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Mixed multiplicities of arbitrary modules

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Abstract

Let $(R, \mathfrak{m})$ be a Noetherian local ring; let two multi-graded R-algebras $G = \oplus_{\mathbf{r} \in \mathbb{N}^q} G_{\mathbf{r}}$ and $G' = \oplus_{\mathbf{r} \in \mathbb{N}^q} G'_{\mathbf{r}}$ with $G' \subseteq G$, that, as usual, are generated as algebras by finitely many elements of degree one; fix M a finitely generated multi-graded G-module and define a G'-module M' as $M' = \oplus_{\mathbf{r} \in \mathbb{N}^q} G'_{\mathbf{r}} M_{\mathbf{0}}$. Let $J \subseteq G_{d,0,\ldots,0}$ be a R-submodule of finite colength. We first define the notion of mixed multiplicities and also the notion of (FC)-sequences associated with (J, G', G, M', M). Then we express mixed multiplicities as j^{th} associated Buchsbaum-Rim multiplicity of a pair $(J, M^{\dagger})$, where $M^{\dagger}$ is a suitable graded module obtained through a quotient of M by a (FC)-sequence. Hence, we extend some important results of [15] to multi-graded algebras. As an application, we get formulas for the mixed multiplicities of arbitrary submodules of a free module, which in some sense generalize similar formulas obtained by Kirby and Rees in [5] and by the authors in [2] for modules of finite colength in a free module.

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1 Introduction

Mixed multiplicities of finitely many zero-dimensional ideals were first defined by Risler and Teissier in [12] and they proved that they could be described as the usual Hilbert-Samuel multiplicity of the ideal generated by an appropriated superficial sequence. This result of Risler and Teissier was later generalized by Rees in [10] who first introduced the notion of joint reduction for a family of ideals and proved that the mixed multiplicities of a family of zero-dimensional ideals could be described as the Hilbert-Samuel multiplicity of the ideal generated by a suitable joint reduction. This Theorem of Rees is known as Rees's mixed multiplicity theorem and it is a crucial result in the theory of mixed multiplicities for zero-dimensional ideals. In order to generalize the above results to the case where not all the ideals are zero-dimensional, Viêt in [15] introduced the notion of (FC)-sequences and showed that mixed multiplicities of a family of arbitrary ideals could be described as the Hilbert-Samuel multiplicity of the ideal generated by a suitable (FC)-sequence. We refer the reader to [13] for a survey on mixed multiplicities of ideals.

The notion of mixed multiplicities for a family $E_1, \dots, E_k$ of R-submodules of R^p of finite colength, where R is a local Noetherian ring, have been described in a purely algebraic form by Kirby and Rees in [5] and in an algebro-geometric form by Kleiman and Thorup in [7] and [8]. The results of Risler and Teissier in [12] and of Rees in [10] where generalized for modules in [5] and [2], where the mixed multiplicities for $E_1, \dots, E_k$ are described as the Buchsbaum-Rim multiplicity of a module generated by a suitable joint reduction of $E_1, \dots, E_k$.

In this work we extend the notion of mixed multiplicities to a family $F, E_1, \dots, E_k$ of R-submodules of R^p with $F \subseteq R^p$ of finite colength. We do this in a more general context. As in the work of Kirby and Rees in [5], we work in the context of multi-graded modules. In order to describe this construction we need to introduce some notations.

Fix $(R, \mathfrak{m})$ an arbitrary Noetherian local ring; fix two multi-graded R-algebras $G = \oplus_{\mathrm{r} \in \mathbb{N}^q} G_{\mathrm{r}}$ and $G' = \oplus_{\mathrm{r} \in \mathbb{N}^q} G'_{\mathrm{r}}$ with $G' \subseteq G$, that, as usual, are generated as algebras by finitely many elements of degree one; fix M a finitely generated multi-graded G-module and define a G'-module M' as $M' = \oplus_{\mathrm{r} \in \mathbb{N}^q} G'_{\mathrm{r}} M_{\mathbf{0}}$. Let J be a finitely generated R-submodules of $G_{d\delta(1)}$ of finite colength and assume that $G'_{\delta(1)} = G_{\delta(1)}$. Set $G'' \subseteq G'$ and $M'' \subseteq M'$ by

$$G'' := \oplus_{\mathbf{r}\in\mathbb{N}^q} J^{r_1} G'^{r_2}_{\delta(2)} \cdots G'^{r_q}_{\delta(q)} \text{ and } M'' := \oplus_{\mathbf{r}\in\mathbb{N}^q} J^{r_1} G'^{r_2}_{\delta(2)} \cdots G'^{r_q}_{\delta(q)} M_{\mathbf{0}}.$$

We denote by $\mathcal{I}(G')$, or just $\mathcal{I}$ if no confusion arise, the ideal of G generated by $G'_{\delta(1)} \cdots G'_{\delta(q)}$, where $\delta(i) = (\delta(i,1), \ldots, \delta(i,q))$, where $\delta(i,j) = 1$ if $i = j$ and 0 otherwise, and we assume that $\mathcal{I}$ is not contained in $\sqrt{\mathrm{Ann} M}$. Set $N = 0_M : \mathcal{I}^\infty$, $N' = N \cap M'$, $M^* := M/N$ and $M'^* := M'/N'$.

Consider the function

$$h(n, \mathbf{r}) := \ell\left(\frac{M'_{\mathbf{r}+dn\delta(1)}}{J^n M'_{\mathbf{r}}}\right).$$

Then we will prove in Proposition 5.1 that $h(n, \mathbf{r})$ is a polynomial of degree $D := \dim(\mathrm{Supp}_{++}(M'^*))$ for all large $(n, \mathbf{r}) \in \mathbb{N}^{q+1}$. If we write the terms of total degree D of this polynomial in the form

$$B(n, \mathbf{r}) = \sum_{s+|\mathbf{k}|=D} \frac{1}{s!\mathbf{k}!} e_{s,\mathbf{k}}(J, G'; M')\mathbf{r}^{\mathbf{k}} n^s.$$

The coefficients $e_{s,\mathbf{k}}(J, G'; M')$ are called the *mixed multiplicity of* (J, G') with respect to M.

The main result of this work is a generalization to this new setting of the results of Viet [15] and Manh and Viet [9]. In order to do so, we build the concept of (FC)-sequence of G' with respect to (M, M') (see Definition 4.2). Precisely we prove the following results, which establish mixed multiplicity formulas by means of Buchsbaum-Rim multiplicity and also the positivity of mixed multiplicities.

Theorem 1.1. *Assume that $\mathcal{I}$ is not contained in $\sqrt{AnnM}$ and let J be a finitely generated R-submodules of $G_{d\delta(1)}$ of finite colength. Let $s, k_1, ..., k_q$ be non-negative integers with sum equal to D. Then*

(i)

$$e_{s,\mathbf{k}}(J, G'; M') = e^{k_1}_{BR}(J; (\overline{M}^*)^{(\delta(1))}),$$

for any sequence $x_1, ..., x_t$ of $t = k_2 + ... + k_q$ elements of G'' satisfying the condition (FC) with respect to (M, M''), consisting of k_2 elements of $G'_{\delta(2)}, \ldots, k_q$ elements of $G'_{\delta(q)}$ where $\overline{M}^* = M/((x_1, ..., x_t)M : \mathcal{I}^\infty)$.

(ii) Assume that $s > 0$. Then $e_{s,\mathbf{k}}(J, G'; M') \neq 0$, if and only if there exist a sequence of $k_2 + \ldots + k_q$ elements of G'' satisfying the condition (FC) with respect to (M, M''), consisting of k_2 elements of $G'_{\delta(2)}, \ldots, k_q$ elements of $G'_{\delta(q)}$.

The most important case for application of the results obtained is the following context. Let $A = \oplus_{n \in \mathbb{N}} A_n$ be a finitely generated standard graded algebra over a Noetherian local ring $(R, \mathfrak{m})$ and $E = \oplus_{n \in \mathbb{N}} E_n$ a finitely generated graded A-module. Set $J \subseteq A_{d_0}$ be a R-submodule of finite colength and $I_1, \ldots, I_s$, with $I_i \subseteq A_{d_i}$ finitely generated R-submodules. We now associate with the sequence of R-modules $I_1, \ldots, I_s$ a pair of graded R-algebras and modules. Define

$$G = \oplus_{(p,\mathbf{r}) \in \mathbb{N}^{s+1}} G_{(p,\mathbf{r})} \quad \text{and} \quad G' = \oplus_{(p,\mathbf{r}) \in \mathbb{N}^{s+1}} G'_{(p,\mathbf{r})}$$

where $G_{(p,\mathbf{r})} := A_{d_1 r_1 + \cdots d_s r_s + p}$ and $G'_{(p,\mathbf{r})} := I_1^{r_1} \cdots I_s^{r_s} A_p$. Notice that in this context $\mathcal{I}(G') = \mathcal{I}_1 \cdots \mathcal{I}_s$, where $\mathcal{I}_i$ denotes the ideal of A generated by I_i. We will assume that $\mathcal{I}(G')$ is not contained in $\sqrt{\mathrm{Ann}(E)}$. Similarly define the graded modules $M_{(p,\mathbf{r})} := E_{d_1 r_1 + \cdots d_s r_s + p}$ and $M'_{(p,\mathbf{r})} := I_1^{r_1} \cdots I_s^{r_s} E_p$. In this context, we have that

$$h(n, p, \mathbf{r}) := \ell \left(\frac{M'_{(p,\mathbf{r}) + d_0 n \delta(1)}}{J^n M'_{(p,\mathbf{r})}} \right) = \ell \left(\frac{I_1^{r_1} \cdots I_s^{r_s} E_{d_0 n + p}}{J^n I_1^{r_1} \cdots I_s^{r_s} E_p} \right).$$

As a consequence of Proposition 5.1 we have that $h(n, p, \mathbf{r})$ is a polynomial of degree $r := \dim(\mathrm{Supp}_{++}(E/0_E : \mathcal{I}(G')^{\infty}))$ for all large $(n, p, \mathbf{r}) \in \mathbb{N}^{s+2}$. If we write the terms of total degree r of the polynomial

$$\ell \left(\frac{I_1^{r_1} \cdots I_s^{r_s} E_{d_0 n + p}}{J^n I_1^{r_1} \cdots I_s^{r_s} E_p} \right)$$

in the form

$$B(n, p, \mathbf{r}) = \sum_{s + j + |\mathbf{k}| = r} \frac{1}{s! j! \mathbf{k}!} e^j (J^{[s]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}; E) n^s p^j \mathbf{r}^{\mathbf{k}}.$$

The coefficients $e^j(J^{[s]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}; E)$ are called the j^{ht} *associated mixed multiplicity of $(J, I_1, \ldots, I_s)$ of the type $(s, k_1, \ldots k_s)$ with respect to E.*

Theorem 1.1 immediately gives the following result

Theorem 1.2. *Let $J \subseteq A_{d_0}$ be an R-submodule of finite colength and $I_1, \dots, I_s$, $I_i \subseteq A_{d_i}$ finitely generated R-submodules. Let $k_0, j, k_1, ..., k_s$ be non-negative integers with sum equal to r. Assume that $r > 0$. Then*

(i)

$$e^j(J^{[k_0]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}; E) = e^j_{BR}(J; \overline{E}^*),$$

for any sequence $x_1, ..., x_t$ of $t = k_1 + ... + k_s$ elements satisfying the condition (FC) with respect to $(J, I_1, \dots, I_s; E)$, consisting of k_1 elements of I_1,...,k_s elements of I_s, where $\overline{E}^* = E/((x_1, ..., x_t) : \mathcal{I}(G')^\infty)$.

