UNIVERSIDADE DE SÃO PAULO

Instituto de Ciências Matemáticas e de Computação

ISSN 0103-2577

HORO-TIGHT CIRCLES IN HYPERBOLIC SPACE

MARCELO BUOSI
SHYUICHI IZUMIYA
MARIA APARECIDA SOARES RUAS

Nº 302

NOTAS

Série Matemática

São Carlos – SP
Out./2008

SYSNO 1698356
DATA ___/___/___
ICMC - SBAB

Horo-tight circles in hyperbolic space

Marcelo Buosi
Shyuichi Izumiya
Maria Aparecida Soares Ruas

October 15, 2008

Abstract

We define a family of real valued functions on the Poincaré model $\mathbb{D}^m$ of hyperbolic space parametrized by $\mathbb{S}^{m-1}$. By means of such functions we provide an elementary proof that horo-tightness and tightness are equivalent properties in the class of immersions from S^1 into hyperbolic space.

2000 Mathematics Subject classification: 53A35, 53C42.

Key Words and Phrases. Horospherical Geometry, Horo-tight immersion

1 Introduction

What are the horo-tight immersions of spheres? This question was proposed by Thomas E. Cecil and Patrick J. Ryan in [7, pg 236]. We address this paper to the simplest case of that question, namely the horo-tight immersions of a simple closed curve. The concept of horo-tightness was introduced in [6], whose main subjects are tight and taut immersions into hyperbolic space. One may regard tightness, tautness and horo-tightness as generalizations of Euclidean tightness. An immersion into $\mathbb{R}^m$ is called Euclidean tight if every non-degenerate linear height function has the minimum number of critical points required by the Morse inequalities. The linear height functions in any Euclidean space are parametrized by the sphere of unit vectors. On the other hand, in hyperbolic space there are three types of spheres, namely, equidistant surfaces, spheres and horospheres. Such trichotomy gives rise to the concepts of tightness, tautness and horo-tightness respectively.

Moreover, it has been discovered a new geometry on hyperbolic space which is called "Horospherical Geometry"recently [9, 1, 10, 11]. This geometry regards that horospheres are the only totally umbilic flat surfaces and other spheres and hyperplanes are not flat. The curvature in this geometry has very nice properties such that the Gauss-Bonnet type theorem and the Chern-Lashof type theorem hold [9, 1]. Therefore the horo-tightness of submanifolds in hyperbolic space is also a quite interesting property from this point of view.

In this paper we define a family of functions in the Poincaré model, which measures the Euclidean radii of the hyperhorospheres tangent to its boundary at a fixed point. This is an interesting family of functions, since we can characterize horo-tightness by requiring that every non-degenerate function in the family has the minimum number of critical points.

The main result (Theorem 3.4), states that an immersion of a simple closed curve into the hyperbolic space is horo-tight if and only if it is tight. In this paper we only give the proof of the main results for codimension greater than one case. In this case, it then follows that tightness, tautness and horo-tightness are all equivalent concepts in the class of such immersions. Moreover, it has been known in ([6], Theorem 4.2) that tight immersions of spheres in higher codimensional hyperbolic space are only metric spheres. However there are examples of (Euclidean) ellipses in hyperbolic plane which are horo-tight (cf., [6]). These examples indicate that the situation of the codimension one case is quite different from that of the higher codimensional case. We give a concrete example of such an ellipse in §4. The codimension one case has been shown in more general setting in [2].

2 Notations and definitions

In this section we prepare some notations and definitions which we will use for the proof of the main theorem. A Morse function φ on a manifold M is called a *perfect function* if

1. $M_r(\varphi) = \{x \in M \mid \varphi(x) \leqslant r\}$ is compact for all real r,
2. there exists a field $\mathbf{F}$ such that for all $r \in \mathbb{R}$ and all integers k, the number of critical points of φ of index k which lie in $M_r(\varphi)$ is equal to the kth Betti number of $M_r(\varphi)$ over $\mathbf{F}$.

