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Rees's Mixed Multiplicity Theorem for Modules

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Abstract

Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring. In this work we introduce the notion of superficial sequences and joint reductions for a family $E_1, \ldots, E_k$ of R-submodules of R^p. This definitions agree with the usual ones when the modules are ideals (i.e., $p = 1$). We prove the existence of both, superficial sequences and joint reductions. We also introduce the notion of mixed multiplicities for a family $E_1, \ldots, E_k$ of R-submodules of R^p of finite colength, which agrees with the one given by Kleiman and Thorup in [KT2] when $k = 2$, and which generalize the notion of mixed multiplicities of $\mathfrak{m}$-primary ideals. The main result of this work is a generalization for modules of the fundamental Rees' mixed multiplicity theorem.

Resumo

Seja $(R, \mathfrak{m})$ um anel local Noetheriano d-dimensional. Neste trabalho introduzimos a noção de sequência superficial e redução simultânea para uma família de R-submódulos $E_1, \ldots, E_k$ de R^p. Estas

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definições coincidem com as definições dadas para ideais (i. e., $p = 1$). Provamos a existência de ambos, sequência superficial e redução simultânea. Também introduzimos a noção de multiplicidades mistas para família de R-submódulos $E_1, \ldots, E_k$ de R^p. de co-comprimento finito, a qual coincide com aquelas multiplicidades mistas dadas por Kleiman e Thorup em [KT2] quando $k = 2$, a qual também generaliza a noção de multiplicidades mistas de ideais $\mathfrak{m}$-primários. O resultado principal deste trabalho é a generalização para módulos do teorema fundamental de Rees para multiplicidades mistas.

1 Introduction

Mixed multiplicities of finitely many zero-dimensional ideals were first defined by Risler and Teissier in [T] and they proved that they could be described as the usual Hilbert-Samuel multiplicity of the ideal generated by an appropriated superficial sequence. This result of Risler and Teissier was later generalized by Rees in [R] who first introduced the notion of joint reduction for a family of ideals and proved that the mixed multiplicities of a family of zero-dimensional ideals could be described as the Hilbert-Samuel multiplicity of the ideal generated by a suitable joint reduction. This Theorem of Rees is known as Rees's mixed multiplicity theorem and it is a crucial result in the theory of mixed multiplicities for zero-dimensional ideals. We refer the reader to [TV] for a survey on mixed multiplicities of ideals.

In this work we introduce first the notion of superficial sequences and joint reductions for a family $E_1, \ldots, E_k$ of R-submodules of R^p, where $(R, \mathfrak{m})$ is a d-dimensional local noetherian ring. This definitions agree with the usual ones when the modules are ideals (i.e., $p = 1$). We prove the existence of both, superficial sequences and joint reductions. We also introduce the notion of mixed multiplicities for a family $E_1, \ldots, E_k$ of R-submodules of R^p of finite colength as follows: let $G = \mathrm{Sym}(R^p) = \oplus_n S_n$ and $\mathcal{R}(E_i) = \oplus_n R_n(E_i)$ be the symmetric and Rees algebra of R^p and E_i respectively. Consider the function

$$h(n_1, \ldots, n_k) := \ell\left(\frac{S_{n_1+\cdots+n_k}}{R_{n_1}(E_1)\cdots R_{n_k}(E_k)}\right).$$

This function is given by a polynomial of total degree $d+p-1$ in k variables $n_1, \ldots, n_k$ for all large values of $n_1, \ldots, n_k$. Write the homogeneous part of this polynomial as

$$\sum_{d_1+\cdots+d_k=d+p-1} \frac{1}{d_1!\cdots d_k!} e(E_1^{[d_1]},\ldots,E_k^{[d_k]}) n_1^{d_1}\cdots n_k^{d_k}.$$

The coefficient $e(E_1^{[d_1]},\ldots,E_k^{[d_k]})$ are called the mixed Buchsbaum-Rim multiplicity of $E_1,\ldots,E_k$. This definition of mixed multiplicities for modules agrees with the one given by Kleiman and Thorup in [KT2] when $k=2$, and generalize the notion of mixed multiplicities of $\mathfrak{m}$-primary ideals.

The main result of this work is a generalization for modules of both, the Risler-Teissier's theorem and Rees' mixed multiplicity theorem. Precisely we prove the following results:

Theorem 1.1. *(Risler-Teissier's theorem for modules) Let $E_1,\ldots,E_k$ be submodules of R^p of finite colength, and $d_1,\ldots,d_k \in \mathbb{N}$ with $d_1+\cdots+d_k = d+p-1$. Let $h_1,\ldots,h_{d+p-1}$ be any superficial sequence for $E_1,\ldots,E_1,\ldots,E_k,\ldots,E_k$, with each E_i listed d_i times. Then,*

$$e(E_1^{[d_1]},\ldots,E_k^{[d_k]}) = e_{BR}(h_1,\ldots,h_{d+p-1}),$$

where $e_{BR}(h_1,\ldots,h_{d+p-1})$ denotes the Buchsbaum-Rim multiplicity of the R-submodule of R^p generated by $h_1,\ldots,h_{d+p-1}$.

Theorem 1.2. *(Rees's mixed multiplicity theorem for modules) Let $E_1,\ldots,E_k$ be submodules of R^p of finite colength, and $d_1,\ldots,d_k \in \mathbb{N}$ with $d_1+\cdots+d_k = d+p-1$. Let $h_1,\ldots,h_{d+p-1}$ be any joint reduction for $E_1,\ldots,E_1,\ldots,E_k,\ldots,E_k$, with each E_i listed d_i times. Then,*

$$e(E_1^{[d_1]},\ldots,E_k^{[d_k]}) = e_{BR}(h_1,\ldots,h_{d+p-1}),$$

where $e_{BR}(h_1,\ldots,h_{d+p-1})$ denotes the Buchsbaum-Rim multiplicity of the R-submodule of R^p generated by $h_1,\ldots,h_{d+p-1}$.

For a precise statement of the above theorems we refer to Theorem 6.1 and Theorem 6.2 respectively.

2 Buchsbaum-Rim multiplicities

Setup: Fix $(R,\mathfrak{m})$ an arbitrary Noetherian local ring; fix two graded D-dimensional R-algebras $G' = \oplus G'_n$ and $G = \oplus G_n$, with $G' \subseteq G$, that, as usual, are generated as algebras by finitely many elements of degree one such that $\ell(G_1/G'_1) < \infty$; and fix M a graded G-module finitely generated in degree zero, that is $G_n M_0 = M_n$ for all $n \in \mathbb{N}$.

As a function of n, the length,

$$h(n) := \ell(M_n/G'_n M_0)$$

is eventually a polynomial in n, denoted by $P_{G',G,M}(n)$, of total degree equal to $\dim M - 1$, which is at most $D-1$, and the coefficient of $n^{D-1}/(D-1)!$ is denoted by $e_{BR}(G', G, M)$ and it is called the *Buchsbaum-Rim multiplicity of the pair* (G', G) *with respect to* M. Notice that $e_{BR}(G', G, M) = 0$ if $\dim M < D$. The notion of Buchsbaum-Rim multiplicity for modules goes back to [BR] and it was carried out in the above generality in [KT1].