(ii) If $k_0 > 0$, then $e^j(J^{[k_0]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}) \neq 0$, if and only if there exist a sequence of $k_1 + ... + k_s$ elements satisfying the condition (FC) with respect to $(J, I_1, \dots, I_s; E)$, consisting of k_1 elements of I_1,...,k_s elements of I_s.

In the case that $\mathrm{ht}(\mathcal{I}(G') + \mathrm{Ann}E/\mathrm{Ann}E) > 0$, there exists positive integers n_0 and $\mathbf{u}$ such that the function

$$\ell\left(\frac{I_1^{r_1} \cdots I_s^{r_s} E_{d_0 n + p}}{J^n I_1^{r_1} \cdots I_s^{r_s} E_p}\right)$$

is, for $n > n_0, \mathrm{r} > \mathbf{u}$ a polynomial of total degree $D := \dim(\mathrm{Supp}_{++}(E))$. We get the following theorem.

Theorem 1.3. *Let $J \subseteq A_{d_0}$ be an R-submodule of finite colength and $I_1, \dots, I_s$, $I_i \subseteq A_{d_i}$ finitely generated R-submodules. Assume $ht(\mathcal{I}(G') + AnnE/AnnE) > 0$. Let $k_0, j, k_1, ..., k_s$ be non-negative integers with sum equal to D, with $k_0 > 0$. Then, if $t = k_1 + ... + k_s < ht(\mathcal{I}(G') + AnnE/AnnE)$ we have that*

$$e^j(J^{[k_0]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}; E) = e^j_{BR}(J; \overline{E}),$$

for any sequence $x_1, ..., x_t$ satisfying the condition (FC) with respect to $(J, I_1, \dots, I_s; E)$ consisting of k_1 elements of I_1,...,k_s elements of I_s where $\overline{E} = E/(x_1, ..., x_t)E$.

As a consequence, we have a result similar to that of Kirby and Rees [5] and Callejas-Bedregal and Perez [2], but in terms of sequences satisfying the condition (FC).

Theorem 1.4. *Suppose that $I_1, \ldots, I_s$ are R-submodules of G_d of finite colength. Let $j, k_1, \ldots, k_s \in \mathbb{N}$ with $j + |\mathbf{k}| = r$. Let $x_1, \ldots, x_r$ be a a weak-(FC)-sequence with respect to $(A_1, I_1, \ldots, I_s)$ consisting of consisting of j elements of A_1, k_1 elements of I_1,...,k_s elements of I_s. Then,*

$$e^j(I_1^{[k_1]}, \ldots, I_s^{[k_s]}) = e_{BR}(x_1, \ldots, x_r).$$

For a precise statement of the above theorems we refer to Theorem 5.10, Theorem 6.2, Theorem 6.3 and Theorem 6.4 respectively.

2 Multigraded algebras

In this section we deal with some general aspects of standard multi-graded structures. We use the following multi-index notation through the remaining of this work. The norm of a multi-index $\mathbf{r} = (r_1, \ldots, r_q)$ is $|\mathbf{r}| = r_1 + \cdots + r_q$ and $\mathbf{r}! = r_1! \cdots r_q!$. If $\mathbf{r}, \mathbf{s}$ are two multi-index then $\mathbf{r}^{\mathbf{s}} = r_1^{s_1} \cdots r_q^{s_q}$ and $\mathbf{r} + \mathbf{s} = (r_1 + s_1, \ldots, r_q + s_q)$. We also use the following notation, $\delta(i) = (\delta(i,1), \ldots, \delta(i,q))$, where $\delta(i,j) = 1$ if $i = j$ and 0 otherwise.

Let $G = \oplus_{\mathbf{r} \in \mathbb{N}^q} G_{\mathbf{r}}$ be a finitely generated multi-graded R-algebra, that, as usual, is generated as algebra by finitely many elements of degree one and $G_{\mathbf{0}} = R$. We write $G_+ = \oplus_{\mathbf{r} \neq \mathbf{0}} G_{\mathbf{r}}$ and $G_{++} = \oplus_{\mathbf{r} > \mathbf{0}} G_{\mathbf{r}}$. An homogeneous prime ideal $\mathfrak{p}$ in G is called relevant if $\mathfrak{p} \supseteq G_{++}$. Denote by $\mathrm{Proj}(G)$ the set of relevant homogeneous prime ideals of G. Observe that $\mathrm{Proj}(G) = \emptyset$ if and only if G_{++} is nilpotent. We consider $\mathrm{Proj}(G)$ as a topological space with the closed subsets $V_{++}(I) = \{\mathfrak{p} \in \mathrm{Proj}(G) | I \subseteq \mathfrak{p}\}$, where $I \subseteq G$ is an homogeneous ideal. Following the definition in [14], we define the relevant dimension of G, denoted by $\mathrm{rdim}(G)$, as follows

$$\mathrm{rdim}(G) = \left\{ \begin{array}{ll} q - 1 & \text{if } \mathrm{Proj}(G) = \emptyset \\ \max\{\dim(G/\mathfrak{p}) | \mathfrak{p} \in \mathrm{Proj}(G)\} & \text{if } \mathrm{Proj}(G) \neq \emptyset. \end{array} \right\}$$

The following result was proved in [3, Lemma 1.2].

$$\dim(\mathrm{Proj}(G)) = \mathrm{rdim}(G) - q.$$

If $M = \oplus_{\mathbf{r} \in \mathbb{N}^q} M_{\mathbf{r}}$ is a finitely generated multi-graded G-module, define the homogeneous support of M as $\mathrm{Supp}_{++}(M) = \{\mathfrak{p} \in \mathrm{Proj}(G) | M_{\mathfrak{p}} \neq 0\}$. Since

$\mathrm{Supp}_{++}(M) = V_{++}(\mathrm{Ann}(M))$, $\mathrm{Supp}_{++}(M)$ is a closed subset of $\mathrm{Proj}(G)$. We define the relevant dimension of M, denoted by $\mathrm{rdim}(M)$, as follows

$$\mathrm{rdim}(M) = \left\{ \begin{array}{ll} q-1 & \text{if } \mathrm{Supp}_{++}(M) = \emptyset \\ \max\{\dim(G/\mathfrak{p}) | \mathfrak{p} \in \mathrm{Supp}_{++}(M)\} & \text{if } \mathrm{Supp}_{++}(M) \neq \emptyset. \end{array} \right\}$$

It is clear from the definition that

$$\dim(\mathrm{Supp}_{++}(M)) = \mathrm{rdim}(M) - q.$$

Given $\mathbf{s} = (s_1, \ldots, s_q) \in \mathbb{N}^q$ define the shifted multi-graded G-module $M(\mathbf{s})$ by $M(\mathbf{s}) := \oplus_{\mathbf{r}\in\mathbb{N}^q} M_{\mathbf{r}+\mathbf{s}}$ and the diagonal graded R-module $M^{(\mathbf{s})}$ by

$$M^{(\mathbf{s})} := \oplus_{n\in\mathbb{N}} M_{(ns_1,\ldots,ns_q)}.$$

Given $x_1, \ldots, x_k$ in G with $x_i \in G_{\mathbf{d}_i}$. Set $\mathbf{d} = \mathbf{d}_1 + \cdots + \mathbf{d}_k$. Define

$$\frac{M}{(x_1, \ldots, x_k)M} := \frac{M(\mathbf{d})}{\Sigma_{i=1}^k x_i M(\mathbf{d} - \mathbf{d}_i)}.$$

3 Buchsbaum-Rim multiplicities

In this section we introduce the notion of Buchsbaum-Rim multiplicities in a generality needed in this work. Fix $(R, \mathfrak{m})$ an arbitrary Noetherian local ring; fix a graded R-algebra $G = \oplus G_n$, that, as usual, is generated as algebra by finitely many elements of degree one; fix J a finitely generated R-submodule of G_d such that $\ell(G_d/J) < \infty$; and fix M a finitely generated graded G-module. Set $r := \dim(\mathrm{Supp}_{++}(M))$. As a function of n, q, the length,

$$h(n,q) := \ell(M_{dn+q}/J^n M_q)$$

is eventually a polynomial in n, q, denoted by $P_{J,M}(n,q)$, of total degree equal to r and the coefficient of $n^{r-j}q^j/(r-j)!j!$ is denoted by $e^j(J,M)$, for all $j = 0, \ldots, r$, and it is called the j^{th} *Associated Buchsbaum-Rim multiplicity of the pair* (J, M). The number $e^0(J,M)$ will be called the *Buchsbaum-Rim multiplicity of the pair* (J,M) and will also be denoted by $e_{BR}(J,M)$. The notion of Buchsbaum-Rim multiplicity for modules goes back to [1] and it was carried out in the above generality in [4], [5], [7], [6], [8] and [11].

4 FC-sequence

In this work, we will link mixed multiplicities of a set of arbitrary modules with j^{th} *Associated Buchsbaum-Rim multiplicity of the pair* (G', G) *with respect to* M. To connect the above multiplicities, we define a (FC)-sequences for multi-graded algebras. The results of this paper will show that a sequence satisfying the conditions (FC) carries important information on mixed multiplicities.

Setup: 1. Fix $(R, \mathfrak{m})$ an arbitrary Noetherian local ring; fix two multi-graded R-algebras $G = \oplus_{\mathbf{r}\in\mathbb{N}^q} G_{\mathbf{r}}$ and $G' = \oplus_{\mathbf{r}\in\mathbb{N}^q} G'_{\mathbf{r}}$ with $G' \subseteq G$, that, as usual, are generated as algebras by finitely many elements of degree one; fix M a finitely generated multi-graded G-module and define a G'-module M' as $M' = \oplus_{\mathbf{r}\in\mathbb{N}^q} G'_{\mathbf{r}} M_{\mathbf{0}}$.

Lemma 4.1. *(Artin-Rees Lemma) Let* $G' \subset G$ *be a multi-graded* R*-algebras and* $M', M'' \subseteq M$ *be multi-graded* G*-modules. Then there exist* $c_1, \ldots, c_q \in \mathbb{N}$ *such that*

$$M''_{\mathbf{r}} \cap M'_{\mathbf{r}} = G'_{\mathbf{r}-\mathbf{c}}(M''_{\mathbf{c}} \cap M'_{\mathbf{c}})$$

for all $\mathbf{r} \geq \mathbf{c}$.

Proof. Let $t_1, \ldots, t_q$ be indeterminates over R and write $G' = \oplus_{\mathbf{r}\in\mathbb{N}^q} G'_{\mathbf{r}} t_1^{r_1} \cdots t_q^{r_q}$ and $M' = \oplus_{\mathbf{r}\in\mathbb{N}^q} M'_{\mathbf{r}} t_1^{r_1} \cdots t_q^{r_q}$. Set $H = \oplus_{\mathbf{r}\in\mathbb{N}^q} (M'_{\mathbf{r}} \cap N_{\mathbf{r}}) t_1^{r_1} \cdots t_q^{r_q}$. Clearly H is a finitely generated multi-graded G'-module. Let c_i be the maximal t_i-degree of elements in a generating set of H. These $c_1, \ldots, c_q$ work, as in the proof of the usual Artin-Rees Lemma. $\square$

We denote by $\mathcal{I}(G')$, or just $\mathcal{I}$ if no confusion arise, the ideal of G generated by $G'_{\delta(1)} \cdots G'_{\delta(q)}$ and we assume that $\mathcal{I}$ is not contained in $\sqrt{\mathrm{Ann} M}$. Set $N = 0_M : \mathcal{I}^\infty$, $N' = N \cap M'$, $M^* := M/N$ and $M'^* := M'/N'$.

