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CONLEY INDEX AND PARABOLIC PROBLEMS WITH LOCALIZED LARGE DIFFUSION AND NONLINEAR BOUNDARY CONDITIONS

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Conley index and parabolic problems with localized large diffusion and nonlinear boundary conditions

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Resumo

Sejam Ω um domínio limitado e regular em $\mathbb{R}^N$ e $\Omega_{0,i}$, $i \in [1..m]$ domínios regulares cujos fechos são dois a dois disjuntos em Ω. Seja $\Omega_0 = \cup_{i=1}^m \Omega_{0,i}$, $\Gamma = \partial\Omega$ e $\Gamma_{0,i} = \partial\Omega_{0,i}$ $i \in [1..m]$. Denote $\Omega_1 = \Omega \setminus \mathrm{cl}\,\Omega_0$ e para cada $\varepsilon > 0$, o seguinte problema parabólico

$$(E_\varepsilon) \qquad \begin{cases} u_t - \mathrm{div}(d_\varepsilon(x)\nabla u) + (\lambda + V_\varepsilon(x))u = \varphi_\varepsilon(x,u), & t > 0,\ x \in \Omega, \\ d_\varepsilon(x)\partial_\nu u + b_\varepsilon(x)u = \psi_\varepsilon(x,u), & t > 0,\ x \in \Gamma. \end{cases}$$

Aqui, $\lambda \in \mathbb{R}$ e ν é a normal exterior em $\partial\Omega$. Além disso, $d_\varepsilon \geq m > 0$, V_ε e b_ε, resp. φ_ε e ψ_ε, são funções definidas em Ω, resp. $\Omega \times \mathbb{R}$ satisfazendo hipóteses de regularidade. Assumimos que, para $\varepsilon \to 0$, $V_\varepsilon \to V_0$, $b_\varepsilon \to b_0$, $\varphi_\varepsilon \to \varphi_0$, $\psi_\varepsilon \to \psi_0$ e $d_\varepsilon|_{\Omega_1} \to d_0$ (em algum sentido) enquanto que $d_\varepsilon|\Omega_0 \to \infty$.

Com condições sobre as funções involvidas, a equação (E_ε) gera um semifluxo local π_ε em $H^1(\Omega)$.

Considere a equação

$$(E_0) \qquad \begin{cases} u_t - \mathrm{div}(d_0(x)\nabla u) + (\lambda + V_0(x))u = \varphi_0(x,u), & t > 0,\ x \in \Omega_1 \\ d_0(x)\partial_\nu u + b_0(x)u = \psi_0(x,u), & t > 0,\ x \in \Gamma, \\ \gamma_{0,i}(u) = u_{\Omega_{0,i}}, & \text{on } \Gamma_{0,i},\ i \in [1..m], \\ \dot u_{\Omega_{0,i}} + |\Omega_{0,i}|^{-1} \int_{\Gamma_{0,i}} d_0(x)\partial_{\nu_{0,i}} u \,\mathrm{d}\sigma & \\ +(\lambda + \widehat{c}_i)u_{\Omega_{0,i}} = |\Omega_{0,i}|^{-1} \int_{\Omega_{0,i}} \varphi_0(x, u_{\Omega_{0,i}})\,\mathrm{d}x, & t > 0,\ i \in [1..m], \end{cases}$$

onde $\gamma_{0,i}$ é o operador traço $\Gamma_{0,i}$, $\nu_{0,i}$ é a normal interior em $\partial\Omega_{0,i}$, $u_{\Omega_{0,i}}$ é o valor de u on $\Omega_{0,i}$ e $\widehat{c}_i := |\Omega_{0,i}|^{-1} \int_{\Omega_{0,i}} V_0\,\mathrm{d}x$.

A equação (E_0) gera um semifluxo local no subespaço fechado $H^1_{\Omega_0}(\Omega)$ de $H^1(\Omega)$. Neste trabalho mostramos que, quando $\varepsilon \to 0$, os semifluxos π_ε convergem, em um sentido singular, ao semifluxo π_0 e estabelecemos um resultado de compacidade singular para a família π_ε, $\varepsilon \geq 0$. Como uma consequência destes resultados, obtemos princípios de continuidade singular do índice de Conley e das tranças índices homológicas para esta família de semifluxos.

Nossos resultados estendem e refinam resultados anteriores de [9] e [1].

CONLEY INDEX AND PARABOLIC PROBLEMS WITH LOCALIZED LARGE DIFFUSION AND NONLINEAR BOUNDARY CONDITIONS

M. C. CARBINATTO AND K. P. RYBAKOWSKI

This paper is dedicated to Jack Hale, with friendship, respect and recognition of his creative pursuit in dynamical systems

ABSTRACT. Let Ω be a bounded smooth domain in $\mathbb{R}^N$ and $\Omega_{0,i}$, $i \in [1..m]$ be smooth domains whose closures are pairwise disjoint and included in Ω. Let $\Omega_0 = \cup_{i=1}^m \Omega_{0,i}$, $\Gamma = \partial\Omega$ and $\Gamma_{0,i} = \partial\Omega_{0,i}$, $i \in [1..m]$. Set $\Omega_1 = \Omega \setminus \operatorname{cl}\Omega_0$. For each $\varepsilon > 0$, consider the following parabolic problem

$$(E_\varepsilon) \qquad \begin{cases} u_t - \operatorname{div}(d_\varepsilon(x)\nabla u) + (\lambda + V_\varepsilon(x))u = \varphi_\varepsilon(x,u), & t > 0,\ x \in \Omega, \\ d_\varepsilon(x)\partial_\nu u + b_\varepsilon(x)u = \psi_\varepsilon(x,u), & t > 0,\ x \in \Gamma. \end{cases}$$

Here, $\lambda \in \mathbb{R}$ and ν is the exterior normal vector field on $\partial\Omega$. Moreover, $d_\varepsilon \geq m > 0$, V_ε and b_ε, resp. φ_ε and ψ_ε, are given functions on Ω, resp. $\Omega \times \mathbb{R}$ satisfying some regularity assumptions. We assume that, for $\varepsilon \to 0$, $V_\varepsilon \to V_0$, $b_\varepsilon \to b_0$, $\varphi_\varepsilon \to \varphi_0$, $\psi_\varepsilon \to \psi_0$ and $d_\varepsilon|_{\Omega_1} \to d_0$ (in some sense) while $d_\varepsilon|\Omega_0 \to \infty$.

Under some general conditions on the functions involved equation (E_ε) generates a local semiflow π_ε on $H^1(\Omega)$.

Consider the equation

$$(E_0) \qquad \begin{cases} u_t - \operatorname{div}(d_0(x)\nabla u) + (\lambda + V_0(x))u = \varphi_0(x,u), & t > 0,\ x \in \Omega_1 \\ d_0(x)\partial_\nu u + b_0(x)u = \psi_0(x,u), & t > 0,\ x \in \Gamma, \\ \gamma_{0,i}(u) = u_{\Omega_{0,i}}, & \text{on } \Gamma_{0,i},\ i \in [1..m], \\ \dot{u}_{\Omega_{0,i}} + |\Omega_{0,i}|^{-1}\int_{\Gamma_{0,i}} d_0(x)\partial_{\nu_{0,i}} u \, d\sigma & \\ +(\lambda + \widehat{c}_i)u_{\Omega_{0,i}} = |\Omega_{0,i}|^{-1}\int_{\Omega_{0,i}} \varphi_0(x, u_{\Omega_{0,i}})\, dx, & t > 0,\ i \in [1..m], \end{cases}$$

where $\gamma_{0,i}$ is the trace operator on $\Gamma_{0,i}$, $\nu_{0,i}$ is the interior normal vector field on $\partial\Omega_{0,i}$, $u_{\Omega_{0,i}}$ is the value of u on $\Omega_{0,i}$ and $\widehat{c}_i := |\Omega_{0,i}|^{-1}\int_{\Omega_{0,i}} V_0\, dx$.

Equation (E_0) generates a local semiflow on a closed subspace $H^1_{\Omega_0}(\Omega)$ of $H^1(\Omega)$. We prove that, as $\varepsilon \to 0$, the semiflows π_ε converge, in a singular sense, to the semiflow π_0 and we establish a singular compactness result for the family π_ε, $\varepsilon \geq 0$. As a consequence of these results, we obtain singular Conley index and homology index braid continuation principles for this family of semiflows.

Our results extend and refine some previous results from [9] and [1].

1. INTRODUCTION

Let N and m be a positive integers and $\widetilde{\varepsilon}_0$ be a positive real number. Let Ω be a bounded smooth domain in $\mathbb{R}^N$ and $\Omega_{0,i}$, $i \in [1..m]$ be smooth domains whose closures

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are pairwise disjoint and are included in Ω. Let $\Omega_0 = \cup_{i=1}^m \Omega_{0,i}$, $\Gamma = \partial\Omega$ and $\Gamma_{0,i} = \partial\Omega_{0,i}$, $i \in [1..m]$. Set $\Omega_1 = \Omega \setminus \operatorname{cl}\Omega_0$.

For each $\varepsilon \in [0, \widetilde{\varepsilon}_0]$, let $d_\varepsilon\colon \Omega \to \mathbb{R}$ be a smooth function such that

$$0 < m_0 \leq d_\varepsilon(x) \leq M_\varepsilon \text{ for all } x \in \Omega,$$

where m_0 and M_ε, $\varepsilon \in [0, \widetilde{\varepsilon}_0]$, are positive constants, and

$$d_\varepsilon(x) \to d_0(x) \text{ as } \varepsilon \to 0 \text{ uniformly for } x \in \Omega_1$$

and

$$d_\varepsilon(x) \to \infty \text{ as } \varepsilon \to 0 \text{ uniformly on compact subsets of } \Omega_0.$$

Let $V_\varepsilon \in L^{q_0}(\Omega)$ and $b_\varepsilon \in L^{q_1}(\Gamma)$ be such that

$$|V_\varepsilon - V_0|_{L^{q_0}(\Omega)} \to 0 \text{ and } |b_\varepsilon - b_0|_{L^{q_1}(\Gamma)} \to 0 \text{ as } \varepsilon \to 0,$$

where

$$q_0 \begin{cases} > 1, & \text{for } N = 1; \\ > 1, & \text{for } N = 2; \\ > N/2, & \text{for } N \geq 3 \end{cases} \quad \text{and } q_1 \begin{cases} > 1, & \text{for } N = 1; \\ > 1, & \text{for } N = 2; \\ > N-1, & \text{for } N \geq 3. \end{cases}$$

Now let $L^2_{\Omega_0}(\Omega)$ be the set of all functions in $L^2(\Omega)$ which are (a.e.) constant on each $\Omega_{0,i}$, $i \in [1..m]$. Set $H^1_{\Omega_0}(\Omega) = H^1(\Omega) \cap L^2_{\Omega_0}(\Omega)$. It follows that $H^1_{\Omega_0}(\Omega)$ (resp. $L^2_{\Omega_0}(\Omega)$) is a closed subspace of $H^1(\Omega)$ (resp. $L^2(\Omega)$).

Let $\gamma\colon H^1(\Omega) \to H^{1/2}(\Gamma)$ be the trace operator. For $\lambda \in \mathbb{R}$ and $\varepsilon \in]0, \widetilde{\varepsilon}_0]$ define the bilinear form $\tau_\varepsilon\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$\tau_\varepsilon(u,v) = \int_\Omega d_\varepsilon \nabla u \nabla v \,\mathrm{d}x + \int_\Omega (\lambda + V_\varepsilon) uv \,\mathrm{d}x + \int_\Gamma b_\varepsilon \gamma(u)\gamma(v)\,\mathrm{d}\sigma, \quad u,\, v \in H^1(\Omega).$$

Here, $\mathrm{d}x$ is the N-Lebesgue measure and $\mathrm{d}\sigma$ is the surface measure on Γ.

It follows from results in [9] that τ_ε is defined and continuous on $H^1(\Omega) \times H^1(\Omega)$. Furthermore, define the bilinear form $\tau_0\colon H^1_{\Omega_0}(\Omega) \times H^1_{\Omega_0}(\Omega) \to \mathbb{R}$ by

$$\tau_0(u,v) = \int_{\Omega_1} d_0 \nabla u \nabla v \,\mathrm{d}x + \int_\Omega (\lambda + V_0) uv \,\mathrm{d}x + \int_\Gamma b_0 \gamma(u)\gamma(v)\,\mathrm{d}\sigma, \quad u,\, v \in H^1_{\Omega_0}(\Omega).$$

Results in [9] imply that τ_0 is defined and continuous on $H^1_{\Omega_0}(\Omega) \times H^1_{\Omega_0}(\Omega)$. Moreover, there are an $\varepsilon_0 \in]0, \widetilde{\varepsilon}_0]$, a $\widetilde{\mu} \in]0, \infty[$ and a $\widetilde{\lambda} \in \mathbb{R}$ such that for all $\lambda \geq \widetilde{\lambda}$

$$\tau_\varepsilon(u,u) \geq \widetilde{\mu}|u|_{H^1(\Omega)}, \quad \varepsilon \in]0, \varepsilon_0],\, u \in H^1(\Omega)$$

$$\tau_0(u,u) \geq \widetilde{\mu}|u|_{H^1_{\Omega_0}(\Omega)}, \quad u \in H^1_{\Omega_0}(\Omega).$$

For $\varepsilon \in]0, \varepsilon_0]$ the pair $(\tau_\varepsilon, \langle\cdot,\cdot\rangle_{L^2(\Omega)})$ defines an operator $A_\varepsilon\colon D(A_\varepsilon) \to H^\varepsilon := L^2(\Omega)$. Specifically, let $D(A_\varepsilon)$ be the set of all $u \in H^1(\Omega)$ such that there is a $w = w_u \in L^2(\Omega)$ with the property that

$$\tau_\varepsilon(u,v) = \langle w, v\rangle_{L^2(\Omega)}$$

for all $v \in H^1(\Omega)$. Then w_u is uniquely determined by u, the set $D(A_\varepsilon)$ is a dense linear subspace both of $H^1(\Omega)$ and of $L^2(\Omega)$, and the map

$$A_\varepsilon\colon D(A_\varepsilon) \to L^2(\Omega), \quad u \mapsto w_u$$

is a linear positive self-adjoint operator in $(L^2, \langle\cdot,\cdot\rangle_{L^2(\Omega)})$ with A_ε^{-1} compact.

Analogously, the pair $(\tau_0, \langle\cdot,\cdot\rangle_{L^2_{\Omega_0}(\Omega)})$ defines a linear positive self-adjoint operator A_0 in $H^0 := L^2_{\Omega_0}(\Omega)$ with A_0^{-1} compact.

It is proved in [9] that, for $\varepsilon \in]0,\varepsilon_0]$, $D(A_\varepsilon)$ is the set of all $u \in H^1(\Omega)$ such that $-\operatorname{div}(d_\varepsilon \nabla u) + V_\varepsilon u \in L^2(\Omega)$ and $d_\varepsilon \partial_\nu u + b_\varepsilon u = 0$ in Γ. Here, ν is the exterior normal vector field on $\partial\Omega$ and $d_\varepsilon \partial_\nu u$ is the conormal derivative of u in some generalized sense. The linear operator A_ε is then given by

$$A_\varepsilon u = -\operatorname{div}(d_\varepsilon \nabla u) + (\lambda + V_\varepsilon)u$$

for $u \in D(A_\varepsilon)$.

Moreover, $D(A_0)$ is the set of all $u \in H^1_{\Omega_0}(\Omega)$ such that $-\operatorname{div}(d_0\nabla u) + V_0 u \in L^2(\Omega_1)$ with $d_0\partial_\nu u + b_0 u = 0$ in Γ. The linear operator A_0 is then given by

$$\begin{aligned} A_0 u =& (-\operatorname{div}(d_0 \nabla u) + (\lambda + V_0)u)\chi_{\Omega_1} \\ &+ \sum_{i=1}^m \left(|\Omega_{0,i}|^{-1} \int_{\Gamma_{0,i}} d_0 \partial_{\nu_{0,i}} u \, \mathrm{d}\sigma + (\lambda + \widehat{c}_i) u_{\Omega_{0,i}} \right) \chi_{\Omega_{0,i}}, \quad u \in D(A_0). \end{aligned}$$

Here, $\nu_{0,i}$ is the interior normal vector field on $\partial\Omega_{0,i}$, $u_{\Omega_{0,i}}$ is the constant value of u on $\Omega_{0,i}$, $\widehat{c}_i := |\Omega_{0,i}|^{-1} \int_{\Omega_{0,i}} V_0 \, dx$, $i \in [1..m]$, and χ_B denotes the characteristic function of a given set B.