Remark 2.1. If I is a finitely generated R-submodule of G_1 such that $\ell(G_1/I) < \infty$ then, we define the *Buchsbaum-Rim multiplicity of the pair* (I, G) *with respect to* M, denoted by $e_{BR}(I, G, M)$, as

$$e_{BR}(I, G, M) := e_{BR}(G', G, M),$$

where $G' = R[I]$ is the R-subalgebra of G generated in degree one by I. In this case, the Hilbert polynomial $P_{G',G,M}(n)$ will be denoted by $P_{I,M}(n)$.

In the remaining of this work, we also use the notation $\mathcal{I}$ for the ideal in G generated by I.

Definition 2.2. *Let I be a finitely generated R-submodule of G_1. We say that $x \in I$ is a* **superficial element** *for I with respect to M if there exist $c \in \mathbb{N}$ such that for all $n \geq c$,*

$$(I^{n+1}M_0 :_{M_n} x) \cap I^c M_{n-c} = I^n M_0.$$

Remark 2.3. Consider the following statements:

(a) $x \in I$ is a superficial element of I with respect to M;

(b) $x \in \mathcal{I}$ is a superficial element of $\mathcal{I}$ with respect to M (in the usual sense).

Then, in general (b) implies (a). But, if M is generated in degree zero then the converse also holds.

Lemma 2.4. *Let I be a finitely generated R-submodule of G_1. Suppose that $x \in I$ is a superficial element of I with respect to M or that x is a non-zerodivisor on M. Assume that $\cap_n \mathcal{I}^n M = 0$. Then there exist $e \in \mathbb{N}$ such that for all $n \geq e$,*

$$(I^n M_0 :_{M_{n-1}} x) \subseteq (0 :_{M_{n-1}} x) + I^{n-e} M_{e-1} \quad \text{and} \quad (0 :_{M_{n-1}} x) \cap I^e M_{n-e-1} = 0$$

If $x \in I$ is a superficial element of I with respect to M, then for all sufficiently large n, $(I^n M_0 :_{M_{n-1}} x) = (0 :_{M_{n-1}} x) + I^{n-1} M_0$.

Proof. By [HS, Lemma 8.5.5] and Remark 2.3 there exist $e \in \mathbb{N}$ such that for all $n \geq e$, we have that

$$(\mathcal{I}^n M :_M x) \subseteq (0 :_M x) + \mathcal{I}^{n-e} M \quad \text{and} \quad (0 :_M x) \cap \mathcal{I}^e M = 0$$

and if $x \in I$ is a superficial element of I with respect to M, then for all sufficiently large n, $(\mathcal{I}^n M :_M x) = (0 :_M x) + \mathcal{I}^{n-1} M$. Hence the result follows by concentrating in degree $n-1$ in the above equalities. □

Lemma 2.5. *Let $I \subseteq G_1$ be a finitely generated R-submodule with $\ell(G_1/I) < \infty$, $x \in I$, and $\dim M = D$. Set $\tilde{M} := M/xM$, $\tilde{G} := G/xG$ or $\tilde{G} := G/Ann(\tilde{M})$ and $\tilde{I} := I\tilde{G}$. Then the following hold.*

1. *If $\dim(\tilde{M}) = D-1$, then $e_{BR}(\tilde{I}, \tilde{G}, \tilde{M}) \geq e_{BR}(I, G, M)$ and for $n \gg 0$, $\ell(\frac{I^n M_0 :_{M_{n-1}} x}{I^{n-1} M_0})$ is a polynomial in n of degree at most $D-2$ with rational coefficients.*

2. *If $\dim(\tilde{G}) = D-1$, then $e_{BR}(\tilde{I}, \tilde{G}, \tilde{M}) \geq e_{BR}(I, G, M)$, and equality holds if and only if $\ell(\frac{I^n M_0 :_{M_{n-1}} x}{I^{n-1} M_0})$ is a polynomial in n of degree at most $D-3$ for all large n.*

Proof. It is enough to consider the following exact sequence of R-modules

$$0 \longrightarrow \frac{I^n M_0 :_{M_{n-1}} x}{I^{n-1} M_0} \longrightarrow \frac{M_{n-1}}{I^{n-1} M_0} \longrightarrow \frac{M_n}{I^n M_0} \longrightarrow \frac{M_n}{x M_{n-1} + I^n M_0} \longrightarrow 0 \qquad (1)$$

The rest of the proof follows by standard Hilbert function arguments. □

Proposition 2.6. *Assume that $ht(\mathcal{I}) > 0$. In the situation of Lemma 2.4 suppose that one of the following conditions hold:*

1. *$x \in I$ is superficial with respect to M and not contained in any minimal prime ideal of $Ann(M)$.*

2. *The dimension of $\tilde{G}$ is $D-1$, I has $D-1$ generators, $x \in I \setminus \mathfrak{m}I$, and x is a non-zerodivisor on M or x is a superficial element for some R-submodule of G_1 of finite colength with respect to M.*

   Then, for large n,

$$e_{BR}(I,M) = \left\{ \begin{array}{ll} e_{BR}(\tilde{I},\tilde{M}) & \text{if } D>2 \\ \ell(\tilde{M}_n) - \ell(0 :_{M_{n-1}} x) & \text{if } D=2. \end{array} \right\}$$

Proof. By the exact sequence (1) we have that

$$\ell(\frac{M_n}{I^n M_0}) - \ell(\frac{M_{n-1}}{I^{n-1}M_0}) = \ell(\frac{M_n}{xM_{n-1}+I^nM_0}) - \ell(\frac{I^nM_0 :_{M_{n-1}} x}{I^{n-1}M_0}).$$

Now, by Lemma 2.4, for large n, say $n \geq c$,

$$(I^nM_0 :_{M_{n-1}} x) = (0 :_{M_{n-1}} x) + I^{n-1}M_0 \quad \text{and} \quad (0 :_{M_{n-1}} x) \cap I^e M_{n-e-1} = 0,$$

so that

$$\frac{(I^nM_0 :_{M_{n-1}} x)}{I^{n-1}M_0} \cong (0 :_{M_{n-1}} x).$$

Thus,

$$\ell(\frac{M_n}{I^nM_0}) = \ell(\frac{M_n}{I^cM_{n-c}}) + \sum_{i=c+1}^{n} \left[\ell(\frac{M_n}{xM_{n-1}+I^iM_{n-i}}) - \ell(0 :_{M_{n-1}} x) \right].$$

The quotient $\frac{M_n}{I^cM_{n-c}}$ is the nth piece of the graded G-module $M/\mathcal{I}^cM$ which could be considered as an $G/\mathcal{I}^c$-module. Since $ht(\mathcal{I}) > 0$, we have that $\dim(G/\mathcal{I}^c) < D$. Hence the function $\lambda(n) := \ell(\frac{M_n}{I^cM_{n-c}})$ is a polynomial in n of degree at most $D-2$. Hence $e_{BR}(I,M) = e_{BR}(\tilde{I},\tilde{M})$ if $D>2$ and $e_{BR}(I,M) = \ell(\tilde{M}_n) - \ell(0 :_{M_{n-1}} x)$ if $D=2$. This proves the case (1), and also (2) if $D=2$.