Definition 4.2. *We say that an element* $x \in G$ *is an* (FC)*-element of* G' *with respect to* (M, M') *if the exists* $i = 1, \ldots, q$ *and an integer* r'_i *such that*

(FC_1) $x \in G'_{\delta(i)} \setminus \mathfrak{m} G'_{\delta(i)}$, *$x$ is non-nilpotent and*

$$M'_{\mathbf{r}} \cap x M_{\mathbf{r}-\delta(i)} = x M'_{\mathbf{r}-\delta(i)}$$

for all $\mathbf{r} \in \mathbb{N}^q$ *with* $r_i \geq r'_i$.

(FC_2) *x is a filter-regular element with respect to $(\mathcal{I}, M')$, i.e., $0_{M'} : x \subseteq 0_{M'} : \mathcal{I}^\infty$.*

(FC_3) *$rdim(M'/xM' : \mathcal{I}^\infty) = rdim(M'^*) - 1$.*

We call x a weak-(FC)-element of G' with respect to (M, M') if x satisfies the conditions (FC_1) and (FC_2).

A sequence of elements $x_1, \ldots, x_k$, of G is said to be an (FC)-sequence of G' with respect to (M, M') if x_{i+1} is an (FC)-element of G' with respect to $(\overline{M}, \overline{M'})$ for each $i = 1, ..., t-1$, where $\overline{M} = M/(x_1, ..., x_i)M$.

$x_1, \ldots, x_k$, of G is said to be a weak-(FC)-sequence of G' with respect to (M, M') if x_{i+1} is a weak-(FC)-element of G' with respect to $(\overline{M}, \overline{M'})$ for each $i = 1, ..., t-1$.

The following lemma will play a crucial role for showing the existence of (FC)-sequences.

Lemma 4.3. *(*Generalized Rees's Lemma) In the setup (1), let Σ be a finite set of prime ideals not containing $\mathcal{I}$. Then for each $i = 1, \ldots, q$, there exist an element $x_i \in G'_{\delta(i)} \setminus \mathfrak{m}G'_{\delta(i)}$, x_i not contained in any prime ideal in Σ, and a positive integer k_i such that

$$M'_{\mathbf{r}} \cap xM_{\mathbf{r}-\delta(i)} = xM'_{\mathbf{r}-\delta(i)}$$

for all $\mathbf{r} \in \mathbb{N}^q$ with $r_i \geq k_i$.

Proof. Set

$$G(U) = \bigoplus_{\mathbf{r} \in \mathbb{Z}^q} G_{\mathbf{r}} t_1^{r_1} \cdots t_q^{r_q},$$

with $G_{\mathbf{r}} = G_{a_1\delta(1)+\cdots+a_q\delta(q)}$, where $a_i = r_i$ if $r_i \geq 0$ and $a_i = 0$ otherwise. Similarly are defined $G'(U), M(U), M'(U)$, with the conditions $M_{\mathbf{0}} = M'_{\mathbf{0}}$. Set $u_1 = t_1^{-1}, \ldots, u_q = t_q^{-1}$. It is easily seen that $u_1 \cdots u_q$ is nonzero-divisor in $M'(U)$. Furthermore, the set of prime associated with $(u_1 \cdots u_q)^n M'(U)$ is independent of n and so is finite. We divide this set into two subsets: $\mathcal{G}_1$ consisting of those containing $G'_{\delta(i)t_i}$ and $\mathcal{G}_2$ those that do not.

Since $G'_{\delta(i)}/\mathfrak{m}G'_{\delta(i)}$ is a vector space over the infinite field k and the sets Σ, $\mathcal{G}_2$ are both finite, we can choose $x_i \in G'_{\delta(i)} \setminus \mathfrak{m}G'_{\delta(i)}$ such that x_i is not contained in any prime ideal belonging to Σ and $x_i t_i$ is not contained in any prime ideal belonging to $\mathcal{G}_2$.

Set

$$M'(U)_n = \frac{((u_1 \cdots u_q)^n M'(U) : x_i t_i)}{(u_1 \cdots u_q)^n M'(U)}.$$

We will show that $M'(U)_n$ is annihilated by $(G'_{\delta(i)} t_i)^N$ if N is sufficiently large. If $\mathfrak{p} \in \mathrm{Ass}_{G'(U)}(M'(U)_n)$, then there exists $z \in ((u_1 \cdots u_q)^n M'(U) : x_i t_i)$ such that $\mathfrak{p} = ((u_1 \cdots u_q)^n M'(U) : z)$. Since $x_i t_i \in \mathfrak{p}$, we have $\mathfrak{p} \in \mathcal{G}_1$. Consequently, $G'_{\delta(i)} t_i$ is contained in $\mathfrak{p}$. This implies that $G'_{\delta(i)} t_i$ is contained in $\sqrt{\mathrm{Ann}_{G'(U)}(M'(U)_n)}$. Since $G'_{\delta(i)}$ is finitely generated, there exists an integer $N > 0$ such that

$$G'_{N\delta(i)} t_i^N \subset \mathrm{Ann}_{G'(U)}(M'(U)_n).$$

In addition, $M'(U)_n$ is a finitely generated graded $G'(U)$-module, this implies that if r_i is sufficiently large then any element of $M'(U)_n$ of degree $(r_1, \ldots, r_q)$ is zero.

Now, let $B(M')$ denoted the submodule of $M'(U)$ consisting of all finite sums:

$$\sum_{\mathbf{r}} c_{\mathbf{r}} t_1^{r_1} \cdots t_q^{r_q}, \text{ where } c_{\mathbf{r}} \in x_i M_{\mathbf{r}-\delta(i)} \cap M'_{\mathbf{r}}.$$

Then $B(M')$ has a finite generating set consisting of elements of the form $x_i b t_1^{r_1} \cdots t_q^{r_q}$ where $b \in M_{\mathbf{r}-\delta(i)}$. We can find an integer k such that

$$(u_1 \cdots u_q)^k b t_1^{r_1} \cdots t_{i-1}^{r_{i-1}} t_i^{r_i - 1} t_{i+1}^{r_{i+1}} \cdots t_q^{r_q} \in M'(U)$$

for all elements $x_i b t_1^{r_1} \cdots t_q^{r_q}$. Hence

$$B(M') \subset x_i t_i M'(U) : (u_1 \cdots u_q)^k.$$

Suppose that $z t_1^{r_1} \cdots t_q^{r_q} \in B(M')$, where $z \in x_i M_{\mathbf{r}-\delta(i)} \cap M'_{\mathbf{r}}$. We have

$$(u_1 \cdots u_q)^k z t_1^{r_1} \cdots t_q^{r_q} = x_i t_i w,$$

where w is homogeneous element of $M'(U)$ whose i-th degree equal to $r_i - k - 1$. By the first part of proof, w will belong to $(u_1 \cdots u_q)^k M'(U)$ for sufficiently large r_i. Since the generating set of $B(M')$ is finite, we choose a positive integers s_i such that if $r_i \geq s_i$, then $w \in (u_1 \cdots u_q)^k M'(U)$. Hence $w = (u_1 \cdots u_q)^k w'$ for $w' \in M'(U)$. Note that

$$(u_1 \cdots u_q)^k z t_1^{r_1} \ldots t_q^{r_q} = x_i t_i w, \quad (u_1 \cdots u_q)^k z t_1^{r_1} \ldots t_q^{r_q} = x_i t_i (u_1 \cdots u_q)^k w'.$$

Since $(u_1 \cdots u_q)^k$ is non-zero-divisor, $zt_1^{r_1} \dots t_q^{r_q} = x_i t_i w'$ and so $zt_1^{r_1} \dots t_q^{r_q} \in x_i t_i M'(U)$. Therefore, $z \in x_i M_{\mathbf{r}-\delta(i)}$. Hence if $r_i \geq s_i$, then

$$x_i M_{\mathbf{r}-\delta(i)} \cap M'_{\mathbf{r}} \subset x_i M'_{\mathbf{r}-\delta(i)} \subset x_i M_{\mathbf{r}-\delta(i)} \cap M'_{\mathbf{r}}$$

for all non-negative integers $r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_q$. That means for any $r_i \geq s_i$, we get

$$M'_{\mathbf{r}} \cap x_i M_{\mathbf{r}-\delta(i)} = x_i M'_{\mathbf{r}-\delta(i)}$$

for all non-negative integers $r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_q$. □

5 Mixed multiplicities

In this section we define the notion of mixed multiplicities of finitely generated multi-graded G-modules. The main result of this section establish mixed multiplicity formulas by means of Buchsbaum-Rim multiplicity and also determines the positivity of mixed multiplicities.

Setup: 2. Retain the setup (1) and let J be a finitely generated R-submodules of $G_{d\delta(1)}$ of finite colength and assume that $G'_{\delta(1)} = G_{\delta(1)}$.

Consider the function

$$h(n, \mathbf{r}) := \ell\left(\frac{M'_{\mathbf{r}+dn\delta(1)}}{J^n M'_{\mathbf{r}}}\right).$$

Proposition 5.1. *In the setup (1), let $\mathcal{I} = \mathcal{I}(G')$ and $N' := (0_{M'} : \mathcal{I}^\infty)$. Set $D = \dim(Supp_{++}(M'^*))$, Then $h(n, \mathbf{r})$ is a polynomial of degree D for all large $(n, \mathbf{r}) \in \mathbb{N}^{q+1}$.*

Proof. We will prove first the following equality

$$\ell\left(\frac{M'_{\mathbf{r}+dn\delta(1)}}{J^n M'_{\mathbf{r}}}\right) = \ell\left(\frac{M'^*_{\mathbf{r}+dn\delta(1)}}{J^n M'^*_{\mathbf{r}}}\right) \tag{5.1}$$

Notice that

$$\begin{aligned}
\ell\left(\frac{M'^*_{\mathbf{r}+dn\delta(1)}}{J^n M'^*_{\mathbf{r}}}\right) &= \ell\left(\frac{N_{\mathbf{r}+dn\delta(1)} + M'_{\mathbf{r}+dn\delta(1)}}{N_{\mathbf{r}+dn\delta(1)} + J^n M'_{\mathbf{r}}}\right) \\
&= \ell\left(\frac{M'_{\mathbf{r}+dn\delta(1)}}{J^n M'_{\mathbf{r}} + N_{\mathbf{r}+dn\delta(1)} \cap M'_{\mathbf{r}+dn\delta(1)}}\right).
\end{aligned}$$

By Artin-Rees Lemma there exist integers $c_1, \ldots, c_q \in \mathbb{N}$ such that

$$N_{\mathbf{r}+dn\delta(1)} \cap M'_{\mathbf{r}+dn\delta(1)} = G'_{\mathbf{r}+dn\delta(1)-\mathbf{c}}(N_{\mathbf{c}} \cap M'_{\mathbf{c}}) \subseteq G'_{\mathbf{r}+dn\delta(1)-\mathbf{c}} N_{\mathbf{c}},$$

for all $\mathbf{r} \geq \mathbf{c}$. For all large t, we have $N\mathcal{I}^t = 0$, it follows that

$$N_{\mathbf{r}+dn\delta(1)} \cap M'_{\mathbf{r}+dn\delta(1)} = 0$$

and hence we prove equality (5.1).