As observed in [6] there are manifolds which do not admit perfect functions.
Let $\mathbb{R}^{m+1} = \{(x_0, x_1, \dots, x_m) \mid x_i \in \mathbb{R}\}$ be an $(m+1)$-dimensional vector space. For any $x = (x_0, x_1, \dots, x_m)$ and $y = (y_0, y_1, \dots, y_m)$ the *Lorentz-Minkowski scalar product* of x and y is defined by $\mathfrak{b}(x, y) = -x_0y_0 + x_1y_1 + \cdots + x_my_m$. The pair $(\mathbb{R}^{m+1}, \mathfrak{b})$ is called the $(m+1)$-*Lorentz-Minkowski space* and is denoted by $\mathbb{R}^{m+1}_1$. A non-zero vector $x \in \mathbb{R}^{m+1}_1$ is called spacelike, timelike or lightlike if $\mathfrak{b}(x, x) > 0, \mathfrak{b}(x, x) < 0$ or $\mathfrak{b}(x, x) = 0$ respectively.
We use two models of hyperbolic space, $\mathbb{H}^m$ and $\mathbb{D}^m$. In the space $\mathbb{R}^{m+1}$, $\mathbb{H}^m$ is given by

$$\mathbb{H}^m = \{x \in \mathbb{R}^{m+1}_1 \mid \mathfrak{b}(x, x) = -1, x_0 \geqslant 1\},$$

on which $\mathfrak{b}$ restricts to a Riemannian metric of constant sectional curvature -1. Let $\mathbb{D}^m$ be the open unit ball at the origin of $\mathbb{R}^m = \{0\} \times \mathbb{R}^m$. The stereographic projection $P : \mathbb{D}^m \to \mathbb{H}^m$ given by

$$P(x) = -e_0 + 2\frac{x + e_0}{1 - \mathfrak{b}(x, x)}$$

induces on $\mathbb{D}^m$ a Riemannian metric $\mathfrak{g}$ such that $(\mathbb{D}^m, \mathfrak{g})$ is isometric to $\mathbb{H}^m$. This is the well known Poincaré ball model.
We now define spheres of spacelike and lightlike vectors in $\mathbb{R}^{m+1}_1$: the *de Sitter m-space* $\mathbb{S}^m_1 = \{x \in \mathbb{R}^{m+1}_1 \mid \mathfrak{b}(x, x) = 1\}$ and the *lightcone $(m-1)$-sphere* $\mathbb{S}^{m-1}_+ = \{x \in \mathbb{R}^{m+1}_1 \mid \mathfrak{b}(x, x) = 0, x_0 = 1\}$.

An immersion $f : M \longrightarrow \mathbb{H}^m$ is called *horo-tight* if every Morse function $\mathcal{H}_v(x) = -\mathfrak{b}(f(x), v)$, $v \in \mathbb{S}^{m-1}_+$, is perfect. The function $L_h(x) = \ln(-\mathfrak{b}(x, v))$, $x \in \mathbb{H}^m$ measures the distance from x to the hyperhorosphere $h = \{x \in \mathbb{H}^m \mid \mathfrak{b}(x, v) = -1\}$.
An immersion $f : M \longrightarrow \mathbb{H}^m$ is called tight if, for every $\sigma \in \mathbb{S}^m_1$, the function

$$x \mapsto L_\pi(x) = \sinh^{-1}(-\mathfrak{b}(x, \sigma))$$

is either degenerate or a perfect function. Recall that $L_\pi(x)$ is the distance from x to the hiperplane $\pi = \{x \in \mathbb{H}^m \mid \mathfrak{b}(x, \sigma) = 0\}$.
An immersion $f : M \longrightarrow \mathbb{H}^m$ is called taut if, for every $p \in \mathbb{H}^m - f(M)$, the function

$$x \mapsto L_p(x) = (\cosh^{-1}(-\mathfrak{b}(x, p)))^2$$

is either degenerate or a perfect function. Recall that $L_p(x)$ is the square of the distance in $\mathbb{H}^m$ from p to x. The above three concepts are invariant under isometries of $\mathbb{H}^m$, this follows from the characterization of such isometries, see for instance [12]. For an immersion $f : M \longrightarrow \mathbb{H}^m$ of a compact manifold, Theorem 5.1 of [6] says that tightness or tautness implies horo-tightness.
We now consider the Poincaré model.

