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LASALLE'S THEOREMS IN IMPULSIVE SEMIDYNAMICAL SYSTEMS

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Resumo

Neste artigo, consideramos sistemas semidinâmicos com efeitos impulsivos em tempo variável e obtemos o Princípio da Invariância de LaSalle e o Teorema da Estabilidade Assintótica para sistemas semidinâmicos impulsivos.

Abstract

We consider a semidynamical system subject to variable impulses and we obtain the LaSalle invariance principle and the asymptotic stability theorem for this semidynamical system.

1 Introduction

Impulsive differential equations (IDE) are an important tool to describe the evolution of systems where the continuous development of a process is interrupted by abrupt changes of state. These equations are modelled by differential equations which describe the period of continuous variation of state and conditions which describe the discontinuities of first kind of the solution or of its derivatives at the moments of impulse. The theory of IDE is an important area of investigation. For details, see Refs. [1], [2] and [11].

In the present paper we apply this theory to semidynamical systems. The aim of this paper is generalize two results of LaSalle's from [10]: the Invariance Principle and the Asymptotic Stability Theorem.

In [10], Saroop K. Kaul considers an impulsive semidynamical system $(\Omega, \widetilde{\pi})$, where $\Omega \subset \mathrm{X}$ is an open set in a metric space X and the continuous impulsive function I is defined from $\partial\Omega$ to X ($\partial\Omega$ is the boundary of Ω in X). The author introduces a continuous Lyapunov function in $(\Omega, \widetilde{\pi})$ given by $V : \overline{G} \to \mathbb{R}$, where $G \subset \Omega$ is a positively invariant closed set under $\widetilde{\pi}$ in Ω, $\overline{G}$ denotes the closure of G in X and V satisfies the following properties:

1. V is continuous,

2. $V(\mathrm{I}(x)) \leq V(x)$ for $x \in \overline{G} \cap \mathrm{M}$, and

3. $\dot{V}(x) \leq 0$ for $x \in G$, where $\dot{V}(x) = \lim\limits_{t \to 0^+} \dfrac{V(\widetilde{\pi}(x,\, t)) - V(x)}{t}$.

Considering $E = \{x \in G : \dot{V}(x) = 0\}$ and $A \subset E$ as being the largest invariant set under $\widetilde{\pi}$, it is proved in [10] the following results:

- (Invariance Principle) There exists an $\alpha \in \mathbb{R}$ such that for $x \in G$, $\widetilde{\mathrm{L}}^+(x) \subset A \cap V^{-1}(\alpha)$.

- (Asymptotic Stability) If $A \subset intG$, then A is an attractor. If, furthermore, $V(A) = \alpha$ for some $\alpha \in \mathbb{R}$, then A is asymptotically stable.

We can observe that the set A does not contain points of $\partial\Omega$ where the discontinuities of the impulsive system occur. Note as well that the limit set is given by

$$\widetilde{\mathrm{L}}^+(x) = \{y \in \Omega : \widetilde{\pi}(x,\, t_n) \overset{n \to +\infty}{\longrightarrow} y,\ \text{for some } t_n \to +\infty\},$$

that is, the elements of $\widetilde{\mathrm{L}}^+(x)$ are taken in Ω and not in X.

In the present paper, we extend the LaSalle's results from [10] in the following manner. We consider impulsive semidynamical systems of type (X, π; M, I) subject to impulse action which varies in time, where X is a metric space, (X, π) is a semidynamical system, M is a non-empty closed subset of X that denotes the impulsive set and $\mathrm{I} : \mathrm{M} \to \mathrm{X}$ is the impulse function. We consider the closure of $\widetilde{\pi}^+(x)$ in X and the elements of $\widetilde{\mathrm{L}}^+(x)$ in X. Due to this apparent slight difference, i.e, when we consider the closure of a set in X rather than in Ω a new phenomenon that is not present in the impulsive systems considered by [10] can occur in our impulsive systems. For instance, Example 3.1 from [5], shows that the limit set is not necessarily positively invariant under $\widetilde{\pi}$.

In the first part of the paper, we present the basis of the theory of impulsive semidynamical systems. In Section 2.1, we give some basic definitions and notations about impulsive semidynamical systems. In Section 2.2, we discuss the continuity of a function which describes the times of reaching the impulsive set. In Section 2.3, we give some additional useful definitions.

The second part of the paper concerns the main results. In Section 3.1, we introduce the concept of a Lyapunov function defined in X and we present a version of the Theorems of the Invariance and Asymptotic stability for our impulsive system.

2 Impulsive semidynamical systems

In this section we present the basic definitions and notation of the theory of impulsive semidynamical systems.