Now let $D>2$ with the setup as in (2). Let $I = (x, x_2, \ldots, x_{D-1})$ and $J = (x_2, \ldots, x_{D-1})$. By Lemma 2.4, there exist $e \in \mathbb{N}$ such that $(0 :_{M_{n-1}} x) \cap I^e M_{n-e-1} = 0$. Since $I = (x) + J$, it follows that

$$(I^nM_0 :_{M_{n-1}} x) = (xI^{n-1}M_0 + J^nM_0 :_{M_{n-1}} x) = I^{n-1}M_0 + (J^nM_0 :_{M_{n-1}} x)$$

By the Artin-Rees Lemma, there exist $c \in \mathbb{N}$ such that for all $n \geq c$, $\mathcal{J}^n M \cap xM \subseteq x\mathcal{J}^{n-c}M$. Hence, $J^n M_0 \cap xM_{n-1} \subseteq xJ^{n-c}M_{c-1}$. Thus for all $n \geq c$, $(J^n M_0 :_{M_{n-1}} x) \subseteq I^{n-1}M_0 + J^{n-c}M_{c-1} + (0 :_{M_{n-1}} x)$, and

$$\frac{(I^n M_0 :_{M_{n-1}} x)}{I^{n-1}M_0} \subseteq \frac{I^{n-1}M_0 + J^{n-c}M_{c-1}}{I^{n-1}M_0} + \frac{I^{n-1}M_0 + (0 :_{M_{n-1}} x)}{I^{n-1}M_0}.$$

For large n, by Lemma 2.4, the second module on the right is isomorphic to $(0 :_{M_{n-1}} x)$. The first module on the right is a module over $R/(I^{c-1}M_0 :_R M_{c-1})$ and its length is bounded above by $\mu(J^{n-c}M_{c-1})\ell(R/(I^{c-1}M_0 :_R M_{c-1}))$. Hence,

$$\begin{aligned} \ell(\tfrac{(I^n M_0 :_{M_{n-1}} x)}{I^{n-1}M_0}) &\leq \mu(J^{n-c}M_{c-1})\ell(R/(I^{c-1}M_0 :_R M_{c-1})) \\ &\leq \mu(M_{c-1}) \binom{n-c+D-3}{D-3} \ell(R/(I^{c-1}M_0 :_R M_{c-1})) \end{aligned}$$

The last term in the above inequalities is a polynomial of degree at most $D-3$, thus by Lemma 2.5, $e_{BR}(I, M) = e_{BR}(\tilde{I}, \tilde{M})$. $\square$

The following Proposition shows that the Buchsbaum-Rim multiplicity is additive in short exact sequence. This result was shown in [KT1, p.198].

Proposition 2.7. *(Additivity) If* $0 \longrightarrow M_0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow 0$ *is an exact sequence of finitely generated graded G-modules. Then,*

$$e_{BR}(I, M_1) = e_{BR}(I, M_0) + e_{BR}(I, M_2).$$

A standard application of the additivity formula is the following so-called associativity formula.

Corollary 2.8. *(Associativity)*

$$e_{BR}(I, M) = \sum_{\mathfrak{p}} e_{BR}(IG/\mathfrak{p}, G/\mathfrak{p})\ \ell(M_{\mathfrak{p}}),$$

where the sum is taken over all prime ideals $\mathfrak{p}$ *of* G *such that* $\dim(G/\mathfrak{p}) = D$.

Proof. We write $\sigma := \sum_{\mathfrak{p}} \ell(M_{\mathfrak{p}})$ where the sum is taken over all prime ideals $\mathfrak{p}$ such that $\dim(G/\mathfrak{p}) = D$, and proceed by induction on σ. If $\sigma = 0$ then $M = 0$ and so the formula is obvious. If $\sigma > 0$, choose a prime ideal $\mathfrak{p}_0$ with $\dim(G/\mathfrak{p}_0) = D$ and $\ell(M_{\mathfrak{p}_0}) > 0$; then $\mathfrak{p}_0 \in \mathrm{Supp}(M)$. Thus M contains a submodule N isomorphic to $G/\mathfrak{p}_0$. If $\dim M/N < D$ then $\sigma = \ell(M_{\mathfrak{p}_0}) = 1$ and by 2.7 $e_{BR}(I, M) = e_{BR}(I, N)$. If $\dim M/N = D$ then $\sigma(M/N) < \sigma(M) = \sigma$ and so by induction hypothesis

$$e_{BR}(I, M/N) = \sum_{\mathfrak{p}} e_{BR}(IG/\mathfrak{p}, G/\mathfrak{p})\, \ell((M/N)_{\mathfrak{p}}).$$

Now $\ell(M_{\mathfrak{p}}) = \ell((M/N)_{\mathfrak{p}})$ for $\mathfrak{p} \neq \mathfrak{p}_0$ and $\ell(M_{\mathfrak{p}_0}) = \ell((M/N)_{\mathfrak{p}_0}) + 1$. Since by Theorem 2.7 $e_{BR}(I, M) = e_{BR}(I, M/N) + e_{BR}(I, G/\mathfrak{p}_0)$, the assertion follows. □

3 Superficial sequence and joint reductions

We present here the definitions and existence for superficial elements and joint reductions for finitely many submodules of G_1.

Definition 3.1. *Let $I_1, \ldots, I_k$ be a finitely generated R-submodule of G_1. We say that $x \in I_1$ is a* **superficial element** *for $I_1, \ldots, I_k$ with respect to M if there exist $c \in \mathbb{N}$ such that for all $n_1 > c$, and all $n_2, \ldots, n_k \geq 0$,*

$$(I_1^{n_1} I_2^{n_2} \cdots I_k^{n_k} M_0 :_{M_{n_1-1+\ldots+n_k}} x) \cap I_1^c I_2^{n_2} \cdots I_k^{n_k} M_{n_1-1-c} = I_1^{n_1-1} I_2^{n_2} \cdots I_k^{n_k} M_0.$$

A sequence of elements $x_1, \ldots, x_k$, with $x_i \in I_i$, is a **superficial sequence** *for $I_1, \ldots, I_k$ with respect to M if for all $i = 1, \ldots, k$, $x_i \in I_i$ is a superficial element with respect to $M/(x_1, \ldots, x_{i-1})M$ for the image of $I_i, \ldots, I_k$ in $G/(x_1, \ldots, x_{i-1})$.*

Remark 3.2. Consider the following statements:

(a) $x \in I_1$ is a superficial element for $I_1, \ldots, I_k$ with respect to M;

(b) $x \in \mathcal{I}_1$ is a superficial element for $\mathcal{I}_1, \ldots, \mathcal{I}_k$ with respect to M (in the sense of [HS, Definition 17.2.1]).

