with impulses.

2.4 Additional definitions

The following result is the Tonelli theorem version for generalized Riemann integrals and it will be useful in the main results. See [21], Theorem 6.6.5, for a proof of it.

Theorem 2.3 *Let J be an interval in $\overline{\mathbb{R}}^n$, with $J = H \times K$, where H and K belong to $\overline{\mathbb{R}}^l$ and $\overline{\mathbb{R}}^m$, $n = l + m$. Let f be a function defined on $\overline{\mathbb{R}}^n$. If*

i) f is measurable on J;

ii) there is a function g such that $|f| \leq g$ on J and either

$$A_1 = \int_H \left(\int_K g(x, y)dy \right) dx < \infty$$

or

$$A_2 = \int_K \left(\int_H g(x, y)dx \right) dy < \infty.$$

Then, f is generalized Riemann integrable (or Henstock-Kurzweil integrable) on J and

$$\int \int_J f = \int_H \left(\int_K g(x, y)dy \right) dx.$$

The next result can be found in [21], Corollary 6.6.7.

Corollary 2.1 *If f is measurable and non-negative, then*

$$\int \int_J f = \int_H \left(\int_K g(x, y)dy \right) dx = \int_K \left(\int_H g(x, y)dx \right) dy.$$

provided that at least one of the three integrals exists and is finite.

3 The Main Result

We divide this section into two parts. The first part is concerned with the properties of the volume function of a process with impulses and with the integrability of this volume function. In the second part, we present an impulsive partial differential equation of Schrödinger type and we show that its solution can be represented by an integral with respect to the volume function of an underlying impulsive process. This is the Feynman-Kac representation.

3.1 The Volume Function of an Impulsive Process

Let $\{x_t\}_{t\geq 0}$ be a Brownian motion. Suppose at time $t_{j-1} > 0$, the displacement is $x_{j-1} = x(t_{j-1})$. For the later time t_j, the increment $x_j - x_{j-1}$ is normally distributed, with mean zero and variance $t_j - t_{j-1}$. Therefore, the probability that $x_j = x(t_j) \in [u_j, v_j[$ is

$$\frac{1}{\sqrt{2\pi(t_j - t_{j-1})}} \int_{u_j}^{v_j} \exp\left(-\frac{1}{2}\frac{(y_j - x_{j-1})^2}{t_j - t_{j-1}}\right) dy_j.$$

Thus, given $x(t_0) = \xi'$, the joint probability that $x_1 \in I_1,..., x_n \in I_n$, where $I_j = [u_j, v_j[$, $1 \leq j \leq n$, is

$$\int_{u_1}^{v_1} \cdots \int_{u_n}^{v_n} \prod_{j=1}^{n} \frac{\exp\left(-\frac{1}{2}\frac{(y_j - y_{j-1})^2}{t_j - t_{j-1}}\right)}{\sqrt{2\pi(t_j - t_{j-1})}} dy_1...dy_n. \tag{6}$$

Therefore, in Brownian motion, we are lead to consider expressions of the form

$$\prod_{j=1}^{n} \frac{\exp\left(-\frac{1}{2}\frac{(y_j - y_{j-1})^2}{t_j - t_{j-1}}\right)}{\sqrt{2\pi(t_j - t_{j-1})}}. \tag{7}$$

In the sequel, we shall present a version for the expressions (6) and (7), when the underlying Brownian process undergoes impulsive changes at some moments of time.

We start by giving some notations in order to define the impulsive process. Let $J : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let τ', τ be real numbers such that $0 < \tau' < \tau$ and $N = \{t_1, ..., t_{n-1}\} \subset]\tau', \tau[$, where $\tau' = t_0$ and $\tau = t_n$. Consider $\mathcal{I} = \{\tau_1, ..., \tau_p\} \subset N$, $\tau_1 < \tau_2 < ... < \tau_p$. Then, $\{\tau_1, \tau_2, ..., \tau_p\} = \{t_{i_1}, t_{i_2}, ..., t_{i_p}\}$ with $i_j \in \{1, 2, ..., n-1\}$ for $1 \leq j \leq p$. Let $\mathcal{N} = \{1, 2, ..., n\}$ and $\mathcal{J} = \{i_1, i_2, ..., i_p\}$.

Given $x \in \mathbb{R}^{]\tau', \tau[}$, consider the function $z \in \mathbb{R}^{]\tau', \tau[}$ such that

$$z(t) = x(t), \quad \text{for} \quad \tau' < t < \tau_1,$$

$$z(t) = x(t) + \sum_{\tau_j \leq t} J(x(\tau_j)), \quad \tau_j \leq t < \tau_{j+1}, \quad j = 1, 2, ..., p,$$

where we take τ_{p+1} to be τ. Figure 1 illustrates the behavior of the impulsive process $z(t)$, $\tau' < t < \tau$, when $x \in C(]\tau', \tau[)$, where $C(]\tau', \tau[)$ denotes the set of those x which are continuous at each t in $]\tau', \tau[$.

We now define a volume function for the impulsive process. First, corresponding to $w(y, N)$ in the Brownian motion, we define $g_{\mathcal{I}}(y, N)$ for the impulsive process by

$$\prod_{j\in\mathcal{N}\setminus\mathcal{J}} \frac{\exp\left(-\frac{1}{2}\frac{(y_j - y_{j-1})^2}{t_j - t_{j-1}}\right)}{\sqrt{2\pi(t_j - t_{j-1})}} \prod_{j\in\mathcal{J}} \frac{\exp\left(-\frac{1}{2}\frac{(y_j - (y_{j-1} - J(y_j)))^2}{t_j - t_{j-1}}\right)}{\sqrt{2\pi(t_j - t_{j-1})}}$$

which is equal to

$$\prod_{j\in\mathcal{N}\setminus\mathcal{J}} \frac{\exp\left(-\frac{1}{2}\frac{(y_j - y_{j-1})^2}{t_j - t_{j-1}}\right)}{\sqrt{2\pi(t_j - t_{j-1})}} \prod_{j\in\mathcal{J}} \frac{\exp\left(-\frac{1}{2}\frac{(J(y_j) + y_j - y_{j-1})^2}{t_j - t_{j-1}}\right)}{\sqrt{2\pi(t_j - t_{j-1})}}. \tag{8}$$

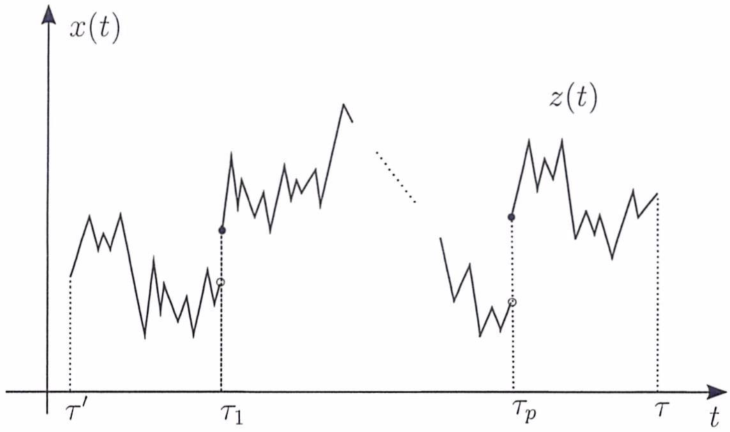

Figure 1: Process $z \in \mathbb{R}^{]\tau',\tau[}$.

Let $I(t_j) = I_j = [u_j,\, v_j[$ and $\Delta I_j = v_j - u_j$, $1 \le j \le n-1$. Recall that $I(N) = I_1 \times \ldots \times I_{n-1}$.

A volume function for the impulsive process is then given by

$$Q_{\mathcal{I}}(I[N]) = Q_{\mathcal{I}}(I[N];\, \tau',\, \tau,\, \xi',\, \xi) = \int_{I(N)} g_{\mathcal{I}}(y,\, N) dy(N).$$

Let $\mathcal{Z} = \left\{ J : \mathbb{R} \to \mathbb{R} : \displaystyle\int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{1}{2}\frac{(J(y_j)+y_j-y_{j-1})^2}{t_j-t_{j-1}}\right)}{\sqrt{2\pi(t_j - t_{j-1})}} dy_j = 1,\ j = 1, \ldots, p \right\}$. Given $\alpha \in \mathbb{R}$, if $J(w) = \alpha$ for all $w \in \mathbb{R}$, then $J \in \mathcal{Z}$. Hence $\mathcal{Z} \neq \emptyset$.

If $J \in \mathcal{Z}$, then $Q_{\mathcal{I}}(I[N])$ is the probability distribution function for the impulsive process which gives the probability that $x_j \in I_j$ for $1 \le j \le n-1$.

If (x, I) is an associated pair, where $I = I[N]$, let

$$G_{\mathcal{I}}(x,\, I[N]) := G_{\mathcal{I}}(x,\, I[N];\, \tau',\, \tau,\, \xi',\, \xi) = Q_{\mathcal{I}}(I[N]). \tag{9}$$

Now we shall prove that $G_{\mathcal{I}}(x,\, I[N])$ is generalized Riemann integrable in $\mathbb{R}^{]\tau',\tau[}$. In order to do that, we need to prove an auxiliary result. So, let us start by introducing some auxiliary functions.

We assume that between τ_i and τ_{i+1}, $i = 1, 2, \ldots, p-1$, there exists a finite number, greater than 2, of t_i's.

Define $\phi_1,\, \phi_2 : \mathbb{R} \times]\tau',\, \tau[\longrightarrow \mathbb{R}$ and $\Phi_j : \mathbb{R} \times \mathbb{R} \times]\tau',\, \tau[\times]\tau',\, \tau[\longrightarrow \mathbb{R}$, $j = 1, 2, \ldots, p-1$, by

$$\phi_1(y_k,\, t_k) = \frac{1}{\sqrt{2\pi(t_k - \tau')}} \exp\left(-\frac{1}{2}\frac{(y_k - \xi')^2}{t_k - \tau'}\right),$$

for $k \in \{1, 2, \ldots, i_1 - 1\}$,

$$\phi_2(y_{i_p},\, t_{i_p}) = \frac{1}{\sqrt{2\pi(\tau - t_{i_p})}} \exp\left(-\frac{1}{2}\frac{(\xi - y_{i_p})^2}{\tau - t_{i_p}}\right)$$

and

$$\Phi_j(y_{i_j}, y_{i_{j+1}-1}, t_{i_j}, t_{i_{j+1}-1}) = \frac{1}{\sqrt{2\pi(t_{i_{j+1}-1} - t_{i_j})}} \exp\left(-\frac{1}{2}\frac{(y_{i_{j+1}-1} - y_{i_j})^2}{t_{i_{j+1}-1} - t_{i_j}}\right),$$

$j = 1, 2, ..., p-1$.

Analogously, define $\phi_1(J(y_k),\, t_k)$, for $k \in \mathcal{J}$, replacing y_k by $J(y_k) + y_k$ in the expression of $\phi_1(y_k,\, t_k)$, and define $\Phi_j(y_{i_j}, J(y_{i_{j+1}}), t_{i_j}, t_{i_{j+1}})$ replacing $y_{i_{j+1}-1}$ by $J(y_{i_{j+1}}) + y_{i_{j+1}}$ and $t_{i_{j+1}-1}$ by $t_{i_{j+1}}$ in the expression of $\Phi_j(y_{i_j}, y_{i_{j+1}-1}, t_{i_j}, t_{i_{j+1}-1})$, for $j \in \{1, 2, ..., p-1\}$.

We can prove the next lemma by completing the square in the exponential.

Lemma 3.1 *If a, b, u, $v \in \mathbb{R}$, with $a > 0$ and $b > 0$, then the function*

$$h(\alpha) = \sqrt{\frac{a}{\pi}}e^{-a(u-\alpha)^2}\sqrt{\frac{b}{\pi}}e^{-b(\alpha-v)^2}$$

is Riemann integrable and

$$\int_{-\infty}^{+\infty} \sqrt{\frac{a}{\pi}}e^{-a(u-\alpha)^2}\sqrt{\frac{b}{\pi}}e^{-b(\alpha-v)^2}d\alpha = \sqrt{\frac{ab}{\pi(a+b)}}\exp\left(-\frac{ab}{a+b}(u-v)^2\right).$$

Proposition 3.1 in the sequel says that, for $y = (y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1}$, the function $g_{\mathcal{I}}(y,\, N)$ defined by equation (8) is generalized Riemann integrable with respect to y in $\mathbb{R}^{n-1}$.