Now, by Lemma [7, Lemma 4.3],

$$h(n, \mathbf{r}) := \ell\left(\frac{M'^{*}_{\mathbf{r}+dn\delta(1)}}{J^n M'^{*}_{\mathbf{r}}}\right)$$

is a polynomial of degree D, where $D := \dim(\mathrm{Supp}_{++}(M'^{*}))$ for all large $n, \mathbf{r}$. Therefore

$$\ell\left(\frac{M'_{\mathbf{r}+dn\delta(1)}}{J^n M'_{\mathbf{r}}}\right)$$

is a polynomial of degree D) for all large $\mathbf{r}$. □

If we write the terms of total degree $D = \dim(\mathrm{Supp}_{++}(M'^{*}))$ of the polynomial

$$\ell\left(\frac{M'_{\mathbf{r}+dn\delta(1)}}{J^n M'_{\mathbf{r}}}\right)$$

in the form

$$B(n, \mathbf{r}) = \sum_{s+|\mathbf{k}|=D} \frac{1}{s!\mathbf{k}!} e_{s,\mathbf{k}}(J, G'; M')\mathbf{r}^{\mathbf{k}} n^s.$$

The coefficients $e_{s,\mathbf{k}}(J, G'; M')$ are called the **mixed multiplicity of** (J, G') with respect to M.

Remark 5.2. *Follows by equation (*5.1*) that*

$$e_{s,\mathbf{k}}(J, G'; M') = e_{s,\mathbf{k}}(J, G'; M'^{*})$$

for all $s + |\mathbf{k}| = D$.

Lemma 5.3. *Assume that* $\mathcal{I}$ *is not contained in* $\sqrt{AnnM}$. *Then* *(i)* $e_{s,j\delta(1)}(J,G';M') \neq 0$ *and* $e_{s,j\delta(1)}(J,G';M') = e^j_{BR}(J;(M^*)^{\delta(1)})$.

(ii) $e_{s,j\delta(1)}(J,G';M') = e^j_{BR}(J;M^{\delta(1)})$ *if* $ht_{G^{\delta(1)}}(Ann((0_M:\mathcal{I}^\infty)\cap M^{\delta(1)}))$ *is positive.*

Proof. We prove first (i). By Proposition 5.1 there exist a positive integer u such that

$$\ell\left(\frac{M'_{\mathbf{r}+dn\delta(1)}}{J^n M'_{\mathbf{r}}}\right) = \ell\left(\frac{M'^*_{\mathbf{r}+dn\delta(1)}}{J^n M'^*_{\mathbf{r}}}\right)$$

is a polynomial of degree D for all $n, r_1, \ldots, r_q \geq u$. We shall denote the above polynomial by $P(n,\mathbf{r})$. Take $r_2 = \ldots = r_q = u$ and fix the u. Set $H(n,r_1) = P(n,r_1,u,\ldots,u)$. Then we get

$$H(n,r_1) = \sum_{s+j=D} \frac{1}{s!j!}\, e_{s,j\delta(1)}(J,G';M') r_1^j n^s + \ldots.$$

Now set $\overline{M} = G'^u_{\delta(2)} \cdots G'^u_{\delta(q)}(M'^*)^{(\delta(1))}$. Clearly $\overline{M}$ is a finitely generated graded $G'^{(\delta(1))}$-module. Since we assume that $J \subseteq G'_{d\delta(1)} = G_{d\delta(1)}$ we can also consider J as contained in $G^{(\delta(1))}$ in degree d. Set $\varphi(n,r) = \ell\left(\frac{\overline{M}_{dn+r}}{J^n\overline{M}_r}\right)$.

For all $n, r \gg 0$ we have that

$$\varphi(n,r) = \sum_{s+j=D} \frac{1}{s!j!}\, e^j_{BR}(J,\overline{M}) r^j n^s + \ldots.$$

But clearly $\varphi(n,r) = P(n,r,u,\ldots,u)$ and hence

$$e_{s,j\delta(1)}(J,G';M') = e^j_{BR}(J,\overline{M}) \neq 0.$$

Since $\mathcal{I}$ is not contained $\sqrt{\mathrm{Ann}M}$, there exist an element $x \in G'^u_{\delta(2)} \cdots G'^u_{\delta(q)}$ which is not a zero-divisor of M'^*.

Consider the exact sequence

$$0 \longrightarrow x(M'^*)^{(\delta(1))} \longrightarrow \overline{M} \longrightarrow \overline{M}/x(M'^*)^{(\delta(1))} \longrightarrow 0.$$

Thus

$$\mathrm{rdim}(\overline{M}/x(M'^*)^{(\delta(1))}) < \mathrm{rdim}(\overline{M}).$$

Therefore, by the additivity property of the associated Buchsbaum-Rim multiplicities, we have that

$$e^j_{BR}(J,\overline{M}) = e^j_{BR}(J,x(M'^*)^{(\delta(1))}) = e^j_{BR}(J,(M'^*)^{(\delta(1))}) = e^j_{BR}(J,M^{*(\delta(1))}).$$

The proof of (ii): Consider the exact sequence

$$0 \longrightarrow (0_M : \mathcal{I}^\infty) \cap M^{(\delta(1))} \longrightarrow M'^{(\delta(1))} \longrightarrow (M'^*)^{(\delta(1))} \longrightarrow 0.$$

It is easily seen that

$$\mathrm{Ass}_{G^{(\delta(1))}}(M'^{(\delta(1))}) = \mathrm{Ass}_{G^{(\delta(1))}}((0_M : \mathcal{I}^\infty) \cap M^{(\delta(1))}) \cup \mathrm{Ass}_{G^{(\delta(1))}}((M'^*)^{(\delta(1))})$$

and

$$\mathrm{Ass}_{G^{(\delta(1))}}((0_M : \mathcal{I}^\infty) \cap M^{(\delta(1))}) \cap \mathrm{Ass}_{G^{(\delta(1))}}((M'^*)^{(\delta(1))}) = \emptyset.$$

From these facts and since $\mathrm{ht}_{G^{(\delta(1))}}(\mathrm{Ann}(0_M : \mathcal{I}^\infty) \cap M^{(\delta(1))}) > 0$, any prime ideal $\mathfrak{p} \in \mathrm{Ass}_{G^{(\delta(1))}}((0_M : \mathcal{I}^\infty) \cap M^{(\delta(1))})$ is a non-minimal element in $\mathrm{Ass}_{G^{(\delta(1))}}(M'^{(\delta(1))})$. From this follows that

$$\mathrm{rdim}((0_M : \mathcal{I}^\infty) \cap M^{(\delta(1))}) < \mathrm{rdim}(M'^{(\delta(1))}) = \mathrm{rdim}((M'^*)^{(\delta(1))}).$$

Therefore, by the additivity property of the associated Buchsbaum-Rim multiplicities, we have that, $e^j_{BR}(J, M'^{(\delta(1))}) = e^j_{BR}(J, (M'^*)^{(\delta(1))})$. Hence by (i), we get

$$e_{s,j\delta(1)}(J, G'; M) = e^j_{BR}(J, (M'^*)^{(\delta(1))}) = e^j_{BR}(J, M'^{(\delta(1))}) = e^j_{BR}(J, M^{(\delta(1))}).$$

The proof is complete. □

Lemma 5.4. *(i) $x \in G'_{\delta(i)}$, $i \geq 2$, is an (FC_2)-element of G' with respect to (M, M') if and only if $(0_{M'} : x)_{\mathbf{r}} = 0$ for all large $\mathbf{r}$.*

(ii) If $x \in G'_{\delta(i)}$, $i \geq 2$, is an (FC_2)-element of G' with respect to (M, M'). Then $xM'_{\mathbf{r}-\delta(i)} \cong M'_{\mathbf{r}-\delta(i)}$ and hence $xM'^*_{\mathbf{r}-\delta(i)} \cong M'^*_{\mathbf{r}-\delta(i)}$, for all large $\mathbf{r}$.

Proof. (i). Suppose that $x \in G'_{\delta(i)}$, $i \geq 2$, is an (FC)-element of G' with respect to (M, M'). Then $(0_{M'} : x) \subset (0_{M'} : \mathcal{I}^\infty)$, therefore $(0_{M'} : x)_{\mathbf{r}} \subset (0_{M'} : \mathcal{I}^\infty)_{\mathbf{r}}$ for all large $\mathbf{r}$. As there exist positive integers $u_1, \ldots, u_q$ such that

$$(0_{M'} : \mathcal{I}^\infty)_{\mathbf{r}} = G'_{\mathbf{r}-\mathbf{u}}(0_{M'} : \mathcal{I}^n)_{\mathbf{u}}$$

for all $\mathbf{r} \geq \mathbf{u}$. From this fact follows that $(0_{M'} : \mathcal{I}^n)_{\mathbf{r}} = 0$ for all $r_i \geq u_i + n$, $i = 1, .., q$ and n integer such that $(0_{M'} : \mathcal{I}^\infty) = (0_{M'} : \mathcal{I}^n)$.

Now assume that $(0_{M'} : x)_{\mathbf{r}} = 0$ for all large $\mathbf{r}$. It is enough to prove that $(0_{M'} : x)_{\mathbf{u}} \subset (0_{M'} : \mathcal{I}^\infty)_{\mathbf{u}}$ for all $\mathbf{u}$. Indeed, if $a \in (0_{M'} : x)_{\mathbf{u}}$ then

$xG'_{\mathbf{s}}a = 0$ for every $\mathbf{s}$. Hence $G'_{\mathbf{s}}a \subset (0_{M'} : x)_{\mathbf{u}+\mathbf{s}} = 0$ for all large $\mathbf{s}$. It implies that $a \in (0_{M'} : \mathcal{I}^n)$ for all large n. Hence $a \in (0_{M'} : \mathcal{I}^\infty)$. So $(0_{M'} : x)_{\mathbf{s}} \subset (0_{M'} : \mathcal{I}^\infty)_{\mathbf{s}}$ for all $\mathbf{s}$.. Hence $x \in G'_{\delta(i)}$, $i \geq 2$, is an (FC_2)-element of G' with respect to (M, M').