For $\varepsilon \in [0,\varepsilon_0]$ the operator A_ε is sectorial so it defines a family of fractional power operators $A^\beta \colon D(A^\beta) \to H^\varepsilon$, $\beta \in [0,\infty[$ and we write, for $\alpha \in [0,\infty[$, $H^\varepsilon_\alpha := D(A^{\alpha/2})$. In particular, $H^\varepsilon_0 = H^\varepsilon$. In a canonical way, H^ε_α is a Hilbert space and we set $H^\varepsilon_{-\alpha}$ to be the dual of H^ε_α.

Now assume the following

Hypothesis 1.1. For $\varepsilon \in [0,\varepsilon_0]$, $\varphi_\varepsilon \colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $\psi_\varepsilon \colon \Gamma \times \mathbb{R} \to \mathbb{R}$, $(x,s) \mapsto \varphi_\varepsilon(x,s)$, $(x,s) \mapsto \psi_\varepsilon(x,s)$, are functions such that

(1) there is a null set N_Ω in Ω with $\varphi_\varepsilon(x,\cdot) \in C^1(\mathbb{R},\mathbb{R})$ for all $x \in \Omega \setminus N_\Omega$;
(2) there is a null set N_Γ in Γ (rel. to the surface measure on Γ) with $\psi_\varepsilon(x,\cdot) \in C^1(\mathbb{R},\mathbb{R})$ for all $x \in \Gamma \setminus N_\Gamma$;
(3) for all $s \in \mathbb{R}$, $\varphi_\varepsilon(\cdot,s)$ and $\partial_s\varphi_\varepsilon(\cdot,s)$ is measurable on Ω;
(4) for all $s \in \mathbb{R}$, $\psi_\varepsilon(\cdot,s)$ and $\partial_s\psi_\varepsilon(\cdot,s)$ is measurable on Γ.

Moreover, $q_2 \in](1-(1/2^*_\Omega))^{-1}, 2^*_\Omega[$ and $q_3 \in](1-(1/2^*_\Gamma))^{-1}, 2^*_\Gamma[$ and

$$r_2 = \frac{2^*_\Omega q_2}{2^*_\Omega - q_2}, \quad r_3 = \frac{2^*_\Gamma q_3}{2^*_\Gamma - q_3}, \quad \beta_2 = \frac{2^*_\Omega}{q_2} - 1, \quad \beta_3 = \frac{2^*_\Gamma}{q_3} - 1.$$

There is a constant $\widetilde{C} \in]0,\infty[$ and functions $a_2 \in L^{r_2}(\Omega)$, $b_2 \in L^{q_2}(\Omega)$, $a_3 \in L^{r_3}(\Gamma)$, $b_3 \in L^{q_3}(\Gamma)$ such that, for all $\varepsilon \in [0,\varepsilon_0]$,

$$\begin{aligned} |\partial_s\varphi_\varepsilon(x,s)| &\le \widetilde{C}(a_2(x) + |s|^{\beta_2}), \text{ for } (x,s) \in (\Omega \setminus N_\Omega) \times \mathbb{R}, \\ |\varphi_\varepsilon(x,0)| &\le b_2(x), \text{ for } x \in \Omega \setminus N_\Omega, \\ |\partial_s\psi_\varepsilon(x,s)| &\le \widetilde{C}(a_3(x) + |s|^{\beta_3}), \text{ for } (x,s) \in (\Gamma \setminus N_\Gamma) \times \mathbb{R}, \\ |\psi_\varepsilon(x,0)| &\le b_3(x), \text{ for } x \in \Gamma \setminus N_\Gamma. \end{aligned} \tag{1.1}$$

Finally, as $\varepsilon \to 0^+$,

$$|\varphi_\varepsilon(x,s) - \varphi_0(x,s)| \to 0, \text{ for } (x,s) \in (\Omega \setminus N_\Omega) \times \mathbb{R} \tag{1.2}$$
$$|\psi_\varepsilon(x,s) - \psi_0(x,s)| \to 0, \text{ for } (x,s) \in (\Gamma \setminus N_\Gamma) \times \mathbb{R}.$$

Under Hypothesis 1.1, there is an $\alpha \in \,]1/2, 1]$ such that whenever $\varepsilon \in \,]0, \varepsilon_0]$, $u \in H_1^\varepsilon$ and $h \in H_\alpha^\varepsilon$, the functions $x \mapsto \varphi_\varepsilon(x, u(x)) \cdot h(x)$ and $x \mapsto \psi_\varepsilon(x, \gamma(u)(x)) \cdot \gamma(h)(x)$ are integrable on Ω and Γ, respectively. (Cf. subsection 4.1 below.) Defining

$$f_\varepsilon(u)(h) = \int_\Omega \varphi_\varepsilon(x, u(x)) \cdot h(x)\,\mathrm{d}x + \int_\Gamma \psi_\varepsilon(x, \gamma(u)(x)) \cdot \gamma(h)(x)\,\mathrm{d}\sigma$$

we obtain a locally Lipschitzian map $f_\varepsilon \colon H_1^\varepsilon \to H_{-\alpha}^\varepsilon$.

The linear isometry $A_\varepsilon \colon D(A_\varepsilon) = H_2^\varepsilon \to H_0^\varepsilon$ can be extended to a unique linear isometry $\widetilde{A}_\varepsilon \colon D(\widetilde{A}_\varepsilon) = H_{2-\alpha}^\varepsilon \to H_{-\alpha}^\varepsilon$. Moreover, $\widetilde{A}_\varepsilon$ is a positive selfadjoint operator on $H_{-\alpha}^\varepsilon$.

Therefore, we may consider the abstract parabolic equation

$$\dot{u} = \widetilde{A}_\varepsilon u + f_\varepsilon(u) \tag{1.3}$$

on H_1^ε. This equation generates a local semiflow π_ε on H_1^ε.

For each $\varepsilon \in \,]0, \varepsilon_0]$, equation (1.3) is an abstract formulation of the following parabolic partial differential equation with localized large diffusion and nonlinear boundary conditions:

$$\begin{cases} u_t - \operatorname{div}(d_\varepsilon(x)\nabla u) + (\lambda + V_\varepsilon(x))u = \varphi_\varepsilon(x,u), & t > 0,\ x \in \Omega \\ d_\varepsilon(x)\partial_\nu u + b_\varepsilon(x)u = \psi_\varepsilon(x,u), & t > 0,\ x \in \partial\Omega. \end{cases}$$

For $\varepsilon = 0$, (1.3) is an abstract formulation of the following boundary value problem:

$$\begin{cases} u_t - \operatorname{div}(d_0(x)\nabla u) + (\lambda + V_0(x))u = \varphi_0(x,u), & t > 0,\ x \in \Omega_1 \\ d_0(x)\partial_\nu u + b_0(x)u = \psi_0(x,u), & t > 0,\ x \in \Gamma, \\ \gamma_{0,i}(u) = u_{\Omega_{0,i}}, & \text{on } \Gamma_{0,i},\ i \in [1\,..\,m], \\ \dot{u}_{\Omega_{0,i}} + |\Omega_{0,i}|^{-1}\int_{\Gamma_{0,i}} d_0(x)\partial_{\nu_{0,i}} u\,\mathrm{d}\sigma & \\ +(\lambda + \widehat{c}_i)u_{\Omega_{0,i}} = |\Omega_{0,i}|^{-1}\int_{\Omega_{0,i}} \varphi_0(x, u_{\Omega_{0,i}})\,\mathrm{d}x, & t > 0,\ i \in [1\,..\,m]. \end{cases}$$

Here, for $i \in [1\,..\,m]$, $\gamma_{0,i}$ is the trace operator on $\Gamma_{0,i}$.

It was proved in [9] that, as $\varepsilon \to 0$, then the spectrum of A_ε converges to the spectrum of A_0. Using this one can obtain results on convergence of e^{-tA_ε} to e^{-tA_0}. Now by using the variation-of-constants formula one suspects that, in some sense, some families of solutions of π_ε converge to solutions of π_0. This was proved in [1] for full bounded solutions under some additional dissipativeness conditions both on the linear and on the nonlinear problem, cf. [1, conditions S, D_ε and D_0]. This latter result also implies existence of global attractors of both π_ε and π_0 and their upper semicontinuity at $\varepsilon = 0$.

In this paper we extend these results. More specifically, we provide in subsection 3.2 various linear singular convergence results on convergence of e^{-tA_ε} to e^{-tA_0} and of $e^{-t\widetilde{A}_\varepsilon}$ to $e^{-t\widetilde{A}_0}$. Then, even without any further dissipativeness condition, we prove in subsection 4.2 that, as $\varepsilon \to 0$, the local semiflows π_ε converge in a singular sense to the local semiflow π_0. More precisely, we establish the following result:

Theorem 1.2. *Let $(\varepsilon_n)_n$ be a sequence in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$ and $(t_n)_n$ be a sequence in $[0,\infty[$ with $t_n \to t_0$, for some $t_0 \in [0,\infty[$. Let $u_0 \in H_1^0$ and $(u_n)_n$ be a sequence with $u_n \in H_1^{\varepsilon_n}$ for every $n \in \mathbb{N}$ and*

$$|u_n - u_0|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty.$$

Assume $u_0\pi_0 t_0$ is defined. Then there exists an $n_0 \in \mathbb{N}$ such that $u_n\pi_{\varepsilon_n}t_n$ is defined for all $n \geq n_0$ and

$$|u_n\pi_{\varepsilon_n}t_n - u_0\pi_0 t_0|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty.$$

We also establish, in section 5, the following singular compactness result for the family π_ε, $\varepsilon \in [0,\varepsilon_0]$.

Theorem 1.3. *Suppose $\kappa \in]0,\infty[$, $(\varepsilon_n)_n$ is a sequence in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$, $(t_n)_n$ is a sequence in $[0,\infty[$ with $t_n \geq \kappa$ for every $n \in \mathbb{N}$ and $(u_n)_n$ is a sequence with $u_n \in H_1^{\varepsilon_n}$ for every $n \in \mathbb{N}$. Assume that there exists a $C \in]0,\infty[$ such that $u_n\pi_{\varepsilon_n}t_n$ is defined and*

$$|u_n\pi_{\varepsilon_n}s|_{H_1^{\varepsilon_n}} \leq C \text{ for all } n \in \mathbb{N} \text{ and for all } s \in [0,t_n].$$

Then there exist a $v \in H_1^0$ and a sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$|u_{n_k}\pi_{\varepsilon_{n_k}}t_{n_k} - v|_{H_1^{\varepsilon_{n_k}}} \to 0 \text{ as } k \to \infty.$$

These results enable us to prove, in section 6, the following singular Conley index continuation result.

Theorem 1.4. *For each $\varepsilon \in]0,\varepsilon_0]$, let I_ε be the identity map on H_1^ε and $Q_\varepsilon \colon H_1^\varepsilon \to H_1^\varepsilon$ be the H_1^ε-orthogonal projection of H_1^ε onto H_1^0.*

Let N be a closed and bounded isolating neighborhood of an invariant set K_0 relative to π_0. For $\varepsilon \in]0,\varepsilon_0]$ and for every $\eta \in]0,\infty[$ set

$$N_{\varepsilon,\eta} := \{\, u \in H_1^\varepsilon \mid Q_\varepsilon u \in N \text{ and } |(I_\varepsilon - Q_\varepsilon)u|_{H_1^\varepsilon} \leq \eta \,\}$$

and $K_{\varepsilon,\eta} := \mathrm{Inv}_{\pi_\varepsilon}(N_{\varepsilon,\eta})$ i.e. $K_{\varepsilon,\eta}$ is the largest π_ε-invariant set in $N_{\varepsilon,\eta}$. Then for every $\eta \in]0,\infty[$ there exists an $\varepsilon^c = \varepsilon^c(\eta) \in]0,\varepsilon_0]$ such that for every $\varepsilon \in]0,\varepsilon^c]$ the set $N_{\varepsilon,\eta}$ is a strongly admissible isolating neighborhood of $K_{\varepsilon,\eta}$ relative to π_ε and

$$h(\pi_\varepsilon, K_{\varepsilon,\eta}) = h(\pi_0, K_0).$$

Furthermore, for every $\eta > 0$, the family $(K_{\varepsilon,\eta})_{\varepsilon\in[0,\varepsilon^c(\eta)]}$ of invariant sets, where $K_{0,\eta} = K_0$, is upper semicontinuous at $\varepsilon = 0$ with respect to the family $|\cdot|_{H_1^\varepsilon}$ of norms i.e.

$$\lim_{\varepsilon\to 0^+} \sup_{w\in K_{\varepsilon,\eta}} \inf_{u\in K_0} |w-u|_{H_1^\varepsilon} = 0.$$

The family $(K_{\varepsilon,\eta})_{\varepsilon\in]0,\varepsilon^c(\eta)]}$ is asymptotically independent of η i.e. whenever η_1 and $\eta_2 \in]0,\infty[$ then there is an $\varepsilon' \in]0,\min(\varepsilon^c(\eta_1),\varepsilon^c(\eta_2))]$ such that $K_{\varepsilon,\eta_1} = K_{\varepsilon,\eta_2}$ for $\varepsilon \in]0,\varepsilon']$.

Finally we also prove the following (co)homology index continuation principle:

Theorem 1.5. *Assume the hypotheses of Theorem 1.4 and for every $\eta \in]0,\infty[$ let $\varepsilon^c(\eta) \in]0,\varepsilon_0]$ be as in that theorem. Let $(M_{p,0})_{p\in P}$ be a $\prec$-ordered Morse decomposition of K_0 relative to π_0. For each $p \in P$, let $V_p \subset N$ be closed in X_0 and such that $M_{p,0} = \mathrm{Inv}_{\pi_0}(V_p) \subset \mathrm{Int}_{H_1^0}(V_p)$. (Such sets V_p, $p \in P$, exist.) For $\varepsilon \in]0,\varepsilon_0]$, for every $\eta \in]0,\infty[$ and $p \in P$ set $M_{p,\varepsilon,\eta} := \mathrm{Inv}_{\pi_\varepsilon}(V_{p,\varepsilon,\eta})$, where*

$$V_{p,\varepsilon,\eta} := \{\, u \in H_1^\varepsilon \mid Q_\varepsilon u \in V_p \text{ and } |(I_\varepsilon - Q_\varepsilon)u|_{H_1^\varepsilon} \leq \eta \,\}.$$

Then for every $\eta \in]0,\infty[$ there is an $\widetilde{\varepsilon} = \widetilde{\varepsilon}(\eta) \in]0, \varepsilon^c(\eta)]$ such that for every $\varepsilon \in]0, \widetilde{\varepsilon}]$ and $p \in P$, $M_{p,\varepsilon,\eta} \subset \mathrm{Int}_{H_1^\varepsilon}(V_{p,\varepsilon,\eta})$ and the family $(M_{p,\varepsilon,\eta})_{p\in P}$ is a $\prec$-ordered Morse decomposition of $K_{\varepsilon,\eta}$ relative to π_ε and the (co)homology index braids of $(\pi_0, K_0, (M_{p,0})_{p\in P})$ and $(\pi_\varepsilon, K_{\varepsilon,\eta}, (M_{p,\varepsilon,\eta})_{p\in P}))$, $\varepsilon \in]0, \widetilde{\varepsilon}]$, are isomorphic and so they determine the same collection of C-connection matrices.

Again, for each $p \in P$, the family $(M_{p,\varepsilon,\eta})_{\varepsilon\in[0,\widetilde{\varepsilon}(\eta)]}$, where $M_{p,0,\eta} = M_{p,0}$ is upper semicontinuous at $\varepsilon = 0$ respect to the family $|\cdot|_{H_1^\varepsilon}$ of norms and the family $(M_{p,\varepsilon,\eta})_{\varepsilon\in]0,\widetilde{\varepsilon}(\eta)]}$ is asymptotically independent of η.

In this paper we proceed as in [2] and keep the presentation of our results at an abstract level. In fact we only assume certain spectral convergence properties and compactness assumptions on a family of linear operators $(A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ (see conditions (Spec) and (Comp) in section 3). We also make an abstract convergence hypothesis (condition (Conv) in section 4) on a family of nonlinear operators $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$.

Our abstract approach permits applications to some other singular systems of reaction-diffusion equations. This will be treated in a subsequent publication.