Then, in general (b) implies (a). But, if M is generated in degree zero then the converse also holds. Same statements for superficial sequences.

Proposition 3.3. *If $(R, \mathfrak{m})$ has infinite residue field, then superficial sequence for $I_1, \ldots, I_k$ with respect to M exist, where $I_1, \ldots, I_k$ are finitely generated R-submodules of G_1. Explicitly, there exist a non-empty Zariski-open subset U of $I_1/\mathfrak{m}I_1$ such that for any $x \in I_1$ with image in U, x is a* *superficial element for $I_1, \ldots, I_k$ with respect to M. Also, if I_1 is not contained in the prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ of G, x can be chosen to avoid the same prime ideals.*

Proof. By [HS, Proposition 17.2.2] there exist a non-empty Zariski-open subset U of $\mathcal{I}_1/\mathfrak{m}\mathcal{I}_1$ such that for any $x \in \mathcal{I}_1$ with image in U, x is a superficial element for $\mathcal{I}_1, \ldots, \mathcal{I}_k$ with respect to M. Such open set could always be chosen consisting of elements of degree one of $\mathcal{I}_1$. Also x can be chosen to avoid the prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ of G. Hence the result follows by Remark 3.5. □

Definition 3.4. *Let $I_1, \ldots, I_k$ be a finitely generated R-submodule of G_1. A sequence of elements $x_1, \ldots, x_k$ with $x_i \in I_i$ is a* **joint reduction** *for $I_1, \ldots, I_k$ with respect to M if there exist $n \in \mathbb{N}$ such that*

$$\left[\sum_{i=1}^{k} x_i I_1 \cdots I_{i-1} I_{i+1} \cdots I_k\right] (I_1 \cdots I_k)^{n-1} M_0 = (I_1 \cdots I_k)^n M_0.$$

Remark 3.5. Consider the following statements:

(a) $x_1, \ldots, x_k$ with $x_i \in I_i$ is a joint reduction for $I_1, \ldots, I_k$ with respect to M;

(b) $x_1, \ldots, x_k$ with $x_i \in \mathcal{I}_i$, of degree one, is a joint reduction for $\mathcal{I}_1, \ldots, \mathcal{I}_k$ with respect to M (in the sense of [HS, Definition 17.1.1]).

Then, in general (*b*) implies (*a*). But, if M is generated in degree zero then the converse also holds. Same statements for superficial sequences.

Lemma 3.6. *Let $I_1, \ldots, I_k$ be a finitely generated R-submodule of G_1. Suppose that $x \in I_1$ is a superficial element for $I_1, \ldots, I_k$ with respect to M. Assume that $\cap_n \mathcal{I}_1^n M = 0$ and that $\mathcal{I}_1 \subseteq \sqrt{\mathcal{I}_2 \cdots \mathcal{I}_k}$. Then for all sufficiently large $n_1, \ldots, n_k$,*

$$(I_1^{n_1} I_2^{n_2} \cdots I_k^{n_k} M_0 :_{M_{n_1-1+\ldots+n_k}} x) = (0 :_{M_{n_1-1+\ldots+n_k}} x) + I_1^{n_1-1} I_2^{n_2} \cdots I_k^{n_k} M_0$$

and

$$(0 :_{M_{n_1-1+\ldots+n_k}} x) \cap I_1^{n_1} I_2^{n_2} \cdots I_k^{n_k} M_0 = 0$$

Proof. By [HS, Lemma 17.2.4] for all sufficiently large $n_1, \ldots, n_k$,

$$(\mathcal{I}_1^{n_1} \mathcal{I}_2^{n_2} \cdots \mathcal{I}_k^{n_k} M_0 :_{M_{n_1-1+\ldots+n_k}} x) = (0 :_{M_{n_1-1+\ldots+n_k}} x) + \mathcal{I}_1^{n_1-1} \mathcal{I}_2^{n_2} \cdots \mathcal{I}_k^{n_k} M_0$$

and

$$(0:_{M_{n_1-1+\ldots+n_k}} x) \cap \mathcal{I}_1^{n_1-1}\mathcal{I}_2^{n_2}\cdots\mathcal{I}_k^{n_k}M_0 = 0.$$

Hence the result follows by Remark 3.5 and by concentrating in degree $n_1 - 1 + \ldots + n_k$ in the above equalities. □

Proposition 3.7. *(Existence of joint reductions) Assume that the residue field $R/\mathfrak{m}$ is infinite. Let $I_1, \ldots, I_{D-1}$ be a finitely generated R-submodule of G_1. Assume that for all i, $\mathcal{I}_i \subseteq \sqrt{\mathcal{I}_{i+1}\cdots\mathcal{I}_{D-1}}$. Then there exist a joint reduction $x_1, \ldots, x_{D-1}$ of $I_1, \ldots, I_{D-1}$ with respect to G.*

Proof. By the argument given at the beginning of the proof of [HS, Theorem 17.3.1] we reduce the proof to the case in which all the ideals $\mathcal{I}_1, \ldots, \mathcal{I}_{D-1}$ are not contained in any associated prime ideal of G.

Let $x_1 \in I_1$ be superficial for $I_1, \ldots, I_{D-1}$ and not contained in any associated prime ideal of G. Such an element x_1 exist by Proposition 3.3. Since $\dim(G/x_1G) = D - 1$, by induction there exist elements $x_i \in I_i$, $i \geq 2$, such that the image of $x_2, \ldots, x_{D-1}$ in G/x_1G is a joint reduction of $I_2, \ldots, I_{D-1}$ with respect to G/x_1G. Thus there exist $n \in \mathbb{N}$ such that $(I_2 \cdots I_{D-1})^n$ is contained in $\left[\sum_{i=2}^{D-1} x_i I_2 \cdots I_{i-1}I_{i+1}\cdots I_{D-1}\right](I_2\cdots I_{D-1})^{n-1}M_0 + x_1 M_{(D-2)n-1}$. Multiplication by I_1^n gives

$$\begin{aligned}(I_1\cdots I_{D-1})^n \subseteq\ & \left[\sum_{i=1}^{D-1} x_i I_1\cdots I_{i-1}I_{i+1}\cdots I_{D-1}\right](I_1\cdots I_{D-1})^{n-1}M_0 \\ & +(x_1 M_{(D-2)n-1}) \cap (I_1\cdots I_{D-1})^n.\end{aligned}$$

As x_1 is superficial for $I_1, \ldots, I_{D-1}$, by Lemma 3.6, for sufficiently large n, we get that

$$\begin{aligned}(I_1\cdots I_{D-1})^n &\subseteq \left[\sum_{i=1}^{D-1} x_i I_1\cdots I_{i-1}I_{i+1}\cdots I_{D-1}\right](I_1\cdots I_{D-1})^{n-1}M_0 \\ &\subseteq (I_1\cdots I_{D-1})^n.\end{aligned}$$

This proves the Proposition. □

Remark 3.8. If $I_1, \ldots, I_{D-1}$, are submodules of G_1 of finite colength then for all $i = 1, \ldots, D-1$, the condition $\mathcal{I}_i \subseteq \sqrt{\mathcal{I}_{i+1}\cdots\mathcal{I}_{D-1}}$ easily holds. Hence, for this kind of submodules joint reduction always exist.