Proposition 3.1 *Let $N = \{t_1, t_2, ..., t_{n-1}\} \subset]\tau',\, \tau[$ be given with $\tau' = t_0$ and $\tau = t_n$. Let $g_{\mathcal{I}}$ be the function defined in (8), where $y(\tau') = y(t_0) = \xi'$ and $y(\tau) = y(t_n) = \xi$. Then, $g_{\mathcal{I}}$ is generalized Riemann integrable with respect to y in $\mathbb{R}^{n-1}$ and*

$$\int_{\mathbb{R}^{n-1}} g_{\mathcal{I}}(y,\, N)dy_1 dy_2 ... dy_{n-1} =$$

$$= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \phi_1(J(y_{i_1}), t_{i_1}) \left(\prod_{j=1}^{p-1} \Phi_j(y_{i_j}, J(y_{i_{j+1}}), t_{i_j}, t_{i_{j+1}})\right) \phi_2(y_{i_p}, t_{i_p}) \prod_{j=1}^{p} dy_{i_j}.$$

PROOF. Let $\mathcal{I} = \{\tau_1, \tau_2, ..., \tau_p\} = \{t_{i_1}, t_{i_2}, ..., t_{i_p}\}$ with $i_j \in \{1, 2, ..., n-1\}$ for $1 \leq j \leq p$. Let

$$\mathcal{N} = \{1, 2, ..., i_1 - 1, i_1, i_1 + 1, ..., i_p - 1, i_p, i_p + 1, ..., n-1, n\}.$$

Define

$$\psi_j(y_j,\, y_{j-1}) = \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}}\exp\left(-\frac{1}{2}\frac{(y_j - y_{j-1})^2}{t_j - t_{j-1}}\right), \quad j \in \mathcal{N} \setminus \mathcal{J},$$

and

$$\varphi_j(y_j,\, y_{j-1}) = \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}}\exp\left(-\frac{1}{2}\frac{(J(y_j) + y_j - y_{j-1})^2}{t_j - t_{j-1}}\right), \quad j \in \mathcal{J}.$$

By Lemma 3.1, we obtain

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \psi_1(y_1,\, y_0) ... \psi_{i_1-1}(y_{i_1-1},\, y_{i_1-2}) dy_1 ... dy_{i_1-2} =$$

$$= \frac{1}{\sqrt{2\pi(t_{i_1-1}-\tau')}} \exp\left(-\frac{1}{2}\frac{(y_{i_1-1}-\xi')^2}{t_{i_1-1}-\tau'}\right) = \phi_1(y_{i_1-1},\, t_{i_1-1}), \tag{10}$$

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \psi_{i_j+1}(y_{i_j+1},\, y_{i_j}) \ldots \psi_{i_{j+1}-1}(y_{i_{j+1}-1},\, y_{i_{j+1}-2}) dy_{i_j+1} \ldots dy_{i_{j+1}-2} =$$

$$= \frac{1}{\sqrt{2\pi(t_{i_{j+1}-1}-t_{i_j})}} \exp\left(-\frac{1}{2}\frac{(y_{i_{j+1}-1}-y_{i_j})^2}{t_{i_{j+1}-1}-t_{i_j}}\right) = \Phi_j(y_{i_j}, y_{i_{j+1}-1}, t_{i_j}, t_{i_{j+1}-1}), \tag{11}$$

$j = 1, 2, \ldots, p-1$, and also

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \psi_{i_p+1}(y_{i_p+1},\, y_{i_p}) \ldots \psi_n(y_n,\, y_{n-1}) dy_{i_p+1} \ldots dy_{n-1} =$$

$$= \frac{1}{\sqrt{2\pi(\tau - t_{i_p})}} \exp\left(-\frac{1}{2}\frac{(\xi - y_{i_p})^2}{\tau - t_{i_p}}\right) = \phi_2(y_{i_p}, t_{i_p}). \tag{12}$$

Thus, taking $t_{i_p+\ell} := t_{n-1}$, $\ell \in \mathbb{N}$, from equations (10), (11) and (12), we have

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} g_{\mathcal{I}}(y,\, N) \left(\prod_{j=1}^{i_1-2} dy_j\right) \left(\prod_{j=1}^{p-1} dy_{i_j+1} \ldots dy_{i_{j+1}-2}\right) \prod_{j=1}^{\ell} dy_{i_p+j} = \tag{13}$$

$$= \phi_1(y_{i_1-1}, t_{i_1-1}) \left(\prod_{j=1}^{p-1} \left[\varphi_{i_j}(y_{i_j},\, y_{i_j-1}) \Phi_j(y_{i_j}, y_{i_{j+1}-1}, t_{i_j}, t_{i_{j+1}-1})\right]\right) \varphi_{i_p}(y_{i_p},\, y_{i_p-1}) \phi_2(y_{i_p}, t_{i_p}).$$

Using Lemma 3.1, we have

$$\int_{-\infty}^{+\infty} \phi_1(y_{i_1-1}, t_{i_1-1}) \varphi_{i_1}(y_{i_1},\, y_{i_1-1}) dy_{i_1-1} = \phi_1(J(y_{i_1}), t_{i_1}) \tag{14}$$

and

$$\int_{-\infty}^{+\infty} \Phi_j(y_{i_j}, y_{i_{j+1}-1}, t_{i_j}, t_{i_{j+1}-1}) \varphi_{i_{j+1}}(y_{i_{j+1}},\, y_{i_{j+1}-1}) dy_{i_{j+1}-1} = \tag{15}$$

$$= \Phi_j(y_{i_j},\, J(y_{i_{j+1}}), t_{i_j}, t_{i_{j+1}}), \quad j = 1, \ldots, p-1.$$

Then, from (13), (14) and (15), it follows that

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} g_{\mathcal{I}}(y,\, N) \prod_{j \in \mathcal{N} \setminus \mathcal{J}} dy_j =$$

$$= \phi_1(J(y_{i_1}), t_{i_1}) \left(\prod_{j=1}^{p-1} \Phi_j(y_{i_j}, J(y_{i_{j+1}}), t_{i_j}, t_{i_{j+1}})\right) \phi_2(y_{i_p}, t_{i_p}). \tag{16}$$

Define the functions $f,\, F : \mathbb{R}^p \to \mathbb{R}$ by

$$f(y_{i_1}, \ldots, y_{i_p}) = \left(\prod_{j=1}^{p-1} \Phi_j(y_{i_j}, J(y_{i_{j+1}}), t_{i_j}, t_{i_{j+1}})\right) \phi_2(y_{i_p}, t_{i_p})$$

and

$$F(y_{i_1}, ..., y_{i_p}) = \phi_1(J(y_{i_1}), t_{i_1}) f(y_{i_1}, ..., y_{i_p}).$$

Thus F is a continuous function, $|F(y_{i_1}, ..., y_{i_p})| \leq \dfrac{f(y_{i_1}, ..., y_{i_p})}{\sqrt{2\pi(t_{i_1} - \tau')}}$ and

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \frac{f(y_{i_1}, ..., y_{i_p})}{\sqrt{2\pi(t_{i_1} - \tau')}} dy_{i_1} ... dy_{i_p} = \frac{1}{\sqrt{2\pi(t_{i_1} - \tau')}}.$$

By Tonelli theorem (Theorem 2.3), the function F is generalized Riemann integrable in $\mathbb{R}^p$ and

$$\int_{\mathbb{R}^p} F(y_{i_1}, ..., y_{i_p}) dy_{i_1} ... dy_{i_p} = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F(y_{i_1}, ..., y_{i_p}) dy_{i_1} ... dy_{i_p}$$

is finite. Hence, from (16),

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} g_{\mathcal{I}}(y, N) dy_1 dy_2 ... dy_{n-1} = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F(y_{i_1}, ..., y_{i_p}) dy_{i_1} ... dy_{i_p} < \infty.$$

Using Corollary 2.1, $g_{\mathcal{I}}(y, N)$ is generalized Riemann integrable with respect to y in $\mathbb{R}^{n-1}$ and

$$\int_{\mathbb{R}^{n-1}} g_{\mathcal{I}}(y, N) dy_1 ... dy_{n-1} =$$

$$= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} g_{\mathcal{I}}(y, N) dy_1 dy_2 ... dy_{n-1} =$$

$$= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \phi_1(J(y_{i_1}), t_{i_1}) \left(\prod_{j=1}^{p-1} \Phi_j(y_{i_j}, J(y_{i_{j+1}}), t_{i_j}, t_{i_{j+1}}) \right) \phi_2(y_{i_p}, t_{i_p}) \prod_{j=1}^{p} dy_{i_j},$$

which completes the proof. ■

The next result says that $G_{\mathcal{I}}(x, I[N])$ given by (9) is generalized Riemann integrable in the function space $\mathbb{R}^{]\tau', \tau[}$.

Theorem 3.1 *The generalized Riemann integral*

$$\int_{\mathbb{R}^{]\tau', \tau[}} G_{\mathcal{I}}(x, I[N])$$

exists.

Proof. Consider a division $\mathcal{E} = \{(x, I[N])\}$ of $\mathbb{R}^{]\tau', \tau[}$, with each N chosen so that $\mathcal{I} \subseteq N \in \mathcal{F}(]\tau', \tau[)$. Then the Riemann sum of $G_{\mathcal{I}}$ is given by

$$\sum_{(x, I[N]) \in \mathcal{E}} G_{\mathcal{I}}(x, I[N]) = \sum_{(x, I[N]) \in \mathcal{E}} Q_{\mathcal{I}}(I[N]).$$

Let $M = \cup\{N : (x, I[N]) \in \mathcal{E}\}$ and enumerate M as $\{t_1, ..., t_{m-1}\}$, where $\tau' = t_0$, $\tau = t_m$ and $t_0 < t_1 < ... < t_{m-1} < t_m$. Each term $Q_{\mathcal{I}}(I[N])$ of the Riemann sum can be rewritten as $Q_{\mathcal{I}}(I[M])$ by inserting additional y_j's in the expression of $g_{\mathcal{I}}$, $j \in \mathcal{N}\backslash\mathcal{J}$, and integrating from $-\infty$ to $+\infty$ on the extra y_j's. Then the Riemann sum becomes

$$\sum_{(x,\, I[M])\in\mathcal{E}} Q_{\mathcal{I}}(I[M]).$$

with M a fixed set of dimensions. So we are now dealing, in effect, with some Riemann sum estimate of an integral in $m-1$ dimensions. Then each term of the Riemann sum is an integral over $I[M] \subset \mathbb{R}^{m-1}$, and, by the finite additivity of these integrals in $\mathbb{R}^{m-1}$,

$$\sum_{(x,\, I[M])\in\mathcal{E}} Q_{\mathcal{I}}(I([M]) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} g_{\mathcal{I}}(y,\, M)dy_1 \dots dy_{m-1}. \tag{17}$$

By Proposition 3.1, the integral (17) exists and we can rewrite $\sum_{(x,\, I[M])\in\mathcal{E}} Q_{\mathcal{I}}(I[M])$ as

$$\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \phi_1(J(y_{i_1}), t_{i_1}) \prod_{j=1}^{p-1} \Phi_j(y_{i_j}, J(y_{i_{j+1}}), t_{i_j}, t_{i_{j+1}}) \phi_2(y_{i_p}, t_{i_p}) \prod_{j=1}^{p} dy_{i_j}. \tag{18}$$

Let β be the integral in (18). Thus, given $\epsilon > 0$, for any gauge γ chosen so that $L(x) \supseteq \mathcal{I}$, we have that for every $(x,\, I[N]) \in \mathcal{E}_\gamma$, $\mathcal{I} \subseteq L(x) \subseteq N$ implies

$$\left| \sum_{(x,\, I[N])\in\mathcal{E}_\gamma} G_{\mathcal{I}}(x,\, I[N]) - \beta \right| < \epsilon.$$

Therefore, $\int_{\mathbb{R}^{]\tau',\tau[}} G_{\mathcal{I}}(x,\, I[N]) = \beta$ and the proof is complete. ∎

Let us show that the expressions $g_{\mathcal{I}}(x,\, N) \prod_{j=1}^{n-1} \Delta I_j$ and $G_{\mathcal{I}}(x,\, I[N])$ are variationally equivalent in $\mathbb{R}^{]\tau',\tau[}$. This result is a consequence of the next proposition.