2. Preliminaries

Suppose H is an infinite dimensional linear space which is complete with respect to the scalar product $\langle\cdot,\cdot\rangle_H$ and let $A\colon D(A) \subset H \to H$ be a (densely defined) positive self-adjoint operator on $(H, \langle\cdot,\cdot\rangle_H)$ with $A^{-1}\colon H \to H$ compact. Let $(\lambda_j)_j$ be *the repeated sequence of eigenvalues of* A, i.e. the uniquely determined nonincreasing sequence $(\lambda_j)_j$ containing exactly the eigenvalues of A and such that the number of occurrences of every eigenvalue of A in this sequence is equal to its multiplicity. Let $(w_j)_j$ be an H-orthonormal sequence of eigenvectors of A corresponding to $(\lambda_j)_j$. For $\alpha \in [0,\infty[$, let $H_\alpha = H_\alpha(A) = D(A^{\alpha/2})$. In particular,

$$H_0 = H.$$

Note that H_α is a Hilbert space under the scalar product

$$\langle u, v\rangle_{H_\alpha} = \langle A^{\alpha/2}u, A^{\alpha/2}v\rangle_H, \quad u,\, v \in H_\alpha.$$

For every $j \in \mathbb{N}$, $w_j \in H_\alpha$ and the sequence $(\lambda_j^{-\alpha/2} w_j)_j$ is H_α-orthonormal and H_α-complete. If $u \in H_\alpha$ we have

$$|u - \sum_{j=1}^{k} \langle u, w_j\rangle_H w_j|_{H_\alpha} \to 0 \text{ as } k \to \infty \tag{2.1}$$

and so

$$|u|^2_{H_\alpha} = \sum_{j=1}^{\infty} \lambda_j^\alpha |\langle u, w_j\rangle_H|^2. \tag{2.2}$$

If $\alpha \in]0,\infty[$, let $H_{-\alpha} = H'_\alpha$ be the dual of H_α. It follows that $H_{-\alpha}$ is a Hilbert space under the dual norm

$$\langle u, v\rangle_{H_{-\alpha}} = \langle F_\alpha^{-1}v, F_\alpha^{-1}u\rangle_{H_\alpha}, \quad u,\, v \in H_{-\alpha},$$

where $F_\alpha\colon H_\alpha \to H_{-\alpha}$, $u \mapsto \langle\cdot, u\rangle_{H_\alpha}$, is the Fréchet-Riesz isomorphism.

Define the map $\psi_\alpha\colon H = H_0 \to H_{-\alpha}$ by $\psi_\alpha(u) = y$, where $y\colon H_\alpha \to \mathbb{K}$ is defined by

$$y(v) = \langle v, u\rangle_H, \quad v \in H_\alpha.$$

ψ_α is an injection so that we can (and will) identify elements $u \in H$ with $\psi_\alpha(u) \in H_{-\alpha}$. We thus consider H as a linear subspace of $H_{-\alpha}$. With this identification, the sequence $(\lambda_j^{\alpha/2} w_j)_j$ is $H_{-\alpha}$-orthonormal and $H_{-\alpha}$-complete. If $u \in H_{-\alpha}$ then

$$|u - \sum_{j=1}^{k} u(w_j)w_j|_{H_{-\alpha}} \to 0 \text{ as } k \to \infty \tag{2.3}$$

and so

$$|u|^2_{H_{-\alpha}} = \sum_{j=1}^{\infty} \lambda_j^{-\alpha}|u(w_j)|^2. \tag{2.4}$$

For $\alpha \in \,]0,\infty[$ there is a unique continuous extension $\widetilde{A}^{-1} = \widetilde{A}_\alpha^{-1}\colon H_{-\alpha} \to H_{2-\alpha}$ of $A^{-1}\colon H \to H_2$. The map $\widetilde{A}^{-1}$ is a bijective linear isometry. Let $\widetilde{A}\colon H_{2-\alpha} \to H_{-\alpha}$ be the inverse of $\widetilde{A}^{-1}$. Then $\widetilde{A}$ is a positive densely defined self-adjoint operator on $H_{-\alpha}$. Moreover, for $\beta \in [0,\infty]$ the β-fractional power space $H_\beta(\widetilde{A})$ of $\widetilde{A}$ is isomorphic (as a Hilbert space) to $H_{\beta-\alpha} = H_{\beta-\alpha}(A)$.

The linear semigroup $e^{-t\widetilde{A}}\colon H_{-\alpha} \to H_{-\alpha}$, $t \in [0,\infty[$, is an extension of the semigroup $e^{-tA}\colon H \to H$, $t \in [0,\infty[$. Since, for every $j \in \mathbb{N}$ and $t \in [0,\infty[$,

$$e^{-tA}w_j = e^{-t\widetilde{A}}w_j = e^{-t\lambda_j}w_j$$

we conclude that, for every $u \in H$, every $\beta \in [0,\infty[$ and every $t \in \,]0,\infty[$

$$|e^{-tA}u - \sum_{j=1}^{k} e^{-t\lambda_j}\langle u, w_j\rangle_H w_j|_{H_\beta} \to 0 \text{ as } k \to \infty. \tag{2.5}$$

We also conclude that, for every $u \in H_{-\alpha}$, every $\beta \in [0,\infty[$ and every $t \in \,]0,\infty[$

$$|e^{-t\widetilde{A}}u - \sum_{j=1}^{k} e^{-t\lambda_j}u(w_j)w_j|_{H_\beta} \to 0 \text{ as } k \to \infty. \tag{2.6}$$

3. Singular convergence of linear semiflows

In this section we introduce two abstract hypotheses, conditions (Spec) and (Comp), and we show that the family of Hilbert spaces and the family of linear operators defined in the Introduction satisfy these conditions. We also show that condition (Spec) enables us to prove some singular convergence results for linear semiflows.

First we introduce the following spectral convergence definition for a family of Hilbert spaces and linear operators.

Definition 3.1. Given $\varepsilon_0 > 0$ we say that the family $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *satisfies condition* (Spec) if the following properties are satisfied:

(1) for every $\varepsilon \in [0, \varepsilon_0]$, $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon})$ is an infinite dimensional Hilbert space and $A_\varepsilon \colon D(A_\varepsilon) \subset H^\varepsilon \to H^\varepsilon$ is a densely defined positive self-adjoint operator on $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon})$ with $A_\varepsilon{}^{-1} \colon H^\varepsilon \to H^\varepsilon$ compact. For $\alpha \in \mathbb{R}$ write $H^\varepsilon_\alpha := H_\alpha(A_\varepsilon)$. In particular, $H^\varepsilon_0 = H^\varepsilon$;

(2) for each $\varepsilon \in]0, \varepsilon_0]$, H^0 is a linear subspace of H^ε and H^0_1 is a linear subspace of H^ε_1;

(3) there exists a constant $C \in]1, \infty[$ such that

$$|u|_{H^\varepsilon_1} \leq C|u|_{H^0_1} \text{ and } |u|_{H^0_1} \leq C|u|_{H^\varepsilon_1}$$

for all $u \in H^0_1$ and all $\varepsilon \in]0, \varepsilon_0]$;

(4) for every $\varepsilon \in]0, \varepsilon_0]$ let $(\lambda_{\varepsilon,j})_j$ be the repeated sequence of eigenvalues of A_ε and $(w_{\varepsilon,j})_j$ be a corresponding H^ε-orthonormal sequence of eigenfunctions. Furthermore, let $(\lambda_{0,j})_j$ be the repeated sequence of eigenvalues of A_0.

Whenever $(\varepsilon_n)_n$ is a sequence in $]0, \varepsilon_0]$ with $\varepsilon_n \to 0$ then

(a) $\lambda_{\varepsilon_n,j} \to \lambda_{0,j}$ as $n \to \infty$, for all $j \in \mathbb{N}$.

Moreover, there is a sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ and there is an H^0-orthonormal sequence of eigenfunctions $(w_{0,j})_j$ of A_0 corresponding to $(\lambda_{0,j})_j$ such that

(b) $|w_{\varepsilon_{n_k},j} - w_{0,j}|_{H_1^{\varepsilon_{n_k}}} \to 0$ as $k \to \infty$, for all $j \in \mathbb{N}$;

(c) $\langle u, w_{\varepsilon_{n_k},j}\rangle_{H^{\varepsilon_{n_k}}} \to \langle u, w_{0,j}\rangle_{H^0}$ as $k \to \infty$, for all $u \in H^0$ and all $j \in \mathbb{N}$.

We also require the following definition.

Definition 3.2. Let the family $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition (Spec). We say that $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *satisfies condition* (Comp) if whenever $(\varepsilon_n)_n$ is a sequence in $]0, \varepsilon_0]$ with $\varepsilon_n \to 0$ and $(\xi_n)_n$ is a sequence with $\xi_n \in H_1^{\varepsilon_n}$ for every $n \in \mathbb{N}$ and

$$\sup_{n\in\mathbb{N}} |\xi_n|_{H_1^{\varepsilon_n}} < \infty,$$

then there exist a $v \in H^0_1$ and a sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$|\xi_{n_k} - v|_{H^{\varepsilon_{n_k}}} \to 0 \text{ as } k \to \infty.$$

3.1. **Application to parabolic problems with large localized diffusion.** Now let $\varepsilon_0 \in]0, \infty[$ and the operators A_ε, $\varepsilon \in [0, \varepsilon_0]$, be as in the Introduction.

For $\varepsilon \in]0, \varepsilon_0[$ set $H^\varepsilon = L^2(\Omega)$ and $\langle\cdot,\cdot\rangle_{H^\varepsilon} = \langle\cdot,\cdot\rangle_{L^2(\Omega)}$. Moreover, write $H^0 = L^2_{\Omega_0}(\Omega)$ and $\langle\cdot,\cdot\rangle_{H^0} = \langle\cdot,\cdot\rangle_{L^2(\Omega)}$. Notice that $H^\varepsilon_0 = H^\varepsilon$ for all $\varepsilon \in [0, \varepsilon_0]$.

For $\varepsilon \in [0, \varepsilon_0]$ and $\alpha \in \mathbb{R}$ write $H^\varepsilon_\alpha = H_\alpha(A_\varepsilon)$. Then, if $\varepsilon \in]0, \varepsilon_0]$, it follows that $H^\varepsilon_1 = H_1(A_\varepsilon) = H^1(\Omega)$ and $\langle\cdot,\cdot\rangle_{H^\varepsilon_1} = \tau_\varepsilon(\cdot,\cdot)$. Furthermore, $H^0_1 = H^1_{\Omega_0}(\Omega)$ and $\langle\cdot,\cdot\rangle_{H^0_1} = \tau_0(\cdot,\cdot)$.

In particular, if $u \in H^0_1$, then, for all $\varepsilon \in]0, \varepsilon_0]$,

$$|u|_{H^\varepsilon_1} = |u|_{H^0_1}.$$

It is now easy to conclude that parts (1), (2) and (3) of Condition (Spec) are satisfied. Moreover the following result holds:

Proposition 3.3. *With the notation introduced above, the family* $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *satisfies conditions* (Spec) *and* (Comp).

Proof. [9, Theorem 5.1] implies that part (4) of Condition (Spec) also holds. Condition (Comp) follows from [9, Theorem 4.4]. □

3.2. **Singular linear convergence.** Now we will show that condition (Spec) allows us to obtain two singular convergence theorems for linear semiflows. We start with following preliminary result.

Proposition 3.4. *Let $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition* (Spec). *Then for every $\varepsilon \in]0, \varepsilon_0]$, the subspace $(H_1^0, |\cdot|_{H_1^0})$ is closed in $(H_1^\varepsilon, |\cdot|_{H_1^\varepsilon})$.*

Proof. Let $\varepsilon \in]0, \varepsilon_0]$ and suppose $(u_n)_n$ is a sequence in H_1^0 with $|u_n - u|_{H_1^\varepsilon} \to 0$ as $n \to \infty$ for some $u \in H_1^\varepsilon$. Part (3) of condition (Spec) implies that

$$|u_n - u_m|_{H_1^0} \le C|u_n - u_m|_{H_1^\varepsilon},$$

so $(u_n)_n$ is a Cauchy sequence in the Banach space $(H_1^0, |\cdot|_{H_1^0})$. Therefore $(u_n)_n$ converges in H_1^0 to some v in H_1^0. But part (3) of condition (Spec) implies that

$$|u_n - v|_{H_1^\varepsilon} \le C|u_n - v|_{H_1^0}.$$

Hence $u = v$ and thus $u \in H_1^0$. This proves the proposition. □

Remark 3.5. Note that, for α, $t \in]0,\infty[$ and $\lambda \in [0,\infty[$

$$\lambda^\alpha e^{-\lambda t} \le C(\alpha)t^{-\alpha} \text{ with } C(\alpha) = (\alpha/e)^\alpha.$$

Let $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition (Spec). Let $\alpha \in [0,\infty[$, $\varepsilon \in [0,\varepsilon_0]$ and $r \in]0,\infty[$. Using the above estimate, we obtain for every $u \in H_{-\alpha}^\varepsilon$

$$\begin{aligned}
|e^{-\tilde{A}_\varepsilon r}u|^2_{H_1^\varepsilon} &= \sum_{j=1}^\infty \lambda_{\varepsilon,j}^{\alpha+1}(e^{-\lambda_{\varepsilon,j}r})^2\lambda_{\varepsilon,j}^{-\alpha}|u(w_{\varepsilon,j})|^2 \\
&= \sum_{j=1}^\infty ((\lambda_{\varepsilon,j})^{(\alpha+1)/2}e^{-\lambda_{\varepsilon,j}r})^2\lambda_{\varepsilon,j}^{-\alpha}|u(w_{\varepsilon,j})|^2 \le (C((\alpha+1)/2)^2 r^{-(\alpha+1)})|u|^2_{H_{-\alpha}^\varepsilon}.
\end{aligned}$$

Consequently, we obtain for every $u \in H_{-\alpha}^\varepsilon$

$$|e^{-\tilde{A}_\varepsilon r}u|_{H_1^\varepsilon} \le C_0 r^{-(\alpha+1)/2}|u|_{H_{-\alpha}^\varepsilon}, \tag{3.1}$$

where $C_0 = C((\alpha+1)/2)$.

We shall need these estimates in the results to follow.

We now prove our first result on the convergence of the linear semiflows.

Theorem 3.6. *Let $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition* (Spec). *Suppose $(\varepsilon_n)_n$ is a sequence in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$. Let $u_0 \in H_1^0$ and $(u_n)_n$ be a sequence such that, for every $n \in \mathbb{N}$, $u_n \in H_1^{\varepsilon_n}$ and*

$$|u_n - u_0|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty.$$

Then

$$\sup_{t\in[0,\infty[} |e^{-tA_{\varepsilon_n}}u_n - e^{-tA_0}u_0|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty.$$

Proof. Since $\lambda_{\varepsilon,j} > 0$ for all $\varepsilon \in]0, \varepsilon_0]$ and for all $j \in \mathbb{N}$, we have

$$|e^{-tA_\varepsilon}v|^2_{H_1^\varepsilon} = \sum_{j=1}^{\infty}(e^{-t\lambda_{\varepsilon,j}})^2\lambda_{\varepsilon,j}|\langle v, w_{\varepsilon,j}\rangle_{H^\varepsilon}|^2 \leq \sum_{j=1}^{\infty}\lambda_{\varepsilon,j}|\langle v, w_{\varepsilon,j}\rangle_{H^\varepsilon}|^2 = |v|^2_{H_1^\varepsilon},$$

for all $v \in H_1^\varepsilon$, $\varepsilon \in]0, \varepsilon_0]$ and $t \in [0, \infty[$. Thus we obtain, for all $n \in \mathbb{N}$ and all $t \in [0, \infty[$,

$$\begin{aligned}|e^{-tA_{\varepsilon_n}}u_n - e^{-tA_0}u_0|_{H_1^{\varepsilon_n}} &\leq |e^{-tA_{\varepsilon_n}}(u_n - u_0)|_{H_1^{\varepsilon_n}} + |e^{-tA_{\varepsilon_n}}u_0 - e^{-tA_0}u_0|_{H_1^{\varepsilon_n}} \\ &\leq |u_n - u_0|_{H_1^{\varepsilon_n}} + |e^{-tA_{\varepsilon_n}}u_0 - e^{-tA_0}u_0|_{H_1^{\varepsilon_n}}.\end{aligned}$$

Therefore we only have to prove that

$$\sup_{t\in[0,\infty[} |e^{-tA_{\varepsilon_n}}u_0 - e^{-tA_0}u_0|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty. \tag{3.2}$$

Suppose (3.2) is not true. Then there are a $\delta_0 > 0$ and a sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\sup_{t\in[0,\infty[} |e^{-tA_{\varepsilon_{n_k}}}u_0 - e^{-tA_0}u_0|_{H_1^{\varepsilon_{n_k}}} \geq \delta_0 \text{ for all } k \in \mathbb{N}. \tag{3.3}$$

Taking a further subsequence, if necessary, and using condition (Spec) we may also assume that there exists an H^0-orthonormal sequence of eigenfunctions $(w_{0,j})_j$ corresponding to $(\lambda_{0,j})_j$ such that, for all $j \in \mathbb{N}$ and $u \in H^0$,

$$|w_{\varepsilon_{n_k},j} - w_{0,j}|_{H_1^{\varepsilon_{n_k}}} \to 0 \text{ and } \langle u, w_{\varepsilon_{n_k},j}\rangle_{H^{\varepsilon_{n_k}}} \to \langle u, w_{0,j}\rangle_{H^0} \text{ as } k \to \infty. \tag{3.4}$$

For each $k \in \mathbb{N}$ and $j \in \mathbb{N}$, let $P_{k,j}\colon H^{\varepsilon_{n_k}} \to H^{\varepsilon_{n_k}}$ be the $H^{\varepsilon_{n_k}}$-orthogonal projection of $H^{\varepsilon_{n_k}}$ onto the span of $\{ w_{\varepsilon_{n_k},1}, \dots, w_{\varepsilon_{n_k},j-1} \}$ and let $P_{0,j}\colon H^0 \to H^0$ be the H^0-orthogonal projection of H^0 onto the span of $\{ w_{0,1}, \dots, w_{0,j-1} \}$.