Definition 2.1 *An immersion $f : M \longrightarrow \mathbb{D}^m$ is horo-tight if $P \circ f : M \longrightarrow \mathbb{H}^m$ is horo-tight.*

Let $\mathbb{S}^{m-1}$ denote the sphere of unit vectors at the origin in $\mathbb{R}^m$. For $w \in \mathbb{S}^{m-1}$, let $\mathcal{R}_w : \mathbb{D}^m \longrightarrow \mathbb{R}$ be the function defined by

$$R_w(p) = 1 - \frac{1 - \langle p, p\rangle}{2(1 - \langle p, w\rangle)} \tag{2.1}$$

where $\langle\, ,\rangle$ is the Euclidean inner product. Geometrically $\mathcal{R}_w(p)$ is the Euclidean radius of the unique hypersphere containing p which is tangent to $\mathbb{S}^{m-1}$ at w. See Figure 1. We consider that R_w a real-valued function on $\mathbb{D}^m$. Therefore, we have the following proposition.

Proposition 2.2 *For any real number $0 < c < 1$, $R_w^{-1}(c)$ is a hyperhorosphere.*

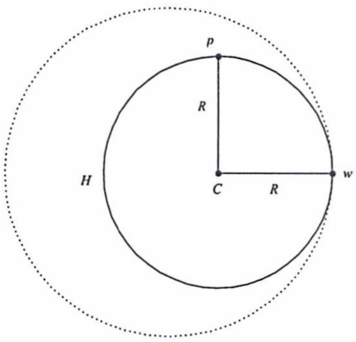

Figure 1: H is the unique hypersphere containing p which is tangent to $\mathbb{S}^{m-1}$ at w. Its Euclidean radius is R.

Under the above notations we have the following lemma.

Lemma 2.3 *Let $f : S^1 \longrightarrow \mathbb{D}^m$ be a horo-tight immersion. Then for any $w \in \mathbb{S}^{m-1}$ either $\mathcal{R}_w \circ f$ is constant or the equation $\mathcal{R}_w(f(x)) = c$ has at most two roots.*

3 Horo-tight immersions of S^1

Let S^1 denote a manifold homeomorphic to the unit circle. In this section we characterize the horo-tight immersions of S^1 into hyperbolic space. As the next lemma says, we can work with embeddings $S^1 \hookrightarrow \mathbb{H}^m$.

Lemma 3.1 *Every horo-tight immersion $f : S^1 \longrightarrow \mathbb{H}^m$ is an embedding.*

Proof. Suppose $f(x_1) = f(x_2) = p$ for two distinct points in S^1. Let $\mathbb{R}_w$ be a Morse function such that the hyperhorosphere $h_s = \{y \in \mathbb{H}^m \mid R_w(y) = s\}$ intersects $f(S^1)$ at p and q, $p \neq q$. It follows that R_w has at least three critical points. A contradiction. □

We treat separately the one codimensional case of the main result. For the case $m = 2$ (i.e, codimension one case), we have shown a more general assertion which implies the following proposition (cf., [2]).

Proposition 3.2 *An immersion $f : S^1 \longrightarrow \mathbb{D}^2$ is horo-tight only if it is tight.*

For the proof of the main theorem in the case when $m \geq 3$, we need the following lemma.

Lemma 3.3 *Let $f : S^1 \longrightarrow \mathbb{D}^m, m \geqslant 3$, be an immersion whose image contains the origin of $\mathbb{R}^m$. Then there exists a hyperhorosphere which intersects $f(S^1)$ in at least three points.*

Proof. We have to find $w \in \mathbb{S}^{m-1}$ and $0 < c < 1$ such that the equation $R_w(x) = c$ has three distinct roots. Since $R_w(\mathbf{0}) = 1/2$ for all $w \in \mathbb{S}^{m-1}$ and $\mathbf{0} \in f(S^1)$, it is sufficient to determine non-zero vectors $f(x), f(y)$ such that the linear system

$$\begin{cases} \langle f(x), w\rangle = \langle f(x), f(x)\rangle \\ \langle f(y), w\rangle = \langle f(y), f(y)\rangle \end{cases} \tag{3.1}$$

has a solution in $\mathbb{S}^{m-1}$. Let $\pi_u \subset \mathbb{R}^m$ be the hyperplane given by the equation $\langle X, u\rangle = \langle u, u\rangle$. For small enough $f(x)$, the hyperplane $\pi_{f(x)}$ contains a non-zero vector $f(y) \neq f(x)$. Thus any $w \in \pi_{f(x)} \cap \pi_{f(y)}$ is a solution of the system (3.1). $\square$

We now prove the main result.