(ii). Consider

$$\lambda_x : M'_{\mathbf{r}-\delta(i)} \to xM'_{\mathbf{r}-\delta(i)}, \; y \mapsto xy.$$

It is clear that λ_x is surjective and $\ker(\lambda_x) = (0_{M'} : x)_{\mathbf{r}-\delta(i)}$. Since $x \in G'_{\delta(i)}$, $i \geq 2$, is an (FC_2)-element of G' with respect to (M, M'). $\ker(\lambda_x) = 0$ for all large $\mathbf{r}$, by item (i). Therefore $xM'_{\mathbf{r}-\delta(i)} \cong M'_{\mathbf{r}-\delta(i)}$, for all large $\mathbf{r}$. $\square$

In the remaining of this work we use the following construction: Given the multi-graded algebras $G' \subseteq G$ and the multi-graded G-module M as in the setup (2) we set $G'' \subseteq G'$ and $M'' \subseteq M'$ by

$$G'' := \oplus_{\mathbf{r}\in\mathbb{N}^q} J^{r_1} G'^{r_2}_{\delta(2)} \cdots G'^{r_q}_{\delta(q)} \text{ and } M'' := \oplus_{\mathbf{r}\in\mathbb{N}^q} J^{r_1} G'^{r_2}_{\delta(2)} \cdots G'^{r_q}_{\delta(q)} M_{\mathbf{0}}.$$

The following proposition will show the existence of weak-(FC)-sequence.

Proposition 5.5. *If $\mathcal{I}$ is not contained in $\sqrt{AnnM}$. For any $i = 1, \ldots, q$, there exist a weak-(FC)-element $x_i \in G'_{\delta(i)}$ of G' with respect to (M, M') and a weak-(FC)-element $y_i \in G''_{\delta(i)}$ of G'' with respect to (M, M'').*

Proof. Set $\mathcal{F} = \mathrm{Ass}(\frac{M}{0_M : \mathcal{I}^\infty})$. Since $\mathcal{I}$ is not contained in $\sqrt{\mathrm{Ann}M}$. We have $\mathcal{F} \neq \emptyset$. it is easily seen that $\mathcal{F}$ is finite and $\mathcal{F} = \{\mathfrak{p} \in \mathrm{Ass}(M) | \mathfrak{p} \not\supseteq \mathcal{I}\}$. By Lemma 4.3, for each $i = 1, \ldots, s$ there exists an element $x_i \in G'_{\delta(i)} \setminus \mathfrak{m}G'_{\delta(i)}$ such that x_i satisfies the condition (FC_1) and $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{F}$. Since $x_i \notin \mathfrak{p}$ for any $\mathfrak{p} \notin \mathcal{F}$, x_i also satisfies the condition (FC_2). Hence x_i is weak-(FC)-element $x_i \in G'_{\delta(i)}$ of G' with respect to (M, M'). The proof of the second statement is analogous. $\square$

Lemma 5.6. *Assume that $\mathcal{I}$ is not contained in $\sqrt{AnnM}$ and let J be a finitely generated R-submodules of $G_{d\delta(1)}$ of finite colength. Suppose $x_i \in G'_{\delta(i)}$ is a (FC)-element of G'' with respect to (M, M''). Then we can replace the condition (FC_2) by the condition*

$$(0_{M'} : x) \subset 0_{M'} : \mathcal{I}^\infty$$

and the condition (FC_3) by the condition

$$rdim\left(\frac{M'}{xM' : \mathcal{I}^\infty}\right) = rdim\left(M'^*\right) - 1.$$

Proof. It is easily seen that if J is a finitely generated R-submodule of $G_{d\delta(1)}$ of finite colength, $\mathcal{I}$ ideal of G generated by $G'_{\delta(2)}\cdots G'_{\delta(q)}$ and $\mathcal{J}$ ideal of G' generated by $JG'_{\delta(2)}\cdots G'_{\delta(q)}$, then $(0_{M'}:\mathcal{J}^\infty)=(0_{M'}:\mathcal{I}^\infty)$ and $(xM':\mathcal{J}^\infty)=(xM':\mathcal{I}^\infty)$. □

Viêt in [16] introduced the concept of generalized joint reduction of ideals in local rings, which is a generalization of the notion of joint reduction given by Rees in [10]. This notion was lately extended to modules coefficients by Manh and Viêt in [9]. Now, we extend the notion of generalized joint reduction of module coefficient to multi-graded modules.

Definition 5.7. *A set $(\mathfrak{I}_1,\ldots,\mathfrak{I}_t)$ of R-modules, with $\mathfrak{I}_i \subseteq G_{\delta(i)}$, $i=1,\ldots,t\leq q$, is called a generalized joint reduction of G' with respect to M if*

$$M'_{\mathbf{r}+\mathbf{1}}=\sum_{j=1}^{t}\mathfrak{I}_jM'_{\mathbf{r}-\delta(j)+\mathbf{1}}$$

for all large $r_1,\ldots,r_q$.

The relationship between maximal weak-(FC)-sequences and generalized joint reduction was determined in [16, Theorem 3.4] and [9, Theorem 2.9] in local rings. We extend this result to multi-graded modules as follows.

Lemma 5.8. *Assume that $\mathcal{I}$ is not contained in $\sqrt{AnnM}$ and let J be a finitely generated R-submodules of $G_{d\delta(1)}$ of finite colength. Suppose*

$$\mathfrak{I}_1=(x_{11},\ldots,x_{1m})\subset G_{\delta(1)},$$

$$\mathfrak{I}_2=(x_{21},\ldots,x_{2n})\subset G_{\delta(2)},$$

$$\cdots$$

$$\mathfrak{I}_t=(x_{t1},\ldots,x_{tp})\subset G_{\delta(t)}$$

and $x_{11},\ldots,x_{1m},x_{21},\ldots,x_{2n}\ldots x_{t1},\ldots,x_{tp}$ is a maximal weak-(FC)-sequence of G'' with respect to (M,M'') in $\cup_{i=1}^{q}G'_{\delta(i)}$. Then the following statements hold.

(i) For any $k\leq t$, we have

$$(\mathfrak{I}_1,\mathfrak{I}_2,\ldots\mathfrak{I}_k)M_{\mathbf{r}-\delta(1)-\cdots-\delta(k)+\mathbf{1}}\cap M'_{\mathbf{r}+\mathbf{1}}=\Sigma_{j=1}^{k}\mathfrak{I}_jM'_{\mathbf{r}+\mathbf{1}-\delta(j)},$$

for all large $r_1,\ldots,r_q$.

(ii) $\mathfrak{I}_1, \mathfrak{I}_2, \ldots \mathfrak{I}_k$ *is a generalized joint reduction of* G' *with respect to* (M, M').

Proof. The proof of (i): Using induction $k \leq t$. For $k = 1$, we will show that

$$(x_{11}, \ldots, x_{1i})M_{\mathbf{r}-\delta(1)+\mathbf{1}} \cap M'_{\mathbf{r}+\mathbf{1}} = (x_{11}, \ldots, x_{1i})M'_{\mathbf{r}+\mathbf{1}-\delta(1)},$$

for all large $r_1, \ldots, r_q$ by induction on $i \leq m$. For $i = 0$, the result is trivial. Set $N = (x_{11}, \ldots, x_{1i-1})M : \mathcal{I}^\infty$. Since $x_{11}, \ldots, x_{1i}$ is a weak-FC-sequence of G'' with respect to (M, M'') in $G'_{\delta(i)}$,

$$(x_{1i}M_{\mathbf{r}-\delta(1)+\mathbf{1}} + N_{\mathbf{r}+\mathbf{1}}) \cap (M'_{\mathbf{r}+\mathbf{1}} + N_{\mathbf{r}+\mathbf{1}}) = x_{1i}M'_{\mathbf{r}-\delta(1)+\mathbf{1}} + N_{\mathbf{r}+\mathbf{1}},$$

for all large $r_1, \ldots, r_q$. Therefore

$$\begin{aligned}
&(x_{1i}M_{\mathbf{r}-\delta(1)+\mathbf{1}} + N_{\mathbf{r}+\mathbf{1}}) \cap M'_{\mathbf{r}+\mathbf{1}} \\
&\quad= M'_{\mathbf{r}+\mathbf{1}} \cap (M'_{\mathbf{r}+\mathbf{1}} + N_{\mathbf{r}+\mathbf{1}}) \cap (x_{1i}M_{\mathbf{r}-\delta(1)+\mathbf{1}} + N_{\mathbf{r}+\mathbf{1}}) \\
&\quad= M'_{\mathbf{r}+\mathbf{1}} \cap (x_{1i}M'_{\mathbf{r}-\delta(1)+\mathbf{1}} + N_{\mathbf{r}+\mathbf{1}}) \\
&\quad= x_{1i}M'_{\mathbf{r}-\delta(1)+\mathbf{1}} + M'_{\mathbf{r}+\mathbf{1}} \cap N_{\mathbf{r}+\mathbf{1}}
\end{aligned}$$

for all large $r_1, \ldots, r_q$. By Artin-Rees Lemma, we have

$$M'_{\mathbf{r}+\mathbf{1}} \cap N_{\mathbf{r}+\mathbf{1}} \subset M'_{\mathbf{r}+\mathbf{1}} \cap (x_{11}, \ldots, x_{1i-1})M_{\mathbf{r}-\delta(1)+\mathbf{1}}$$

for all large $r_1, \ldots, r_q$. By inductive assumption,

$$M'_{\mathbf{r}+\mathbf{1}} \cap N_{\mathbf{r}+\mathbf{1}} \subset (x_{11}, \ldots, x_{1i-1})M_{\mathbf{r}-\delta(1)+\mathbf{1}}$$

for all large $r_1, \ldots, r_q$. Consequently,

$$M'_{\mathbf{r}+\mathbf{1}} \cap N_{\mathbf{r}+\mathbf{1}} = (x_{11}, \ldots, x_{1i-1})M'_{\mathbf{r}-\delta(1)+\mathbf{1}}$$

for all large $r_1, \ldots, r_q$. Hence we get

$$\begin{aligned}
(x_{1i}M_{\mathbf{r}-\delta(1)+\mathbf{1}} + N_{\mathbf{r}+\mathbf{1}}) \cap M'_{\mathbf{r}+\mathbf{1}} &= x_{1i}M'_{\mathbf{r}-\delta(1)+\mathbf{1}} + (x_{11}, \ldots, x_{1i-1})M'_{\mathbf{r}-\delta(1)+\mathbf{1}} \\
&= (x_{11}, \ldots, x_{1i})M'_{\mathbf{r}-\delta(1)+\mathbf{1}}
\end{aligned}$$

for all large $r_1, \ldots, r_q$. By Artin-Rees Lemma,

$$(x_{1i}M_{\mathbf{r}-\delta(1)+\mathbf{1}} + N_{\mathbf{r}+\mathbf{1}}) \cap M'_{\mathbf{r}+\mathbf{1}} = (x_{11}, \ldots, x_{1i})M_{\mathbf{r}-\delta(1)+\mathbf{1}} \cap M'_{\mathbf{r}+\mathbf{1}}$$

for all large $r_1, \ldots, r_q$. Thus,

$$(x_{11}, \ldots, x_{1i})M_{\mathbf{r}-\delta(1)+\mathbf{1}} \cap M'_{\mathbf{r}+\mathbf{1}} = (x_{11}, \ldots, x_{1i})M'_{\mathbf{r}-\delta(1)+\mathbf{1}}$$

for all large $r_1, \ldots, r_q$ and $i \le m$. From this it follows that

$$\mathfrak{I}_1 M_{\mathbf{r}-\delta(1)+\mathbf{1}} \cap M'_{\mathbf{r}+\mathbf{1}} = \mathfrak{I}_1 M'_{\mathbf{r}-\delta(1)+\mathbf{1}}$$

for all large $r_1, \ldots, r_q$. The result is proved for $k = 1$.