Let $t \in [0, \infty[$ be arbitrary. Then for each $j \in \mathbb{N}$ and each $k \in \mathbb{N}$ we have

$$\begin{aligned}|e^{-tA_{\varepsilon_{n_k}}}u_0 - e^{-tA_0}u_0|_{H_1^{\varepsilon_{n_k}}} &\leq |P_{k,j}e^{-tA_{\varepsilon_{n_k}}}u_0 - P_{0,j}e^{-tA_0}u_0|_{H_1^{\varepsilon_{n_k}}} \\ &\quad + |(I - P_{k,j})e^{-tA_{\varepsilon_{n_k}}}u_0|_{H_1^{\varepsilon_{n_k}}} + |(I - P_{0,j})e^{-tA_0}u_0|_{H_1^{\varepsilon_{n_k}}}.\end{aligned}$$

Notice that for each $j \in \mathbb{N}$,

$$|P_{k,j}u_0 - P_{0,j}u_0|_{H_1^{\varepsilon_{n_k}}} \to 0 \text{ as } k \to \infty. \tag{3.5}$$

Indeed, for each $k \in \mathbb{N}$ we have

$$\begin{aligned}|P_{k,j}u_0 - P_{0,j}u_0|_{H_1^{\varepsilon_{n_k}}} &= \left|\sum_{i=1}^{j-1}\langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}} w_{\varepsilon_{n_k},i} - \sum_{i=1}^{j-1}\langle u_0, w_{0,i}\rangle_{H^0} w_{0,i}\right|_{H_1^{\varepsilon_{n_k}}} \\ &\leq \sum_{i=1}^{j-1} |\langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}}|\,|w_{\varepsilon_{n_k},i} - w_{0,i}|_{H_1^{\varepsilon_{n_k}}} \\ &\quad + \sum_{i=1}^{j-1} |\langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}} - \langle u_0, w_{0,i}\rangle_{H^0}|\,|w_{0,i}|_{H_1^{\varepsilon_{n_k}}}.\end{aligned}$$

Condition (Spec) and (3.4) now imply (3.5).

Let $\delta > 0$ be arbitrary. By (2.1), $|(I - P_{0,j})u_0|_{H_1^0} \to 0$ as $j \to \infty$, so there is a $j_0 \in \mathbb{N}$ such that

$$|(I - P_{0,j_0})u_0|_{H_1^0} < \delta.$$

Since $|(I - P_{k,j})u_0 - (I - P_{0,j})u_0|_{H_1^{\varepsilon_{n_k}}} = |P_{k,j}u_0 - P_{0,j}u_0|_{H_1^{\varepsilon_{n_k}}}$ for all $j \in \mathbb{N}$ and for all $k \in \mathbb{N}$, it follows from (3.5) that there is a $k_0 \in \mathbb{N}$ such that

$$|(I - P_{k,j_0})u_0 - (I - P_{0,j_0})u_0|_{H_1^{\varepsilon_{n_k}}} < \delta \text{ for all } k \geq k_0. \tag{3.6}$$

Hence for all $k \geq k_0$,

$$\begin{aligned} |(I - P_{k,j_0})e^{-tA_{\varepsilon_{n_k}}}u_0|_{H_1^{\varepsilon_{n_k}}} &= |e^{-tA_{\varepsilon_{n_k}}}(I - P_{k,j_0})u_0|_{H_1^{\varepsilon_{n_k}}} \\ &\leq |(I - P_{k,j_0})u_0|_{H_1^{\varepsilon_{n_k}}} \leq \delta + |(I - P_{0,j_0})u_0|_{H_1^{\varepsilon_{n_k}}} \\ &\leq \delta + C|(I - P_{0,j_0})u_0|_{H_1^0} \leq (1 + C)\delta. \end{aligned} \tag{3.7}$$

Moreover,

$$\begin{aligned} |(I - P_{0,j_0})e^{-tA_0}u_0|_{H_1^{\varepsilon_{n_k}}} &\leq C|(I - P_{0,j_0})e^{-tA_0}u_0|_{H_1^0} = C|e^{-tA_0}(I - P_{0,j_0})u_0|_{H_1^0} \\ &\leq C|(I - P_{0,j_0})u_0|_{H_1^0} \leq C\delta. \end{aligned} \tag{3.8}$$

We further have

$$\begin{aligned} &|P_{k,j_0}e^{-tA_{\varepsilon_{n_k}}}u_0 - P_{0,j_0}e^{-tA_0}u_0|_{H_1^{\varepsilon_{n_k}}} \\ &\leq \sum_{i=1}^{j_0-1} |e^{-t\lambda_{\varepsilon_{n_k},i}}\langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}} w_{\varepsilon_{n_k},i} - e^{-t\lambda_{0,i}}\langle u_0, w_{0,i}\rangle_{H^0} w_{0,i}|_{H_1^{\varepsilon_{n_k}}} \\ &\leq \sum_{i=1}^{j_0-1} |e^{-t\lambda_{\varepsilon_{n_k},i}}\langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}} (w_{\varepsilon_{n_k},i} - w_{0,i})|_{H_1^{\varepsilon_{n_k}}} \\ &+ \sum_{i=1}^{j_0-1} |e^{-t\lambda_{\varepsilon_{n_k},i}}\langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}} w_{0,i} - e^{-t\lambda_{0,i}}\langle u_0, w_{0,i}\rangle_{H^0} w_{0,i}|_{H_1^{\varepsilon_{n_k}}} \\ &\leq \sum_{i=1}^{j_0-1} |\langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}}|\, |w_{\varepsilon_{n_k},i} - w_{0,i}|_{H_1^{\varepsilon_{n_k}}} \\ &+ C\sum_{i=1}^{j_0-1} |e^{-t\lambda_{\varepsilon_{n_k},i}}\langle u_0, w_{\varepsilon_{n_k},i}\rangle_{H^{\varepsilon_{n_k}}} - e^{-t\lambda_{0,i}}\langle u_0, w_{0,i}\rangle_{H^0}|\, |w_{0,i}|_{H_1^0}. \end{aligned}$$

Since for every $i \in \mathbb{N}$,

$$\sup_{t\in[0,\infty[} |e^{-t\lambda_{\varepsilon_{n_k},i}} - e^{-t\lambda_{0,i}}| \to 0 \text{ as } k \to \infty,$$

it follows that

$$\sup_{t\in[0,\infty[} |P_{k,j_0}e^{-tA_{\varepsilon_{n_k}}}u_0 - P_{0,j_0}e^{-tA_0}u_0|_{H_1^{\varepsilon_{n_k}}} \to 0 \text{ as } k \to \infty. \tag{3.9}$$

Since $\delta > 0$ is arbitrary, (3.7), (3.8) and (3.9) imply that

$$\sup_{t\in[0,\infty[} |e^{-tA_{\varepsilon_{n_k}}}u_0 - e^{-tA_0}u_0|_{H_1^{\varepsilon_{n_k}}} \to 0 \text{ as } k \to \infty, \tag{3.10}$$

but this contradicts (3.3). The proof is complete. □

We also require a second, more technical, theorem on the convergence of the linear semiflows.

Theorem 3.7. *Let $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition* (Spec). *Suppose $(\varepsilon_n)_n$ is a sequence in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$. Let $\alpha \in [0,\infty[$, $u_0 \in H^0_{-\alpha}$ be arbitrary and let $(u_n)_n$ and $(v_n)_n$ be sequences such that u_n and $v_n \in H^{\varepsilon_n}_{-\alpha}$ for $n \in \mathbb{N}$. Suppose that*

(1) $|u_n - v_n|_{H^{\varepsilon_n}_{-\alpha}} \to 0$ *as* $n \to \infty$.
(2) *For all* $j \in \mathbb{N}$, $v_n(w_{\varepsilon_n,j}) \to u_0(w_{0,j})$ *as* $n \to \infty$.
(3) $\sup_{n\in\mathbb{N}} |v_n|_{H^{\varepsilon_n}_{-\alpha}} < \infty$.

For every $\varepsilon \in [0,\varepsilon_0]$, let $\widetilde{A}_\varepsilon = \widetilde{A}_{\varepsilon,-\alpha}\colon H^\varepsilon_{2-\alpha} \to H^\varepsilon_{-\alpha}$ be the extension of A_ε to $H^\varepsilon_{-\alpha}$. Then for every $\beta \in]0,\infty[$

$$\sup_{t\in[\beta,\infty[} \left| e^{-t\widetilde{A}_{\varepsilon_n}} u_n - e^{-t\widetilde{A}_0} u_0 \right|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty.$$

Proof. Fix $\beta \in]0,\infty[$. Suppose the theorem is not true. Then there are a $\delta_0 > 0$ and a sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\sup_{t\in[\beta,\infty[} \left| e^{-t\widetilde{A}_{\varepsilon_{n_k}}} u_{n_k} - e^{-t\widetilde{A}_0} u_0 \right|_{H_1^{\varepsilon_{n_k}}} \geq \delta_0 \text{ for all } k \in \mathbb{N}. \tag{3.11}$$

Taking a further subsequence, if necessary, and using condition (Spec) we may also assume that there exists an H^0-orthonormal sequence of eigenfunctions $(w_{0,j})_j$ corresponding to $(\lambda_{0,j})_j$ such that, for all $j \in \mathbb{N}$ and $u \in H^0$,

$$|w_{\varepsilon_{n_k},j} - w_{0,j}|_{H_1^{\varepsilon_{n_k}}} \to 0 \text{ and } \langle u, w_{\varepsilon_{n_k},j}\rangle_{H^{\varepsilon_{n_k}}} \to \langle u, w_{0,j}\rangle_{H^0} \text{ as } k \to \infty. \tag{3.12}$$

Let $\delta > 0$ be arbitrary. By Remark 3.5 there is an $s_0 = s_0(\delta,\beta) > 0$ such that $s^{(\alpha+1)/2}e^{-st} < \delta$ for $s \geq s_0$ and $t \geq \beta$. Since $\lambda_{0,j} \to \infty$ as $j \to \infty$, there is a $j_0 = j_0(\delta,\beta) \in \mathbb{N}$ such that $\lambda_{0,j_0} > s_0$ for all $j \geq j_0$. Thus there is an $k_0 = k_0(\delta,\beta) \in \mathbb{N}$ such that $\lambda_{\epsilon_{n_k},j_0} > s_0$ for $k \geq k_0$. Therefore we obtain

$$\lambda_{\epsilon_{n_k},j} \geq s_0(\delta,\beta) \text{ for } k \geq k_0(\delta,\beta) \text{ and } j \geq j_0(\delta,\beta). \tag{3.13}$$

Formula (2.6) implies that, for all $\varepsilon \in [0,\varepsilon_0]$, all $t \in]0,\infty[$ and all $u \in H^\varepsilon_{-\alpha}$,

$$|e^{-t\widetilde{A}_\varepsilon} u - \sum_{j=1}^{k} e^{-t\lambda_{\varepsilon,j}} u(w_{\varepsilon,j}) w_{\varepsilon,j}|_{H_1^\varepsilon} \to 0 \text{ as } k \to \infty. \tag{3.14}$$

Let $t \geq \beta$ be arbitrary. Then

(3.15)
$$\begin{aligned}
|e^{-t\widetilde{A}_{\varepsilon_{n_k}}} u_{n_k} - e^{-t\widetilde{A}_0} u_0|_{H_1^{\varepsilon_{n_k}}} &\leq \sum_{j=1}^{j_0-1} |e^{-t\lambda_{\varepsilon_{n_k},j}} u_{n_k}(w_{\varepsilon_{n_k},j}) w_{\varepsilon_{n_k},j} - e^{-t\lambda_{0,j}} u_0(w_{0,j}) w_{0,j}|_{H_1^{\varepsilon_{n_k}}} \\
&\quad + |e^{-t\widetilde{A}_{\varepsilon_{n_k}}} u_{n_k} - \sum_{j=1}^{j_0-1} e^{-t\lambda_{\varepsilon_{n_k},j}} u_{n_k}(w_{\varepsilon_{n_k},j}) w_{\varepsilon_{n_k},j}|_{H_1^{\varepsilon_{n_k}}} \\
&\quad + |e^{-t\widetilde{A}_0} u_0 - \sum_{j=1}^{j_0-1} e^{-t\lambda_{0,j}} u_0(w_{0,j}) w_{0,j}|_{H_1^{\varepsilon_{n_k}}}.
\end{aligned}$$

Now (3.13) implies that

$$\begin{aligned}
&|e^{-t\widetilde{A}_{\varepsilon_{n_k}}}u_{n_k} - \sum_{j=1}^{j_0-1} e^{-t\lambda_{\varepsilon_{n_k},j}}u_{n_k}(w_{\varepsilon_{n_k},j})w_{\varepsilon_{n_k},j}|^2_{H_1^{\varepsilon_{n_k}}} \\
&= \sum_{j=j_0}^{\infty} (\lambda_{\varepsilon_{n_k},j}^{(\alpha+1)/2} e^{-t\lambda_{\varepsilon_{n_k},j}})^2 \lambda_{\varepsilon_{n_k},j}^{-\alpha} |u_{n_k}(w_{\varepsilon_{n_k},j})|^2 \\
&\le \delta^2 \sum_{j=j_0}^{\infty} \lambda_{\varepsilon_{n_k},j}^{-\alpha} |u_{n_k}(w_{\varepsilon_{n_k},j})|^2 \le \delta^2 |u_{n_k}|^2_{H_{-\alpha}^{\varepsilon_{n_k}}} \le \delta^2 \widetilde{C}^2,
\end{aligned} \tag{3.16}$$

where $\widetilde{C} := \sup_{k\in\mathbb{N}} |u_{n_k}|^2_{H_{-\alpha}^{\varepsilon_{n_k}}}$. Note that $\widetilde{C} < \infty$ by our assumptions (1) and (3). Analogously,

$$\begin{aligned}
&|e^{-t\widetilde{A}_0}u_0 - \sum_{j=1}^{j_0-1} e^{-t\lambda_{0,j}}u_0(w_{0,j})w_{0,j}|^2_{H_1^{\varepsilon_{n_k}}} \\
&\le C^2 |e^{-t\widetilde{A}_0}u_0 - \sum_{j=1}^{j_0-1} e^{-t\lambda_{0,j}}u_0(w_{0,j})w_{0,j}|^2_{H_1^0} \\
&= C^2 \sum_{j=j_0}^{\infty} (\lambda_{0,j}^{(\alpha+1)/2} e^{-t\lambda_{0,j}})^2 \lambda_{0,j}^{-\alpha} |u_0(w_{0,j})|^2 \\
&\le C^2\delta^2 \sum_{j=j_0}^{\infty} \lambda_{0,j}^{-\alpha} |u_0(w_{0,j})|^2 \le C^2\delta^2 |u_0|^2_{H_{-\alpha}^0}.
\end{aligned} \tag{3.17}$$

Let $j \in [1\,..\,j_0-1]$ be arbitrary. Then

$$\begin{aligned}
&|e^{-t\lambda_{\varepsilon_{n_k},j}}u_{n_k}(w_{\varepsilon_{n_k},j})w_{\varepsilon_{n_k},j} - e^{-t\lambda_{0,j}}u_0(w_{0,j})w_{0,j}|_{H_1^{\varepsilon_{n_k}}} \\
&\le |e^{-t\lambda_{\varepsilon_{n_k},j}}(u_{n_k} - v_{n_k})(w_{\varepsilon_{n_k},j})w_{\varepsilon_{n_k},j}|_{H_1^{\varepsilon_{n_k}}} \\
&+ |e^{-t\lambda_{\varepsilon_{n_k},j}}v_{n_k}(w_{\varepsilon_{n_k},j})(w_{\varepsilon_{n_k},j} - w_{0,j})|_{H_1^{\varepsilon_{n_k}}} \\
&+ |e^{-t\lambda_{\varepsilon_{n_k},j}}(v_{n_k}(w_{\varepsilon_{n_k},j}) - u_0(w_{0,j}))w_{0,j}|_{H_1^{\varepsilon_{n_k}}} \\
&+ |(e^{-t\lambda_{\varepsilon_{n_k},j}} - e^{-t\lambda_{0,j}})u_0(w_{0,j})w_{0,j}|_{H_1^{\varepsilon_{n_k}}} \\
&\le |u_{n_k} - v_{\varepsilon_{n_k},j}|_{H_{-\alpha}^{\varepsilon_{n_k}}} |w_{\varepsilon_{n_k},j}|_{H_\alpha^{\varepsilon_{n_k}}} |w_{\varepsilon_{n_k},j}|_{H_1^{\varepsilon_{n_k}}} \\
&+ |v_{n_k}|_{H_{-\alpha}^{\varepsilon_{n_k}}} |w_{\varepsilon_{n_k},j}|_{H_\alpha^{\varepsilon_{n_k}}} \cdot |w_{\varepsilon_{n_k},j} - w_{0,j}|_{H_1^{\varepsilon_{n_k}}} \\
&+ |v_{n_k}(w_{\varepsilon_{n_k},j}) - u_0(w_{0,j})| \cdot |w_{0,j}|_{H_1^{\varepsilon_{n_k}}} \\
&+ |e^{-t\lambda_{\varepsilon_{n_k},j}} - e^{-t\lambda_{0,j}}| \cdot |u_0(w_{0,j})| \cdot |w_{0,j}|_{H_1^{\varepsilon_{n_k}}}.
\end{aligned} \tag{3.18}$$