Theorem 3.4 *An immersion $f : S^1 \longrightarrow \mathbb{D}^m$ is horo-tight if and only if it is tight.*

Proof. We have shown that horo-tightness of $S^1 \hookrightarrow \mathbb{D}^2$ implies tightness, so let us assume $m \geqslant 3$. We will actually prove that f is taut and tight. In view of Theorem 2.1 of [6], it is enough to show that f is taut in the Euclidean sense.

For any point $p \in \mathbb{H}^m$ one can find an isometry $\Phi : \mathbb{H}^m \longrightarrow \mathbb{H}^m$ such that $\Phi(p) = e_0$, see [12]. Hence we may assume that $\mathbf{0} \in f(S^1)$ and thus, by Lemmas 3.3 and 2.3 there exists a hyperhorosphere h which contains $f(S^1)$. For almost all hyperplanes π in $\mathbb{R}^m$ we have a hyperhorosphere h' such that $\pi \cap h = h' \cap h$. Since f is horo-tight it is also tight in the Euclidean sense. The result now follows from Theorem 2.1 of [3]. The converse is a particular case of Theorem 5.1 of [6]. $\square$

From the above proof we see that tautness, tightness, and horo-tightness are equivalent concepts in the class of immersions from S^1 into $\mathbb{D}^m, m \geqslant 3$. Whether the above results hold for higher dimensional spheres is an interesting question. For the codimension one case, we have already characterized horo-tightness in [2]. For the higher codimensional case, it is still an open problem.

4 Example

In this section we give a concrete example which indicates the difference between horo-tightness and tautness in the case when the codimension is one.

We begin with the following result.

Proposition 4.1 *An embedding $f : S^1 \to \mathbb{H}^2$ is horo-tight if and only if its geodesic curvature κ_g satisfy $|\kappa_g| \geqslant 1$ on S^1.*

The proof follows from [6, Theorem 4.3] and Proposition 3.2.

In order to compute the curvature of a curve in $\mathbb{H}^2$ we need some preliminary. For any $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3_1$, the pseudo vector product of x and y is defined as follows:

$$x \wedge y = \begin{vmatrix} -e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

Let $\gamma : I \longrightarrow \mathbb{H}^2$ be a curve parametrized by arc-length, ie, $\|t(s)\| = 1, \forall s \in I$, where $t(s) = \gamma'(s)$. We now set a unit vector $e(s) = \gamma(s) \wedge t(s)$. Then we have a pseudo-orthonormal frame $\{y(s), t(s), e(s)\}$ along γ. By the similar arguments as those as in the ordinary Frenet-Serret formula for the Euclidean space curve, the following Frenet-Serret type formula holds [8]:

$$\left\{ \begin{array}{rcl} \gamma'(s) & = & t(s) \\ t'(s) & = & \gamma(s) + \kappa_g(s)e(s) \\ e'(s) & = & -\kappa_g(s)t(s) \end{array} \right.$$

where $\kappa_g(s)$ is the geodesic curvature of the curve γ in $\mathbb{H}^2$ which is given by

$$\kappa_g(s) = \det(\gamma(s), t(s), t'(s)).$$

If γ is not parametrized by arc-length then

$$\kappa_g(u) = \frac{\det(\gamma(u), t(u), t'(u))}{\|\gamma'(u)\|^3}.$$

Now we can show that the ellipse in $\mathbb{D}^2$ given by $f(u) = (0.4\cos(u), 0.3\sin(u))$ is horo-tight. By Propösittion 4.1, it is enough to show that the curvature k of $\gamma = P \circ f$ satisfy $|k| \geqslant 1$ in S^1, where P is the stereographic projection $P : \mathbb{D}^2 \longrightarrow \mathbb{H}^2$. In the Figure 2 we plot the graph of the curvature of γ and we see that it satisfies the required inequality.

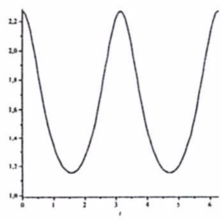

Figure 2: Graph of the geodesic curvature of γ.