Set $\mathfrak{N} = (\mathfrak{I}_1, \ldots, \mathfrak{I}_{k-1})M : \mathcal{I}^\infty$. By Artin-Rees Lemma, we have

$$\mathfrak{N}_{\mathbf{r}+\mathbf{1}} \cap M'_{\mathbf{r}+\mathbf{1}} \subset M'_{\mathbf{r}+\mathbf{1}} \cap (\mathfrak{I}_1, \ldots, \mathfrak{I}_{k-1})M_{\mathbf{r}-\delta(1)-\cdots-\delta(k-1)+\mathbf{1}}$$

and

$$(\mathfrak{N}_{\mathbf{r}+\mathbf{1}} + \mathfrak{I}_k M_{\mathbf{r}-\delta(k)+\mathbf{1}}) \cap M'_{\mathbf{r}+\mathbf{1}} \subseteq M'_{\mathbf{r}+\mathbf{1}} \cap (\mathfrak{I}_1, \ldots, \mathfrak{I}_k)M_{\mathbf{r}-\delta(1)-\cdots-\delta(k)+\mathbf{1}}$$

for all large $r_1, \ldots, r_q$. Hence

$$\mathfrak{N}_{\mathbf{r}+\mathbf{1}} \cap M'_{\mathbf{r}+\mathbf{1}} = M'_{\mathbf{r}+\mathbf{1}} \cap (\mathfrak{I}_1, \ldots, \mathfrak{I}_{k-1})M_{\mathbf{r}-\delta(1)-\cdots-\delta(k-1)+\mathbf{1}}$$

and

$$(\mathfrak{N}_{\mathbf{r}+\mathbf{1}} + \mathfrak{I}_k M_{\mathbf{r}-\delta(k)+\mathbf{1}}) \cap M'_{\mathbf{r}+\mathbf{1}} = M'_{\mathbf{r}+\mathbf{1}} \cap (\mathfrak{I}_1, \ldots, \mathfrak{I}_k)M_{\mathbf{r}-\delta(1)-\cdots-\delta(k)+\mathbf{1}} \quad (5.2)$$

for all large $r_1, \ldots, r_q$. By the result is true for $k = 1$,

$$(\mathfrak{N}_{\mathbf{r}+\mathbf{1}} + \mathfrak{I}_k M_{\mathbf{r}-\delta(k)+\mathbf{1}}) \cap (M'_{\mathbf{r}+\mathbf{1}} + \mathfrak{N}_{\mathbf{r}+\mathbf{1}}) = \mathfrak{I}_k M'_{\mathbf{r}-\delta(k)+\mathbf{1}} + \mathfrak{N}_{\mathbf{r}+\mathbf{1}}$$

for all large $r_1, \ldots, r_q$. From the above facts, we get

$$\begin{aligned}
&(\mathfrak{N}_{\mathbf{r}+\mathbf{1}} + \mathfrak{I}_k M_{\mathbf{r}-\delta(k)+\mathbf{1}}) \cap M'_{\mathbf{r}+\mathbf{1}} \\
&= M'_{\mathbf{r}+\mathbf{1}} \cap (\mathfrak{N}_{\mathbf{r}+\mathbf{1}} + \mathfrak{I}_k M_{\mathbf{r}-\delta(k)+\mathbf{1}}) \cap (M'_{\mathbf{r}+\mathbf{1}} + \mathfrak{N}_{\mathbf{r}+\mathbf{1}}) \\
&= M'_{\mathbf{r}+\mathbf{1}} \cap (\mathfrak{I}_k M_{\mathbf{r}-\delta(k)+\mathbf{1}} + \mathfrak{N}_{\mathbf{r}+\mathbf{1}}) \\
&= \mathfrak{I}_k M_{\mathbf{r}-\delta(k)+\mathbf{1}} + \mathfrak{N}_{\mathbf{r}+\mathbf{1}} \cap M'_{\mathbf{r}+\mathbf{1}} \\
&= \mathfrak{I}_k M_{\mathbf{r}-\delta(k)+\mathbf{1}} + (\mathfrak{I}_1, \ldots, \mathfrak{I}_{k-1})M_{\mathbf{r}-\delta(1)-\cdots-\delta(k-1)+\mathbf{1}} \cap M'_{\mathbf{r}+\mathbf{1}} \qquad *
\end{aligned}$$

for all large $r_1, \ldots, r_q$. By inductive assumption applied to $k-1$, we see that

$$(\mathfrak{I}_1, \ldots, \mathfrak{I}_{k-1})M_{\mathbf{r}-\delta(1)-\cdots-\delta(k-1)+\mathbf{1}} \cap M'_{\mathbf{r}+\mathbf{1}} = \sum_{j=1}^{k-1} \mathfrak{I}_j M'_{\mathbf{r}-\delta(j)+\mathbf{1}} \quad (5.3)$$

for all large $r_1, \ldots, r_q$. Hence by 5.2, $*$ and 5.3, we get

$$(\mathfrak{I}_1, \ldots, \mathfrak{I}_k)M_{\mathbf{r}-\delta(1)-\cdots-\delta(k)+\mathbf{1}} \cap M'_{\mathbf{r}+\mathbf{1}} = \sum_{j=1}^{k} \mathfrak{I}_j M'_{\mathbf{r}-\delta(j)+\mathbf{1}}$$

for all large $r_1, \ldots, r_q$.

The proof of (ii): Since $x_{11}, \ldots, x_{1m}, x_{21}, \ldots, x_{2n} \ldots x_{t1}, \ldots, x_{tp}$ is a maximal weak-(FC)-sequence of G'' with respect to (M, M'') in $\cup_{i=1}^{q} G'_{\delta(i)}$, by Proposition 5.5, we have that

$$M'_{\mathbf{r}+\mathbf{1}} \subset (\mathfrak{J}_1, \ldots, \mathfrak{J}_t)M_{\mathbf{r}-\delta(1)-\cdots-\delta(t)+\mathbf{1}}$$

for all large $r_1, \ldots, r_q$. By applying part (i), we get (ii). $\square$

Proposition 5.9. *Assume that $\mathcal{I}$ is not contained in $\sqrt{AnnM}$ and let J be a finitely generated R-submodules of $G_{d\delta(1)}$ of finite colength. Then the following statements hold.*

(i) If $x \in G'_{\delta(i)}$, $i \geq 2$, is an (FC)-element of G'' with respect to (M, M''), then

$$e_{s,\mathbf{k}}(J, G'; M') = e_{s,\mathbf{k}-\delta(i)}(J, G'; \overline{M'}),$$

where k_i is a positive integer and $\overline{M'} := M'/xM'$.

(ii) If $e_{s,\mathbf{k}}(J, G'; M') \neq 0$, then for any $i \geq 2$ such that $k_i > 0$, there exists an (FC)-element of of G'' with respect to (M, M'').

Proof. By Proposition 5.1 that the function

$$\ell\left(\frac{{M'}^*_{\mathbf{r}+dn\delta(1)}}{J^n {M'}^*_{\mathbf{r}}}\right)$$

is, for large $n, \mathbf{r}$, a polynomial of total degree $D = \dim(\mathrm{Supp}_{++}({M'}^*))$.

Set $B := \overline{M'}^*$. Then for all $n > v, \mathbf{r} > \mathbf{v}$, we have

$$\begin{aligned}
\ell\left(\frac{B_{\mathbf{r}+dn\delta(1)}}{J^n B_{\mathbf{r}}}\right) &= \ell\left(\frac{{M'}^*_{\mathbf{r}+dn\delta(1)}+x{M'}^*_{\mathbf{r}-\delta(i)+dn\delta(1)}}{J^n {M'}^*_{\mathbf{r}}+x{M'}^*_{\mathbf{r}-\delta(i)+dn\delta(1)}}\right)\\
&= \ell\left(\frac{{M'}^*_{\mathbf{r}+dn\delta(1)}}{J^n {M'}^*_{\mathbf{r}}+{M'}^*_{\mathbf{r}+dn\delta(1)}\cap x{M'}^*_{\mathbf{r}-\delta(i)+dn\delta(1)}}\right)\\
&= \ell\left(\frac{{M'}^*_{\mathbf{r}+dn\delta(1)}}{J^n {M'}^*_{\mathbf{r}}+x{M'}^*_{\mathbf{r}-\delta(i)+dn\delta(1)}}\right)\\
&= \ell\left(\frac{{M'}^*_{\mathbf{r}+dn\delta(1)}}{J^n {M'}^*_{\mathbf{r}}}\right) - \ell\left(\frac{J^n {M'}^*_{\mathbf{r}}+x{M'}^*_{\mathbf{r}-\delta(i)+dn\delta(1)}}{J^n {M'}^*_{\mathbf{r}}}\right)\\
&= \ell\left(\frac{{M'}^*_{\mathbf{r}+dn\delta(1)}}{J^n {M'}^*_{\mathbf{r}}}\right) - \ell\left(\frac{x{M'}^*_{\mathbf{r}-\delta(i)+dn\delta(1)}}{J^n {M'}^*_{\mathbf{r}}\cap x{M'}^*_{\mathbf{r}-\delta(i)+dn\delta(1)}\cap {M'}^*_{\mathbf{r}+dn\delta(1)}}\right)\\
&= \ell\left(\frac{{M'}^*_{\mathbf{r}+dn\delta(1)}}{J^n {M'}^*_{\mathbf{r}}}\right) - \ell\left(\frac{x{M'}^*_{\mathbf{r}-\delta(i)+dn\delta(1)}}{xJ^n {M'}^*_{\mathbf{r}-\delta(i)}}\right).
\end{aligned}$$

Since $x \in G'_{\delta(i)}$, $i \geq 2$, is an (FC_2)-element of G'' with respect to (M, M''), it follows Lemma 5.4(ii) and Lemma 5.6 that

$$M'^*_{\mathbf{r}-\delta(i)+dn\delta(1)} \cong xM'^*_{\mathbf{r}-\delta(i)+dn\delta(1)}$$

and

$$J^n M'^*_{\mathbf{r}-\delta(i)} \cong xJ^n M'^*_{\mathbf{r}-\delta(i)}.$$

Thus we have an isomorphism of G-modules

$$\frac{M'^*_{\mathbf{r}-\delta(i)+dn\delta(1)}}{J^n M'^*_{\mathbf{r}-\delta(i)}} \cong \frac{xM'^*_{\mathbf{r}-\delta(i)+dn\delta(1)}}{xJ^n M'^*_{\mathbf{r}-\delta(i)}}.$$

for all large $n, \mathbf{r}$. So

$$\ell\left(\frac{xM'^*_{\mathbf{r}-\delta(i)+dn\delta(1)}}{xJ^n M'^*_{\mathbf{r}-\delta(i)}}\right) = \ell\left(\frac{M'^*_{\mathbf{r}-\delta(i)+dn\delta(1)}}{J^n M'^*_{\mathbf{r}-\delta(i)}}\right).$$

for all n and all large $\mathbf{r}$.
Hence

$$\ell\left(\frac{B_{\mathbf{r}+dn\delta(1)}}{J^n B_{\mathbf{r}}}\right) = \ell\left(\frac{M'^*_{\mathbf{r}+dn\delta(1)}}{J^n M'^*_{\mathbf{r}}}\right) - \ell\left(\frac{M'^*_{\mathbf{r}-\delta(i)+dn\delta(1)}}{J^n M'^*_{\mathbf{r}-\delta(i)}}\right).$$