Proposition 3.9. *For $i = 1, \dots, D-1$, let $I_i = (a_{i1}, \dots, a_{il_i})$ be submodules of G_1 and let $x_i \in I_i$. Write $x_i = \sum_{j=1}^{l_i} u_{ij}a_{ij}$ for some $u_{ij} \in R$. Set $l = \sum_i l_i$. Then there exists a (possibly empty) Zariski open set $U \subseteq k^l$ such that $x_1, \dots, x_{D-1}$ is a joint reduction of $I_1, \dots, I_{D-1}$ if and only if the image of $(u_{11}, \dots, u_{1l_1}, \dots, u_{D-11}, \dots, u_{D-1l_{D-1}})$ in k^l lies in U.*

Proof. The Proof is analogous to the one given in [HS, Proposition 17.3.2]. □

4 Mixed multiplicities

Let $I_1, \dots, I_k$, be submodules of G_1 of finite colength Then, as a function of $n_1, \dots, n_k$, the length,

$$h_M(n_1, \dots, n_k) := \ell(M_{n_1+\dots+n_k}/I_1^{n_1} \cdots I_k^{n_k})M_0$$

is for sufficiently large $n_1, \dots, n_k$ a polynomial of total degree $D-1$, whose leading term could be written as

$$\sum_{d_1+\dots+d_k=D-1} \frac{1}{d_1! \cdots d_k!} e(I_1^{[d_1]}, \dots, I_k^{[d_k]}; M) n_1^{d_1} \cdots n_k^{d_k}.$$

The coefficient $e(I_1^{[d_1]}, \dots, I_k^{[d_k]}; M)$ are called the **mixed Buchsbaum-Rim multiplicity of** $I_1, \dots, I_k$ with respect to M. This definition agrees with the one given in [KT2, p. 568] when $k = 2$. When $k = D-1$ and $d_1 = \dots = d_k = 1$ we denote $e(I_1^{[d_1]}, \dots, I_k^{[d_k]}; M)$ by $e(I_1, \dots, I_{D-1}; M)$.

Lemma 4.1. *Let $I_1, \dots, I_k$, be submodules of G_1, and $M, N \subseteq T$ finitely generated graded G-modules. Then there exist integers $c_1, \dots, c_k$ such that for all $n_i \geq c_i$, $i = 1, \dots, k$,*

$$I_1^{n_1} I_2^{n_2} \cdots I_k^{n_k} M_0 \cap N_{n_1+\dots+n_k} = I_1^{n_1-c_1} \cdots I_k^{n_k-c_k}(I_1^{c_1} \cdots I_k^{c_k} M_0 \cap N_{c_1+\dots+c_k}).$$

Proof. By [HS, Theorem 17.1.6] there exist integers $c_1, \dots, c_k$ such that for all $n_i \geq c_i$, $i = 1, \dots, k$,

$$\mathcal{I}_1^{n_1} \cdots \mathcal{I}_k^{n_k} M \cap N = \mathcal{I}_1^{n_1-c_1} \cdots \mathcal{I}_k^{n_k-c_k}(\mathcal{I}_1^{c_1} \cdots \mathcal{I}_k^{c_k} M \cap N).$$

Hence, the result follows by taking degree $n_1 + \dots + n_k$ in the above equality. □

Proposition 4.2. *(Additivity) If* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ *is an exact sequence of finitely generated graded G-modules. Then,*

$$e(I_1^{[d_1]}, \ldots, I_k^{[d_k]}; M) = e(I_1^{[d_1]}, \ldots, I_k^{[d_k]}; M') + e(I_1^{[d_1]}, \ldots, I_k^{[d_k]}; M'').$$

Proof. By Lemma 4.1 there exist integers $c_1, \ldots, c_k$ such that for all $n_i \geq c_i$, $i = 1, \ldots, k$,

$$\begin{aligned} I_1^{n_1} I_2^{n_2} \cdots I_k^{n_k} M_0 \cap M'_{n_1+\cdots+n_k} &= I_1^{n_1-c_1} \cdots I_k^{n_k-c_k}(I_1^{c_1} \cdots I_k^{c_k} M_0 \cap M'_{c_1+\cdots+c_k}) \\ &\subseteq I_1^{n_1-c_1} \cdots I_k^{n_k-c_k} M'_{c_1+\cdots+c_k}. \end{aligned}$$

Thus

$$\begin{aligned} \ell\left(\frac{M'_{n_1+\cdots+n_k}}{I_1^{n_1} \cdots I_k^{n_k} M'_0}\right) &\geq \ell\left(\frac{M_{n_1+\cdots+n_k}}{I_1^{n_1} \cdots I_k^{n_k} M_0 \cap M'_{n_1+\cdots+n_k}}\right) \\ &\geq \ell\left(\frac{M'_{n_1+\cdots+n_k}}{I_1^{n_1-c_1} \cdots I_k^{n_k-c_k} M'_{c_1+\cdots+c_k}}\right). \end{aligned}$$

The result follows by standard Hilbert function arguments. □

Lemma 4.3. *Suppose that* $\dim M = \dim G = D > 2$, *and* $(x_1, \ldots, x_{D-1})$ *and* $(I_1, \ldots, I_{D-1})$ *submodules of* G_1 *of finite colength with* $ht\mathcal{I} > 0$, *where* $x_i \in I_i$ *for each* i. *Set* $\tilde{M} = M/x_1 M$, *and for any* $l \in \mathbb{N}$, *set* $N_l = \bigoplus_{n \geq 0} \frac{I_1^l M_n + x_1 M_{n+l-1}}{x_1 M_{n+l-1}}$. *Then* $\dim N_l = \dim \tilde{M}$.

Set $\tilde{G} = G/x_1 G$ *or* $\tilde{G} = G/Ann_G \tilde{M}$, *and assume that* $\dim \tilde{G} = D - 1 = \dim \tilde{M}$. *Then*

$$e(I_2 \tilde{G}, \ldots, I_{D-1} \tilde{G}; \tilde{M}) = e(I_2 \tilde{G}, \ldots, I_{D-1} \tilde{G}; N_l)$$

and

$$e_{BR}((x_2, \ldots, x_{D-1})\tilde{G}; \tilde{M}) = e_{BR}((x_2, \ldots, x_{D-1})\tilde{G}; N_l).$$

Proof. Set $M_l = \bigoplus_{n \geq 0} \frac{M_{n+l}}{I_1^l M_n + x_1 M_{n+l-1}}$ and $\tilde{M}_l = \bigoplus_{n \geq 0} \frac{M_{n+l}}{x_1 M_{n+l-1}}$.