2.1 Basic definitions and terminology

Let X be a metric space and $\mathbb{R}_+$ be the set of non-negative real numbers. The triple $(\mathrm{X}, \pi, \mathbb{R}_+)$ is called a *semidynamical system*, if the function $\pi : \mathrm{X} \times \mathbb{R}_+ \longrightarrow \mathrm{X}$ is continuous with $\pi(x, 0) = x$ and $\pi(\pi(x, t), s) = \pi(x, t + s)$, for all $x \in \mathrm{X}$ and $t, s \in \mathbb{R}_+$. We denote such system by $(\mathrm{X}, \pi, \mathbb{R}_+)$ or simply (X, π). For every $x \in \mathrm{X}$, we consider the continuous function $\pi_x : \mathbb{R}_+ \longrightarrow \mathrm{X}$ given by $\pi_x(t) = \pi(x, t)$ and we call it the *motion* passing through the point x at the moment $t = 0$ and the set $\sum_x^+ = \pi(x, \mathbb{R}_+)$ is called the positive semi-trajectory of this motion.

Let (X, π) be a semidynamical system. Given $x \in \mathrm{X}$, the *positive orbit* of x is given by $\pi^+(x) = \{\pi(x, t) : t \in \mathbb{R}_+\}$. For $t \geq 0$ and $x \in \mathrm{X}$, we define $\mathrm{F}(x, t) = \{y : \pi(y, t) = x\}$ and, for $\triangle \subset [0, +\infty)$ and $D \subset \mathrm{X}$, we define $\mathrm{F}(D, \triangle) = \bigcup\{\mathrm{F}(x, t) : x \in D \text{ and } t \in \triangle\}$. Then a point $x \in \mathrm{X}$ is called an *initial point*, if $\mathrm{F}(x, t) = \emptyset$ for all $t > 0$.

Now we define semidynamical systems with impulse action. An *impulsive semidynamical system* $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ consists of a semidynamical system, (X, π), a non-empty closed subset M of X such that for every $x \in \mathrm{M}$, there exists $\varepsilon_x > 0$ such that

$$\mathrm{F}(x, (0, \varepsilon_x)) \cap \mathrm{M} = \emptyset \qquad \text{and} \qquad \pi(x, (0, \varepsilon_x)) \cap \mathrm{M} = \emptyset,$$

and a continuous function $\mathrm{I} : \mathrm{M} \to \mathrm{X}$ whose action we explain below in the description of the impulsive semi-trajectory of an impulsive semidynamical system. The points of M are isolated in every trajectory of the system (X, π). The set M is called the *impulsive set*, the function I is called *impulse function* and we write $\mathrm{N} = \mathrm{I}(\mathrm{M})$. We also define $\mathrm{M}^+(x) = (\pi^+(x) \cap \mathrm{M}) \setminus \{x\}$.

Given an impulsive semidynamical systems $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ and $x \in \mathrm{X}$ with $\mathrm{M}^+(x) \neq \emptyset$, it is always possible to find a smallest number s such that the semi-trajectory $\pi_x(t)$ for $0 < t < s$ does not intercept the set M. This result is stated next and a proof of it can be found in [3].

Lemma 2.1 *Let $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ be an impulsive semidynamical system. Then for every $x \in \mathrm{X}$, there is a positive number s, $0 < s \leq +\infty$, such that $\pi(x, t) \notin \mathrm{M}$, whenever $0 < t < s$, and $\pi(x, s) \in \mathrm{M}$ if $\mathrm{M}^+(x) \neq \emptyset$.*

2.2 Semicontinuity and continuity of ϕ

The result of this section is borrowed from [7]. It concerns the function ϕ defined previously which indicates the moments of impulse action of a trajectory in an impulsive system.

Let (X, π) be a semidynamical system. Any closed set $\mathrm{S} \subset \mathrm{X}$ containing x ($x \in \mathrm{X}$) is called a *section* or a $\lambda-$*section* through x, with $\lambda > 0$, if there exists a closed set $\mathrm{L} \subset \mathrm{X}$ such that

a) $\mathrm{F}(\mathrm{L}, \lambda) = \mathrm{S}$;

b) $\mathrm{F}(\mathrm{L}, [0, 2\lambda])$ is a neighborhood of x;

c) $\mathrm{F}(\mathrm{L}, \mu) \cap \mathrm{F}(\mathrm{L}, \nu) = \emptyset$, for $0 \leq \mu < \nu \leq 2\lambda$.

The set $\mathrm{F}(\mathrm{L}, [0, 2\lambda])$ is called a *tube* or a λ-*tube* and the set L is called a *bar*. Let (X, π) be a semidynamical system. We now present the conditions TC and STC for a tube.

Any tube $\mathrm{F}(\mathrm{L}, [0, 2\lambda])$ given by a section S through $x \in \mathrm{X}$ such that $\mathrm{S} \subset \mathrm{M} \cap \mathrm{F}(\mathrm{L}, [0, 2\lambda])$ is called *TC-tube* on x. We say that a point $x \in \mathrm{M}$ fulfills the *Tube Condition* and we write (TC), if there exists a TC-tube $\mathrm{F}(\mathrm{L}, [0, 2\lambda])$ through x. In particular, if $\mathrm{S} = \mathrm{M} \cap \mathrm{F}(\mathrm{L}, [0, 2\lambda])$ we have a *STC-tube* on x and we say that a point $x \in \mathrm{M}$ fulfills the *Strong Tube Condition* (we write (STC)), if there exists a STC-tube $\mathrm{F}(\mathrm{L}, [0, 2\lambda])$ through x.