Now if x satisfies the condition (FC_3), that mean

$$\mathrm{rdim}\overline{M'}^* = \mathrm{rdim}(M'^*) - 1$$

(see Lemma 5.6), then it can be verified that

$$\ell\left(\frac{B_{\mathbf{r}+dn\delta(1)}}{J^n B_{\mathbf{r}}}\right).$$

is a polynomial of degree $D-1$ for all large $n, \mathbf{r}$, and the terms of total degree $D-1$ in this polynomial

$$\ell\left(\frac{B_{\mathbf{r}+dn\delta(1)}}{J^n B_{\mathbf{r}}}\right).$$

is equal to the terms of total degree $D-1$ in the polynomial

$$\ell\left(\frac{M'^*_{\mathbf{r}+dn\delta(1)}}{J^n M'^*_{\mathbf{r}}}\right) - \ell\left(\frac{M'^*_{\mathbf{r}-\delta(i)+dn\delta(1)}}{J^n M'^*_{\mathbf{r}-\delta(i)}}\right).$$

The above facts show that

$$e_{s,\mathbf{k}}(J,G';M') = e_{s,\mathbf{k}-\delta(i)}(J,G';\overline{M'}).$$

The Proof (ii); as $\mathcal{I}$ is not contained in $\sqrt{\mathrm{Ann}M}$, by (Proposition 5.5), there exists an element $x \in G'_{\delta(i)}$, $i \geq 2$, satisfying a (FC_1) and (FC_2) of FC sequences of G'' with respect to (M, M''). From the proof of (i) we have

$$\ell\left(\frac{B_{\mathrm{r}+dn\delta(1)}}{J^nB_{\mathrm{r}}}\right) = \ell\left(\frac{{M'}^*_{\mathrm{r}+dn\delta(1)}}{J^n{M'}^*_{\mathrm{r}}}\right) - \ell\left(\frac{{M'}^*_{\mathrm{r}-\delta(i)+dn\delta(1)}}{J^n{M'}^*_{\mathrm{r}-\delta(i)}}\right).$$

Since $e_{s,\mathbf{k}}(J,G';M') \neq 0$, it follows that

$$\ell\left(\frac{B_{\mathrm{r}+dn\delta(1)}}{J^nB_{\mathrm{r}}}\right)$$

is a polynomial of degree $D-1$. Thus, $\mathrm{rdim}(M'/xM' : \mathcal{I}^\infty) = \mathrm{rdim}(M'^*) - 1$ and x satisfies the conditions (FC)-sequences. The proof of propositions is compleat. $\square$

The following theorem establish mixed multiplicity formulas by means of Buchsbaum-Rim multiplicity and also determines the positivity of mixed multiplicities.

Theorem 5.10. *Assume that $\mathcal{I}$ is not contained in $\sqrt{AnnM}$ and let J be a finitely generated R-submodules of $G_{d\delta(1)}$ of finite colength. Let $s, k_1, ..., k_q$ be non-negative integers with sum equal to D. Then*

(i)
$$e_{s,\mathbf{k}}(J,G';M') = e_{BR}^{k_1}(J;(\overline{M}^*)^{(\delta(1))}),$$
for any sequence $x_1, ..., x_t$ of $t = k_2 + ... + k_q$ elements of G'' satisfying the condition (FC) with respect to (M, M''), consisting of k_2 elements of $G'_{\delta(2)}$,...,k_q elements of $G'_{\delta(q)}$ where $\overline{M}^* = M/((x_1, ..., x_t)M : \mathcal{I}^\infty)$.

(ii) Assume that $s > 0$. Then $e_{s,\mathbf{k}}(J,G';M') \neq 0$, if and only if there exist a sequence of k_2+...+k_q elements of G'' satisfying the condition (FC) with respect to (M, M''), consisting of k_2 elements of $G'_{\delta(2)}$,...,k_q elements of $G'_{\delta(q)}$.

Proof. The proof of (i) : We shall begin with showing that $\mathcal{I}$ is not contained in $\sqrt{\mathrm{Ann}M}$ and $x_1, x_2, ..., x_t$ is a sequence of $t = k_2 + ... + k_q$ elements of G'' satisfying the condition (FC) with respect to (M, M''), consisting of k_2 elements of $G'_{\delta(2)}$,...,k_q elements of $G'_{\delta(q)}$ then

$$e_{s,\mathbf{k}}(J, G'; M') = e_{s,k_1\delta(1)}(J, G'; \overline{M_t}) \neq 0,$$

where $\overline{M_t} = M/(x_1, ..., x_t)M$. The proof is by induction on $t = k_2 + ... + k_q$. For $t = 0$, since $\mathcal{I}$ is not contained in $\sqrt{\mathrm{Ann}M}$, by Lemma 5.3

$$e_{s,\mathbf{k}}(J, G'; M') = e_{s,j\delta(1)}(J, G'; \overline{M}) \neq 0,$$

The result is true.

Suppose that the result has been proved for $t - 1$ we need show that the result is true for t. now, assume that $x_1, x_2, ..., x_t$ is a sequence of $t = k_2 + ... + k_q$ elements of G'' satisfying the condition (FC) with respect to (M, M''), consisting of k_2 elements of $G'_{\delta(2)}$,...,k_q elements of $G'_{\delta(q)}$. Since $t > 0$, there exists i $(2 \leq i \leq q)$ such that $k_i > 0$ and $x_i \in G_{\delta(i)}$. By Proposition 5.9, we get

$$e_{s,\mathbf{k}}(J, G'; M') = e_{s,\mathbf{k}-\delta(i)}(J, G'; \overline{M_1}),$$

where $\overline{M_1} := M/x_1M$. Since $x_1, \ldots, x_t$ elements of G'' satisfying the condition (FC) with respect to (M, M''), it follows that $x_2, \ldots, x_t$ are elements of G'' satisfying the condition (FC) with respect to (M, M''). Since $k_2 + \cdots + (k_i - 1) + \cdots + k_q = t - 1$, then it follows by inductive assumption applied to $(t-1)$ that

$$e_{s,\mathbf{k}-\delta(i)}(J, G'; \overline{M_1}) = e_{s,k_1\delta(1)}(J, G'; \overline{M_t}) \neq 0.$$

The induction is complete. We now turn to the proof of (i). Since $\mathcal{I}$ is not contained in $\sqrt{\mathrm{Ann}M}$ and $x_1, x_2, ..., x_t$ is a sequence of $t = k_2 + ... + k_q$ elements of G'' satisfying the condition (FC) with respect to (M, M''), consisting of k_2 elements of $G'_{\delta(2)}$,...,k_q elements of $G'_{\delta(q)}$, as shown, we get

$$e_{s,\mathbf{k}}(J, G'; M') = e_{s,k_1\delta(1)}(J, G'; \overline{M_t}) \neq 0.$$

Since $e_{s,k_1\delta(1)}(J, G'; \overline{M_t}) \neq 0$, it follows that $\mathcal{I}$ is not contained in $\sqrt{\mathrm{Ann}\overline{M_t}}$. Now, from Lemma 5.3 we have

$$e_{s,k_1\delta(1)}(J,G';\overline{M_t}) = e^{k_1}_{BR}(J,(\overline{M_t}^*)^{(\delta(1))}) = e^{k_1}_{BR}(J,(\overline{M}^*)^{(\delta(1))}).$$

With the above facts, we immediately see

$$e_{s,\mathbf{k}}(J,G';M') = e^{k_1}_{BR}(J,(\overline{M}^*)^{(\delta(1))}).$$

The proof of (i) is complete.

The proof of (ii) : We prove first the necessity. The proof is by induction on $(t = k_2 + ... + k_q)$. If $t = 0$, the result is trivial. For suppose the result has been proved for $t-1$. As the next step, we claim that the result is true for t. On the one hand $t = k_2 + ... + k_q > 0$ on the other hand $e_{s,\mathbf{k}}(J,G';M') \neq 0$, by Proposition 5.9, there exists i $(2 \leq i \leq q)$ such that $k_i > 0$ and an element $x_i \in G'_{\delta(i)}$ satisfying the conditions (FC) with respect to (M, M'') and

$$e_{s,\mathbf{k}}(J,G';M') = e_{s,\mathbf{k}-\delta(i)}(J,G';\overline{M'}),$$

where $\overline{M'} := M'/x_iM'$. From this equality we have

$$e_{s,\mathbf{k}-\delta(i)}(J,G';\overline{M'}) \neq 0.$$

On the other hand $k_2 + \cdots + (k_i - 1) + \cdots + k_q = t-1$, then it follows by our inductive assumption applied to $(t-1)$ that we choose $(t-1)$ elements $x_2, ..., x_t$ consisting of k_2 elements of $G'_{\delta(2)}$,...,$k_i - 1$ elements of $G'_{\delta(i)}$,..., e k_q elements of $G'_{\delta(q)}$ satisfying the conditions (FC) with respect to $(\overline{M}, \overline{M''})$. Since x_1 satisfies the conditions (FC) with respect (M, M''), it follows that $x_1, x_2, ..., x_t$ is a sequence satisfying the conditions (FC) with respect to (M, M'').

We turn to the proof of sufficiency. Suppose that there exists a sequence $x_1, x_2, ..., x_t$ of $t = k_2 + ... + k_q$ elements satisfying the conditions (FC) with respect to (M, M'') consisting of k_2 elements of $G'_{\delta(2)}$,...,k_q elements of $G'_{\delta(q)}$. By (i),

$$e_{s,\mathbf{k}}(J,G';M') = e_{s,k_1\delta(1)}(J,G';(\overline{M})^*).$$

Since $x_1, x_2, ..., x_t$ is a sequence of $t = k_2 + ... + k_q$ elements satisfying the conditions (FC) with respect to (M, M''), it follows that

$$\operatorname{rdim}(M'/((x_1, ..., x_t)M' : \mathcal{I}^\infty) = s + k_1 > 0.$$

Hence, it follows that $\mathcal{I}$ is not contained in $\sqrt{\operatorname{Ann} M}$. By Lemma 5.3, $e_{s,k_1\delta(1)}(J,G';(\overline{M})^*) \neq 0$. Thus, $e_{s,\mathbf{k}}(J,G';M') \neq 0$. The proof of theorem is complete.

□

6 Application

This section is devoted to the discussion of applications of the results of Section 5. Here the multi-graded module will be replaced by a collection of finitely generated ideals of a standard graded Noetherian algebra. We deal with the positivity and the relationship between mixed multiplicities and Buchsbaum-Rim multiplicities in this new context.

Let $A = \oplus_{n\in\mathbb{N}} A_n$ be a finitely generated standard graded algebra over a Noetherian local ring $(R, \mathfrak{m})$ of Krull dimension d and $E = \oplus_{n\in\mathbb{N}} E_n$ a finitely generated graded A-module.