Define the auxiliary function $q_{\mathcal{I}}(I(N))$ by

$$q_{\mathcal{I}}(I(N)) = g_{\mathcal{I}}(x,\, N) \prod_{j=1}^{n-1} \Delta I_j.$$

If (x, I) is an associated pair, where $I = I[N]$, let

$$q_{\mathcal{I}}(x,\, I[N]) = q_{\mathcal{I}}(I(N)).$$

Proposition 3.2 *Let $k(x(N)) = k(x(t_1), ..., x(t_{n-1}))$ be a real-valued function which depends on $(x(t_1), ..., x(t_{n-1}))$. If k is jointly continuous in x_j, $1 \leq j \leq n-1$, then the expressions $k(x(N))q_{\mathcal{I}}(x,\, I[N])$ and $\int_{I(N)} k(y(N))g_{\mathcal{I}}(y,\, N)dy(N)$ are variational equivalent in $\mathbb{R}^{]\tau',\tau[}$, provided at least one the two integrals exists.*

PROOF. Let $\epsilon > 0$ be given. Since J is a continuous functions, given $x \in \overline{\mathbb{R}}^{]\tau',\tau[}$, we can choose $L(x)$ and $\delta(x(N))$ such that, if $N \supseteq L(x) \supseteq \mathcal{I}$, then $(I(N),\, x(N))$ is $\delta-$fine, and if $y \in I(N)$, then

$$|k(x(N))g_{\mathcal{I}}(x,\, N) - k(y(N))g_{\mathcal{I}}(y,\, N)| < \frac{\epsilon}{4}\sqrt{2\pi(t_{i_1} - \tau')}g_{\mathcal{I}}(x,\, I)$$

and

$$g_{\mathcal{I}}(y,\, N) > \frac{1}{2}g_{\mathcal{I}}(x,\, N).$$

Thus,

$$\left|k(x(N))q_{\mathcal{I}}(x,\, I[N]) - \int_{I(N)} k(y(N))g_{\mathcal{I}}(y,\, N)dy(N)\right| =$$

$$= \left|k(x(N))g_{\mathcal{I}}(x,\, N)\prod_{j=1}^{n-1}\Delta I_j - \int_{I(N)} k(y(N))g_{\mathcal{I}}(y,\, N)dy(N)\right| =$$

$$= \left|\int_{I(N)} \left[k(x(N))g_{\mathcal{I}}(x,\, N) - k(y(N))g_{\mathcal{I}}(y,\, N)\right] dy(N)\right| \leq$$

$$\leq \frac{\epsilon}{2}\sqrt{2\pi(t_{i_1} - \tau')}\int_{I(N)} g_{\mathcal{I}}(y,\, N)dy(N).$$

Now, we can choose a gauge γ such that for each division $\mathcal{E}_\gamma$,

$$\sum_{(x,\, I[N])\in\mathcal{E}_\gamma} \left|k(x(N))q_{\mathcal{I}}(x,\, I[N]) - \int_{I(N)} k(y(N))g_{\mathcal{I}}(y,\, N)dy(N)\right| \leq$$

$$\leq \frac{\epsilon}{2}\sqrt{2\pi(t_{i_1} - \tau')}\sum_{(x,\, I[N])\in\mathcal{E}_\gamma}\int_{I(N)} g_{\mathcal{I}}(y,\, N)dy(N) =$$

$$= \frac{\epsilon}{2}\sqrt{2\pi(t_{i_1} - \tau')}\int_{\mathbb{R}^{n-1}} g_{\mathcal{I}}(y,\, N)dy(N) < \epsilon.$$

Therefore,

$$\int_{\mathbb{R}^{]\tau',\tau[}} k(x(N))q_{\mathcal{I}}(x,\, I[N]) = \int_{\mathbb{R}^{]\tau',\tau[}}\int_{I(N)} k(y(N))g_{\mathcal{I}}(y,\, N)dy(N),$$

which completes the proof. ∎

As a direct consequence of Proposition 3.2, we have the next corollary.

Corollary 3.1 *The expressions $q_{\mathcal{I}}(x,\, I[N])$ and $G_{\mathcal{I}}(x,\, I[N])$ are variationally equivalent in $\mathbb{R}^{]\tau',\tau[}$.*

Let $\tau' < T_1 < \tau$ and $D_1 = \{x \in \mathbb{R}^{]\tau',\tau[} : x$ is discontinuous at $T_1\}$. We intend to prove that $\displaystyle\int_{D_1} G_{\mathcal{I}}(x,\, I[N])$ exists and equals zero, because this result will be useful in the next section. We need to show some auxiliary results in order to prove this fact.

Let $M = \{T_1, ..., T_m\} \subset]\tau', \tau[$ be fixed and suppose a functional h satisfies $h(x) = h(x(M))$ for all $x \in \mathbb{R}^{]\tau',\tau[}$. Then h is called a cylinder functional. Note that h depends only on the values taken by x at $T_1, ..., T_m$, and we can treat it as a function of $x(M) \in \mathbb{R}^m$ or as a function of $x \in \mathbb{R}^{]\tau',\tau[}$. Thus, consider the particular case when $M = \{T_1, T_2\}$ and $h(x) = h(x(M))$. Let $H_{\mathcal{I}}(I[N])$ be given by

$$H_{\mathcal{I}}(I[N]) = \int_{I(N)} h(x(M))g_{\mathcal{I}}(x,\, N)dx_1...dx_{n-1}.$$

If $\tau_i < T_1 < T_2 < \tau_{i+1}$ for some $i \in \{0, 1, 2, ..., p\}$, $\tau_0 = \tau'$ and $\tau_{p+1} = \tau$, define

$$H_1(x, M) = h(x(M)) \left(\prod_{j=1}^{i} \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_j} + J(x_{\tau_j}) - x_{\tau_{j-1}})^2}{\tau_j - \tau_{j-1}}\right)}{\sqrt{2\pi(\tau_j - \tau_{j-1})}} \right) \times$$

$$\times \frac{\exp\left(-\frac{1}{2}\frac{(x_{T_1} - x_{\tau_i})^2}{T_1 - \tau_i}\right)}{\sqrt{2\pi(T_1 - \tau_i)}} \frac{\exp\left(-\frac{1}{2}\frac{(x_{T_2} - x_{T_1})^2}{T_2 - T_1}\right)}{\sqrt{2\pi(T_2 - T_1)}} \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_{i+1}} + J(x_{\tau_{i+1}}) - x_{T_2})^2}{\tau_{i+1} - T_2}\right)}{\sqrt{2\pi(\tau_{i+1} - T_2)}} \times$$

$$\times \left(\prod_{j=i+2}^{p} \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_j} + J(x_{\tau_j}) - x_{\tau_{j-1}})^2}{\tau_j - \tau_{j-1}}\right)}{\sqrt{2\pi(\tau_j - \tau_{j-1})}} \right) \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau} - x_{\tau_p})^2}{\tau - \tau_p}\right)}{\sqrt{2\pi(\tau - \tau_p)}}$$

and, if $T_1 = \tau_i$ and $T_1 < T_2 < \tau_{i+1}$ for some $i \in \{1, 2, ..., p\}$, define

$$H_2(x, M) = h(x(M)) \left(\prod_{j=1}^{i} \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_j} + J(x_{\tau_j}) - x_{\tau_{j-1}})^2}{\tau_j - \tau_{j-1}}\right)}{\sqrt{2\pi(\tau_j - \tau_{j-1})}} \right) \times$$

$$\times \frac{\exp\left(-\frac{1}{2}\frac{(x_{T_2} - x_{\tau_i})^2}{T_2 - \tau_i}\right)}{\sqrt{2\pi(T_2 - \tau_i)}} \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_{i+1}} + J(x_{\tau_{i+1}}) - x_{T_2})^2}{\tau_{i+1} - T_2}\right)}{\sqrt{2\pi(\tau_{i+1} - T_2)}} \times$$

$$\times \left(\prod_{j=i+2}^{p} \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_j} + J(x_{\tau_j}) - x_{\tau_{j-1}})^2}{\tau_j - \tau_{j-1}}\right)}{\sqrt{2\pi(\tau_j - \tau_{j-1})}} \right) \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau} - x_{\tau_p})^2}{\tau - \tau_p}\right)}{\sqrt{2\pi(\tau - \tau_p)}}.$$

The next theorem states conditions on $H_1(x,\, M)$ or $H_2(x,\, M)$ so that the function $H_{\mathcal{I}}(I[N])$ is generalized Riemann integrable in $\mathbb{R}^{]\tau',\tau[}$.

Theorem 3.2 *Suppose h, considered as a function of $x(M) = (x(T_1),\, x(T_2)) \in \mathbb{R}^2$, is almost everywhere continuous.*

1. *If* $\tau_i < T_1 < T_2 < \tau_{i+1}$ *for some* $i \in \{0, 1, ..., p\}$ *and* $H_1(x, M)$ *is generalized Riemann integrable in* $\mathbb{R}^{p+2}$ *with respect to the variables* $x_{\tau_1}, ..., x_{\tau_i}, x_{T_1}, x_{T_2}, x_{\tau_{i+1}}, ..., x_{\tau_p}$, *then* $H_{\mathcal{I}}(I[N])$ *is generalized Riemann integrable in* $\mathbb{R}^{]\tau', \tau[}$, *and*

$$\int_{\mathbb{R}^{]\tau',\tau[}} H_{\mathcal{I}}(I[N]) = \int_{\mathbb{R}^{p+2}} H_1(x, M) dx_{\tau_1} ... dx_{\tau_i} dx_{T_1} dx_{T_2} dx_{\tau_{i+1}} ... dx_{\tau_p}.$$

2. *If* $T_1 = \tau_i$ *and* $T_1 < T_2 < \tau_{i+1}$ *for some* $i \in \{1, 2, ..., p\}$ *and* $H_2(x, M)$ *is generalized Riemann integrable in* $\mathbb{R}^{p+1}$ *with respect to the variables* $x_{\tau_1}, ..., x_{\tau_i}, x_{T_2}, x_{\tau_{i+1}}, ..., x_{\tau_p}$, *then* $H_{\mathcal{I}}(I[N])]$ *is generalized Riemann integrable in* $\mathbb{R}^{]\tau', \tau[}$, *and*

$$\int_{\mathbb{R}^{]\tau',\tau[}} H_{\mathcal{I}}(I[N]) = \int_{\mathbb{R}^{p+1}} H_2(x, M) dx_{\tau_1} ... dx_{\tau_i} dx_{T_2} dx_{\tau_{i+1}} ... dx_{\tau_p}.$$

PROOF. Let us prove item 1. Let $\mathcal{E} = \{(x, I[N])\}$ be a division of $\mathbb{R}^{]\tau', \tau[}$ with each N satisfying $\mathcal{I} \subseteq N \in \mathcal{F}(]\tau', \tau[)\}$. We recall that

$$H_{\mathcal{I}}(I[N]) = \int_{I(N)} h(x(M)) g_{\mathcal{I}}(x, N) dx_1 ... dx_{n-1}.$$

Let $\mathcal{O} = \cup\{N : (x, I[N]) \in \mathcal{E}\}$ and enumerate $\mathcal{O}$ as $\{t_1, ..., t_{r-1}\}$, where $\tau' = t_0$, $\tau = t_r$ and $t_0 < t_1 < ... < t_{r-1} < t_r$. As in the proof of Theorem 3.1, each term $H_{\mathcal{I}}(I[N])$ of the Riemann sum can be rewritten as $H_{\mathcal{I}}(I[\mathcal{O}])$. Then, by the finite additivity of these integrals, the Riemann sum becomes

$$\sum_{(x, I[N]) \in \mathcal{E}} H_{\mathcal{I}}(I[N]) = \sum_{(x, I[N]) \in \mathcal{E}} H_{\mathcal{I}}(I[\mathcal{O}]) = \sum_{(x, I[N]) \in \mathcal{E}} \int_{I[\mathcal{O}]} h(x(M)) g_{\mathcal{I}}(x, \mathcal{O}) dx_1 ... dx_{r-1}.$$

But,

$$\sum_{(x, I[\mathcal{O}]) \in \mathcal{E}} \int_{I[\mathcal{O}]} h(x(M)) g_{\mathcal{I}}(x, \mathcal{O}) dx_1 ... dx_{r-1} =$$

$$= \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} h(x(M)) g_{\mathcal{I}}(x, \mathcal{O}) dx_1 ... dx_{r-1} =$$

$$= \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} H_1(x, M) dx_{\tau_1} ... dx_{\tau_i} dx_{T_1} dx_{T_2} dx_{\tau_{i+1}} ... dx_{\tau_p},$$

where the last equality follows from Lemma 3.1.