The following theorem concerns the continuity of ϕ which is accomplished outside M for M satisfying the condition TC. See [7], Theorem 3.8.

Theorem 2.1 *Consider an impulsive semidynamical system* $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$. *Assume that no initial point in* (X, π) *belongs to the impulsive set* M *and that each element of* M *satisfies the condition* (TC). *Then* ϕ *is continuous at* x *if and only if* $x \notin \mathrm{M}$.

2.3 Additional definitions

Let us consider a metric space X with metric ρ. By $B(x, \delta)$ we mean the open ball with center at $x \in \mathrm{X}$ and ratio δ. Let $B(A, \delta) = \{x \in \mathrm{X} : \rho_A(x) < \delta\}$, where $\rho_A(x) = \inf\{\rho(x, y) : y \in A\}$. Throughout this paper, we use the notation $int(A)$ and $\overline{A}$ to denote respectively the interior and closure of A in X.

In what follows, $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ is an impulsive semidynamical system and $x \in \mathrm{X}$.

We define the *limit set* of x in $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ by

$$\widetilde{\mathrm{L}}^{+}(x) = \{y \in \mathrm{X} : \widetilde{\pi}(x, t_n) \overset{n \to +\infty}{\longrightarrow} y, \text{ for some } t_n \overset{n \to +\infty}{\longrightarrow} +\infty\};$$

and the *prolongation set* of x in $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ by

$$\widetilde{\mathrm{D}}^{+}(x) = \{y \in \mathrm{X} : \widetilde{\pi}(x_n, t_n) \overset{n \to +\infty}{\longrightarrow} y, \text{ for some } x_n \overset{n \to +\infty}{\longrightarrow} x \text{ and } t_n \in [0, +\infty)\}.$$

For a set $A \subset \mathrm{X}$ we consider $\widetilde{\mathrm{D}}^{+}(A) = \bigcup\{\widetilde{\mathrm{D}}^{+}(x) : x \in A\}$.

If $\widetilde{\pi}^{+}(A) \subset A$, we say that A is *positively* $\widetilde{\pi}-$*invariant.*

Let $A \subset \mathrm{X}$. If for every neighborhood U of A, there is a positively $\tilde{\pi}-$invariant neighborhood V of A such that $V \subset U$, then A is called orbitally $\tilde{\pi}-$stable. We define the sets

$$\widetilde{\mathrm{P}}^{+}_{\mathrm{W}}(\mathrm{A}) = \{x \in \mathrm{X} : \text{ for every neighborhood } U \text{ of A, there is a sequence}$$

$$\{t_n\} \subset \mathbb{R}_+,\ t_n \to +\infty \text{ such that } \tilde{\pi}(x,\, t_n) \in U\},$$

and

$$\widetilde{\mathrm{P}}^{+}(\mathrm{A}) = \{x \in \mathrm{X} : \text{ for every neighborhood } U \text{ of A, there is } \tau \in \mathbb{R}_+$$

$$\text{such that } \tilde{\pi}(x,\, [\tau,\, +\infty)) \subset U\}.$$

The set $\widetilde{\mathrm{P}}^{+}_{\mathrm{W}}(\mathrm{A})$ is called a region of weak attraction of A with respect to $\tilde{\pi}$ and the set $\widetilde{\mathrm{P}}^{+}(\mathrm{A})$ is called *region of attraction* of A with respect to $\tilde{\pi}$. If $x \in \widetilde{\mathrm{P}}^{+}_{\mathrm{W}}(\mathrm{A})$ or $x \in \widetilde{\mathrm{P}}^{+}(\mathrm{A})$, then we say that x is $\tilde{\pi}-$weakly attracted or $\tilde{\pi}-$*attracted* to A, respectively. A set $\mathrm{A} \subset \mathrm{X}$ is called a weak $\tilde{\pi}$-attractor, if $\widetilde{\mathrm{P}}^{+}_{\mathrm{W}}(\mathrm{A})$ is a neighborhood of A, and it is called a $\tilde{\pi}$-attractor, if $\widetilde{\mathrm{P}}^{+}(\mathrm{A})$ is a neighborhood of A. A subset $\mathrm{A} \subset \mathrm{X}$ is called *asymptotically* $\tilde{\pi}-$*stable*, if it is both a weak $\tilde{\pi}-$attractor and orbitally $\tilde{\pi}-$stable. By Corollary 3.1 from [3], a set $A \subset \mathrm{X}$ is $\tilde{\pi}$-asymptotically stable if and only if it is both orbitally $\tilde{\pi}$-stable and a $\tilde{\pi}$-attractor.

3 The Main Results

Let $(\mathrm{X}, \pi;\ \mathrm{M}, \mathrm{I})$ be an impulsive semidynamical system on a metric space X. We assume the following additional hypotheses:

- no initial point in $(\mathrm{X},\ \pi)$ belongs to the impulsive set M, that is, given $x \in \mathrm{M}$ there are $y \in \mathrm{X}$ and $t \in \mathbb{R}_+$ such that $\pi(y,\, t) = x$.
- ϕ is continuous on $\mathrm{X} \setminus \mathrm{M}$.