From the short exact sequence

$$0 \to N_l \to \tilde{M}_l \to M_l \to 0,$$

as $ht\mathcal{I} > 0$ we have that $\ell(\frac{M_{n+l}}{I_1^l M_n + x_1 M_{n+l-1}})$ is eventually a polynomial in n of degree at most $\dim \tilde{M} - 2$. Thus $\dim M_l < \dim \tilde{M}_l = \dim \tilde{M}$. Hence $\dim \tilde{M} = \dim N_l$. The result follows now by Proposition 2.8 and Proposition 4.2. □

5 Main results

Theorem 5.1. *Suppose that* $\dim M = \dim G = D$, *and* $I_1, \ldots, I_k$ *submodules of* G_1 *of finite colength and that* $ht\mathcal{I}_1 > 0$. *Let* $x \in I_1$ *is superficial for* $I_1, \ldots, I_k$ *with respect to* M. *Set* $\tilde{G} = G/x_1G$ *or* $\tilde{G} = G/Ann_G\tilde{M}$. *Then for any integers* $d_1, \ldots, d_k \in \mathbb{N}$ *with* $d_1 + \cdots + d_k = D-1$ *and* $d_1 > 0$, *and for large* n

$$e(I_1^{[d_1]}, \ldots, I_k^{[d_k]}; M) = \left\{ \begin{array}{ll} e(I_1^{[d_1-1]}\tilde{G}, \ldots, I_k^{[d_k]}\tilde{G}; \tilde{M}) & \textit{if } D > 2 \\ \ell(\tilde{M}_n) - \ell(0 :_{M_{n-1}} x) & \textit{if } D = 2. \end{array} \right\}.$$

Proof. From the exact sequence

$$0 \longrightarrow \frac{I_1^{n_1} \cdots I_k^{n_k} M_0 :_{M_{n_1-1+n_2+\cdots n_k}} x}{I_1^{n_1-1} I_2^{n_2} \cdots I_k^{n_k} M_0} \longrightarrow \frac{M_{n_1-1+n_2+\cdots n_k}}{I_1^{n_1-1} I_2^{n_2} \cdots I_k^{n_k} M_0} \longrightarrow$$

$$\longrightarrow \frac{M_{n_1+\cdots n_k}}{I_1^{n_1} \cdots I_k^{n_k} M_0} \longrightarrow \frac{M_{n_1+\cdots n_k}}{xM_{n_1-1+n_2+\cdots n_k} + I_1^{n_1} \cdots I_k^{n_k} M_0} \longrightarrow 0$$

we have, for large $n_1, \ldots, n_k$,

$$\begin{aligned} \ell\left(\frac{M_{n_1+\cdots n_k}}{I_1^{n_1}\cdots I_k^{n_k} M_0}\right) - \ell\left(\frac{M_{n_1-1+n_2+\cdots n_k}}{I_1^{n_1-1} I_2^{n_2}\cdots I_k^{n_k} M_0}\right) &= \ell\left(\frac{M_{n_1+\cdots n_k}}{xM_{n_1-1+n_2+\cdots n_k} + I_1^{n_1}\cdots I_k^{n_k} M_0}\right) \\ &\quad -\ell\left(\frac{I_1^{n_1}\cdots I_k^{n_k} M_0 :_{M_{n_1-1+n_2+\cdots n_k}} x}{I_1^{n_1-1} I_2^{n_2}\cdots I_k^{n_k} M_0}\right). \end{aligned}$$

By Lemma 3.6, for large $n_1, \ldots, n_k$, say $n_1 \geq c_1, \ldots, n_k \geq c_k$,

$$\frac{(I_1^{n_1} \cdots I_k^{n_k} M_0 :_{M_{n_1-1+n_2+\cdots+n_k}} x)}{I_1^{n_1-1} \cdots I_k^{n_k} M_0} \cong (0 :_{M_{n_1-1+n_2+\cdots+n_k}} x).$$

Both sides can be expressed in terms of the multi-graded Hilbert polynomials for such $n_1, \ldots, n_k$. The highest degree part on the left-hand side is then the highest degree in

$$\sum_{d_1+\cdots+d_k=D-1} \frac{1}{d_1! \cdots d_k!} e(I_1^{[d_1]}, \ldots, I_k^{[d_k]}; M)(n_1^{d_1} - (n_1-1)^{d_1}) n_2^{d_2} \cdots n_k^{d_k}$$

which is

$$\sum_{d_1+\cdots+d_k=D-1} \frac{1}{(d_1-1)!d_2!\cdots d_k!} e(I_1^{[d_1]},\ldots,I_k^{[d_k]};M)n_1^{d_1-1}n_2^{d_2}\cdots n_k^{d_k}.$$

On the right-hand side the highest degree then equals, in case $D > 2$,

$$\sum_{d_1+\cdots+d_k=D-2} \frac{1}{d_1!\cdots d_k!} e(I_1^{[d_1]}\tilde{G},\ldots,I_k^{[d_k]}\tilde{G};\tilde{M})n_1^{d_1}\cdots n_k^{d_k},$$

and if $D = 2$, the highest degree on the right is the constant

$$\begin{aligned} e(I_1^{[0]}\tilde{G},\ldots,I_k^{[0]}\tilde{G};\tilde{M}) - \ell(0:_{M_{n_1-1+n_2+\cdots+n_k}} x) &= \ell(\tilde{M}_{n_1+\cdots+n_k}) - \\ &\quad \ell(0:_{M_{n_1-1+n_2+\cdots+n_k}} x). \end{aligned}$$

With this, the result follows by reading off the leading coefficients. □

Corollary 5.2. *Suppose that* $\dim M = \dim G = D$, *and* $I_1,\ldots,I_k$ *submodules of* G_1 *of finite colength and that* $ht\mathcal{I}_i > 0$. *Let* $d_1,\ldots,d_k \in \mathbb{N}$ *with* $d_1+\cdots+d_k = D-1$. *Let* $x_1,\ldots,x_{D-1}$ *be any superficial sequence for* $I_1,\ldots,I_1,\ldots,I_k,\ldots,I_k$ *with respect to* M, *with each* I_i *listed* d_i *times, and each* x_i *not in any minimal prime ideal over* $x_1,\ldots,x_{i-1}$. *Then, for large* n,

$$\begin{aligned} e(I_1^{[d_1]},\ldots,I_k^{[d_k]};M) &= \ell\left(\frac{M_n}{(x_1,\ldots,x_{D-1})M_{n-1}}\right) \\ &\quad -\ell((x_1,\ldots,x_{D-2})M_n :_{M_n} x_{D-1}), \end{aligned}$$

which equal $e_{BR}(x_1,\ldots,x_{D-1};M)$.

Proposition 5.3. *(Associativity) Suppose that* $\dim M = \dim G = D$, *and* $I_1,\ldots,I_k$ *submodules of* G_1 *of finite colength and that* $ht\mathcal{I}_i > 0$. *Then for any integers* $d_1,\ldots,d_k \in \mathbb{N}$ *with* $d_1+\cdots+d_k = D-1$ *and* $d_1 > 0$,

$$e(I_1^{[d_1]},\ldots,I_k^{[d_k]};M) = \sum_{\mathfrak{p}} e(I_1^{[d_1]}G/\mathfrak{p},\ldots,I_k^{[d_k]}G/\mathfrak{p};G/\mathfrak{p})\ \ell(M_{\mathfrak{p}}),$$

where the sum is taken over all prime ideals $\mathfrak{p}$ *of* G *such that* $\dim(G/\mathfrak{p}) = D$.