Let β be the value of $\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} H_1(x, M) dx_{\tau_1} ... dx_{\tau_i} dx_{T_1} dx_{T_2} dx_{\tau_{i+1}} ... dx_{\tau_p}$. Given $\epsilon > 0$, we can choose a gauge γ, with $\mathcal{I} \subset L(x) \subseteq N$, such that

$$\left| \sum_{(x, I[N]) \in \mathcal{E}_\gamma} H_{\mathcal{I}}(I[N]) - \beta \right| < \epsilon.$$

for every division $\mathcal{E}_\gamma$. Therefore $\int_{\mathbb{R}^{]\tau', \tau[}} H_{\mathcal{I}}(I[N]) = \beta$.

Analogously, we prove item 2. ■

Let $\tau' < T_1 < \tau$ and let $D_1 = \{x \in \mathbb{R}^{]\tau',\tau[} : x \text{ is discontinuous at } T_1\}$. Let $\tau' < T_2 < \tau$, $T_2 \neq T_1$, and let

$$X^1 = \left\{ x \in \mathbb{R}^{]\tau',\tau[} : \lim_{T_2 \to T_1} \sup |x(T_2) - x(T_1)|^2 \geq 1 \right\},$$

$$X^j = \left\{ x \in \mathbb{R}^{]\tau',\tau[} : \frac{1}{j} \leq \lim_{T_2 \to T_1} \sup |x(T_2) - x(T_1)|^2 \leq \frac{1}{j-1} \right\},$$

$j = 2, 3, \ldots$. Let $D^r = \bigcup_{j=1}^{r} X^j$. Then, $D_1 = \bigcup_{r=1}^{+\infty} D^r$.

In the next lines, we prove that $G_{\mathcal{I}}(x,\, I[N])$ is generalized Riemann integrable in D^r with integral zero. Then we conclude that this function is generalized Riemann integrable in D_1 with integral zero.

Lemma 3.2 *For $r = 1, 2, 3, \ldots$, $\int_{D^r} G_{\mathcal{I}}(x,\, I[N])$ exists and equals zero.*

Proof. At first, suppose $T_1 \notin \{\tau_1, \ldots, \tau_p\}$. We can suppose, without loss of generality, that $\tau_i < T_1 < T_2 < \tau_{i+1}$ for some $i \in \{0, 1, 2, \ldots, p+1\}$, $\tau_0 = \tau'$ and $\tau_{p+1} = \tau$. Note that

$$\frac{1}{\sqrt{2\pi(T_2 - T_1)}} \int_{-\infty}^{+\infty} (x_{T_2} - x_{T_1})^2 \exp\left(-\frac{1}{2}\frac{(x_{T_2} - x_{T_1})^2}{T_2 - T_1}\right) dx_{T_1} =$$

$$= \frac{2(T_2 - T_1)}{\sqrt{\pi}} \int_{-\infty}^{+\infty} u^2 \exp(-u^2) du = \frac{2(T_2 - T_1)}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) =$$

$$= \frac{2(T_2 - T_1)}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = T_2 - T_1 = |T_2 - T_1|.$$

Then,

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \frac{|x_{T_2} - x_{T_1}|^2}{\sqrt{2\pi(\tau_1 - \tau')}} \prod_{j=2}^{i} \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_j} + J(x_{\tau_j}) - x_{\tau_{j-1}})^2}{\tau_j - \tau_{j-1}}\right)}{\sqrt{2\pi(\tau_j - \tau_{j-1})}} \times$$
$$\times \frac{\exp\left(-\frac{1}{2}\frac{(x_{T_1} - x_{\tau_i})^2}{T_1 - \tau_i}\right)}{\sqrt{2\pi(T_1 - \tau_i)}} \frac{\exp\left(-\frac{1}{2}\frac{(x_{T_2} - x_{T_1})^2}{T_2 - T_1}\right)}{\sqrt{2\pi(T_2 - T_1)}} \times$$
$$\times \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_{i+1}} + J(x_{\tau_{i+1}}) - x_{T_2})^2}{\tau_{i+1} - T_2}\right)}{\sqrt{2\pi(\tau_{i+1} - T_2)}} \prod_{j=i+2}^{p} \frac{\exp\left(-\frac{1}{2}\frac{(J(x_{\tau_j}) + x_{\tau_j} - x_{\tau_{j-1}})^2}{\tau_j - \tau_{j-1}}\right)}{\sqrt{2\pi(\tau_j - \tau_{j-1})}} \times$$
$$\times \frac{\exp\left(-\frac{1}{2}\frac{(x_\tau - x_{\tau_p})^2}{\tau - \tau_p}\right)}{\sqrt{2\pi(\tau - \tau_p)}} dx_{\tau_1} \ldots dx_{\tau_i} dx_{T_1} dx_{T_2} dx_{\tau_{i+1}} \ldots dx_{\tau_p} = \frac{|T_2 - T_1|}{\sqrt{2\pi(\tau_1 - \tau')}}.$$

Let ς be the last integral. Let $h(x(M)) = (x_{T_2} - x_{T_1})^2$ in the expression of $H_1(x, M)$. Then, by Theorem 3.2, item 1., we have

$$\int_{\mathbb{R}^{]\tau',\tau[}} H_{\mathcal{I}}(x, I[N]) =$$

$$= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} |x_{T_2} - x_{T_1}|^2 \prod_{j=1}^{i} \left(\frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_j} + J(x_{\tau_j}) - x_{\tau_{j-1}})^2}{\tau_j - \tau_{j-1}}\right)}{\sqrt{2\pi(\tau_j - \tau_{j-1})}} \right) \times$$

$$\times \frac{\exp\left(-\frac{1}{2}\frac{(x_{T_1} - x_{\tau_i})^2}{T_1 - \tau_i}\right)}{\sqrt{2\pi(T_1 - \tau_i)}} \frac{\exp\left(-\frac{1}{2}\frac{(x_{T_2} - x_{T_1})^2}{T_2 - T_1}\right)}{\sqrt{2\pi(T_2 - T_1)}} \times$$

$$\times \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_{i+1}} + J(x_{\tau_{i+1}}) - x_{T_2})^2}{\tau_{i+1} - T_2}\right)}{\sqrt{2\pi(\tau_{i+1} - T_2)}} \prod_{j=i+2}^{p} \left(\frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_j} + J(x_{\tau_j}) - x_{\tau_{j-1}})^2}{\tau_j - \tau_{j-1}}\right)}{\sqrt{2\pi(\tau_j - \tau_{j-1})}} \right) \times$$

$$\times \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau} - x_{\tau_p})^2}{\tau - \tau_p}\right)}{\sqrt{2\pi(\tau - \tau_p)}} dx_{\tau_1} \dots dx_{\tau_i} dx_{T_1} dx_{T_2} dx_{\tau_{i+1}} \dots dx_{\tau_p}$$

$$\leq \varsigma = \frac{|T_2 - T_1|}{\sqrt{2\pi(\tau_1 - \tau')}},$$

where the last inequality follows from the Tonelli theorem (Theorem 2.3).

Given $\epsilon > 0$ and $j \in \mathbb{N}$, we can choose T_2 and a division $\mathcal{E}_\gamma$ such that

$$\frac{\epsilon}{j} > \sum_{(x,\, I[N]) \in \mathcal{E}_\gamma} H_{\mathcal{I}}(x, I[N]) \overset{(*)}{=} \sum_{(x,\, I[N]) \in \mathcal{E}_\gamma} \int_{I[\mathcal{O}]} (x_{T_2} - x_{T_1})^2 g_{\mathcal{I}}(x, \mathcal{O}) dx_1 \dots dx_{r-1} \geq$$

$$\geq \sum_{(x,\, I[N]) \in \mathcal{E}_\gamma} \chi(X^j, x) \int_{I[\mathcal{O}]} (x_{T_2} - x_{T_1})^2 g_{\mathcal{I}}(x, \mathcal{O}) dx_1 \dots dx_{r-1} \geq$$

$$\geq \frac{1}{j} \sum_{(x,\, I[N]) \in \mathcal{E}_\gamma} \chi(X^j, x) \int_{I[\mathcal{O}]} g_{\mathcal{I}}(x, \mathcal{O}) dx_1 \dots dx_{r-1} =$$

$$= \frac{1}{j} \sum_{(x,\, I[N]) \in \mathcal{E}_\gamma} \chi(X^j, x) G_{\mathcal{I}}(x, I[N]).$$

The symbol $\mathcal{O}$ after the first equality $(*)$ denotes $\mathcal{O} = \cup\{N : (x, I[N]) \in \mathcal{E}_\gamma\}$. Since $\epsilon > 0$ is arbitrary,

$$\int_{\mathbb{R}^{]\tau',\tau[}} \chi(X^j, x) G_{\mathcal{I}}(x, I[N]) = 0$$

for every $j = 1, 2, \ldots$. Then, by the finite additivity of the integral,

$$\int_{\mathbb{R}^{]\tau',\tau[}} \chi(D^r, x) G_{\mathcal{I}}(x,\, I[N]) = 0.$$

If $T_1 \in \{\tau_1, \ldots, \tau_p\}$, then $T_1 = \tau_i$ for some $i \in \{1, 2, \ldots, p\}$. Considering $T_1 < T_2 < \tau_{i+1}$, since $\dfrac{1}{\sqrt{2\pi(T_2 - \tau_i)}} \displaystyle\int_{-\infty}^{+\infty} |x_{T_2} - x_{\tau_i}|^2 \exp\left(-\frac{1}{2}\frac{(x_{T_2} - x_{\tau_i})^2}{T_2 - \tau_i}\right) dx_{\tau_i} = |T_2 - T_1|$, then

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \frac{|x_{T_2} - x_{T_1}|^2}{\sqrt{2\pi(\tau_1 - \tau')}} \prod_{j=2}^{i} \left(\frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_j} + J(x_{\tau_j}) - x_{\tau_{j-1}})^2}{\tau_j - \tau_{j-1}}\right)}{\sqrt{2\pi(\tau_j - \tau_{j-1})}} \right) \times$$

$$\times \frac{\exp\left(-\frac{1}{2}\frac{(x_{T_2} - x_{\tau_i})^2}{T_2 - \tau_i}\right)}{\sqrt{2\pi(T_2 - \tau_i)}} \frac{\exp\left(-\frac{1}{2}\frac{(J(x_{\tau_{i+1}}) + x_{\tau_{i+1}} - x_{T_2})^2}{\tau_{i+1} - T_2}\right)}{\sqrt{2\pi(\tau_{i+1} - T_2)}} \times$$

$$\times \prod_{j=i+2}^{p} \frac{\exp\left(-\frac{1}{2}\frac{(x_{\tau_j} + J(x_{\tau_j}) - x_{\tau_{j-1}})^2}{\tau_j - \tau_{j-1}}\right)}{\sqrt{2\pi(\tau_j - \tau_{j-1})}} \frac{\exp\left(-\frac{1}{2}\frac{(x_\tau - x_{\tau_p})^2}{\tau - \tau_p}\right)}{\sqrt{2\pi(\tau - \tau_p)}} \times$$

$$\times dx_{\tau_1} \ldots dx_{\tau_i} dx_{T_2} dx_{\tau_{i+1}} \ldots dx_{\tau_p} = \frac{|T_2 - T_1|}{\sqrt{2\pi(\tau_1 - \tau)}}.$$

Thus,

$$\int_{\mathbb{R}^{]\tau',\tau[}} H_{\mathcal{I}}(x,\, I[N]) \leq \frac{|T_2 - T_1|}{\sqrt{2\pi(\tau_1 - \tau)}},$$

where

$$\int_{\mathbb{R}^{]\tau',\tau[}} H_{\mathcal{I}}(I[N]) = \int_{\mathbb{R}^{p+1}} H_2(x,\, M) dx_{\tau_1} \ldots dx_{\tau_i} dx_{T_2} dx_{\tau_{i+1}} \ldots dx_{\tau_p}$$

with $h(x(M)) = (x_{T_2} - x_{T_1})^2$ in the expression of $H_2(x,\, M)$.