In what follows we prove the Invariance Principle and the Asymptotic Stability Theorem for the system $(\mathrm{X}, \pi;\ \mathrm{M}, \mathrm{I})$. We start by introducing some fundamental concepts in the next lines.

A function $V : \mathrm{X} \to \mathbb{R}$ is said to be a Lyapunov function, if

1. V is continuous in X,
2. $V(\mathrm{I}(x)) \leq V(x)$ if $x \in \mathrm{M}$,
3. $\dot{V}(x) \leq 0$ for $x \in \mathrm{X}$, where $\dot{V}(x) = \lim\limits_{t \to 0+} \dfrac{V(\pi(x,\, t)) - V(x)}{t}$.

Item 3. says that $V(\pi(x,\, t)) \leq V(x)$ for every $t \geq 0$ and $x \in \mathrm{X}$. In the next lemma we prove this fact for an impulsive semidynamical system.

Lemma 3.1 *Let* $(\mathrm{X},\ \pi;\ \mathrm{M},\ \mathrm{I})$ *be an impulsive semidynamical system and* $x \in \mathrm{X}$. *Then* $V(\tilde{\pi}(x,\, t)) \leq V(x)$ *for all* $t \geq 0$.

Proof. It is enough to prove the case when $\mathrm{M}^+(x_k^+) \neq \emptyset$ for all integer $k \geq 0$.

By item 3. of the definition of V, we have $V(\pi(x,\,t)) \leq V(x)$ for all $t \geq 0$. In particular, $V(\pi(x,\,t)) \leq V(x)$ for $0 \leq t \leq \phi(x)$. From this and item 2.,

$$V(x_1^+) = V(\mathrm{I}(x_1)) \overset{2.}{\leq} V(x_1) = V(\pi(x,\,\phi(x))) \leq V(x).$$

Hence, $V(\tilde{\pi}(x,\,t)) \leq V(x)$ whenever $0 \leq t \leq \phi(x)$.

Now, if $\phi(x) < t < \phi(x) + \phi(x_1^+)$, it follows that

$$V(\tilde{\pi}(x,\,t)) = V(\pi(x_1^+,\,t - \phi(x))) \leq V(x_1^+) = V(\tilde{\pi}(x,\,\phi(x))) \leq V(x).$$

For $t = \phi(x) + \phi(x_1^+)$,

$$V(\tilde{\pi}(x,\,t)) = V(x_2^+) = V(\mathrm{I}(x_2)) \leq V(x_2) = V(\pi(x_1^+,\,\phi(x_1^+))) \leq V(x_1^+) \leq V(x),$$

and so on. Therefore, $V(\tilde{\pi}(x,\,t)) \leq V(x)$ for all $t \geq 0$ and $x \in \mathrm{X}$. $\square$

Now, let $G \subset \mathrm{X}$ be a positively $\tilde{\pi}$-invariant closed set and $E = \{x \in G : \dot{V}(x) = 0\}$. Consider $A \subset E$ as being the largest positively invariant set under $\tilde{\pi}$.

The next result is our first version for the LaSalle invariance principle.

Theorem 3.1 (Invariance Principle 1) *Let $(\mathrm{X}, \pi;\, \mathrm{M}, \mathrm{I})$ be an impulsive semidynamical system. Suppose $\tilde{\mathrm{L}}^+(x) \cap \mathrm{M} = \emptyset$ for each $x \in G$. Then there is a real number α such that $\tilde{\mathrm{L}}^+(x) \subset A \cap V^{-1}(\alpha)$.*

Proof. Let $x \in G$. If $\tilde{\mathrm{L}}^+(x) = \emptyset$, the result follows. Suppose $\tilde{\mathrm{L}}^+(x) \neq \emptyset$. Since G is a positively $\tilde{\pi}-$invariant closed set, we have $\tilde{\mathrm{L}}^+(x) \subset G$.

We have two cases to consider: when $\tilde{\mathrm{L}}^+(x)$ is a singleton and otherwise.

Suppose $\tilde{\mathrm{L}}^+(x)$ is a singleton, that is, $\tilde{\mathrm{L}}^+(x) = \{s_0\}$ for some $s_0 \in G$. Let us show that $\tilde{\mathrm{L}}^+(x) \subset A$. Since $s_0 \notin \mathrm{M}$, then $\tilde{\mathrm{L}}^+(x)$ is positively $\tilde{\pi}-$invariant ([4], Lemma 3.3). Consequently $\dot{V}(s_0) = 0$ and $\tilde{\mathrm{L}}^+(x) \subset E$. Since A is the largest positively invariant set under $\tilde{\pi}$ in E, it follows that $\tilde{\mathrm{L}}^+(x) \subset A$. Set $\alpha = V(s_0)$. Hence, $\tilde{\mathrm{L}}^+(x) \subset A \cap V^{-1}(\alpha)$.