Proof. By passing to $R[X]_{\mathfrak{m}R[X]}$, where X is a variable over R, neither the hypotheses nor the conclusion change, so we may assume that $R/\mathfrak{m}$ is infinite. By Proposition 3.3, there is a superficial sequence $x_1,\ldots,x_{D-1}$ for

$I_1, \ldots, I_1, \ldots, I_k, \ldots, I_k$ with respect to M and with respect to each of the finitely many $G/\mathfrak{p}$, with each I_i listed d_i times, and each x_i not in any minimal prime ideal over $x_1, \ldots, x_{i-1}$. Then if $J = (x_1, \ldots, x_{D-1})$, by Corollary 5.2, $e(I_1^{[d_1]}, \ldots, I_k^{[d_k]}; M) = e_{BR}(J; M)$, and for each $\mathfrak{p}$, $e(I_1^{[d_1]}, \ldots, I_k^{[d_k]}; M) = e_{BR}(J; G/\mathfrak{p})$. Now the proposition follows from the Associativity Formula for Buchsbaum-Rim multiplicities, Proposition 2.8. □

Theorem 5.4. *(Rees's mixed multiplicity theorem) Suppose that* $\dim M = \dim G = D$, *and* $I_1, \ldots, I_k$ *submodules of* G_1 *of finite colength and that* $ht\mathcal{I}_i > 0$. *Let* $d_1, \ldots, d_k \in \mathbb{N}$ *with* $d_1 + \cdots + d_k = D - 1$. *Let* $x_1, \ldots, x_{D-1}$ *be a joint reduction for* $I_1, \ldots, I_1, \ldots, I_k, \ldots, I_k$ *with respect to* M, *with each* I_i *listed* d_i *times. Then,*

$$e(I_1^{[d_1]}, \ldots, I_k^{[d_k]}; M) = e_{BR}(x_1, \ldots, x_{D-1}; M).$$

Proof. Let J be the R-submodule of G_1 generated by $x_1, \ldots, x_{D-1}$. Because $x_1, \ldots, x_{D-1}$ is a joint reduction for $I_1, \ldots, I_1, \ldots, I_k, \ldots, I_k$ with respect to M, J has finite colength in G_1, and therefore $e_{BR}(J; M)$ makes sense. As (mixed) multiplicities and joint reductions are preserved under passage to the faithfully flat extension $R[X]_{\mathfrak{m}R[X]}$, where X is a variable over R, without lost of generality we may assume that the residue field is infinite. Since $e(I_1^{[d_1]}, \ldots, I_k^{[d_k]}; M) = e(I_1, \ldots, I_1, \ldots, I_k, \ldots, I_k; M)$, we may simplify notation and set $k = D - 1$ and all $d_i = 1$.

If $D = 1$, then $e(I_1, \ldots, I_{D-1}; M) = \ell(M_n) = e_{BR}(0; M)$ for large n, so the theorem holds.

Let $D = 2$. Then x_1 being a joint reduction of I_1 with respect to M says that there exists an integer l such that $x_1 I_1^l M_0 = I_1^{l+1} M_0$. Thus for all $n > l$, $x_1^n M_0 \subseteq x_1^{n-l} I_1^l M_0 = I_1^n M_0$ and $\ell(M_n/x_1^n M_0) \geq \ell(M_n/I_1^n M_0) \geq \ell(M_n/x_1^{n-l} M_l) \geq \ell(M_{n-l}/x_1^{n-l} M_0)$. Thus the Hilbert polynomial of I and x_1 with respect to M must have the same leading coefficient, namely

$$e_{BR}((x_1); M) = e_{BR}(I_1; M).$$

Now let $D > 2$. If $x_1, \ldots, x_{D-1}$ is a joint reduction of $I_1, \ldots, I_{D-1}$ with respect to M, then it is so with respect to each $G/\mathfrak{p}$, as $\mathfrak{p}$ varies over the minimal prime ideals in G that contain $\mathrm{Ann}M$. Then by the Associativity Formulas for (mixed) multiplicities (Propositions 2.8 and 5.3), it suffices to prove the theorem in case $M = G = G/\mathfrak{p}$ is an integral domain.

Set $l = \sum_i \mu(I_i)$. With notation as in Proposition 3.9, let $U \subseteq (R/\mathfrak{m})^l$ be a Zariski-open subset that determines all the joint reductions of the

$I_1, \ldots, I_{D-1}$; by assumption, U is non-empty. Let U' be a non-empty Zariski-open subset of $I_1/\mathfrak{m}I_1$ such that any preimage in I_1 of any element of U' is a non-zero superficial element for $I_1, \ldots, I_{D-1}$. Such U' exists and is non-empty by Proposition 3.3. By [HS, Lemma 8.5.12], there exists $y \in U'$ such that $y, x_2, \ldots, x_{D-1} \in U$. Set $\tilde{G} = G/yG$. By Theorem 5.1, $e(I_1, \ldots, I_{D-1}; G) = e(I_2\tilde{G}, \ldots, I_{D-1}\tilde{G}; \tilde{G})$, and by Proposition 2.6,

$$e_{BR}(y, x_2, \ldots, x_{D-1}; G) = e_{BR}((x_2, \ldots, x_{D-1})\tilde{G}; \tilde{G}).$$

Set $N_l = \bigoplus_{n \geq 0} \frac{I_1^l G_n + yG_{n+l-1}}{yG_{n+l-1}}$. Since $y, x_2, \ldots, x_{D-1}$ is a joint reduction of $I_1, \ldots, I_{D-1}$, for all large l, $x_2, \ldots, x_{D-1}$ is a joint reduction of $I_2, \ldots, I_{D-1}$, with respect to N_l. By induction on D then

$$e(I_2\tilde{G}, \ldots, I_{D-1}\tilde{G}; N_l) = e_{BR}((x_2, \ldots, x_{D-1})\tilde{G}; N_l).$$

By Lemma 4.3,

$$\begin{aligned} e(I_1, \ldots, I_{D-1}; G) &= e(I_2\tilde{G}, \ldots, I_{D-1}\tilde{G}; \tilde{G}) \\ &= e(I_2\tilde{G}, \ldots, I_{D-1}\tilde{G}; N_l) \\ &= e_{BR}((x_2, \ldots, x_{D-1})\tilde{G}; N_l) \\ &= e_{BR}((x_2, \ldots, x_{D-1})\tilde{G}; \tilde{G}) \\ &= e_{BR}((y, x_2, \ldots, x_{D-1}); G). \end{aligned}$$