References

[1] M. Buosi, S. Izumiya and M. A. S. Ruas, *The total absolute horospherical curvature of submanifolds in hyperbolic space*, to appear in Advances in Geometry.

[2] M. Buosi, S. Izumiya and M. A. S. Ruas, *Horo-tight hyperspheres in hyperbolic space*, preprint (2008).

[3] S. Carter and A. West, *Tight ant taut immersions*, Proc. London Math. Soc, (3) **25** (1972), 701-720.

[4] T. E. Cecil, *A characterization of metric spheres in hyperbolic space by Morse theory*, Tohôku Math. J., **26** (1974), 341-351.

[5] T. E. Cecil and P. J. Ryan, *Distance functions and umbilic submanifolds of hyperbolic space*, Nagoya Math. J., **74** (1979), 67-75.

[6] T. E. Cecil and P. J. Ryan, *Tight ant taut immersions into hyperbolic space*, J. London Math. Soc., **19** (1979), 561-572.

[7] T. E. Cecil and P. J. Ryan, *Tight ant taut immersions of manifolds*, Research Notes in Mathematics, **107** (1985).

[8] S. Izumiya, D. Pei, T. Sano and E. Torii, *Evolutes of hyperbolic plane curves*, Acta. Math. Sinica. **20** (2004) 543-550.

[9] S. Izumiya and M. C. Romero Fuster, *The horospherical Gauss-Bonnet type theorem in hyperbolic space*, J. of Mathematical Society of Japan, **58** (2006), 965-984

[10] S. Izumiya, *Horospherical geometry in hyperbolic space*, in Noncommutativity and Singularities-Proceedings of French-Japanese symposia held at IHES in 2006, to appear in Advanced Studies in Pure Mathematics.

[11] S. Izumiya, K. Saji and M. Takahashi, Horospherical flat surfaces in Hyperbolic 3-space, Hokkaido University preprint series **838**, (2007)

[12] R. G. Ratcliffe, *Foundations of hyperbolic manifolds*, Graduate Texts in Math., **149** (1994).

M. Buosi
Universidade Federal do Espírito Santo,
Departamento de Matemática
Av. Fernando Ferrari, 514, Goiabeiras, 29075-910, Vitória, ES, Brazil.
e-mail:pmarpceplo@yahoo.com.br

S. Izumiya
Department of Mathematics,
Hokkaido University,
Sapporo 060-0810, Japan.
e-mail:izumiya@math.sci.hokudai.ac.jp

M. A. S. Ruas
Instituto de Ciências Matemáticas e de Computação
Universidade de São Paulo, Departamento de Matemática
Caixa Postal 668 13560-970, São Carlos, SP, Brazil.
e-mail:maasruas@icmc.usp.br

NOTAS DO ICMC

SÉRIE MATEMÁTICA

301/08 ARRIETA, J.M.; CARVALHO, A.N.; LOZADA-CRUZ, G. – Dynamics in dumbbell domains III. Continuity of attractors.

300/08 ARRIETA, J.M.; CARVALHO, A.N.; LOZADA-CRUZ, G. – Dynamics in dumbbell domains II. The limiting problem.

299/08 DEMUNER, D.P.; FEDERSON, M.; GUTIERREZ, C. – The Poincaré-Bendixson theorem on the Klein bottle for continous vector fields.

298/08 BONOTTO, E.M.; GIMENES, L.P.; FEDERSON, M.; – Oscillation for a second-order neutral differential equation with impulses.

297/08 CALLEJAS-BEDREGAL, R.; JORGE PÉREZ, V.H. – Rees's mixed multiplicity theorem for modules.

296/08 BONOTTO, E.M. – LaSalle's theorems in impulsive semidynamical systems.

295/08 CHALLAPA, L.S.; RUAS, M.A.S. – Index of an implicit differential equations.

294/08 BUOSI, M.; IZUMIYA, S.; RUAS, M.A.S. – Total absolute horospherical curvature of submanifolds in hyperbolic space.

293/08 BAROTTO, E.M.; FEDERSON, M.; MULDOWNEY, P. – A Feynman-kac solution to an impulsive equation of schrödinger type.

292/08 RODRIGUES, H.M.; SOLÀ-MORALES, J. – A note on the relationship between spectral radius and norms of bounded linear operators.