Now, suppose $\tilde{\mathrm{L}}^+(x)$ is not a singleton. Let $y_1,\, y_2 \in \tilde{\mathrm{L}}^+(x)$, $y_1 \neq y_2$. Then, there are sequences $\{t_n\}_{n\geq 1}$ and $\{s_n\}_{n\geq 1}$ in $\mathbb{R}_+$ with $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$, $s_n \overset{n\to+\infty}{\longrightarrow} +\infty$ such as

$$\tilde{\pi}(x,\,t_n) \overset{n\to+\infty}{\longrightarrow} y_1 \quad \text{and} \quad \tilde{\pi}(x,\,s_n) \overset{n\to+\infty}{\longrightarrow} y_2.$$

Since t_n and s_n tend to $+\infty$, we can choose a sequence ℓ_k such that $\{t_{\ell_k}\}_{k\geq 1}$ is a subsequence of $\{t_n\}_{n\geq 1}$ and $\{s_{\ell_k}\}_{k\geq 1}$ is a subsequence of $\{s_n\}_{n\geq 1}$ with $t_{\ell_k} > s_{\ell_k}$ for each $k = 1, 2, 3, \ldots$. Since $V(\tilde{\pi}(x,\,t)) \leq V(x)$ for all $t \geq 0$ and $x \in \mathrm{X}$ (Lemma 3.1), we have

$$V(\tilde{\pi}(x,\,t_{\ell_k})) \leq V(\tilde{\pi}(x,\,s_{\ell_k})).$$

Since V is a continuous function, when $k \to +\infty$ we have

$$V(y_1) \leq V(y_2).$$

Analogously, we can choose subsequences $\{t_{\jmath_k}\}_{k\geq 1}$ and $\{s_{\jmath_k}\}_{k\geq 1}$ with $s_{\jmath_k} > t_{\jmath_k}$. Thus

$$V(\widetilde{\pi}(x,\, s_{\jmath_k})) \leq V(\widetilde{\pi}(x,\, t_{\jmath_k})).$$

Then, when $k \to +\infty$ we have

$$V(y_2) \leq V(y_1).$$

Hence, $V(y_1) = V(y_2)$. Consequently, $V(y) = \alpha$ for every $y \in \widetilde{\mathrm{L}}^+(x)$ and some $\alpha \in \mathbb{R}$. Since $\widetilde{\mathrm{L}}^+(x)$ is positively $\widetilde{\pi}-$invariant because $\widetilde{\mathrm{L}}^+(x) \cap \mathrm{M} = \emptyset$ ([4], Lemma 3.3), it follows that $\dot{V}(y) = 0$ for every $y \in \widetilde{\mathrm{L}}^+(x)$. Therefore, $\widetilde{\mathrm{L}}^+(x) \subset A \cap V^{-1}(\alpha)$. □

The next lemma is a result about invariance which will be important to the next theorem.

Lemma 3.2 *Let* $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ *be an impulsive semidynamical system and* $x \in \mathrm{X}$. *Suppose* $\widetilde{\mathrm{L}}^+(x) \neq \emptyset$, *then* $\widetilde{\mathrm{L}}^+(x) \setminus \mathrm{M}$ *is positively* $\widetilde{\pi}-$*invariant.*

Proof. Let $y \in \widetilde{\mathrm{L}}^+(x) \setminus \mathrm{M}$. Since $\{y\}$ is compact, M is closed and $y \notin \mathrm{M}$, there exists an $\eta > 0$ such that $B(y,\, \eta) \cap \mathrm{M} = \emptyset$. Thus, we can apply the same proof given by Kaul in [10], Lemma 2.6 to get the result. □

Now, we present a second version for the LaSalle invariance principle, when the limit set of an orbit intercepts the impulsive set M.

Theorem 3.2 (Invariance Principle 2) *Let* $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ *be an impulsive semidynamical system. Suppose* $V : X \to \mathbb{R}_+$ *and* $V(y) = 0$ *for every* $y \in \mathrm{M}$. *If* $\widetilde{\mathrm{L}}^+(x) \cap \mathrm{M} \neq \emptyset$ *for some* $x \in G$, *then there is a real number* α *such that* $\widetilde{\mathrm{L}}^+(x) \subset A \cap V^{-1}(\{0\} \cup \{\alpha\})$.

Proof. Suppose $\widetilde{\mathrm{L}}^+(x) \cap \mathrm{M} \neq \emptyset$ for some $x \in G$. Consequently, $\widetilde{\mathrm{L}}^+(x) \cap \mathrm{M} = \{s_k\}_{k\in\Lambda}$ where $\Lambda \subseteq \mathbb{N}$. Since $\widetilde{\mathrm{L}}^+(x) \setminus \mathrm{M}$ is positively $\widetilde{\pi}-$invariant (Lemma 3.2) then $\widetilde{\mathrm{L}}^+(x) \bigcup_{k\in\Lambda} \widetilde{\pi}^+(s_k)$ is also positively $\widetilde{\pi}$-invariant. Since $\widetilde{\mathrm{L}}^+(x) \subset G$ and $\widetilde{\pi}^+(s_k) \subset G$ for each $k \in \Lambda$, then we can conclude that

$$\widetilde{\mathrm{L}}^+(x) \bigcup_{k\in\Lambda} \widetilde{\pi}^+(s_k) \subset G.$$