It remains to prove that $e_{BR}((y, x_2, \ldots, x_{D-1}); G) = e_{BR}((x_1, \ldots, x_{D-1}); G)$. Set $G'' = G/x_{D-1}G$, $N_l'' = \bigoplus_{n \geq 0} \frac{I_1^l G_n + x_{D-1}G_{n+l-1}}{x_{D-1}G_{n+l-1}}$. By Proposition 2.6 and by Lemma 4.3,

$$\begin{aligned} e_{BR}((y, x_2, \ldots, x_{D-1}); G) &= e_{BR}((y, x_2, \ldots, x_{D-1})G''; G'') \\ &= e_{BR}((y, x_2, \ldots, x_{D-1})G''; N_l''). \end{aligned}$$

Similarly,

$$\begin{aligned} e_{BR}((x_1, \ldots, x_{D-1}); G) &= e_{BR}((x_1, \ldots, x_{D-1})G''; G'') \\ &= e_{BR}((x_1, \ldots, x_{D-1})G''; N_l''). \end{aligned}$$

By the same argument as before $x_1, \ldots, x_{D-1}$ is a joint reduction of $I_1, \ldots, I_{D-1}$ with respect to N_l''. Thus by induction on D, using G'' and N_l'', y can in addition be chosen sufficiently general in I_1 such that

$$e_{BR}((y, x_2, \ldots, x_{D-1})G''; G'') = e_{BR}((x_1, \ldots, x_{D-1})G''; N_l'').$$

This proves the theorem. $\square$

6 Mixed multiplicities for modules

We begin by recalling the notion of the Buchsbaum-Rim multiplicity for submodules of finite colength in a free module as introduced and developed in [BR].

Let $(R, \mathfrak{m})$ be a Noetherian local ring and E a submodule of the free R-module R^p. The symmetric algebra $G := \mathrm{Sym}(R^p) = \oplus S_n(R^p)$ of R^p is a polynomial ring $R[T_1, \ldots, T_p]$. If $h = (h_1, \ldots, h_p) \in R^p$, then we define the element $w(h) = h_1T_1 + \ldots + h_pT_p \in S_1(R^p) =: G_1$. We denote by $\mathcal{R}(E) := \oplus\mathcal{R}_n(E)$ the subalgebra of G generated in degree one by $\{w(h) : h \in E\}$ and call it *the Rees algebra* of E. Then $\mathcal{R}(E)$ has dimension $d+p$. Given any finitely generated R-module N of dimension d, consider the graded G-module $M := G \otimes_R N$. Notice that M is a finitely generated graded G-module of dimension $d+p$ that is generated in degree zero.

If the R-submodule E of R^p has finite colength, the Buchsbaum-Rim multiplicity, $e_{BR}(E; N)$, of E with respect to N could be defined as $e_{BR}(E; N) := e(G', G; M)$, where $G' = \mathcal{R}(E)$.

We are now ready to introduce the main object of this paper, the mixed multiplicities for finitely many submodules of finite colength in R^p. Here the linear submodules of G_1 of the previous section will be replaced by a module E.

Let $E_1, \ldots, E_k$ be R-submodules of R^p and denote by I_i the R-submodule of G_1 generated by $\mathcal{R}_1(E_i)$. We translate into this context the basic definitions of the previous sections.

A sequence of elements $h_1, \ldots, h_k$, with $h_i \in E_i$, is a **superficial sequence** for $E_1, \ldots, E_k$ with respect to N if for all $i = 1, \ldots, k$, $h_i \in E_i$ are such that the sequence $w(h_1), \ldots, w(h_k)$ is a superficial sequence for $I_1, \ldots, I_k$ with respect to M.

A sequence of elements $h_1, \ldots, h_k$, with $h_i \in E_i$, is a **joint reduction** for $E_1, \ldots, E_k$ with respect to N if for all $i = 1, \ldots, k$, $h_i \in E_i$ are such that the sequence $w(h_1), \ldots, w(h_k)$ is a joint reduction for $I_1, \ldots, I_k$ with respect to M.

The **mixed Buchsbaum-Rim multiplicities**, $e(I_1^{[d_1]}, \ldots, I_k^{[d_k]}; M)$, for the modules $E_1, \ldots, E_k$ with respect to N, are defined as

$$e(E_1^{[d_1]}, \ldots, E_k^{[d_k]}; N) := e(I_1^{[d_1]}, \ldots, I_k^{[d_k]}; M)$$

for all $d_1, \ldots, d_k \in \mathbb{N}$ with $d_1 + \cdots + d_k = d + p - 1$.

The main results of the previous section are now described in this new setting.

Corollary 5.2 immediately gives the following result which says that the mixed Buchsbaum-Rim multiplicities associated to a family of modules $E_1, \ldots, E_k$ of finite colength in R^p with respect to N could be described as the Buchsbaum-Rim multiplicity of a R-module generated by a superficial sequence.

Theorem 6.1. *(Risler-Teissier's theorem for modules) Suppose that* $\dim N = \dim R = d$, *and* $E_1, \ldots, E_k$ *submodules of* R^p *of finite colength. Let* $d_1, \ldots, d_k \in \mathbb{N}$ *with* $d_1 + \cdots + d_k = d+p-1$. *Let* $h_1, \ldots, h_{d+p-1}$ *be any superficial sequence for* $E_1, \ldots, E_1, \ldots, E_k, \ldots, E_k$ *with respect to* N, *with each* E_i *listed* d_i *times, and each* $w(h_i)$ *not in any minimal prime ideal over* $w(h_1), \ldots, w(h_{i-1})$. *Then,*

$$e(E_1^{[d_1]}, \ldots, E_k^{[d_k]}; N) = e_{BR}(h_1, \ldots, h_{d+p-1}; N).$$

Theorem 5.4 immediately gives the following result which says that the mixed Buchsbaum-Rim multiplicities associated to a family of modules $E_1, \ldots, E_k$ of finite colength in R^p with respect to N could be described as the Buchsbaum-Rim multiplicity of a R-module generated by a joint reduction.

Theorem 6.2. *(Rees's mixed multiplicity theorem for modules) Suppose that* $\dim N = \dim R = d$, *and* $E_1, \ldots, E_k$ *submodules of* R^p *of finite colength such that the ideals* $\mathcal{I}_i$ *of* $G = Sym(R^p)$ *generated by* $\mathcal{R}_1(E_i)$ *have positive height in* G. *Let* $d_1, \ldots, d_k \in \mathbb{N}$ *with* $d_1 + \cdots + d_k = d + p - 1$. *Let* $h_1, \ldots, h_{d+p-1}$ *be any joint reduction for* $E_1, \ldots, E_1, \ldots, E_k, \ldots, E_k$ *with respect to* N, *with each* E_i *listed* d_i *times. Then,*

$$e(E_1^{[d_1]}, \ldots, E_k^{[d_k]}; N) = e_{BR}(h_1, \ldots, h_{d+p-1}; N).$$

Remark 6.3. *If* $\dim R = d > 0$ *and* E *is a submodule of* R^p *of finite colength then the ideal* $\mathcal{I}$ *of* $G = Sym(R^p)$ *generated by* $\mathcal{R}_1(E)$ *has positive height in* G.

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