Note that, for every $\gamma \in [0,\infty[$, $|w_{\varepsilon_{n_k},j}|_{H_\gamma^{\varepsilon_{n_k}}} = \lambda_{\varepsilon_{n_k},j}^{\gamma/2}$. Moreover, $|w_{0,j}|_{H_1^{\varepsilon_{n_k}}} \le C|w_{0,j}|_{H_1^0}$ and

$$\sup_{t\in[\beta,\infty[} |e^{-t\lambda_{\varepsilon_{n_k},j}} - e^{-t\lambda_{0,j}}| \to 0 \text{ as } k \to \infty.$$

Hence, our assumptions and (3.18) show that

$$\sup_{t\in[\beta,\infty[} |e^{-t\lambda_{\varepsilon_{n_k},j}} u_{n_k}(w_{\varepsilon_{n_k},j}) w_{\varepsilon_{n_k},j} - e^{-t\lambda_{0,j}} u_0(w_{0,j}) w_{0,j}|_{H_1^{\varepsilon_{n_k}}} \to 0 \text{ as } k \to \infty. \tag{3.19}$$

Thus formulas (3.15), (3.16), (3.17), (3.19) and the fact that $\delta > 0$ is arbitrary imply that

$$\sup_{t\in[\beta,\infty[} \left| e^{-t\widetilde{A}_{\varepsilon_{n_k}}} u_{n_k} - e^{-t\widetilde{A}_0} u_0 \right|_{H_1^{\varepsilon_{n_k}}} \to 0 \text{ as } k \to \infty$$

which contradicts (3.11). The theorem is proved. □

Corollary 3.8. *Let $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition* (Spec). *Suppose $(\varepsilon_n)_n$ is a sequence in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$. Let $u_0 \in H^0$ be arbitrary and let $(u_n)_n$ be a sequence such that $u_n \in H^{\varepsilon_n}$ for $n \in \mathbb{N}$. Suppose that*

$$|u_n - u_0|_{H^{\varepsilon_n}} \to 0 \text{ as } n \to \infty.$$

Then for every $\beta \in]0,\infty[$

$$\sup_{t\in[\beta,\infty[} |e^{-tA_{\varepsilon_n}} u_n - e^{-tA_0} u_0|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty.$$

Proof. Use Theorem 3.7 with $\alpha = 0$ and $v_n = u_0$ for all $n \in \mathbb{N}$. □

4. Singular convergence of nonlinear semiflows

We now introduce a natural condition on a family of nonlinearities, condition (Conv), and show that the family of maps defined in the Introduction satisfies this condition. We also show that conditions (Spec) and (Conv) imply a general singular convergence theorem for semiflows. The proof of Theorem 1.2 concludes this section.

Definition 4.1. Let $\varepsilon_0 > 0$ be arbitrary and $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ be a family satisfying condition (Spec). Let $\alpha \in [0,1[$ be given and for every $\varepsilon \in [0,\varepsilon_0]$ let $\widetilde{A}_\varepsilon = \widetilde{A}_{\varepsilon,-\alpha} : H^\varepsilon_{2-\alpha} \to H^\varepsilon_{-\alpha}$ be the extension of A_ε to $H^\varepsilon_{-\alpha}$. We say that the family $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ of maps satisfies condition (Conv) if the following properties are satisfied:

(1) $f_\varepsilon : H_1^\varepsilon \to H^\varepsilon_{-\alpha}$ for every $\varepsilon \in [0,\varepsilon_0]$.
(2) $\lim_{\varepsilon\to 0^+} |e^{-t\widetilde{A}_\varepsilon} f_\varepsilon(u) - e^{-t\widetilde{A}_0} f_0(u)|_{H_1^\varepsilon} = 0$ for every $u \in H_1^0$ and every $t \in]0,\infty[$.
(3) For every $M \in [0,\infty[$ there is an $L = L_M \in [0,\infty[$ such that

$$|f_\varepsilon(u) - f_\varepsilon(v)|_{H^\varepsilon_{-\alpha}} \le L|u-v|_{H_1^\varepsilon}$$

for all $\varepsilon \in [0,\varepsilon_0]$ and $u, v \in H_1^\varepsilon$ satisfying $|u|_{H_1^\varepsilon}, |v|_{H_1^\varepsilon} \le M$.
(4) For every $u \in H_1^0$ there is an $\varepsilon_0' \in]0,\varepsilon_0]$ such that $\sup_{\varepsilon\in[0,\varepsilon_0']} |f_\varepsilon(u)|_{H^\varepsilon_{-\alpha}} < \infty$.

4.1. **Application to parabolic problems with large localized diffusion.** Let ε_0, the Hilbert spaces H^ε and the operators A_ε, $\varepsilon \in [0,\varepsilon_0]$, be as in subsection 3.1. Furthermore, let

$$2^*_\Omega = \begin{cases} \frac{2N}{N-2}, & \text{if } N \ge 3; \\ \text{an arbitrary } p^* \in]0,\infty[, & \text{if } N = 2; \\ \infty, & \text{if } N = 1 \end{cases}$$

and

$$2_\Gamma^* = \begin{cases} \frac{2(N-1)}{N-2}, & \text{if } N \geq 3; \\ \text{an arbitrary } p^{**} \in]0, \infty[, & \text{if } N = 2; \\ \infty, & \text{if } N = 1. \end{cases}$$

By interpolation theory (cf. [12]) for every $\theta \in [0,1]$ and every $\varepsilon \in [0, \varepsilon_0]$ there is a continuous imbedding from H_θ^ε to $H^\theta(\Omega)$ with imbedding constant $C_{1,\theta} \in]0, \infty[$ independent of $\varepsilon \in [0, \varepsilon_0]$. Furthermore, there is a continuous imbedding from $H^\theta(\Omega)$ into $L^{p_{\theta,\Omega}}(\Omega)$ with imbedding constant $C_{2,\theta} \in]0, \infty[$. Here,

$$p_{\theta,\Omega} = \left(\theta \frac{1}{2_\Omega^*} + (1-\theta)\frac{1}{2}\right)^{-1}.$$

Moreover, for every $\rho \in [0,1]$ there is a continuous imbedding from $H^{\rho/2}(\Gamma)$ into $L^{p_{\rho,\Gamma}}(\Gamma)$ with imbedding constant $C_{3,\rho} \in]0, \infty[$. Here,

$$p_{\rho,\Gamma} = \left(\rho \frac{1}{2_\Gamma^*} + (1-\rho)\frac{1}{2}\right)^{-1}.$$

Finally, by [8], for every $\theta \in]1/2, 1]$ there is a bounded linear trace operator $\gamma = \gamma_\theta \colon H^\theta(\Omega) \to H^{\theta-(1/2)}(\Gamma)$ with a bound $C_{4,\theta} \in]0, \infty[$. Now the continuity of the functions $\theta \mapsto p_{\theta,\Omega}$ and $\theta \mapsto p_{2\theta-1,\Gamma}$ at $\theta = 1$ implies the following result.

Lemma 4.2. *Let $q_2 \in](1-(1/2_\Omega^*))^{-1}, \infty[$ and $q_3 \in](1-(1/2_\Gamma^*))^{-1}, \infty[$ be arbitrary. Then there is a $\theta \in]1/2, 1]$ such that*

$$p_2 = \frac{q_2}{q_2 - 1} < p_{\theta,\Omega} \text{ and } p_3 = \frac{q_3}{q_3 - 1} < p_{2\theta-1,\Gamma}.$$

Set $\alpha = \theta$ and let $C_5 \in]0, \infty[$ (resp. $C_6 \in]0, \infty[$) be a bound of the imbedding $L^{p_{\alpha,\Omega}}(\Omega) \to L^{p_2}(\Omega)$ (resp. $L^{p_{2\alpha-1,\Gamma}}(\Gamma) \to L^{p_3}(\Gamma)$). Then, whenever $\Phi \in L^{q_2}(\Omega)$, $\Psi \in L^{q_3}(\Gamma)$, $\varepsilon \in [0, \varepsilon_0]$ and $h \in H_\alpha^\varepsilon$, then $\Phi \cdot h \in L^1(\Omega)$, $\Psi \cdot \gamma(h) \in L^1(\Gamma)$,

$$\int_\Omega |\Phi \cdot h| \, \mathrm{d}x \leq C_{1,\alpha} C_{2,\alpha} C_5 |\Phi|_{L^{q_2}(\Omega)} |h|_{H_\alpha^\varepsilon}, \text{ and}$$

$$\int_\Gamma |\Psi \cdot \gamma(h)| \, \mathrm{d}\sigma \leq C_{1,\alpha} C_{4,\alpha} C_{3,2\alpha-1} C_6 |\Psi|_{L^{q_3}(\Gamma)} |h|_{H_\alpha^\varepsilon}.$$

In particular, there is a unique $f_\varepsilon \in H_{-\alpha}^\varepsilon$ such that

$$f_\varepsilon(h) = \int_\Omega \Phi \cdot h \, \mathrm{d}x + \int_\Gamma \Psi \cdot \gamma(h) \, \mathrm{d}\sigma, \quad h \in H_\alpha^\varepsilon.$$

Moreover,

$$|f_\varepsilon|_{H_{-\alpha}^\varepsilon} \leq C_{7,\alpha}(|\Phi|_{L^{q_2}(\Omega)} + |\Psi|_{L^{q_3}(\Gamma)})$$

where $C_{7,\alpha} = \max(C_{1,\alpha} C_{2,\alpha} C_5, C_{1,\alpha} C_{4,\alpha} C_{3,2\alpha-1} C_6)$.

The next two theorems describe how to obtain a family of maps that satisfies hypothesis (Conv).

Theorem 4.3. *For $\varepsilon \in [0, \varepsilon_0]$, let $\Phi_\varepsilon \colon H^1(\Omega) \to L^{q_2}(\Omega)$ and $\Psi_\varepsilon \colon H^{1/2}(\Gamma) \to L^{q_3}(\Gamma)$ be maps satisfying the following assumptions:*

(1) *For all $M \in [0, \infty[$ there is an $L = L_M \in [0, \infty[$ such that*

(a) *for all* $\varepsilon \in [0, \varepsilon_0]$ *and all* u, $v \in H^1(\Omega)$ *with* $|u|_{H^1(\Omega)}$, $|v|_{H^1(\Omega)} \leq M$,

$$|\Phi_\varepsilon(u) - \Phi_\varepsilon(v)|_{L^{q_2}(\Omega)} \leq L|u - v|_{H^1(\Omega)}$$

(b) *for all* $\varepsilon \in [0, \varepsilon_0]$ *and all* u, $v \in H^{1/2}(\Gamma)$ *with* $|u|_{H^{1/2}(\Gamma)}$, $|v|_{H^{1/2}(\Gamma)} \leq M$,

$$|\Psi_\varepsilon(u) - \Psi_\varepsilon(v)|_{L^{q_3}(\Gamma)} \leq L|u - v|_{H^{1/2}(\Gamma)}.$$

(2) *For every* $u \in H^1_{\Omega_0}(\Omega)$,

$$|\Phi_\varepsilon(u) - \Phi_0(u)|_{L^{q_2}(\Omega)} \to 0 \text{ as } \varepsilon \to 0^+.$$

(3) *For every* $u \in H^{1/2}(\Gamma)$,

$$|\Psi_\varepsilon(u) - \Psi_0(u)|_{L^{q_3}(\Gamma)} \to 0 \text{ as } \varepsilon \to 0^+.$$

Let $\alpha \in]1/2, 1]$ *be as in Lemma 4.2. For* $\varepsilon \in [0, \varepsilon_0]$ *and* $u \in H^\varepsilon_1$ *define, for* $h \in H^\varepsilon_\alpha$,

$$f_\varepsilon(u)(h) = \int_\Omega \Phi_\varepsilon(u) \cdot h \,\mathrm{d}x + \int_\Gamma \Psi_\varepsilon(\gamma(u)) \cdot \gamma(h) \,\mathrm{d}\sigma.$$

Then $f_\varepsilon(u) \in H^\varepsilon_{-\alpha}$ *and the family* $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ *of maps satisfies condition* (Conv).

Remark 4.4. By the definition of 2^*_Ω and 2^*_Γ we may, for $N = 1$, 2, take q_2 and q_3 arbitrary in $]1, \infty[$.

Proof of Theorem 4.3. Lemma 4.2 implies that the family $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfies (1) of condition (Conv). Let $M \in [0, \infty[$ be arbitrary and $L = L_M$ be as in assumption (1). If $\varepsilon \in [0, \varepsilon_0]$ and u, $v \in H^\varepsilon_1$ with $|u|_{H^\varepsilon_1}$, $|v|_{H^\varepsilon_1} \leq M/C_{1,1}$ then u, $v \in H^1(\Omega)$ with $|u|_{H^1(\Omega)}$, $|v|_{H^1(\Omega)} \leq M$ so

$$\begin{aligned}|f_\varepsilon(u) - f_\varepsilon(v)|_{H^\varepsilon_{-\alpha}} &\leq C_{7,\alpha}(|\Phi_\varepsilon(u) - \Phi_\varepsilon(v)|_{L^{q_2}(\Omega)} + |\Psi_\varepsilon(\gamma(u)) - \Psi_\varepsilon(\gamma(v))|_{L^{q_3}(\Gamma)}) \\ &\leq C_{7,\alpha}(L + LC_{4,1})|u - v|_{H^1(\Omega)} \leq C_{7,\alpha}(L + LC_{4,1})C_{1,1}|u - v|_{H^\varepsilon_1}.\end{aligned}$$

This together with assumption (1) implies part (3) of condition (Conv). If $u \in H^0_1$ then

$$|f_\varepsilon(u)|_{H^\varepsilon_{-\alpha}} \leq C_{7,\alpha}(|\Phi_\varepsilon(u)|_{L^{q_2}(\Omega)} + |\Psi_\varepsilon(\gamma(u))|_{L^{q_3}(\Gamma)}).$$

This together with assumptions (2) and (3) easily implies part (4) of condition (Conv).

Now let $w \in H^0_1$ be arbitrary and $(\varepsilon_n)_n$ be a sequence in $]0, \varepsilon_0]$ with $\varepsilon_n \to 0$. Let $t \in]0, \infty[$ be arbitrary. We will show that

$$\lim_{n\to\infty} |e^{-t\tilde{A}_{\varepsilon_n}} f_{\varepsilon_n}(w) - e^{-t\tilde{A}_0} f_0(w)|_{H^{\varepsilon_n}_1} = 0, \tag{4.1}$$

proving (2) of condition (Conv).

It follows from Proposition 3.3 that the families $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon})_{\varepsilon\in[0,\varepsilon_0]}$ and $(A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition (Spec). For $n \in \mathbb{N}$ set $u_n = f_{\varepsilon_n}(w)$ and define $v_n \in H^{\varepsilon_n}_{-\alpha}$ by

$$v_n(h) = \int_\Omega \Phi_0(w) \cdot h \,\mathrm{d}x + \int_\Gamma \Psi_0(\gamma(w)) \cdot \gamma(h) \,\mathrm{d}\sigma, \quad h \in H^{\varepsilon_n}_\alpha.$$

Finally, set $u = f_0(w)$. Then

$$|u_n - v_n|_{H^{\varepsilon_n}_{-\alpha}} \leq C_{7,\alpha}(|\Phi_{\varepsilon_n}(w) - \Phi_0(w)|_{L^{q_2}(\Omega)} + |\Psi_{\varepsilon_n}(\gamma(w)) - \Psi_0(\gamma(w))|_{L^{q_3}(\Gamma)}) \to 0 \text{ as } n \to \infty.$$

Thus assumption (1) of Theorem 3.7 is satisfied.