Since $V(s_k) = 0$ and $V(\widetilde{\pi}(s_k,\, t)) \leq V(s_k)$, for each $k \in \Lambda$ and $t \geq 0$, it follows that

$$\dot{V}(y) = 0 \quad \text{for all} \quad y \in \bigcup_{k\in\Lambda} \widetilde{\pi}^+(s_k).$$

By the proof of Theorem 3.1, we can conclude that there is $\alpha \in \mathbb{R}_+$ such that $V(z) = \alpha$ for every $z \in \widetilde{\mathrm{L}}^+(x) \setminus \mathrm{M}$. Consequently, $\dot{V}(w) = 0$ for every $w \in \widetilde{\mathrm{L}}^+(x) \bigcup_{k\in\Lambda} \widetilde{\pi}^+(s_k)$. Since A is the largest positively invariant set under $\widetilde{\pi}$ in E, it follows that

$$\widetilde{\mathrm{L}}^+(x) \subset \widetilde{\mathrm{L}}^+(x) \bigcup_{k\in\Lambda} \widetilde{\pi}^+(s_k) \subset A.$$

Therefore, $\widetilde{\mathrm{L}}^+(x) \subset A \cap V^{-1}(\{0\} \cup \{\alpha\})$. □

Corollary 3.1 *If* $\widetilde{L}^+(x)$ *is connected in the conditions of Theorem 3.2, then* $\widetilde{L}^+(x) \subset A \cap V^{-1}(0)$.

Proof. By assumption, no initial point in (X, π) belongs to the impulsive set M, that is, given $x \in M$ there are $y \in X$ and $t \in \mathbb{R}_+$ such that $\pi(y, t) = x$. Then, given $z \in \widetilde{L}^+(x) \cap M$, there is a sequence $\{w_n\}_{n\geq 1} \subset \widetilde{L}^+(x) \setminus M$ such that

$$w_n \stackrel{n\to+\infty}{\longrightarrow} z.$$

From the continuity of V

$$\alpha = V(w_n) \stackrel{n\to+\infty}{\longrightarrow} V(z) = 0.$$

Hence, $\alpha = 0$. Therefore, $\widetilde{L}^+(x) \subset A \cap V^{-1}(0)$. □

Now we are going to present an asymptotic stability theorem. The stability theory for impulsive semidynamical systems can be found in [3], [4], [6], [8] and [10].

In order to prove Theorem 3.4, we need the following theorem from [8].

Theorem 3.3 *Let* $(X, \pi; M, I)$ *be an impulsive semidynamical system on a locally compact space* X *and let* K *be a compact subset of* X*. Then* K *is orbitally* $\widetilde{\pi}$*-stable if and only if* $\widetilde{D}^+(K) = K$.

Our asymptotic stability theorem follows next.

Theorem 3.4 (Asymptotic Stability) *Let* $(X, \pi; M, I)$ *be an impulsive semidynamical system on a locally compact space* X*. Assume* G *compact in* X*.*

1. *Suppose* $\widetilde{L}^+(x) \cap M = \emptyset$ *for each* $x \in G$*, then:*

a) *If* $A \subset intG$*, then* A *is a* $\widetilde{\pi}$*-attractor.*

b) *If, furthermore,* $V^{-1}(\alpha) = A$ *for some* $\alpha \in \mathbb{R}_+$*, then* A *is asymptotically* $\widetilde{\pi}$*-stable.*

2. *Suppose* $V : X \to \mathbb{R}_+$*,* $V(y) = 0$ *for every* $y \in M$ *and* $\widetilde{L}^+(x) \cap M \neq \emptyset$ *for each* $x \in G$*, then:*

a) *If* $A \subset intG$*, then* A *is a* $\widetilde{\pi}$*-attractor.*

b) *If, furthermore,* $V^{-1}(\{0\} \cup \{\alpha\}) = A$ *for some* $\alpha \in \mathbb{R}_+$*, then* A *is asymptotically* $\widetilde{\pi}$*-stable.*

Proof.

1. a) Let $x \in G$. Since G is positively $\widetilde{\pi}-$invariant, we have $\widetilde{\pi}^+(x) \subset G$. Let $\{t_n\}_{n\geq 1}$ be a sequence such that $t_n \stackrel{n\to+\infty}{\longrightarrow} +\infty$. Since G is compact, the sequence $\{\widetilde{\pi}(x, t_n)\}_{n\geq 1} \subset G$ admits a convergent subsequence. Let $\{\widetilde{\pi}(x, t_{n_k})\}_{k\geq 1}$ be this subsequence and a its limit. Then

$$\widetilde{\pi}(x, t_{n_k}) \stackrel{k\to+\infty}{\longrightarrow} a \in \widetilde{L}^+(x).$$