The rest of the proof follows by analogous argument. ∎

Theorem 3.3 *The integral* $\displaystyle\int_{D_1} G_{\mathcal{I}}(x,\, I[N])$ *exists and equals zero.*

Proof. We have $D_1 = \displaystyle\bigcup_{r=1}^{+\infty} D^r$ and $D^r \subset D^{r+1}$. For each associated pair $(x,\, I[N])$, define

$$f_k(x,\, I[N]) = \chi(D^k, x) G_{\mathcal{I}}(x,\, I[N]), \quad k = 1, 2, 3 \ldots.$$

Given $x \in D_1$, there exists a positive integer k_0 such that $x \in D^k$ for $k \geq k_0$. Thus, $\chi(D^k, x) = \chi(D_1, x)$ for $k \geq k_0$. Consequently, for each associated pair $(x,\, I[N])$, we have

$$f_k(x,\, I[N]) \overset{k \to +\infty}{\longrightarrow} f_0(x,\, I[N]),$$

where $f_0(x,\, I[N]) = \chi(D^1, x)G_{\mathcal{I}}(x,\, I[N])$. Note that

$$|f_0(x,\, I[N])| \leq G_{\mathcal{I}}(x,\, I[N]) \quad \text{and} \quad |f_k(x,\, I[N])| \leq G_{\mathcal{I}}(x,\, I[N]), \quad k = 1, 2, 3, \ldots.$$

Given $\epsilon > 0$, there exists $k_1 > 0$ such that

$$|f_0(x,\, I[N]) - f_k(x,\, I[N])| < \epsilon G_{\mathcal{I}}(x,\, I[N]),$$

for $k > k_1$ and all associated pairs $(x,\, I[N])$. By Theorem 3.1, $G_{\mathcal{I}}(x,\, I[N])$ is generalized Riemann integrable in $\mathbb{R}^{]\tau',\,\tau[}$. By Theorem 2.2, f_0 is generalized Riemann integrable in $\mathbb{R}^{]\tau',\,\tau[}$ and

$$\int_{\mathbb{R}^{]\tau',\,\tau[}} f_0(x,\, I[N]) = \lim_{k \to +\infty} \int_{\mathbb{R}^{]\tau',\,\tau[}} f_k(x,\, I[N]).$$

Using Lemma 3.2, we obtain

$$\int_{\mathbb{R}^{]\tau',\,\tau[}} \chi(D^1, x)G_{\mathcal{I}}(x,\, I[N]) = 0$$

and the proof is complete. ∎

3.2 An equation of Schrödinger type with impulses

We start by defining the solution of an impulsive partial differential equation. The idea of this definition is borrowed from [10].

Suppose $0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_p < \tau$ are given numbers and $\tau \in]0,\, +\infty[$. Define

$$\Delta = \mathbb{R} \times [0,\, \tau],$$

$$\begin{aligned}
\Gamma_k &= \{(\psi,\, t) : \psi \in \mathbb{R},\, t \in]\tau_k,\, \tau_{k+1}[\}, \quad 0 \leq k \leq p-1, \\
\overline{\Gamma}_k &= \{(\psi,\, t) : \psi \in \mathbb{R},\, t \in [\tau_k,\, \tau_{k+1}[\}, \quad 0 \leq k \leq p-1, \\
\Gamma_p &= \{(\psi,\, t) : \psi \in \mathbb{R},\, t \in]\tau_p,\, \tau[\}, \\
\overline{\Gamma}_p &= \{(\psi,\, t) : \psi \in \mathbb{R},\, t \in [\tau_p,\, \tau[\}, \\
\Gamma &= \bigcup_{k=0}^{p} \Gamma_k \quad \text{and} \quad \overline{\Gamma} = \bigcup_{k=0}^{p} \overline{\Gamma}_k.
\end{aligned}$$

Let $\mathcal{K}(\Delta,\, \mathbb{R})$ be the class of all functions $u : \Delta \to \mathbb{R}$ such that

i) the functions $u|_{\Gamma_k}$, $k = 0, 1, \ldots, p$, are continuous.

ii) for each k, $k = 1, \ldots, p$, the limit $\displaystyle\lim_{(\nu,\,t) \to (\psi,\,\tau_k^-)} u(\nu,\, t) = u(\psi,\, \tau_k^-)$, $\psi \in \mathbb{R}$, exists.

iii) for each k, $k = 1, \ldots, p$, the limit $\displaystyle\lim_{(\nu,\,t) \to (\psi,\,\tau_k^+)} u(\nu,\, t) = u(\psi,\, \tau_k^+)$, $\psi \in \mathbb{R}$, exists.

We consider the equation of Schrödinger type in Γ

$$\frac{\partial}{\partial t}u(\psi,\, t) - \frac{1}{2}\frac{\partial^2}{\partial \psi^2}u(\psi,\, t) + V(\psi)u(\psi,\, t) = 0, \tag{19}$$

subject to the impulse condition

$$u(\psi,\, \tau_k) - u(\psi,\, \tau_k^-) = I(\psi,\, \tau_k,\, u(\psi,\, \tau_k)), \tag{20}$$

where $k = 1, 2, ..., p$, and $V : \mathbb{R} \to \mathbb{R}$ and $I : \mathbb{R}^3 \to \mathbb{R}$ are functions taking real values.

Definition 3.1 *The function $u : \Delta \to \mathbb{R}$ is called a solution of the problem* (19) – (20) *if:*

i) $u \in \mathcal{K}(\Delta,\, \mathbb{R})$*;*

ii) the derivatives $u_t(\psi,\, t)$ and $u_{\psi\psi}(\psi,\, t)$ exist, for $(\psi,\, t) \in \Gamma$;

iii) u satisfies (19) *in Γ and* (20) *at each τ_k, $k = 1, 2, ..., p$.*

Let $U_{\mathcal{I}}$ be a real-valued function defined on $\mathbb{R}$ that admits second order derivatives.

Given $s \in]\tau',\, \tau[$ and $\varsigma \in \mathbb{R}$, let N_s be the set $\{t_1, ..., t_{r-1}\}$, where $t_0 = \tau'$ and $t_r = s$. Then define

$$v_{\mathcal{I}}(N_s,\, I_s;\, \varsigma,\, s) = \int_{I(N_s)} g_{\mathcal{I}}(y,\, N_s)e^{-U_{\mathcal{I}}(x_{r-1})(s-\tau')}dy(N_s)$$

and

$$q_{\mathcal{I}}(x,\, N_s,\, I_s) = g_{\mathcal{I}}(x,\, N_s)\prod_{j=1}^{r-1}\Delta I_j.$$

Let $W_{\mathcal{I}}(x,\, N_s,\, I_s;\, \varsigma,\, s) = q_{\mathcal{I}}(x,\, N_s,\, I_s)e^{-U_{\mathcal{I}}(x_{r-1})(s-\tau')}$. If $W_{\mathcal{I}}(x,\, N_s,\, I_s; \varsigma,\, s)$ is generalized Riemann integrable in $\mathbb{R}^{]\tau',\, s[}$, define

$$\phi_{\mathcal{I}}(\varsigma,\, s) = \int_{\mathbb{R}^{]\tau',\, s[}} W_{\mathcal{I}}(x,\, N_s,\, I_s; \varsigma,\, s).$$

The proof given in [11], shows that the expressions $W_{\mathcal{I}}(x,\, N_s,\, I_s; \varsigma,\, s)$, $v_{\mathcal{I}}(N_s,\, I_s;\, \varsigma,\, s)$ and $e^{-U_{\mathcal{I}}(x_{r-1})(s-\tau')}G_{\mathcal{I}}(x,\, I[N_s])$ are variationally equivalent in $\mathbb{R}^{]\tau',\, s[}$. Therefore we have the following result.

Proposition 3.3 *The following equalities hold*

$$\phi_{\mathcal{I}}(\varsigma,\, s) = \int_{\mathbb{R}^{]\tau',\, s[}} v_{\mathcal{I}}(N_s\, I_s;\, \varsigma,\, s) = \int_{\mathbb{R}^{]\tau',\, s[}} e^{-U_{\mathcal{I}}(x_{r-1})(s-\tau')}G_{\mathcal{I}}(x,\, I[N_s]),$$

whenever either of the integrals exists.

Note that, here, the domain of integration is $\mathbb{R}^{]\tau',\, s[}$ rather than $\mathbb{R}^{]\tau',\, \tau[}$, and the elements x, N_s and I_s are taken in $\mathbb{R}^{]\tau',\, s[}$.

We shall show that $\phi_{\mathcal{I}}(\varsigma,\, s)$ satisfies the equation of Schrödinger type in Γ

$$\frac{\partial}{\partial s}u(\varsigma,\, s) - \frac{1}{2}\frac{\partial^2}{\partial \varsigma^2}u(\varsigma,\, s) + V(\varsigma)u(\varsigma,\, s) = 0, \tag{21}$$

where $V(\varsigma) = \frac{1}{2}\left[\ddot{U}_{\mathcal{I}}(\varsigma)(s-\tau') - [\dot{U}_{\mathcal{I}}(\varsigma)(s-\tau')]^2 - U_{\mathcal{I}}(\varsigma)\right]$, subject to the impulse condition

$$u(\xi_k,\, \tau_k) - u(\xi_k,\, \tau_k^-) = I(\xi_k,\, \tau_k,\, u(\xi_k,\, \tau_k)), \tag{22}$$

where $x(\tau_k) = \xi_k$, $k = 1, 2, ..., p$, and

$$I(\xi_1,\, \tau_1,\, u(\xi_1,\, \tau_1)) = \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} A_1 dx_{i_1}...dx_{i_{k-1}},$$

with

$$A_1 = \frac{\exp\left(-\frac{1}{2}\frac{(x_{i_1}+J(x_{i_1})-\xi')^2}{\tau_1-\tau'} - U_{\mathcal{I}}(\xi_1)(\tau_1-\tau')\right)}{\sqrt{2\pi(\tau_1-\tau')}} - \frac{\exp\left(-\frac{1}{2}\frac{(x_{i_1}-\xi')^2}{\tau_1-\tau'} - U_{\mathcal{I}}(\xi_1)(\tau_1-\tau')\right)}{\sqrt{2\pi(\tau_1-\tau')}}$$

and

$$I(\xi_k,\, \tau_k,\, u(\xi_k,\, \tau_k)) = \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} A \times B dx_{i_1}...dx_{i_{k-1}}, \quad \text{if} \quad 2 \le k \le p$$

with

$$A = \prod_{j=1}^{k-1}\left(\frac{\exp\left(-\frac{1}{2}\frac{(x_{i_j}+J(x_{i_j})-x_{i_{j-1}})^2}{t_{i_j}-t_{i_{j-1}}}\right)}{\sqrt{2\pi(t_{i_j}-t_{i_{j-1}})}}\right),$$

and

$$B = \frac{\exp\left(-\frac{1}{2}\frac{(x_{i_k}+J(x_{i_k})-x_{i_{k-1}})^2}{t_{i_k}-t_{i_{k-1}}} - U_{\mathcal{I}}(\xi_k)(\tau_k-\tau')\right)}{\sqrt{2\pi(t_{i_k}-t_{i_{k-1}})}} +$$
$$-\frac{\exp\left(-\frac{1}{2}\frac{(x_{i_k}-x_{i_{k-1}})^2}{t_{i_k}-t_{i_{k-1}}} - U_{\mathcal{I}}(\xi_k)(\tau_k-\tau')\right)}{\sqrt{2\pi(t_{i_k}-t_{i_{k-1}})}}.$$

Our intention is to prove that $\phi_{\mathcal{I}}(\varsigma,\, s)$ is a solution of the problem (21) − (22), for $\tau' < s < \tau$ and $\varsigma \in \mathbb{R}$.