Let $C_8 \in]0,\infty[$ be a bound for the imbedding $H^1(\Omega) \to H^\alpha(\Omega)$. Then, for every $j \in \mathbb{N}$,

$$|v_n(w_{\varepsilon_n,j}) - u(w_{0,j})| \leq |\Phi_0(w)|_{L^{q_2}(\Omega)}|w_{\varepsilon_n,j} - w_{0,j}|_{L^{p_2}(\Omega)} + |\Psi_0(\gamma(w))|_{L^{q_3}(\Gamma)}|\gamma(w_{\varepsilon_n,j} - w_{0,j})|_{L^{p_3}(\Gamma)} \leq \widetilde{C}|w_{\varepsilon_n,j} - w_{0,j}|_{H_1^{\varepsilon_n}},$$

where $\widetilde{C} := C_5C_{2,\alpha}C_8C_{1,1}|\Phi_0(w)|_{L^{q_2}(\Omega)} + C_6C_{3,2\alpha-1}C_{4,\alpha}C_8C_{1,1}|\Psi_0(\gamma(w))|_{L^{q_3}(\Gamma)}$. Hence $|v_n(w_{\varepsilon_n,j}) - u(w_{0,j})| \to 0$ as $n \to \infty$. Thus assumption (2) of Theorem 3.7 is satisfied.

Now, for all $n \in \mathbb{N}$,

$$|v_n|_{H^{\varepsilon_n}_{-\alpha}} \leq C_{7,\alpha}(|\Phi_0(w)|_{L^{p_2}(\Omega)} + |\Psi_0(\gamma(w))|_{L^{p_3}(\Gamma)}).$$

Thus assumption (3) of Theorem 3.7 is satisfied. Now (4.1) follows from Theorem 3.7. The proof is complete. □

Theorem 4.5. *Assume Hypothesis 1.1. For $\varepsilon \in [0,\varepsilon_0]$ and $u \in H^1(\Omega)$ (resp. $u \in H^{1/2}(\Gamma)$) define $\Phi_\varepsilon(u)(x) = \varphi_\varepsilon(x,u(x))$ (resp. $\Psi_\varepsilon(u)(x) = \psi_\varepsilon(x,u(x))$) for $x \in \Omega$ (resp. $x \in \Gamma$).*

Then $\Phi_\varepsilon\colon H^1(\Omega) \to L^{q_2}(\Omega)$ and $\Psi_\varepsilon\colon H^{1/2}(\Gamma) \to L^{q_3}(\Gamma)$ are defined and satisfy the assumptions of Theorem 4.3.

Proof. Use results and arguments in [6, Chapter 2]. □

Finally we obtain the following

Corollary 4.6. *For $\varepsilon \in [0,\varepsilon_0]$ let $\varphi_\varepsilon\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $\psi_\varepsilon\colon \Gamma \times \mathbb{R} \to \mathbb{R}$, $(x,s) \mapsto \varphi_\varepsilon(x,s)$, $(x,s) \mapsto \psi_\varepsilon(x,s)$, be functions as in Theorem 4.5. For $\varepsilon \in [0,\varepsilon_0]$ and $u \in H^1(\Omega)$ (resp. $u \in H^{1/2}(\Gamma)$) define $\Phi_\varepsilon(u)(x) = \varphi_\varepsilon(x,u(x))$ (resp. $\Psi_\varepsilon(u)(x) = \psi_\varepsilon(x,u(x))$) for $x \in \Omega$ (resp. $x \in \Gamma$) and let $\alpha \in]1/2,1]$ be as in Lemma 4.2. For $\varepsilon \in [0,\varepsilon_0]$ and $u \in H_1^\varepsilon$ define, for $h \in H_\alpha^\varepsilon$,*

$$f_\varepsilon(u)(h) = \int_\Omega \Phi_\varepsilon(u)\cdot h\,\mathrm{d}x + \int_\Gamma \Psi_\varepsilon(\gamma(u))\cdot\gamma(h)\,\mathrm{d}\sigma.$$

Then $f_\varepsilon(u) \in H_{-\alpha}^\varepsilon$ and the family $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ of maps satisfies condition (Conv).

Proof. This follows from Theorems 4.5 and 4.3. □

4.2. **Singular convergence of semiflows.** In the sequel, if $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfies condition (Spec) and $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfies condition (Conv) then we will write, for every $\varepsilon \in [0,\varepsilon_0]$, $\pi_\varepsilon := \pi_{A_\varepsilon,f_\varepsilon}$ to denote the local semiflow on H_1^ε generated by the abstract parabolic equation

$$\dot{u} = -\widetilde{A}_\varepsilon u + f_\varepsilon(u).$$

To prove the theorems of this section we will need the following auxiliary result.

Lemma 4.7. *Let $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition* (Spec) *and $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition* (Conv). *For every $\overline{u} \in H_1^0$ there exist a $\delta > 0$ and a $\tau > 0$ such that for every $a_0 \in H_1^0$ with $|a_0 - \overline{u}|_{H_1^0} \leq \delta$, $a_0\pi_0 s$, $s \in [0,\tau]$, is defined and whenever $(\varepsilon_n)_n$ is a sequence in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$ and $(a_n)_n$ is a sequence with $a_n \in H_1^{\varepsilon_n}$ for every $n \in \mathbb{N}$ and*

$$|a_n - a_0|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty,$$

then there is an $n_0 \in \mathbb{N}$ such that $a_n\pi_{\varepsilon_n}s$, $s \in [0,\tau]$, is defined for $n \geq n_0$. Moreover there exists an $M' \in [0,\infty[$ such that $|a_n\pi_{\varepsilon_n}s|_{H_1^{\varepsilon_n}} \leq M'$ for all $n \geq n_0$ and for all $s \in [0,\tau]$.

Proof. Let $\overline{u} \in H_1^0$ be arbitrary and $C_1 \in]0, \infty[$ be such that $|\overline{u}|_{H_1^0} \le C_1$. Let C be as in part (3) of condition (Spec), set

$$M' := 3C_1 + 3CC_1 \tag{4.2}$$

and let $L := L_{M'}$ be as in Definition 4.1 with M replaced by M'.

Part (4) of Definition 4.1 implies that there is an $\varepsilon_0' \in]0, \varepsilon_0]$ such that

$$C_2 = \sup_{\varepsilon \in [0, \varepsilon_0']} |f_\varepsilon(\overline{u})|_{H^\varepsilon} < \infty$$

Now choose τ and $\delta \in]0, \infty[$ such that

$$2(1-\alpha)^{-1} C_0 L \tau^{(1-\alpha)/2} \le 1/2, \tag{4.3}$$

$$2(1-\alpha)^{-1} C_0 \tau^{(1-\alpha)/2} (2LC_1 + C_2) \le C_1/4, \tag{4.4}$$

$$2C\delta \le C_1/4 \tag{4.5}$$

and

$$C|e^{-tA_0}\overline{u} - \overline{u}|_{H_1^0} \le C_1/4 \text{ for } t \in [0, \tau], \tag{4.6}$$

where the constant C_0 is as in Remark 3.5.

For every $\varepsilon \in [0, \varepsilon_0']$ and $a \in H_1^\varepsilon$ with

$$|a - \overline{u}|_{H_1^\varepsilon} \le C_1, \tag{4.7}$$

define

$$S_{\varepsilon,a} := \{\, u \mid u \colon [0, \tau] \to H_1^\varepsilon \text{ is continuous and } |u(t) - a|_{H_1^\varepsilon} \le C_1 \text{ for all } t \in [0, \tau] \,\}.$$

For $u \in S_{\varepsilon,a}$ define the map $T_{\varepsilon,a}(u) \colon [0, \tau] \to H_1^\varepsilon$ by

$$\begin{aligned} T_{\varepsilon,a}(u)(t) :&= e^{-t\widetilde{A}_\varepsilon} a + \int_0^t e^{-(t-s)\widetilde{A}_\varepsilon} f_\varepsilon(u(s))\, \mathrm{d}s \\ &= e^{-tA_\varepsilon} a + \int_0^t e^{-(t-s)\widetilde{A}_\varepsilon} f_\varepsilon(u(s))\, \mathrm{d}s. \end{aligned}$$

The map $T_{\varepsilon,a}(u)$ is continuous. Moreover, whenever $u \in S_{\varepsilon,a}$, then for all $t \in [0, \tau]$

$$|u(t)|_{H_1^\varepsilon} \le C_1 + |a|_{H_1^\varepsilon} \le C_1 + C_1 + |\overline{u}|_{H_1^\varepsilon} \le 2C_1 + C_1 C \le M',$$

where the last inequality follows from (4.2). Thus for all $u, v \in S_{\varepsilon,a}$ arbitrary and for all $t \in [0, \tau]$, we have, by (3.1),

$$\begin{aligned} |T_{\varepsilon,a}(u)(t) - T_{\varepsilon,a}(v)(t)|_{H_1^\varepsilon} &= \left| \int_0^t e^{-(t-s)\widetilde{A}_\varepsilon} (f_\varepsilon(u(s)) - f_\varepsilon(v(s)))\, \mathrm{d}s \right|_{H_1^\varepsilon} \\ &\le C_0 \int_0^t (t-s)^{-(\alpha+1)/2} |f_\varepsilon(u(s)) - f_\varepsilon(v(s))|_{H_{-\alpha}^\varepsilon}\, \mathrm{d}s \\ &\le C_0 L \int_0^t (t-s)^{-(\alpha+1)/2}\, \mathrm{d}s \sup_{s \in [0,\tau]} |u(s) - v(s)|_{H_1^\varepsilon} \\ &= 2(1-\alpha)^{-1} C_0 L \tau^{(1-\alpha)/2} \sup_{s \in [0,\tau]} |u(s) - v(s)|_{H_1^\varepsilon} \\ &\le 1/2 \sup_{s \in [0,\tau]} |u(s) - v(s)|_{H_1^\varepsilon}. \end{aligned} \tag{4.8}$$

The last inequality follows from (4.3). Moreover for all $u \in S_{\varepsilon,a}$ and $t \in [0,\tau]$

$$|T_{\varepsilon,a}(u)(t) - a|_{H_1^\varepsilon} \le |e^{-tA_\varepsilon}a - a|_{H_1^\varepsilon} + \left| \int_0^t e^{-(t-s)\widetilde{A}_\varepsilon} f_\varepsilon(u(s))\, \mathrm{d}s \right|_{H_1^\varepsilon}.$$

Since for $\varepsilon \in [0, \varepsilon_0']$ and $s \in [0,\tau]$ we have

$$\begin{aligned} |f_\varepsilon(u(s))|_{H_{-\alpha}^\varepsilon} &\le |f_\varepsilon(u(s)) - f_\varepsilon(a)|_{H_{-\alpha}^\varepsilon} + |f_\varepsilon(a)|_{H_{-\alpha}^\varepsilon} \\ &\le L|u(s) - a|_{H_1^\varepsilon} + |f_\varepsilon(a) - f_\varepsilon(\overline{u})|_{H_{-\alpha}^\varepsilon} + |f_\varepsilon(\overline{u})|_{H_{-\alpha}^\varepsilon} \\ &\le LC_1 + LC_1 + C_2 = 2LC_1 + C_2, \end{aligned}$$

we obtain, by (3.1),

$$\begin{aligned} \left| \int_0^t e^{-(t-s)\widetilde{A}_\varepsilon} f_\varepsilon(u(s))\, \mathrm{d}s \right|_{H_1^\varepsilon} &\le C_0 \int_0^t (t-s)^{-(\alpha+1)/2} |f_\varepsilon(u(s))|_{H_{-\alpha}^\varepsilon}\, \mathrm{d}s \\ &\le 2(1-\alpha)^{-1} C_0 \tau^{(1-\alpha)/2}(2LC_1 + C_2) \le C_1/4. \end{aligned}$$

In the previous computation we used the fact that $|a|_{H_1^\varepsilon} \le C_1 + |\overline{u}|_{H_1^\varepsilon} \le C_1 + CC_1 \le M'$ and $|\overline{u}|_{H_1^\varepsilon} \le M'$. If $\widetilde{a}_0 \in H_1^0$ satisfies $|\widetilde{a}_0 - \overline{u}|_{H_1^0} \le \delta$, then

$$\begin{aligned} |e^{-tA_\varepsilon}a - a|_{H_1^\varepsilon} &\le |e^{-tA_\varepsilon}a - e^{-tA_0}\widetilde{a}_0|_{H_1^\varepsilon} + |e^{-tA_0}\widetilde{a}_0 - e^{-tA_0}\overline{u}|_{H_1^\varepsilon} + |e^{-tA_0}\overline{u} - \overline{u}|_{H_1^\varepsilon} \\ &\quad + |\overline{u} - \widetilde{a}_0|_{H_1^\varepsilon} + |a - \widetilde{a}_0|_{H_1^\varepsilon} \\ &\le |e^{-tA_\varepsilon}a - e^{-tA_0}\widetilde{a}_0|_{H_1^\varepsilon} + C|e^{-tA_0}\widetilde{a}_0 - e^{-tA_0}\overline{u}|_{H_1^0} + C|e^{-tA_0}\overline{u} - \overline{u}|_{H_1^0} \\ &\quad + C|\overline{u} - \widetilde{a}_0|_{H_1^0} + |a - \widetilde{a}_0|_{H_1^\varepsilon} \\ &\le |e^{-tA_\varepsilon}a - e^{-tA_0}\widetilde{a}_0|_{H_1^\varepsilon} + |a - \widetilde{a}_0|_{H_1^\varepsilon} + C|e^{-tA_0}\overline{u} - \overline{u}|_{H_1^0} + 2C\delta \\ &\le |e^{-tA_\varepsilon}a - e^{-tA_0}\widetilde{a}_0|_{H_1^\varepsilon} + |a - \widetilde{a}_0|_{H_1^\varepsilon} + C_1/2, \end{aligned}$$

where the last inequality follows from (4.5) and (4.6).

Thus putting things together, we obtain for all $\varepsilon \in [0, \varepsilon_0']$, all $a \in H_1^\varepsilon$ satisfying (4.7) and all $\widetilde{a}_0 \in H_1^0$ with $|\widetilde{a}_0 - \overline{u}|_{H_1^0} \le \delta$

$$|T_{\varepsilon,a}(u)(t) - a|_{H_1^\varepsilon} \le 3C_1/4 + |e^{-tA_\varepsilon}a - e^{-tA_0}\widetilde{a}_0|_{H_1^\varepsilon} + |a - \widetilde{a}_0|_{H_1^\varepsilon},\quad u \in S_{\varepsilon,a},\ t \in [0,\tau]. \tag{4.9}$$

In particular, for all $a \in H_1^0$ satisfying $|a - \overline{u}|_{H_1^0} \le \delta$ and all $u \in S_{0,a}$ we have

$$|T_{0,a}(u)(t) - a|_{H_1^0} \le 3C_1/4 \le C_1 \text{ for all } t \in [0,\tau].$$

Hence we conclude that $T_{0,a}(S_{0,a}) \subset S_{0,a}$ and so, by Banach Fixed Point Theorem, there is a unique fixed point of $T_{0,a}$ in $S_{0,a}$. In particular $a\pi_0 s$ is defined for all $s \in [0,\tau]$.

Now let $(\varepsilon_n)_n$ be a sequence in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$. Suppose $a_0 \in H_1^0$ satisfies $|a_0 - \overline{u}|_{H_1^0} \le \delta$ and $(a_n)_n$ is a sequence with $a_n \in H_1^{\varepsilon_n}$ for every $n \in \mathbb{N}$ and $|a_n - a_0|_{H_1^{\varepsilon_n}} \to 0$. By what we just proved, it follows that $a_0\pi_0 s$ is defined for all $s \in [0,\tau]$.

Theorem 3.6 implies that there is an $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$

$$\sup_{t \in [0,\tau]} |e^{-tA_{\varepsilon_n}}a_n - e^{-tA_0}a_0|_{H_1^{\varepsilon_n}} + |a_n - a_0|_{H_1^{\varepsilon_n}} \le C_1/4. \tag{4.10}$$

Thus, it follows from (4.9) that $T_{\varepsilon_n,a_n}(S_{\varepsilon_n,a_n}) \subset S_{\varepsilon_n,a_n}$ for all $n \ge n_0$. Hence T_{ε_n,a_n} has a fixed point in S_{ε_n,a_n}. In particular $a_n\pi_{\varepsilon_n}s$ is defined for all $s \in [0,\tau]$ and for all $n \ge n_0$.

Moreover, (4.9) and (4.10) imply that, for $s \in [0, \tau]$ and $n \geq n_0$,

$$|a_n \pi_{\varepsilon_n} s|_{H_1^{\varepsilon_n}} \leq |a_n \pi_{\varepsilon_n} s - a_n|_{H_1^{\varepsilon_n}} + |a_n - \overline{u}|_{H_1^{\varepsilon_n}} + |\overline{u}|_{H_1^{\varepsilon_n}} \leq 2C_1 + CC_1 \leq M'.$$

The lemma is proved. □

We can now state our first singular convergence result for semiflows.