Hence, $\widetilde{\mathrm{L}}^+(x) \neq \emptyset$. By Theorem 3.1, $\widetilde{\mathrm{L}}^+(x) \subset A$. Suppose there are an open set $\mathcal{O}$, with $A \subset \mathcal{O} \subset intG$, and a sequence $\{\tau_n\}_{n\geq 1} \subset \mathbb{R}_+$ such that

$$\widetilde{\pi}(x, \tau_n) \in G \setminus \mathcal{O}.$$

Since $G \setminus \mathcal{O}$ is compact, there exists a subsequence $\{\widetilde{\pi}(x, \tau_{n_k})\}_{k\geq 1}$ that converges to a point $w \in G \setminus \mathcal{O}$. But this is a contradiction because $\widetilde{\mathrm{L}}^+(x) \subset A$. Hence, for every open set $\mathcal{A}$ such that $A \subset \mathcal{A}$, there is a positive number $\tau_{\mathcal{A}} > 0$ such that

$$\widetilde{\pi}(x, [\tau_{\mathcal{A}}, +\infty)) \subset \mathcal{A}.$$

Therefore, $intG \subset \widetilde{\mathrm{P}}^+(A)$ and the result follows.

b) Note that A is compact in X because V is continuous, $V^{-1}(\alpha)$ is closed and G is compact. By Theorem 3.3, it is enough to prove that $\widetilde{\mathrm{D}}^+(A) = A$. Since $A \subset \widetilde{\mathrm{D}}^+(A)$, let us prove the other inclusion. Let $y \in \widetilde{\mathrm{D}}^+(A)$. Then, there exists $x \in A$ such that $y \in \widetilde{\mathrm{D}}^+(x)$. Thus, there are sequences $\{x_n\}_{n\geq 1} \subset$ X and $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+$, such that $x_n \overset{n\to+\infty}{\longrightarrow} x$ and $\widetilde{\pi}(x_n, t_n) \overset{n\to+\infty}{\longrightarrow} y$. Since $V(\widetilde{\pi}(x_n, t_n)) \leq V(x_n)$ and V is continuous, then when $n \to +\infty$ we have

$$V(y) \leq V(x) = \alpha.$$

We assert that $y \in A$. In fact. Suppose $y \notin A$. We have two cases to consider: when $V(y) = \alpha$ and when $V(y) < \alpha$.

First, suppose $V(y) = \alpha$. Since $x_n \overset{n\to+\infty}{\longrightarrow} x \in A$ and $A \subset intG \subset \widetilde{\mathrm{P}}^+(A)$ (see the proof of item 1.), there is an integer $n_0 > 0$ such that $x_n \in int(G)$ for all $n > n_0$. Since G is positively $\widetilde{\pi}$−invariant, we have $\widetilde{\pi}^+(x_n) \subset G$ for all $n > n_0$. Hence $y \in G$ because G is a closed set. By Theorem 3.1, $\widetilde{\mathrm{L}}^+(y) \subset A \cap V^{-1}(\gamma)$ for some $\gamma \in \mathbb{R}$. Since $V(A) = \alpha$, we have $\alpha = \gamma$. As $V(\widetilde{\pi}(y, t)) \leq V(y)$ for all $t \geq 0$ and $\widetilde{\mathrm{L}}^+(y) \subset A \cap V^{-1}(\alpha)$, it follows that

$$V(\widetilde{\pi}(y, t)) = \alpha \quad \text{for all} \quad t \geq 0.$$

Thus, $A \cup \widetilde{\pi}^+(y)$ is positively $\widetilde{\pi}$−invariant and $A \cup \widetilde{\pi}^+(y) \subset G \cap V^{-1}(\alpha)$. This is a contradiction, because A is the largest positively $\widetilde{\pi}$−invariant set in E with this property.

Now, suppose $V(y) < \alpha$. Set $V(y) = \beta$. Since A is compact and V is continuous, there are $\delta > 0$ and $\eta > 0$ such that

$$B(A, \delta) \cap B(y, \eta) = \emptyset,$$

$$|V(z) - \alpha| < \frac{\alpha - \beta}{3} \quad \text{for all} \quad z \in B(A, \delta)$$

and

$$|V(w) - \beta| < \frac{\alpha - \beta}{3} \quad \text{for all} \quad w \in B(y, \eta).$$

Since $\tilde{\pi}(x_n, t_n) \overset{n\to+\infty}{\longrightarrow} y$, there is an integer $n_1 > 0$ such that $\tilde{\pi}(x_n, t_n) \in B(y, \eta)$ for all $n > n_1$. On the other hand, A is a $\tilde{\pi}-$attractor. Thus $\tilde{\mathrm{P}}^+(A)$ is a neighborhood of A. Then, there exists an integer $n_2 > 0$, $n_2 > n_1$, such that $x_n \in \tilde{\mathrm{P}}^+(A)$ for all $n > n_2$ (because $x_n \overset{n\to+\infty}{\longrightarrow} x \in A$). Therefore, there exists $\tau^n > t_n > 0$ such that $\tilde{\pi}(x_n, [\tau^n, +\infty)) \subset B(A, \delta)$, $n > n_2$. Then,

$$\alpha - \frac{\alpha-\beta}{3} < V(\tilde{\pi}(x_n, \tau^n)) \leq V(\tilde{\pi}(x_n, t_n)) < \beta + \frac{\alpha-\beta}{3},$$

$n > n_2$. Hence,

$$\frac{\alpha-\beta}{2} < \frac{\alpha-\beta}{3}$$

which is a contradiction. Hence, $y \in A$ and $\tilde{\mathrm{D}}^+(A) = A$. Then by Theorem 3.3, A is orbitally $\tilde{\pi}$-stable. Therefore, A is asymptotically $\tilde{\pi}$-stable.