In the next result, we establish the integrability of the function $W_{\mathcal{I}}(x,\, N_s,\, I_s;\, \varsigma,\, s)$.

Proposition 3.4 *Let $\tau' < s < \tau$ and $x(s) = \varsigma$. Then $W_{\mathcal{I}}(x,\, N_s,\, I_s;\, \varsigma,\, s)$ is generalized Riemann integrable in $\mathbb{R}^{]\tau',\, s[}$ and*

$$\int_{\mathbb{R}^{]\tau',\, s[}} W_{\mathcal{I}}(x,\, N_s,\, I_s; \varsigma,\, s) =$$

$$= e^{-U_{\mathcal{I}}(\varrho(\varsigma))(s-\tau')} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \left(\prod_{j=1}^{r(s)+1} \frac{\exp\left(-\frac{1}{2}\frac{(x_{i_j}+J(x_{i_j})-x_{i_{j-1}})^2}{t_{i_j}-t_{i_{j-1}}}\right)}{\sqrt{2\pi(t_{i_j}-t_{i_{j-1}})}} \right) dx_{i_1}\dots dx_{i_{r(s)}}.$$

where $t_{i_0} = t_0 = \tau'$, $r(s) = \max\left\{i_j : j \in \{1, 2, \dots, p\} \text{ and } t_{i_j} < s\right\}$ *and* $t_{i_{r(s)+1}} = s$.

Proof. Since $U_{\mathcal{I}}$ is continuous, given $\epsilon > 0$, Theorem 3.3 says that for $x \in \mathbb{R}^{]\tau', s[}$ continuous at s, we can choose $L(x)$ such that

$$N_s = \{t_1, \dots, t_{r-1}\} \supseteq L(x) \supseteq \mathcal{I}$$

implies

$$\left| e^{-U_{\mathcal{I}}(x_{r-1})(s-\tau')} - e^{-U_{\mathcal{I}}(\varsigma)(s-\tau')} \right| < \frac{\epsilon}{\varphi(\varsigma,\, s)},$$

where

$$\varphi(\varsigma,\, s) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \left(\prod_{j=1}^{r(s)+1} \frac{\exp\left(-\frac{1}{2}\frac{(x_{i_j}+J(x_{i_j})-x_{i_{j-1}})^2}{t_{i_j}-t_{i_{j-1}}}\right)}{\sqrt{2\pi(t_{i_j}-t_{i_{j-1}})}} \right) dx_{i_1}\dots dx_{i_{r(s)}}.$$

By Proposition 3.1, $0 < \varphi(\varsigma,\, s) < +\infty$, for every $s \in]\tau', \tau[$ and $\varsigma \in \mathbb{R}$. By Theorem 3.1, we have

$$\int_{\mathbb{R}^{]\tau',\, s[}} G_{\mathcal{I}}(x,\, I[N_s]) = \varphi(\varsigma,\, s).$$

Then, we can choose a gauge γ such that, for every division $\mathcal{E}_\gamma$,

$$\left| \sum_{(x, I[N_s]) \in \mathcal{E}_\gamma} \left[e^{-U_{\mathcal{I}}(x_{r-1})(s-\tau')} G_{\mathcal{I}}(x,\, I[N_s]) - e^{-U_{\mathcal{I}}(\varsigma)(s-\tau')} G_{\mathcal{I}}(x,\, I[N_s]) \right] \right| <$$

$$< \frac{\epsilon}{\varphi(\varsigma,\, s)} \sum_{(x, I[N_s]) \in \mathcal{E}_\gamma} G_{\mathcal{I}}(x,\, I[N_s]) < \frac{\epsilon}{\varphi(\varsigma,\, s)}(\epsilon + \varphi(\varsigma,\, s)).$$

Hence, we have the result. ■

As a consequence of Proposition 3.4, we have the following result.

Corollary 3.2 *The function $\phi_{\mathcal{I}}$ satisfies the condition given by* (22).

Now, we give a result which establishes conditions for the continuity of the function $\phi_{\mathcal{I}}$ at intervals with no impulse action.

Proposition 3.5 *Let $s \in]\tau', \tau[\backslash \{\tau_1, \dots, \tau_p\}$ and $\varsigma \in \mathbb{R}$. Given $\epsilon > 0$, there exists $\delta > 0$ such that, if $|s_1 - s| < \delta$ and $|\varsigma_1 - \varsigma| < \delta$, then $|\phi_{\mathcal{I}}(\varsigma_1,\, s_1) - \phi_{\mathcal{I}}(\varsigma,\, s)| < \epsilon$.*

PROOF. Let $s \in]\tau', \tau[\backslash\{\tau_1, ..., \tau_p\}$. We can suppose, without loss of generality, that $\tau_k < s < \tau_{k+1}$, for some $k \in \{0, 1, ..., p\}$, where $\tau_0 = t_0 = \tau'$ and $\tau_{p+1} = \tau$. Then, there exits $\delta > 0$ such that $]s - \delta, s + \delta[\subset]\tau_k, \tau_{k+1}[$. By Proposition 3.4, we have

$$\phi_{\mathcal{I}}(\psi, \beta) = e^{-U_{\mathcal{I}}(\psi)(\beta-\tau')} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left(\prod_{j=1}^{k+1} \frac{\exp\left(-\frac{1}{2}\frac{(x_{i_j}+J(x_{i_j})-x_{i_{j-1}})^2}{t_{i_j}-t_{i_{j-1}}}\right)}{\sqrt{2\pi(t_{i_j}-t_{i_{j-1}})}} \right) dx_{i_1}...dx_{i_k},$$

for every $\beta \in]s-\delta, s+\delta[$ and every $\psi \in \mathbb{R}$, where $t_{i_0} = \tau'$, $t_{i_{k+1}} = \beta$, $x(\tau') = \xi'$, $x(\beta) = \psi$ and $J(x_{i_{k+1}}) = 0$. Given $\beta \in]s - \delta, s + \delta[$, $\beta \neq s$, consider the following expressions

$$\omega_{\mathcal{I}}(\varsigma, s) = e^{-U_{\mathcal{I}}(\varsigma)(s-\tau')} \left(\prod_{j=1}^{k} \frac{\exp\left(-\frac{1}{2}\frac{(x_{i_j}+J(x_{i_j})-x_{i_{j-1}})^2}{t_{i_j}-t_{i_{j-1}}}\right)}{\sqrt{2\pi(t_{i_j}-t_{i_{j-1}})}} \right) \frac{\exp\left(-\frac{1}{2}\frac{(\varsigma-x_{i_k})^2}{s-t_{i_k}}\right)}{\sqrt{2\pi(s-t_{i_k})}}$$

and

$$\omega_{\mathcal{I}}(\psi, \beta) = e^{-U_{\mathcal{I}}(\psi)(\beta-\tau')} \left(\prod_{j=1}^{k} \frac{\exp\left(-\frac{1}{2}\frac{(x_{i_j}+J(x_{i_j})-x_{i_{j-1}})^2}{t_{i_j}-t_{i_{j-1}}}\right)}{\sqrt{2\pi(t_{i_j}-t_{i_{j-1}})}} \right) \frac{\exp\left(-\frac{1}{2}\frac{(\psi-x_{i_k})^2}{\beta-t_{i_k}}\right)}{\sqrt{2\pi(\beta-t_{i_k})}}.$$

Since $U_{\mathcal{I}}$ is continuous, we have $\omega_{\mathcal{I}}(\psi, \beta) \longrightarrow \omega_{\mathcal{I}}(\varsigma, s)$ as $\psi \longrightarrow \varsigma$ and $\beta \longrightarrow s$. Note that $\omega_{\mathcal{I}}(\varsigma, s) > 0$. Then, given $\epsilon > 0$, there exists $\delta_1 > 0$, with $\delta_1 < \delta$, such that

$$|\omega_{\mathcal{I}}(\psi, \beta) - \omega_{\mathcal{I}}(\varsigma, s)| < \epsilon\omega_{\mathcal{I}}(\varsigma, s).$$

whenever $0 < |\beta - s| < \delta_1$ and $0 < |\psi - \varsigma| < \delta_1$. By the dominated convergence test (Theorem 2.2, see also [19], Theorem 9.2), we have

$$\phi_{\mathcal{I}}(\psi, \beta) \longrightarrow \phi_{\mathcal{I}}(\varsigma, s)$$

as $\psi \longrightarrow \varsigma$ and $\beta \longrightarrow s$. Therefore the result is proved. ∎

Theorem 3.4 below says the lateral limits of the function $\phi_{\mathcal{I}}$ at the moments $\{\tau_1, ..., \tau_p\}$ exist.

Theorem 3.4 *Let $x(\tau_k) = \xi_k$, $k = 1, 2, ..., p$. Then, the limits*

$$\lim_{(\varsigma, s)\to(\xi_k, \tau_k^+)} \phi_{\mathcal{I}}(\varsigma, s) \quad \textit{and} \quad \lim_{(\varsigma, s)\to(\xi_k, \tau_k^-)} \phi_{\mathcal{I}}(\varsigma, s)$$

exist for $k = 1, 2, ..., p$.

PROOF. Consider a fixed τ_k for some $k \in \{1, 2, ..., p\}$. Let $\delta > 0$ be arbitrarily small. By Proposition 3.4, we have

$$\phi_{\mathcal{I}}(\varsigma, \tau_k + \delta) = e^{-U_{\mathcal{I}}(\varsigma)(\tau_k+\delta-\tau')} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left(\prod_{j=1}^{k} \frac{\exp\left(-\frac{1}{2}\frac{(x_{i_j}+J(x_{i_j})-x_{i_{j-1}})^2}{t_{i_j}-t_{i_{j-1}}}\right)}{\sqrt{2\pi(t_{i_j}-t_{i_{j-1}})}} \right) \times$$

$$\times \frac{\exp\left(-\frac{1}{2}\frac{(\varsigma-\xi_{i_k})^2}{\delta}\right)}{\sqrt{2\pi\delta}} dx_{i_1}...dx_{i_k}.$$

Now, note that

$$\left(\prod_{j=1}^{k} \frac{\exp\left(-\frac{1}{2}\frac{(x_{i_j}+J(x_{i_j})-x_{i_{j-1}})^2}{t_{i_j}-t_{i_{j-1}}}\right)}{\sqrt{2\pi(t_{i_j}-t_{i_{j-1}})}}\right) \frac{\exp\left(-\frac{1}{2}\frac{(\varsigma-\xi_{i_k})^2}{\delta}\right)}{\sqrt{2\pi\delta}} \leq$$

$$\leq \left(\frac{1}{\sqrt{2\pi(t_{i_1}-t_{i_0})}}\right)\left(\prod_{j=2}^{k} \frac{\exp\left(-\frac{1}{2}\frac{(x_{i_j}+J(x_{i_j})-x_{i_{j-1}})^2}{t_{i_j}-t_{i_{j-1}}}\right)}{\sqrt{2\pi(t_{i_j}-t_{i_{j-1}})}}\right) \frac{\exp\left(-\frac{1}{2}\frac{(\varsigma-\xi_{i_k})^2}{\delta}\right)}{\sqrt{2\pi\delta}}.$$

Define α as being the right hand side of the inequality above. Then, the integral $\int_{-\infty}^{+\infty}...\int_{-\infty}^{+\infty}\alpha dx_{i_1}...dx_{i_k}$ exists and it is equal to $\dfrac{1}{\sqrt{2\pi(t_{i_1}-t_{i_0})}} = \dfrac{1}{\sqrt{2\pi(\tau_1-\tau')}}$. Thus, by Tonelli theorem (Theorem 2.3), we have

$$\phi_{\mathcal{I}}(\varsigma,\, \tau_k+\delta) \leq \frac{e^{-U_{\mathcal{I}}(\varsigma)(\tau_k+\delta-\tau')}}{\sqrt{2\pi(\tau_1-\tau')}}.$$

Hence the limit $\lim\limits_{(\varsigma,\, s)\to(\xi_k,\, \tau_k^+)} \phi_{\mathcal{I}}(\varsigma,\, s)$ exist.