Theorem 4.8. *Let $(H^\varepsilon, \langle \cdot, \cdot \rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon \in [0, \varepsilon_0]}$ satisfy condition* (Spec), *and $(f_\varepsilon)_{\varepsilon \in [0,\varepsilon_0]}$ satisfy condition* (Conv). *Let $(\varepsilon_n)_n$ be a sequence in $]0, \varepsilon_0]$ with $\varepsilon_n \to 0$. Let $u_0 \in H_1^0$ and $(u_n)_n$ be a sequence with $u_n \in H_1^{\varepsilon_n}$ for every $n \in \mathbb{N}$ and*

$$|u_n - u_0|_{H^{\varepsilon_n}} \to 0 \text{ as } n \to \infty.$$

Let $b \in]0, \infty[$ and suppose that $u_n \pi_{\varepsilon_n} t$ and $u \pi_0 t$ are defined for all $n \in \mathbb{N}$ and $t \in [0, b]$. Moreover suppose there exists an $M' \in [0, \infty[$ such that $|u_n \pi_{\varepsilon_n} s|_{H_1^{\varepsilon_n}} \leq M'$ for all $n \in \mathbb{N}$ and for all $s \in [0, b]$. Then for every $t \in]0, b]$ and every sequence $(t_n)_n$ in $]0, b]$ converging to t

$$|u_n \pi_{\varepsilon_n} t_n - u_0 \pi_0 t_n|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty.$$

Proof. For every $t \in [0, b]$ we have, by the variation-of-constants formula,

$$\begin{aligned} u_n \pi_{\varepsilon_n} t - u_0 \pi_0 t &= e^{-tA_{\varepsilon_n}} u_n - e^{-tA_0} u_0 \\ &+ \int_0^t e^{-(t-s)\widetilde{A}_{\varepsilon_n}} (f_{\varepsilon_n}(u_n \pi_{\varepsilon_n} s) - f_{\varepsilon_n}(u_0 \pi_0 s)) \, ds \\ &+ \int_0^t (e^{-(t-s)\widetilde{A}_{\varepsilon_n}} f_{\varepsilon_n}(u_0 \pi_0 s) - e^{-(t-s)\widetilde{A}_0} f_0(u_0 \pi_0 s)) \, ds. \end{aligned}$$

Define the function $g_n \colon [0, b] \times [0, b] \to \mathbb{R}$ as follows: If $0 < s < t$ then set

$$g_n(t, s) = |e^{-(t-s)\widetilde{A}_{\varepsilon_n}} f_{\varepsilon_n}(u_0 \pi_0 s) - e^{-(t-s)\widetilde{A}_0} f_0(u_0 \pi_0 s)|_{H_1^{\varepsilon_n}}$$

and set $g_n(t, s) = 0$ otherwise. The function g_n restricted to the set of (s, t) with $0 < s < t$ is continuous. Thus g_n is measurable on $[0, b] \times [0, b]$. By Fubini's theorem the function

$$c_n(t) := \int_0^b g_n(t, s) \, ds = \int_0^t g_n(t, s) \, ds$$

is a.e. defined and measurable on $[0, b]$. Set

$$a_n(t) := |e^{-tA_{\varepsilon_n}} u_n - e^{-tA_0} u_0|_{H_1^{\varepsilon_n}} + c_n(t) \text{ for } t \in]0, b].$$

It follows that a_n is measurable on $[0, b]$. Using (3.1) we obtain

(4.11)

$$|g_n(t, s)| \leq C_2 C_0 (t - s)^{-(\alpha+1)/2} + CC_2 C_0 (t - s)^{-(\alpha+1)/2} =: C_3 (t - s)^{-(\alpha+1)/2} \text{ for } 0 < s < t,$$

where

$$C_2 := \max\{ \sup_{s \in [0,b]} \sup_{n \in \mathbb{N}} |f_{\varepsilon_n}(u_0 \pi_0 s)|_{H_{-\alpha}^{\varepsilon_n}}, \sup_{s \in [0,b]} |f_0(u_0 \pi_0 s)|_{H_{-\alpha}^0} \}.$$

Notice that condition (Conv) implies that $C_2 < \infty$. Now let $t \in]0, b]$ be arbitrary. If $0 < s < t$ then by part (2) of condition (Conv) we have $g_n(t, s) \to 0$ as $n \to \infty$. If $0 < t < s$ then $g_n(t, s) = 0$ for all $n \in \mathbb{N}$. Again $g_n(t, s) \to 0$ as $n \to \infty$. It follows from (4.11) and the dominated convergence theorem that

$$c_n(t) \to 0 \text{ as } n \to \infty.$$

Thus, using Corollary 3.8 we obtain

$$a_n(t) \to 0, \text{ as } n \to \infty, \text{ for all } t \in]0,b]. \tag{4.12}$$

Now, for all $n \in \mathbb{N}$ and $t \in]0,b]$

$$a_n(t) \le M' + C|u_0|_{H_1^0} + 2C_3(1-\alpha)^{-1}b^{(1-\alpha)/2} =: C_4.$$

Notice that $\widetilde{M} := \sup_{s\in[0,b]} |u_0\pi_0 s|_{H_1^0} < \infty$. Hence $|u_0\pi_0 s|_{H_1^{\varepsilon_n}} \le C\widetilde{M}$ for all $s \in [0,b]$. Set $M'' := \max\{M', C\widetilde{M}\}$ and let $L := L_{M''}$ be as in Definition 4.1 with M replaced by M''. We have

$$\begin{aligned}
|u_n\pi_{\varepsilon_n}t - u_0\pi_0 t|_{H_1^{\varepsilon_n}} &\le |e^{-tA_{\varepsilon_n}}u_n - e^{-tA_0}u_0|_{H_1^{\varepsilon_n}} \\
&\quad + \int_0^t |e^{-(t-s)\widetilde{A}_{\varepsilon_n}}(f_{\varepsilon_n}(u_n\pi_{\varepsilon_n}s) - f_{\varepsilon_n}(u_0\pi_0 s))|_{H_1^{\varepsilon_n}}\, ds \\
&\quad + \int_0^t |(e^{-(t-s)\widetilde{A}_{\varepsilon_n}}f_{\varepsilon_n}(u_0\pi_0 s) - e^{-(t-s)\widetilde{A}_0}f_0(u_0\pi_0 s))|_{H_1^{\varepsilon_n}}\, ds \\
&\le a_n(t) + C_0\int_0^t (t-s)^{-(\alpha+1)/2}|f_{\varepsilon_n}(u_n\pi_{\varepsilon_n}s) - f_{\varepsilon_n}(u_0\pi_0 s))|_{H_{-\alpha}^{\varepsilon_n}}\, ds \\
&\le a_n(t) + C_0 L\int_0^t (t-s)^{-(\alpha+1)/2}|u_n\pi_{\varepsilon_n}s - u_0\pi_0 s|_{H_1^{\varepsilon_n}}\, ds.
\end{aligned}$$

An application of Henry's Inequality [7, Lemma 7.1.1] implies that

$$|u_n\pi_{\varepsilon_n}t - u_0\pi_0 t|_{H_1^{\varepsilon_n}} \le a_n(t) + \int_0^t \rho(t-s)a_n(s)\, ds \text{ for } t \in]0,b],$$

where

$$\rho(x) := \sum_{n=1}^{\infty} \frac{(C_0 L\Gamma(\beta))^n}{\Gamma(n\beta)} x^{n\beta-1}$$

with $\beta := (1-\alpha)/2$.

The function $\rho\colon]0,\infty[\to]0,\infty[$ is well defined and continuous on $]0,\infty[$ and it satisfies the estimate

$$\rho(x) \le C_5 x^{-(\alpha+1)/2} + C_5 \quad \text{for } x \in]0,b].$$

Let $(t_n)_n$ be any sequence in $]0,b]$ converging to some $t \in]0,b]$. Fix a $\delta_0 \in]0,t[$ and let $\delta \in]0,\delta_0/2[$ be arbitrary. There is an $n_0 = n_0(\delta) \in \mathbb{N}$ such that $|t_n - t| < \delta$ for $n \ge n_0$. Therefore for all such $n \in \mathbb{N}$ and all $s \in [0, t-2\delta]$ it follows that $t_n - s > \delta$ so $\rho(t_n - s) \le C_5\delta^{-(\alpha+1)/2} + C_5$. Thus

$$\rho(t_n - s)a_n(s) \le C_6 \quad \text{for } s \in]0, t-2\delta].$$

Therefore (4.12) and the dominated convergence theorem show that

$$\int_0^{t-2\delta} \rho(t_n - s)a_n(s)\, ds \to 0 \text{ as } n \to \infty.$$

On the other hand,

$$\int_{t-2\delta}^{t_n} \rho(t_n - s)a_n(s)\, ds \le C_7(\delta^{(1-\alpha)/2} + \delta).$$

Since $\delta \in]0, \delta_0/2[$ is arbitrary, it follows that

$$\int_0^{t_n} \rho(t_n - s)a_n(s)\, \mathrm{d}s \to 0 \text{ as } n \to \infty.$$

Consequently,

$$|u_n\pi_{\varepsilon_n}t_n - u_0\pi_0 t_n|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty.$$

The theorem is proved. □

Our second convergence result reads as follows:

Theorem 4.9. *Let $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition* (Spec) *and $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition* (Conv)*. Let $(\varepsilon_n)_n$ be a sequence in $]0, \varepsilon_0]$ with $\varepsilon_n \to 0$ and let $(t_n)_n$ be a sequence in $[0, \infty[$ with $t_n \to 0$. Let $u_0 \in H_1^0$ and $(u_n)_n$ be a sequence with $u_n \in H_1^{\varepsilon_n}$ for every $n \in \mathbb{N}$ and*

$$|u_n - u_0|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty.$$

Then there exists an $n_0 \in \mathbb{N}$ such that $u_0\pi_0 t_n$ and $u_n\pi_{\varepsilon_n}t_n$ are defined for all $n \geq n_0$ and

$$|u_n\pi_{\varepsilon_n}t_n - u_0\pi_0 t_n|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty.$$

Proof. Set $\overline{u} := u_0$ in Lemma 4.7 and let $\tau > 0$ be as in that lemma. It follows from Lemma 4.7 that $u_0\pi_0 s$, $s \in [0, \tau]$, is defined and there is an $n_0 \in \mathbb{N}$ such that $u_n\pi_{\varepsilon_n}s$, $s \in [0, \tau]$, is defined for $n \geq n_0$. Moreover there exists a $M' \geq 0$ such that $|u_n\pi_{\varepsilon_n}s|_{H_1^{\varepsilon_n}} \leq M'$ for all $n \geq n_0$ and for all $s \in [0, \tau]$.

Since $t_n \to 0$ as $n \to \infty$, we may assume that $t_n \in [0, \tau]$ for all $n \in \mathbb{N}$. For every $t \in [0, \tau]$ we have

$$u_n\pi_{\varepsilon_n}t - u_0 = e^{-tA_{\varepsilon_n}}u_n - u_0 + \int_0^t e^{-(t-s)\widetilde{A}_{\varepsilon_n}} f_{\varepsilon_n}(u_n\pi_{\varepsilon_n}s)\, \mathrm{d}s.$$

Notice that $\widetilde{M} := \sup_{s\in[0,b]} |u_0\pi_0 s|_{H_1^0} < \infty$. Hence $|u_0\pi_0 s|_{H_1^{\varepsilon_n}} \leq C\widetilde{M}$ for all $s \in [0, \tau]$. Set $M'' := \max\{M', C\widetilde{M}\}$ and let $L := L_{M''}$ be as in Definition 4.1 with M replaced by M''. It follows that for all $n \geq n_0$ and for every $s \in [0, \tau]$

$$\begin{aligned}
|f_{\varepsilon_n}(u_n\pi_{\varepsilon_n}s)|_{H_{-\alpha}^{\varepsilon_n}} &\leq |f_{\varepsilon_n}(u_n\pi_{\varepsilon_n}s) - f_{\varepsilon_n}(u_0)|_{H_{-\alpha}^{\varepsilon_n}} + |f_{\varepsilon_n}(u_0)|_{H_{-\alpha}^{\varepsilon_n}} \\
&\leq L|u_n\pi_{\varepsilon_n}s - u_0|_{H_1^{\varepsilon_n}} + |f_{\varepsilon_n}(u_0)|_{H_{-\alpha}^{\varepsilon_n}} \\
&\leq L(M' + C|u_0|_{H_1^0}) + |f_{\varepsilon_n}(u_0)|_{H_{-\alpha}^{\varepsilon_n}}.
\end{aligned}$$

Part (4) of condition (Conv) now implies

$$|f_{\varepsilon_n}(u_n\pi_{\varepsilon_n}s)|_{H_{-\alpha}^{\varepsilon_n}} \leq \widetilde{C}, \text{ for all } s \in [0, \tau] \text{ and for all } n \geq n_0,$$

for some positive constant $\widetilde{C}$. Therefore for all $n \geq n_0$

$$\begin{aligned}
|u_n\pi_{\varepsilon_n}t_n - u_0\pi_0 t_n|_{H_1^{\varepsilon_n}} &\leq |u_n\pi_{\varepsilon_n}t_n - u_0|_{H_1^{\varepsilon_n}} + C|u_0\pi_0 t_n - u_0|_{H_1^0} \\
&\leq |e^{-t_nA_{\varepsilon_n}}u_n - e^{-t_nA_0}u_0|_{H_1^{\varepsilon_n}} + C|e^{-t_nA_0}u_0 - u_0|_{H_1^0} \\
&\quad + C_0\widetilde{C}\int_0^{t_n} (t_n - s)^{-(\alpha+1)/2}\, \mathrm{d}s + C|u_0\pi_0 t_n - u_0|_{H_1^0}.
\end{aligned}$$

Since $|e^{-t_nA_0}u_0 - u_0|_{H_1^0} \to 0$ and $|u_0\pi_0 t_n - u_0|_{H_1^0} \to 0$ as $n \to \infty$, an application of Theorem 3.6 completes the proof. □

Theorem 4.8 and Theorem 4.9 imply the following corollary.

Corollary 4.10. *Let $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition* (Spec) *and $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition* (Conv). *Then for every $\overline{u} \in H_1^0$ there exist a $\delta > 0$ and a $\tau > 0$ such that for every $a_0 \in H_1^0$ with $|a_0 - \overline{u}|_{H_1^0} \leq \delta$, $a_0\pi_0 s$, $s \in [0,\tau]$, is defined and whenever $(\varepsilon_n)_n$ is a sequence in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$ and $(a_n)_n$ is a sequence with $a_n \in H_1^{\varepsilon_n}$ for every $n \in \mathbb{N}$ and*

$$|a_n - a_0|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty,$$

then there is an $n_0 \in \mathbb{N}$ such that $a_n\pi_{\varepsilon_n} s$, $s \in [0,\tau]$, is defined for $n \geq n_0$ and

$$\sup_{s\in[0,\tau]} |a_n\pi_{\varepsilon_n} s - a_0\pi_0 s|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty.$$

Proof. Lemma 4.7 implies that for every $\overline{u} \in H_1^0$ there exist a $\delta > 0$ and a $\tau > 0$ such that for every $a_0 \in H_1^0$ with $|a_0 - \overline{u}|_{H_1^0} \leq \delta$, $a_0\pi_0 s$, $s \in [0,\tau]$, is defined and whenever $(\varepsilon_n)_n$ is a sequence in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$ and $(a_n)_n$ is a sequence with $a_n \in H_1^{\varepsilon_n}$ for every $n \in \mathbb{N}$ and

$$|a_n - a_0|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty,$$

then there is an $n_0 \in \mathbb{N}$ such that $a_n\pi_{\varepsilon_n} s$, $s \in [0,\tau]$, is defined for $n \geq n_0$. Moreover there exists a $M' \geq 0$ such that $|a_n\pi_{\varepsilon_n} s|_{H_1^{\varepsilon_n}} \leq M'$ for all $n \geq n_0$ and for all $s \in [0,\tau]$.