2. a) The proof follows the steps of item 1., a), using Theorem 3.2 instead of Theorem 3.1 to conclude that $\tilde{\mathrm{L}}^+(x) \subset A$.

   b) The idea of the proof follows the steps of item 1., b), with suitable changes. Since $V^{-1}(\{0\} \cup \{\alpha\})$ is closed in X, A is compact in X. It is enough to prove that $\tilde{\mathrm{D}}^+(A) \subset A$. Let $y \in \tilde{\mathrm{D}}^+(A)$. Then, there exists $x \in A$ such that $y \in \tilde{\mathrm{D}}^+(x)$. Thus, there are sequences $\{x_n\}_{n\geq 1} \subset \mathrm{X}$ and $\{t_n\}_{n\geq1} \subset \mathbb{R}_+$, such that $x_n \overset{n\to+\infty}{\longrightarrow} x$ and $\tilde{\pi}(x_n, t_n) \overset{n\to+\infty}{\longrightarrow} y$. Since $V(\tilde{\pi}(x_n, t_n)) \leq V(x_n)$ and V is continuous, then when $n \to +\infty$ we have

$$V(y) \leq V(x).$$

   We have two cases to consider: when $V(x) = 0$ and when $V(x) = \alpha$.

   First, suppose $V(x) = 0$. Since $V(\mathrm{X}) \subseteq \mathbb{R}_+$, we have $V(y) = 0$. Hence, $y \in A$ because $V^{-1}(\{0\} \cup \{\alpha\}) = A$.

   Now, if $V(x) \leq \alpha$, the proof follows as we did in 1., item b), where the γ defined in this item is equal to 0 or α.

   Therefore $\tilde{\mathrm{D}}^+(A) = A$ and A is asymptotically $\tilde{\pi}$-stable.

$\square$

The next result concerns the global asymptotic stability for impulsive systems of $Ch0^+$. Recall that a point $x \in \mathrm{X}$ is of characteristic 0^+ or simply $x \in \mathrm{X}$ is of $Ch0^+$, if $\tilde{\mathrm{D}}^+(x) = \overline{\tilde{\pi}^+(x)}$. We say that the impulsive system $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ is of $Ch0^+$, if $\tilde{\mathrm{D}}^+(x) = \overline{\tilde{\pi}^+(x)}$ for all $x \in \mathrm{X}$.

Corollary 3.2 *Let* $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ *be an impulsive semidynamical system of* $Ch0^+$ *on a connected locally compact space* X.

1. *Suppose* $\tilde{\mathrm{L}}^+(x) \cap \mathrm{M} = \emptyset$ *for each* $x \in G$. *If* G *is a compact,* $\mathrm{M} \subset A$ *and* $A \subset intG$, *then* A *is globally asymptotically* $\tilde{\pi}$*-stable.*

2. *Suppose* $V : \mathrm{X} \to \mathbb{R}_+$, $V(y) = 0$ *for every* $y \in \mathrm{M}$ *and* $\widetilde{\mathrm{L}}^+(x) \cap \mathrm{M} \neq \emptyset$ *for each* $x \in G$. *If* G *is a compact,* $\mathrm{M} \subset A$ *and* $A \subset intG$*, then* A *is globally asymptotically* $\widetilde{\pi}$*-stable.*

Proof. By Theorem 3.4, A is $\widetilde{\pi}-$attractor and by [4] Theorem 3.4, A is globally asymptotically $\widetilde{\pi}$-stable. □

Now, we consider a particular impulsive semidynamical system, $(\mathbb{R}^2, \pi; \Omega, \mathrm{M}, \mathrm{I})$, where $(\mathbb{R}^2, \pi)$ is a semidynamical system, Ω is an open set in X, $\mathrm{M} = \partial\Omega$ and $\mathrm{I} : \mathrm{M} \to \Omega$. For details, see [6].

Corollary 3.3 *Let* $(\mathbb{R}^2, \pi; \Omega, \mathrm{M}, \mathrm{I})$ *be an impulsive semidynamical system and* $\overline{\Omega}$ *be compact. Assume that for every* $x \in \Omega$, $\widetilde{\mathrm{L}}^+(x)$ *admits neither rest points nor initial points. If* $A \subset intG \cap \Omega$*, then* A *is asymptotically* $\widetilde{\pi}-$*stable.*

Proof. By the Poincaré-Bendixson Theorem ([6] Theorem 3.9), $\widetilde{\mathrm{L}}^+(x) \cap \mathrm{M} = \emptyset$ and $\widetilde{\mathrm{L}}^+(x)$ is a periodic orbit. Hence, $V|_{\widetilde{\mathrm{L}}^+(x)}$ is constant, that is, there is an $\alpha \in \mathbb{R}$ such that $V(y) = \alpha$ for all $y \in \widetilde{\mathrm{L}}^+(x)$. By item 1., Theorem 3.4, A is asymptotically $\widetilde{\pi}-$stable. □

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