From now on we set $J \subseteq A_{d_0}$ be a R-submodule of finite colength and $I_1, \ldots, I_s$, with $I_i \subseteq A_{d_i}$ finitely generated R-submodules. We now associate with the sequence of R-modules $I_1, \ldots, I_s$ a pair of graded R-algebras and modules. Define

$$G = \oplus_{(p,\mathbf{r})\in\mathbb{N}^{s+1}} G_{(p,\mathbf{r})} \quad \text{and} \quad G' = \oplus_{(p,\mathbf{r})\in\mathbb{N}^{s+1}} G'_{(p,\mathbf{r})}$$

where $G_{(p,\mathbf{r})} := A_{d_1r_1+\cdots d_sr_s+p}$ and $G'_{(p,\mathbf{r})} := I_1^{r_1}\cdots I_s^{r_s}A_p$. Notice that in this context $\mathcal{I}(G') = \mathcal{I}_1\cdots\mathcal{I}_s$, where $\mathcal{I}_i$ denotes the ideal of A generated by I_i. We will assume that $\mathcal{I}(G')$ is not contained in $\sqrt{\operatorname{Ann}(E)}$. Similarly define the graded modules $M_{(p,\mathbf{r})} := E_{d_1r_1+\cdots d_sr_s+p}$ and $M'_{(p,\mathbf{r})} := I_1^{r_1}\cdots I_s^{r_s}E_p$. In this context, we have that

$$h(n,p,\mathbf{r}) := \ell\left(\frac{M'_{(p,\mathbf{r})+d_0n\delta(1)}}{J^nM'_{(p,\mathbf{r})}}\right) = \ell\left(\frac{I_1^{r_1}\cdots I_s^{r_s}E_{d_0n+p}}{J^nI_1^{r_1}\cdots I_s^{r_s}E_p}\right).$$

By Proposition 5.1, we have that $h(n,p,\mathbf{r})$ is a polynomial of degree $r := \dim(\operatorname{Supp}_{++}(E/0_E : \mathcal{I}(G')^\infty))$ for all large $(n,p,\mathbf{r}) \in \mathbb{N}^{s+2}$. If we write the terms of total degree r of the polynomial

$$\ell\left(\frac{I_1^{r_1}\cdots I_s^{r_s}E_{d_0n+p}}{J^nI_1^{r_1}\cdots I_s^{r_s}E_p}\right)$$

in the form

$$B(n,p,\mathbf{r}) = \sum_{s+j+|\mathbf{k}|=q} \frac{1}{s!j!\mathbf{k}!} e^j(J^{[s]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}; E) n^s p^j \mathbf{r}^{\mathbf{k}}.$$

The coefficients $e^j(J^{[s]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}; E)$ are called the j^{ht} *associated mixed multiplicity of* $(J, I_1, \ldots, I_s)$ *of the type* $(s, k_1, \ldots k_s)$ *with respect to* E.

Let $I_1, \ldots, I_s$ be as in the beginning of this section. We say that an element $x \in I_i \setminus \mathfrak{m}I_i$ is an (weak-) (FC)-element of I_i with respect to $(I_1, \ldots, I_s; E)$ if it so with respect to (M, M'). Similarly we define the notion of (weak-) (FC)-sequence of I_i with respect to $(I_1, \ldots, I_s; E)$.

Proposition 5.9 immediately gives the following result.

Proposition 6.1. *Let $J, I_1, \ldots, I_s$ be as in the beginning of this section. Then the following statements hold.*

(i) If $x \in I_i$, $i \geq 1$, is an (FC)-element of G with respect to $(J, I_1, \ldots, I_s; E)$, then

$$e^j(J^{[s]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}; E) = e^j(J^{[s]}, I_1^{[k_1]}, \ldots, I_i^{[k_i-1]} \cdots, I_s^{[k_s]}; \overline{E}),$$

where k_i is a positive integer and $\overline{E} := E/xE$.

(ii) If $e^j(J^{[s]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}; E) \neq 0$, then for any $i \geq 1$ such that $k_i > 0$, there exists an (FC)-element of I_i with respect to $(J, I_1, \ldots, I_s; E)$.

Theorem 5.10 immediately gives the following result.

Theorem 6.2. *Let $J, I_1, \ldots, I_s$ be as in the beginning of this section. Let $k_0, j, k_1, ..., k_s$ be non-negative integers with sum equal to r. Assume that $r > 0$. Then*

(i)

$$e^j(J^{[k_0]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}; E) = e^j_{BR}(J; \overline{E}^*),$$

for any sequence $x_1, ..., x_t$ of $t = k_1 + ... + k_s$ elements satisfying the condition (FC) with respect to $(J, I_1, \ldots, I_s; E)$, consisting of k_1 elements of I_1,...,k_s elements of I_s, where $\overline{E}^* = E/((x_1, ..., x_t) : \mathcal{I}(G')^\infty)$.

(ii) If $k_0 > 0$, then $e^j(J^{[k_0]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}) \neq 0$, if and only if there exist a sequence of $k_1 + ... + k_s$ elements satisfying the condition (FC) with respect to $(J, I_1, \ldots, I_s; E)$, consisting of k_1 elements of I_1,...,k_s elements of I_s.

In the case that $\mathrm{ht}(\mathcal{I}(G') + \mathrm{Ann}E/\mathrm{Ann}E) > 0$, there exists positive integers n_0 and $\mathbf{u}$ such that the function

$$\ell\left(\frac{I_1^{r_1} \cdots I_s^{r_s} E_{d_0 n+p}}{J^n I_1^{r_1} \cdots I_s^{r_s} E_p}\right)$$

is, for $n > n_0, \mathbf{r} > \mathbf{u}$ a polynomial of total degree $D := \dim(\mathrm{Supp}_{++}(E))$. We get the following theorem.

Theorem 6.3. *Let $J, I_1, \dots, I_s$ be as in the beginning of this section. Assume $ht(\mathcal{I}(G') + AnnE/AnnE) > 0$. Let $k_0, j, k_1, ..., k_s$ be non-negative integers with sum equal to D, with $k_0 > 0$. Then, if $t = k_1 + ... + k_s < ht(\mathcal{I}(G') + AnnE/AnnE)$ we have that*

$$e^j(J^{[k_0]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}; E) = e^j_{BR}(J; \overline{E}),$$

for any sequence $x_1, ..., x_t$ satisfying the condition (FC) with respect to $(J, I_1, \dots, I_s; E)$ consisting of k_1 elements of I_1,...,k_s elements of I_s where $\overline{E} = E/(x_1, ..., x_t)E$.

Proof. Set $\overline{E}_1^* = \overline{E}/(0_{\overline{E}} : \mathcal{I}(G')^\infty)$. Then $\overline{E}_1^* = \overline{E}^*$. By Theorem 6.2 (i),

$$e^j(J^{[k_0]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}; E) = e^j_{BR}(J; \overline{E}^*) = e^j_{BR}(J; \overline{E}_1^*).$$

It is clear that $\mathrm{ht}(\mathcal{I}(G') + \mathrm{Ann}(E)/\mathrm{Ann}(E)) > 0$. Thus, applying Lemma 5.3, we have

$$e^j_{BR}(J; \overline{E}_1^*) = e^j_{BR}(J; \overline{E}).$$

From the facts, we get

$$e^j(J^{[k_0]}, I_1^{[k_1]}, \dots, I_s^{[k_s]}; E) = e^j_{BR}(J; \overline{E}).$$

□

In particular, when $I_i \subseteq A_{d_i}$, $i = 1, \dots, s$ are R-submodules of finite colength, Kleiman and Thorup in [7], Kirby and Rees in [5] and the authors in [2], defined the following notions of mixed multiplicities of $(I_1, \dots, I_s)$, which agrees with the similar notion defined in this work. As a function of $r_1, \dots, r_s, p$, the length,

$$h(r_1, \dots, r_s, p) := \ell(A_{d_1r_1+\dots+d_sr_s+p}/\mathbf{I}^{|\mathbf{r}|}A_p)$$

is, for sufficiently large $r_1, \dots, r_s, p$, a polynomial of total degree at most $r := \dim(\mathrm{Proj}(A))$, whose leading term could be written as

$$\sum_{j+|\mathbf{k}|=r} \frac{1}{j!\mathbf{k}!} e^j(I_1^{[k_1]}, \dots, I_s^{[k_s]})\mathbf{r}^{\mathbf{k}} p^j.$$

The coefficients $e^j(I_1^{[k_1]}, \dots, I_s^{[k_s]})$ are called the j^{th} *associated mixed Buchsbaum-Rim multiplicity of* $I_1, \dots, I_s$. The terms $e^0(I_1^{[k_1]}, \dots, I_s^{[k_s]})$, with $|\mathbf{k}| = r$, are called the **mixed Buchsbaum-Rim multiplicity of** $I_1, \ldots, I_s$ of type $(k_1, \ldots, k_s)$, and are also denoted by $e_{BR}(I_1^{[k_1]}, \ldots, I_s^{[k_s]})$.

Assume now that $I_i \subseteq A_d$, $i = 1, \ldots, s$. Let $j, k_1, \ldots, k_s$ be non-negative integers with sum equal to r. Suppose that $x_1, x_2, ..., x_r$ is a weak-(FC)-sequence with respect to $(A_1, I_1, \ldots, I_s)$ consisting of consisting of j elements of A_1, k_1 elements of I_1,...,k_s elements of I_s. By Lemma 5.8, $x_1, x_2, ..., x_r$ is a joint reduction for

$$A_1, \ldots, A_1, I_1, \ldots, I_1, \ldots, I_s, \ldots, I_s,$$

consisting of j copies of A_1 and k_i copies of I_i with $i = 1, \ldots, q$. Consequently, we get

$$e^j(I_1^{[k_1]}, \ldots, I_s^{[k_s]}) = e_{BR}(x_1, \ldots, x_r).$$

From these facts, we have a result similar to that of Kirby and Rees [5] and Callejas-Bedregal and Perez [2], but in terms of sequences satisfying the condition (FC).

Theorem 6.4. *Suppose that $I_1, \ldots, I_s$ are R-submodules of G_d of finite colength. Let $j, k_1, \ldots, k_s \in \mathbb{N}$ with $j + |\mathrm{k}| = r$. Let $x_1, \ldots, x_r$ be a a weak-(FC)-sequence with respect to $(A_1, I_1, \ldots, I_s)$ consisting of consisting of j elements of A_1, k_1 elements of I_1,...,k_s elements of I_s. Then,*

$$e^j(I_1^{[k_1]}, \ldots, I_s^{[k_s]}) = e_{BR}(x_1, \ldots, x_r).$$

Remark 6.5. The most important case for application of the results obtained in this section is the following context. Let $(R, \mathfrak{m})$ be a Noetherian local ring and N a finitely generated R-module. Given a finitely generated R-submodule M of the free module R^p we associate two standard graded R-algebras, the symmetric algebra $S := \mathrm{Sym}(R^p) = \oplus S_n(R^p)$ of R^p, which is a polynomial ring $R[T_1, \ldots, T_p]$, and the Rees algebra of M, $\mathcal{R}(M) := \oplus \mathcal{R}_n(M)$, which is the subalgebra of S generated in degree one by $\{h_1T_1 + \ldots + h_pT_p \in S_1(R^p) \,|\, (h_1, \ldots, h_p) \in M\}$. Consider the graded S-module $E := S \otimes_R N$.

Now, let $M, N_1, \ldots, N_s$ be finitely generated R-submodules of R^p such that $M \subseteq R^p$ has finite colength. In this context we consider $J := \mathcal{R}_1(M) \subseteq S_1(R^p)$ and $I_i := \mathcal{R}_1(N_i) \subseteq S_1(R^p)$, $i = 1, \ldots, s$, which are R-submodules of $S_1(R^p)$ finitely generated by linear forms.

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