Now, we note that

$$\phi_{\mathcal{I}}(\varsigma,\, \tau_k-\delta) = e^{-U_{\mathcal{I}}(\varsigma)(\tau_k-\delta-\tau')} \int_{-\infty}^{+\infty}...\int_{-\infty}^{+\infty} \left(\prod_{j=1}^{k-1} \frac{\exp\left(-\frac{1}{2}\frac{(x_{i_j}+J(x_{i_j})-x_{i_{j-1}})^2}{t_{i_j}-t_{i_{j-1}}}\right)}{\sqrt{2\pi(t_{i_j}-t_{i_{j-1}})}}\right) \times$$

$$\times \frac{\exp\left(-\frac{1}{2}\frac{(\varsigma-\xi_{i_{k-1}})^2}{(\tau_k-\delta-\tau_{k-1})}\right)}{\sqrt{2\pi(\tau_k-\delta+\tau_{k-1})}} dx_{i_1}...dx_{i_{k-1}}.$$

Analogously, we have

$$\phi_{\mathcal{I}}(\varsigma,\, \tau_k-\delta) \leq \frac{e^{-U_{\mathcal{I}}(\varsigma)(\tau_k-\delta-\tau')}}{\sqrt{2\pi(\tau_1-\delta-\tau')}} \quad \text{if} \quad k=1.$$

and

$$\phi_{\mathcal{I}}(\varsigma,\, \tau_k-\delta) \leq \frac{e^{-U_{\mathcal{I}}(\varsigma)(\tau_k-\delta-\tau')}}{\sqrt{2\pi(\tau_1-\tau')}} \quad \text{if} \quad k=2,3,...,p,$$

Thus, the limit $\lim\limits_{(\varsigma,\, s)\to(\xi_k,\, \tau_k^-)} \phi_{\mathcal{I}}(\varsigma,\, s)$ also exists. ∎

In the proof of Proposition 3.5, we defined $\omega_{\mathcal{I}}(\varsigma,\, s)$ by

$$\omega_{\mathcal{I}}(\varsigma,\, s) = e^{-U_{\mathcal{I}}(\varsigma)(s-\tau')} \left(\prod_{j=1}^{r(s)} \frac{\exp\left(-\frac{1}{2}\frac{(x_{i_j}+J(x_{i_j})-x_{i_{j-1}})^2}{t_{i_j}-t_{i_{j-1}}}\right)}{\sqrt{2\pi(t_{i_j}-t_{i_{j-1}})}}\right) \frac{\exp\left(-\frac{1}{2}\frac{(\varsigma-x_{i_{r(s)}})^2}{s-t_{i_{r(s)}}}\right)}{\sqrt{2\pi(s-t_{i_{r(s)}})}},$$

for $\tau' < s < \tau$ such that $s \neq \tau_j$ for each $j = 1, 2, ..., p$ and $\varsigma \in \mathbb{R}$. Note that

$$\phi_{\mathcal{I}}(\varsigma,\, s) = \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} \omega_{\mathcal{I}}(\varsigma,\, s) dx_{i_1} ... dx_{i_{r(s)}}.$$

By differentiation, we get

$$\frac{\partial \omega_{\mathcal{I}}(\varsigma,\, s)}{\partial s} = -U_{\mathcal{I}}(\varsigma)\omega_{\mathcal{I}}(\varsigma,\, s) - \frac{1}{2(s - t_{i_{r(s)}})}\omega_{\mathcal{I}}(\varsigma,\, s) + \frac{1}{2}\left(\frac{\varsigma - x_{i_{r(s)}}}{s - t_{i_{r(s)}}}\right)^2 \omega_{\mathcal{I}}(\varsigma,\, s),$$

$$\frac{\partial \omega_{\mathcal{I}}(\varsigma,\, s)}{\partial \varsigma} = -\dot{U}_{\mathcal{I}}(\varsigma)(s - \tau')\omega_{\mathcal{I}}(\varsigma,\, s) - \frac{(\varsigma - x_{i_{r(s)}})}{s - t_{i_{r(s)}}}\omega_{\mathcal{I}}(\varsigma,\, s)$$

and

$$\frac{\partial^2 \omega_{\mathcal{I}}(\varsigma,\, s)}{\partial \varsigma^2} = \left[-\ddot{U}_{\mathcal{I}}(\varsigma)(s - \tau') + [\dot{U}_{\mathcal{I}}(\varsigma)(s - \tau')]^2 + 2\dot{U}_{\mathcal{I}}(\varsigma)(s - \tau')\left(\frac{\varsigma - x_{i_{r(s)}}}{s - t_{i_{r(s)}}}\right)\right] \omega_{\mathcal{I}}(\varsigma,\, s) +$$

$$- \frac{1}{s - t_{i_{r(s)}}}\omega_{\mathcal{I}}(\varsigma,\, s) + \left(\frac{\varsigma - x_{i_{r(s)}}}{s - t_{i_{r(s)}}}\right)^2 \omega_{\mathcal{I}}(\varsigma,\, s).$$

Thus,

$$\frac{\partial \omega_{\mathcal{I}}(\varsigma,\, s)}{\partial s} - \frac{1}{2}\frac{\partial^2 \omega_{\mathcal{I}}(\varsigma,\, s)}{\partial \varsigma^2} + \rho(\varsigma,\, s)\omega_{\mathcal{I}}(\varsigma,\, s) = 0, \tag{23}$$

where $\rho(\varsigma,\, s) = \dfrac{1}{2}\left[\ddot{U}_{\mathcal{I}}(\varsigma)(s - \tau') - [\dot{U}_{\mathcal{I}}(\varsigma)(s - \tau')]^2 - 2\dot{U}_{\mathcal{I}}(\varsigma)(s - \tau')\left(\dfrac{\varsigma - x_{i_{r(s)}}}{s - t_{i_{r(s)}}}\right) - U_{\mathcal{I}}(\varsigma)\right].$

The next result says that $\rho(\varsigma,\, s)\omega_{\mathcal{I}}(\varsigma,\, s)$ is generalized Riemann integrable.

Proposition 3.6 *Let $\tau' < s < \tau$, $s \neq \tau_j$ for every $j = 1, 2, ..., p$ and $x(s) = \varsigma$. Then the function $\rho(\varsigma,\, s)\omega_{\mathcal{I}}(\varsigma,\, s)$ is generalized Riemann integrable and*

$$\int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} \rho(\varsigma,\, s)\omega_{\mathcal{I}}(\varsigma,\, s) dx_{i_1} ... dx_{i_{r(s)}} =$$

$$= \frac{1}{2}\left[\ddot{U}_{\mathcal{I}}(\varsigma)(s - \tau') - [\dot{U}_{\mathcal{I}}(\varsigma)(s - \tau')]^2 - U_{\mathcal{I}}(\varsigma)\right] \phi_{\mathcal{I}}(\varsigma,\, s).$$

PROOF. It is enough to prove that

$$\int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} \left(\frac{\varsigma - x_{i_{r(s)}}}{s - t_{i_{r(s)}}}\right) \omega_{\mathcal{I}}(\varsigma,\, s) dx_{i_1} ... dx_{i_{r(s)}} = 0.$$

Indeed. Note that

$$\int_{-\infty}^{+\infty} \left(\frac{\varsigma - x_{i_{r(s)}}}{s - t_{i_{r(s)}}}\right) \frac{\exp\left(-\frac{1}{2}\frac{(\varsigma - x_{i_{r(s)}})^2}{s - t_{i_{r(s)}}}\right)}{\sqrt{2\pi(s - t_{i_{r(s)}})}} dx_{i_{r(s)}} = 0.$$

Now, we define

$$\kappa(\varsigma,\, s) = e^{-U_{\mathcal{I}}(\varsigma)(s-\tau')} \left(\frac{\varsigma - x_{i_{r(s)}}}{s - t_{i_{r(s)}}}\right) \left(\prod_{j=1}^{r(s)-1} \frac{\exp\left(-\frac{1}{2}\frac{(x_{i_j}+J(x_{i_j})-x_{i_{j-1}})^2}{t_{i_j}-t_{i_{j-1}}}\right)}{\sqrt{2\pi(t_{i_j}-t_{i_{j-1}})}}\right) \times$$

$$\times \frac{1}{\sqrt{2\pi(t_{r(s)} - t_{i_{r(s)-1}})}} \frac{\exp\left(-\frac{1}{2}\frac{(\varsigma - x_{i_{r(s)}})^2}{s-t_{i_{r(s)}}}\right)}{\sqrt{2\pi(s - t_{i_{r(s)}})}}.$$

Then,

$$\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \kappa(\varsigma,\, s) dx_{i_1} \dots dx_{i_{r(s)}} = 0.$$

Since

$$0 \le \left(\frac{\varsigma - x_{i_{r(s)}}}{s - t_{i_{r(s)}}}\right) \omega_{\mathcal{I}}(\varsigma,\, s) \le \kappa(\varsigma,\, s) \quad \text{or} \quad \kappa(\varsigma,\, s) \le \left(\frac{\varsigma - x_{i_{r(s)}}}{s - t_{i_{r(s)}}}\right) \omega_{\mathcal{I}}(\varsigma,\, s) \le 0,$$

by the Tonelli theorem (Theorem 2.3), we have

$$\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \left(\frac{\varsigma - x_{i_{r(s)}}}{s - t_{i_{r(s)}}}\right) \omega_{\mathcal{I}}(\varsigma,\, s) dx_{i_1} \dots dx_{i_{r(s)}} = 0$$

and the proof is complete. ■

By Proposition 3.6, the expression $\frac{\partial \omega_{\mathcal{I}}(\varsigma,\, s)}{\partial s} - \frac{1}{2}\frac{\partial^2 \omega_{\mathcal{I}}(\varsigma,\, s)}{\partial \varsigma^2}$ in equation (23) is integrable and

$$\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \left(\frac{\partial \omega_{\mathcal{I}}(\varsigma,\, s)}{\partial s} - \frac{1}{2}\frac{\partial^2 \omega_{\mathcal{I}}(\varsigma,\, s)}{\partial \varsigma^2}\right) dx_{i_1} \dots dx_{i_{r(s)}} =$$

$$= -\frac{1}{2}\left[\ddot{U}_{\mathcal{I}}(\varsigma)(s-\tau') - [\dot{U}_{\mathcal{I}}(\varsigma)(s-\tau')]^2 - U_{\mathcal{I}}(\varsigma)\right] \phi_{\mathcal{I}}(\varsigma,\, s).$$

The problem now is to prove the following equalities

$$\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \frac{\partial \omega_{\mathcal{I}}(\varsigma,\, s)}{\partial s} dx_{i_1} \dots dx_{i_{r(s)}} = \frac{\partial}{\partial s} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \omega_{\mathcal{I}}(\varsigma,\, s) dx_{i_1} \dots dx_{i_{r(s)}}$$

and

$$\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \frac{\partial^2 \omega_{\mathcal{I}}(\varsigma,\, s)}{\partial \varsigma^2} dx_{i_1} \dots dx_{i_{r(s)}} = \frac{\partial^2}{\partial \varsigma^2} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \omega_{\mathcal{I}}(\varsigma,\, s) dx_{i_1} \dots dx_{i_{r(s)}}$$

and to conclude that

$$\frac{\partial \phi_{\mathcal{I}}}{\partial s}(\varsigma,\, s) - \frac{1}{2}\frac{\partial^2 \phi_{\mathcal{I}}}{\partial \varsigma^2}(\varsigma,\, s) + \frac{1}{2}\left[\ddot{U}_{\mathcal{I}}(\varsigma)(s-\tau') - [\dot{U}_{\mathcal{I}}(\varsigma)(s-\tau')]^2 - U_{\mathcal{I}}(\varsigma)\right] \phi_{\mathcal{I}}(\varsigma,\, s) = 0$$

for $(\varsigma,\, s) \in \Gamma$. Thus, in order to prove the equalities, we introduce some additional notations below.

Given $f(\varsigma,\, s)$, let

$$D_{abc}f(\varsigma,\, s) = \frac{1}{a}f_a(\varsigma,\, s) - \frac{1}{2bc}f_{bc}(\varsigma,\, s),$$

where

$$f_a(\varsigma,\, s) = f(\varsigma,\, s+a) - f(\varsigma,\, s)$$

and

$$f_{bc}(\varsigma,\, s) = f(\varsigma+b+c,\, s) - f(\varsigma+b,\, s) - f(\varsigma+c,\, s) + f(\varsigma,\, s)$$

for non-zero real numbers a, b, c. Then the limit

$$\lim_{a,b,c\to 0} D_{abc}f(\varsigma,\, s)$$

exists and equals

$$\frac{\partial f}{\partial s}(\varsigma,\, s) - \frac{1}{2}\frac{\partial^2 f}{\partial \varsigma^2}(\varsigma,\, s),$$

if and only if the partial derivatives

$$\frac{\partial f}{\partial s},\ \frac{\partial^2 f}{\partial \varsigma^2}$$

exist.

In our case, since the derivatives $\dfrac{\partial \omega_{\mathcal{I}}}{\partial s}(\varsigma,\, s)$ and $\dfrac{\partial^2 \omega_{\mathcal{I}}(\varsigma,\, s)}{\partial \varsigma^2}$ exist, we have

$$\lim_{a,b,c\to 0} D_{abc}\omega_{\mathcal{I}}(\varsigma,\, s) = \frac{\partial \omega_{\mathcal{I}}}{\partial s}(\varsigma,\, s) - \frac{1}{2}\frac{\partial^2 \omega_{\mathcal{I}}}{\partial \varsigma^2}(\varsigma,\, s) = -\rho(\varsigma,\, s)\omega_{\mathcal{I}}(\varsigma,\, s). \tag{24}$$

By Proposition 3.6, the limit $\lim\limits_{a,b,c\to 0} D_{abc}\omega_{\mathcal{I}}(\varsigma,\, s)$ is generalized Riemann integrable and

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \lim_{a,b,c\to 0} D_{abc}\omega_{\mathcal{I}}(\varsigma,\, s)dx_{i_1}...dx_{i_{r(s)}} =$$

$$= -\frac{1}{2}\left[\ddot{U}_{\mathcal{I}}(\varsigma)(s-\tau') - [\dot{U}_{\mathcal{I}}(\varsigma)(s-\tau')]^2 - U_{\mathcal{I}}(\varsigma)\right]\phi_{\mathcal{I}}(\varsigma,\, s). \tag{25}$$

Thus, if we prove that

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \lim_{a,b,c\to 0} D_{abc}\omega_{\mathcal{I}}(\varsigma,\, s)dx_{i_1}...dx_{i_{r(s)}} =$$

$$= \lim_{a,b,c\to 0} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} D_{abc}\omega_{\mathcal{I}}(\varsigma,\, s)dx_{i_1}...dx_{i_{r(s)}},$$

then we can conclude that $\dfrac{\partial \phi_{\mathcal{I}}(\varsigma,\, s)}{\partial s}$ and $\dfrac{\partial^2 \phi_{\mathcal{I}}(\varsigma,\, s)}{\partial \varsigma^2}$ exist.

The next theorem shows the existence of the derivatives $\dfrac{\partial \phi_{\mathcal{I}}}{\partial s}$ and $\dfrac{\partial^2 \phi_{\mathcal{I}}}{\partial \varsigma^2}$.

Theorem 3.5 *Let $\tau' < s < \tau$, $s \neq \tau_j$ for every $j = 1, 2, ..., p$ and $x(s) = \varsigma$. Then the partial derivatives $\dfrac{\partial \phi_{\mathcal{I}}}{\partial s}(\varsigma,\, s)$ and $\dfrac{\partial^2 \phi_{\mathcal{I}}}{\partial \varsigma^2}(\varsigma,\, s)$ exist for $(\varsigma,\, s) \in \Gamma$.*

PROOF. Let $\epsilon > 0$ be given. By equation (24), we can choose $\mu > 0$ such that $0 < |\alpha| < \mu$, $0 < |\beta| < \mu$ and $0 < |\gamma| < \mu$ imply

$$|D_{\alpha\beta\gamma}\omega_{\mathcal{I}}(\varsigma,\, s) + \rho(\varsigma,\, s)\omega_{\mathcal{I}}(\varsigma,\, s)| < \omega_{\mathcal{I}}(\varsigma,\, s)\epsilon.$$

Given x, N, I, choose α_0, β_0 and γ_0 satisfying $0 < \alpha_0 < \mu$, $0 < \beta_0 < \mu$ and $0 < \gamma_0 < \mu$ such that

$$\sup_{\substack{0<|\alpha|<\alpha_0 \\ 0<|\beta|<\beta_0 \\ 0<|\gamma|<\gamma_0}} |D_{\alpha\beta\gamma}\omega_{\mathcal{I}}(\varsigma,\, s) + \rho(\varsigma,\, s)\omega_{\mathcal{I}}(\varsigma,\, s)| < \omega_{\mathcal{I}}(\varsigma,\, s).$$

Since $0 < |\alpha| < \alpha_0$, $0 < |\beta| < \beta_0$, $0 < |\gamma| < \gamma_0$, we have

$$-\omega_{\mathcal{I}}(\varsigma,\, s) \leq D_{\alpha\beta\gamma}\omega_{\mathcal{I}}(\varsigma,\, s) + \rho(\varsigma,\, s)\omega_{\mathcal{I}}(\varsigma,\, s) \leq \omega_{\mathcal{I}}(\varsigma,\, s).$$

By the dominated convergence test (Theorem 2.2, see also [19], Theorem 9.2), we have

$$\lim_{\alpha,\,\beta,\,\gamma\to 0} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} D_{\alpha\beta\gamma}\omega_{\mathcal{I}}(\varsigma,\, s)dx_{i_1}...dx_{i_{r(s)}} =$$

$$= -\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \rho(\varsigma,\, s)\omega_{\mathcal{I}}(\varsigma,\, s)dx_{i_1}...dx_{i_{r(s)}} =$$

$$= -\frac{1}{2}\left[\ddot{U}_{\mathcal{I}}(\varsigma)(s-\tau') - [\dot{U}_{\mathcal{I}}(\varsigma)(s-\tau')]^2 - U_{\mathcal{I}}(\varsigma)\right]\phi_{\mathcal{I}}(\varsigma,\, s) =$$

$$\stackrel{(25)}{=} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \lim_{\alpha,\,\beta,\,\gamma\to 0} D_{\alpha\beta\gamma}\omega_{\mathcal{I}}(\varsigma,\, s)dx_{i_1}...dx_{i_{r(s)}}.$$

Then, since

$$\lim_{\alpha,\,\beta,\,\gamma\to 0} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} D_{\alpha\beta\gamma}\omega_{\mathcal{I}}(\varsigma,\, s)dx_{i_1}...dx_{i_{r(s)}} = \frac{\partial \phi_{\mathcal{I}}(\varsigma,\, s)}{\partial s} - \frac{1}{2}\frac{\partial^2 \phi_{\mathcal{I}}(\varsigma,\, s)}{\partial \varsigma^2},$$

the result is proved. ∎

Thus we conclude the following result.

Theorem 3.6 *The function $\phi_{\mathcal{I}}(\varsigma,\, s) = \int_{\mathbb{R}^{]\tau',\, s[}} W_{\mathcal{I}}(x,\, N_s,\, I_s;\varsigma,\, s)$ satisfies the partial differential equation of Schrödinger type in Γ*

$$\frac{\partial}{\partial s}u(\varsigma,\, s) - \frac{1}{2}\frac{\partial^2}{\partial \varsigma^2}u(\varsigma,\, s) + V(\varsigma)u(\varsigma,\, s) = 0,$$

where $V(\varsigma) = \dfrac{1}{2}\left[\ddot{U}_{\mathcal{I}}(\varsigma)(s-\tau') - [\dot{U}_{\mathcal{I}}(\varsigma)(s-\tau')]^2 - U_{\mathcal{I}}(\varsigma)\right]$, subject to the impulse condition

$$= I(\xi_k,\, \tau_k,\, u(\xi_k,\, \tau_k)),$$

where $x(\tau_k) = \xi_k$ and $I(\xi_k,\, \tau_k,\, u(\xi_k,\, \tau_k))$ is given by (22), $k = 1, 2, ..., p$.

3.3 Example

Now, we illustrate the theory by explicit evaluation of $\phi_{\mathcal{I}}$, when each of the impulses is a constant and the function $U_{\mathcal{I}}(t) = \beta$ for every $t \in \mathbb{R}$, with $\beta \in \mathbb{R}$. Consider the continuous function $J : \mathbb{R} \to \mathbb{R}$ given by $J(x(\tau_j)) = \alpha_j$, $1 \leq j \leq p$. Let $s \in]\tau', \tau[$, $\varsigma \in \mathbb{R}$ and $N_s = \{t_1, ..., t_{r-1}\}$, with $t_0 = \tau'$ and $t_r = s$.

Consider the following auxiliary function $\varrho : \mathbb{R} \to \mathbb{R}$ given by

$$\varrho(x(t)) = \begin{cases} x(t), & \text{if } \ t \neq \tau_j \ \text{ for all } \ j = 1..., p, \\ x(t) + J(x(t)), & \text{if } \ t = \tau_j \ \text{ for any } \ j = 1, ..., p. \end{cases}$$

Then

$$\omega_{\mathcal{I}}(\varsigma,\, s) = e^{-\beta(s-\tau')} \left(\prod_{j=1}^{r(s)} \frac{\exp\left(-\frac{1}{2}\frac{(x_{i_j}+\alpha_j - x_{i_{j-1}})^2}{t_{i_j}-t_{i_{j-1}}}\right)}{\sqrt{2\pi(t_{i_j} - t_{i_{j-1}})}} \right) \frac{\exp\left(-\frac{1}{2}\frac{(\varrho(\varsigma)-x_{r(s)})^2}{s-t_{r(s)}}\right)}{\sqrt{2\pi(s - t_{r(s)})}}.$$

Then, by using Lemma 3.1, we obtain

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \omega_{\mathcal{I}}(\varsigma,\, s) dx_{\tau_1} ... dx_{\tau_{r(s)}} =$$

$$= \frac{e^{-\beta(s-\tau')}}{\sqrt{2\pi(s-\tau')}} \exp\left(-\frac{1}{2}\frac{(\varrho(\varsigma) - \xi' + \alpha_1 + ... + \alpha_{r(s)})^2}{s - \tau'}\right)$$

and by Theorem 3.6, we have

$$\phi_{\mathcal{I}}(\varsigma,\, s) = \frac{e^{-\beta(s-\tau')}}{\sqrt{2\pi(s-\tau')}} \exp\left(-\frac{1}{2}\frac{\left(\varrho(\varsigma) - \xi' + \sum\limits_{s \geq t_k} \alpha_k\right)^2}{s - \tau'} \right)$$

is a solution of the partial differential equation of Schrödinger type in Γ

$$\frac{\partial}{\partial \tau} u(\varsigma,\, s) - \frac{1}{2}\frac{\partial^2}{\partial \xi^2} u(\varsigma,\, s) + \beta u(\varsigma,\, s) = 0,$$

subject to the impulse condition

$$u(\xi_k,\, \tau_k) - u(\xi_k,\, \tau_k^-) =$$

$$= \frac{1}{\sqrt{2\pi(\tau_k - \tau')}} \left[\exp\left(-\frac{1}{2}\frac{\left(\xi_k - \xi' + \sum\limits_{i=1}^{k}\alpha_i\right)^2}{\tau_k - \tau'} \right) - \exp\left(-\frac{1}{2}\frac{\left(\xi_k - \xi' + \sum\limits_{i=1}^{k-1}\alpha_i\right)^2}{\tau_k - \tau'} \right) \right].$$

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