To complete the proof of the corollary we need to show that

$$\sup_{s\in[0,\tau]} |a_n\pi_{\varepsilon_n} s - a_0\pi_0 s|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty.$$

Suppose this is not true. Then there are a $\delta_0 > 0$ and a sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\sup_{s\in[0,\tau]} |a_{n_k}\pi_{\varepsilon_{n_k}} s - a_0\pi_0 s|_{H_1^{\varepsilon_{n_k}}} \geq \delta_0 \text{ for all } k \in \mathbb{N}.$$

Thus for each $k \in \mathbb{N}$ there exists an $s_k \in [0,\tau]$ such that

$$|a_{n_k}\pi_{\varepsilon_{n_k}} s_k - a_0\pi_0 s_k|_{H_1^{\varepsilon_{n_k}}} \geq \delta_0. \tag{4.13}$$

Without loss of generality we can assume that there is an $s_0 \in [0,\tau]$ such that $s_k \to s_0$ as $k \to \infty$. If $s_0 = 0$, it follows from Theorem 4.9 that $|a_{n_k}\pi_{\varepsilon_{n_k}} s_k - a_0\pi_0 s_k|_{H_1^{\varepsilon_{n_k}}} \to 0$ as $k \to \infty$ which contradicts (4.13). If $s_0 > 0$, then Theorem 4.8 implies that $|a_{n_k}\pi_{\varepsilon_{n_k}} s_k - a_0\pi_0 s_k|_{H_1^{\varepsilon_{n_k}}} \to 0$ as $k \to \infty$ which again contradicts (4.13). □

We conclude this section proving our main convergence result for semiflows.

Theorem 4.11. *Let $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition* (Spec) *and $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition* (Conv). *Let $(\varepsilon_n)_n$ be a sequence in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$ and let $(t_n)_n$ be a sequence in $[0,\infty[$ with $t_n \to t_0$, for some $t_0 \in [0,\infty[$. Let $u_0 \in H_1^0$ and $(u_n)_n$ be a sequence with $u_n \in H_1^{\varepsilon_n}$ for every $n \in \mathbb{N}$ and*

$$|u_n - u_0|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty.$$

Assume $u_0\pi_0 t_0$ is defined. Then there exists an $n_0 \in \mathbb{N}$ such that $u_n\pi_{\varepsilon_n} t_n$ is defined for all $n \geq n_0$ and

$$|u_n\pi_{\varepsilon_n} t_n - u_0\pi_0 t_0|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty.$$

Proof. Since $u_0\pi_0 t_0$ is defined, there is a $b > t_0$, $b \in]0,\infty[$, such that $u_0\pi_0 t$ is defined for all $t \in [0, b[$. Define

$$I := \{\, t \in [0,b[\mid \text{there exists an } n_0 \in \mathbb{N} \text{ such that } u_n\pi_{\varepsilon_n} t \text{ is defined for } n \geq n_0 \text{ and } \sup_{s\in[0,t]} |u_n\pi_{\varepsilon_n} s - u_0\pi_0 s|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty \,\}.$$

It is clear that $0 \in I$. Furthermore if $0 \leq t' < t$ and $t \in I$, then $t' \in I$. Let

$$\bar{t} := \sup I.$$

It follows that $\bar{t} \leq b$ and so $[0, \bar{t}[\subset I$. An application of Corollary 4.10 with $\overline{u} := u_0$ shows that $\bar{t} > 0$. We claim that $\bar{t} = b$. Suppose, on the contrary, that $\bar{t} < b$. It follows that $\overline{u} := u_0\pi_0\bar{t}$ is defined. Let $\delta > 0$ and $\tau > 0$ be as in Corollary 4.10 with respect to this choice of $\overline{u}$.

Choose $t \in \mathbb{R}$ with $0 < t < \bar{t} < t + \tau$ and $|u_0\pi_0 t - u_0\pi_0\bar{t}|_{H_1^0} < \delta$. We have that $t \in I$ so there exists an $n_0 \in \mathbb{N}$ such that $u_n\pi_{\varepsilon_n} t$ is defined for all $n \geq n_0$ and

$$\sup_{s\in[0,t]} |u_n\pi_{\varepsilon_n} s - u_0\pi_0 s|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty. \tag{4.14}$$

Set $\widetilde{u}_n := u_n\pi_{\varepsilon_n} t$ and $\widetilde{u} := u_0\pi_0 t$. Applying Corollary 4.10 with a_n replaced by $\widetilde{u}_n$ and a_0 replaced by $\widetilde{u}$ we thus have that $\widetilde{u}\pi_0\tau$ is defined and we obtain the existence of an $n_1 \geq n_0$ such that $\widetilde{u}_n\pi_{\varepsilon_n}\tau$ is defined for all $n \geq n_1$ and

$$\sup_{s\in[0,\tau]} |\widetilde{u}_n\pi_{\varepsilon_n} s - \widetilde{u}\pi_0 s|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty. \tag{4.15}$$

Formulas (4.14) and (4.15) imply that $u_0\pi_0(t+\tau)$ is defined, $u_n\pi_{\varepsilon_n}(t+\tau)$ is also defined for all $n \geq n_1$ and

$$\sup_{s\in[0,t+\tau]} |u_n\pi_{\varepsilon_n} s - u_0\pi_0 s|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty.$$

Thus $t + \tau \in I$, but $t + \tau > \bar{t}$, a contradiction, which proves that $\bar{t} = b$.

Since $t_0 \in [0, b[$, it follows that there is a $t \in [0, b[$ with $t_0 < t$ and $t_n < t$ for all n large enough. In particular $u_0\pi_0 t_n$ and $u_n\pi_{\varepsilon_n} t_n$ are defined for all n large enough and

$$|u_n\pi_{\varepsilon_n} t_n - u_0\pi_0 t_n|_{H_1^{\varepsilon_n}} \to 0 \text{ as } n \to \infty.$$

Since

$$|u_0\pi_0 t_n - u_0\pi_0 t_0|_{H_1^{\varepsilon_n}} \leq C|u_0\pi_0 t_n - u_0\pi_0 t_0|_{H_1^0}$$

and $|u_0\pi_0 t_n - u_0\pi_0 t_0|_{H_1^0} \to 0$ as $n \to \infty$, the theorem follows. □

We can now give a

Proof of Theorem 1.2. The theorem follows from Proposition 3.3, Corollary 4.6 and Theorem 4.11. □

5. Singular compactness

In this section we shall prove that under the abstract compactness hypothesis, condition (Comp), on the family $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ the corresponding family of semiflows satisfies a singular compactness property. This property, together with the singular convergence result obtained in the previous section, is crucial for establishing the singular continuation principle for Conley index and for (co)homology index braid. We conclude this section by proving Theorem 1.3 stated in the Introduction.

We start with the following result.

Theorem 5.1. *Let $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition* (Spec) *and $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy condition* (Conv). *Let $\varepsilon \in [0,\varepsilon_0]$ be arbitrary. Then every closed and bounded set in H_1^ε is strongly π_ε-admissible.*

Proof. Let N be a closed and bounded set in H_1^ε and let $M > 0$ such that for all $u \in N$ we have $|u|_{H_1^\varepsilon} \le M$. Let $L := L_M$ be as in Definition 4.1. Let $u_0 \in N$. Hence for all $u \in N$

$$\begin{aligned} |f_\varepsilon(u)|_{H^\varepsilon_{-\alpha}} &\le |f_\varepsilon(u) - f_\varepsilon(u_0)|_{H^\varepsilon_{-\alpha}} + |f_\varepsilon(u_0)|_{H^\varepsilon_{-\alpha}} \\ &\le L|u-u_0|_{H_1^\varepsilon} + |f_\varepsilon(u_0)|_{H^\varepsilon_{-\alpha}} \le 2ML + |f_\varepsilon(u_0)|_{H^\varepsilon_{-\alpha}}. \end{aligned}$$

Now the result follows from [11, Theorem III 4.4]. □

We can now state the following singular compactness theorem.

Theorem 5.2. *Let $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy conditions* (Spec) *and* (Comp) *and suppose that $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfies condition* (Conv). *Suppose $\kappa \in]0,\infty[$, $(\varepsilon_n)_n$ is a sequence in $]0,\varepsilon_0]$ with $\varepsilon_n \to 0$, $(t_n)_n$ is a sequence in $[0,\infty[$ with $t_n \ge \kappa$ for every $n \in \mathbb{N}$ and $(u_n)_n$ is a sequence with $u_n \in H_1^{\varepsilon_n}$ for every $n \in \mathbb{N}$. Assume that there exists a $C'' \in]0,\infty[$ such that $u_n\pi_{\varepsilon_n}t_n$ is defined and*

$$|u_n\pi_{\varepsilon_n}s|_{H_1^{\varepsilon_n}} \le C'' \text{ for all } n \in \mathbb{N} \text{ and for all } s \in [0,t_n].$$

Then there exist a $v \in H_1^0$ and a sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$|u_{n_k}\pi_{\varepsilon_{n_k}}t_{n_k} - v|_{H_1^{\varepsilon_{n_k}}} \to 0 \text{ as } k \to \infty.$$

Proof. Set $\xi_n = u_n\pi_{\varepsilon_n}(t_n - \kappa)$ for $n \in \mathbb{N}$. Hence

$$|\xi_n|_{H_1^{\varepsilon_n}} \le C'' \text{ for all } n \in \mathbb{N}.$$

Condition (Comp) implies that there exist a $\widetilde{v} \in H_1^0$ and a sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$|\xi_{n_k} - \widetilde{v}|_{H^{\varepsilon_{n_k}}} \to 0 \text{ as } k \to \infty.$$

We claim that $\widetilde{v}\pi_0\kappa$ is defined. Suppose our claim is not true. Let $\beta \in [0,\infty[$ be such that $CC'' < \beta$ and $|\widetilde{v}|_{H_1^0} < \beta$. By Theorem 5.1 with $\varepsilon = 0$ and $N = \{\, u \in H_1^0 \mid |u|_{H_1^0} \le \beta \,\}$, there exists a $t_0 \in]0,\kappa[$ such that $\widetilde{v}\pi_0 t_0$ is defined and $|\widetilde{v}\pi_0 t_0|_{H_1^0} > \beta$. Condition (Spec) implies that

$$|\widetilde{v}\pi_0 t_0|_{H_1^\varepsilon} \ge \beta/C > C'', \ \varepsilon \in [0,\varepsilon_0]. \tag{5.1}$$

It follows that there exists a $k_0 \in \mathbb{N}$ such that $\widetilde{v}\pi_0 s$ and $\xi_{n_k}\pi_{\varepsilon_{n_k}} s$ are defined for all $s \in [0, t_0]$, for all $k \geq k_0$ and

$$|\xi_{n_k}\pi_{\varepsilon_{n_k}} s|_{H_1^{\varepsilon_{n_k}}} \leq C'' \text{ for all } s \in [0, t_0] \text{ and for all } k \geq k_0.$$

Hence Theorem 4.8 implies that

$$|\xi_{n_k}\pi_{\varepsilon_{n_k}} t_0 - \widetilde{v}\pi_0 t_0|_{H_1^{\varepsilon_{n_k}}} \to 0 \text{ as } k \to \infty.$$

This together with formula (5.1) implies that

$$|\xi_{n_k}\pi_{\varepsilon_{n_k}} t_0|_{H_1^{\varepsilon_{n_k}}} > C'' \text{ for all } k \in \mathbb{N} \text{ large enough}$$

which is a contradiction. Thus $\widetilde{v}\pi_0\kappa$ is defined. Another application of Theorem 4.8 shows that

$$|\xi_{n_k}\pi_{\varepsilon_{n_k}} \kappa - \widetilde{v}\pi_0\kappa|_{H_1^{\varepsilon_{n_k}}} \to 0 \text{ as } k \to \infty.$$

Set $v := \widetilde{v}\pi_0\kappa \in H_1^0$. Since $\xi_{n_k}\pi_{\varepsilon_{n_k}}\kappa = u_{n_k}\pi_{\varepsilon_{n_k}} t_{n_k}$ for all $k \in \mathbb{N}$, the proof is complete. □

Recall that Proposition 3.4 implies that for every $\varepsilon \in]0, \varepsilon_0]$, the set H_1^0 is a closed subspace of H_1^ε. For each $\varepsilon \in]0, \varepsilon_0]$, let $Q_\varepsilon \colon H_1^\varepsilon \to H_1^\varepsilon$ be the H_1^ε-orthogonal projection of H_1^ε onto H_1^0.

Theorem 5.2 easily implies the following corollary:

Corollary 5.3. *Let $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy conditions* (Spec) *and* (Comp) *and suppose the family of maps $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfies condition* (Conv). *Suppose $N \subset H_1^0$ is a closed and bounded set and $(\varepsilon_n)_n$ is a sequence in $]0, \varepsilon_0]$ with $\varepsilon_n \to 0$. Let $\eta > 0$ be arbitrary and define*

$$N_n = N_{n,\eta} := \{\, u \in H_1^{\varepsilon_n} \mid Q_{\varepsilon_n} u \in N \text{ and } |(I - Q_{\varepsilon_n})u|_{H_1^{\varepsilon_n}} \leq \eta \,\}.$$

Suppose $(t_n)_n$ is a sequence in $[0, \infty[$ with $t_n \to \infty$ and $(u_n)_n$ is a sequence with $u_n \in H_1^{\varepsilon_n}$ for every $n \in \mathbb{N}$ and

$$u_n\pi_{\varepsilon_n}[0, t_n] \subset N_n \text{ for all } n \in \mathbb{N}.$$

Then there exist a $v \in N$ and a sequence $(n_k)_k$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ such that

$$|u_{n_k}\pi_{\varepsilon_{n_k}} t_{n_k} - v|_{H_1^{\varepsilon_{n_k}}} \to 0 \text{ as } k \to \infty. \quad \square$$

We can now give a

Proof of Theorem 1.3. The theorem follows from Proposition 3.3, Corollary 4.6 and Corollary 5.3. □

6. Singular continuation principle for the Conley index and for (co)homology index braids

In this section, under the conditions (Spec), (Conv) and (Comp), we obtain a singular continuation principle for the Conley index and for (co)homology index braids for the class of parabolic equations described in subsection 4.2. We also prove Theorem 1.4 and Theorem 1.5 from the Introduction.

This section is not self-contained, in particular, we use some results established in the papers [3, 4, 5]. We will also assume that the reader is familiar with the Conley index theory for semiflows on (not necessarily locally compact) metric spaces, as expounded in [10] or [11].

Let $(H^\varepsilon, \langle\cdot,\cdot\rangle_{H^\varepsilon}, A_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfy conditions (Spec) and (Comp) and suppose the family of maps $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfies condition (Conv).

Set $X_0 := H_1^0$. For every $\varepsilon \in \,]0,\varepsilon_0]$, define $Y_\varepsilon := (I - Q_\varepsilon)H_1^\varepsilon$ and endow Y_ε with the norm $|\cdot|_{H_1^\varepsilon}$ restricted to Y_ε. Define on $Z_\varepsilon = X_0 \times Y_\varepsilon$ the following norm:

$$||(u,v)||_\varepsilon := \max\{\,|u|_{H_1^0}, |v|_{H_1^\varepsilon}\,\} \text{ for } (u,v) \in Z_\varepsilon.$$

We will denote by Γ_ε the metric on Z_ε induced by the norm $||\cdot||_\varepsilon$. For each $\varepsilon \in \,]0,\varepsilon_0]$, define $\theta_\varepsilon := 0$.

Let $\Psi_\varepsilon \colon H_1^\varepsilon \to Z_\varepsilon$ be the linear map defined by

$$\Psi_\varepsilon(w) := (Q_\varepsilon w, (I - Q_\varepsilon)w) \text{ for } w \in H_1^\varepsilon.$$

It follows that Ψ_ε is a bijective linear map and its inverse map is given by

$$\Psi_\varepsilon{}^{-1}(u,v) = u + v \text{ for } (u,v) \in Z_\varepsilon.$$

Moreover both Ψ_ε and $\Psi_\varepsilon{}^{-1}$ are continuous maps. This fact is a consequence of the following inequalities:

$$||\Psi_\varepsilon(w)||_\varepsilon \le C|w|_{H_1^\varepsilon} \text{ for } w \in H_1^\varepsilon, \tag{6.1}$$

$$|\Psi_\varepsilon{}^{-1}(u,v)|_{H_1^\varepsilon} \le (1 + C^2)^{1/2}||(u,v)||_\varepsilon \text{ for } (u,v) \in Z_\varepsilon, \tag{6.2}$$

where the constant $C \in \,]1,\infty[$ was defined in hypothesis (Spec).

Given $(u,v) \in Z_\varepsilon$ and $t \in [0,\infty[$ define

$$(u,v)\tilde{\pi}_\varepsilon t := \Psi_\varepsilon(\Psi_\varepsilon{}^{-1}(u,v)\pi_\varepsilon t)$$

whenever $\Psi_\varepsilon{}^{-1}(u,v)\pi_\varepsilon t$ is defined. It follows that $\tilde{\pi}_\varepsilon$ is a local semiflow on Z_ε, the *conjugate to π_ε via Ψ_ε*. Theorem 4.11 and inequalities (6.1) and (6.2) immediately imply the following

Corollary 6.1. *Under the above hypotheses the family $(\tilde{\pi}_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ converges singularly to π_0.* □

Theorem 5.1, Corollary 5.3 and inequalities (6.1) and (6.2) imply the following:

Corollary 6.2. *Let N be a closed and bounded subset of X_0. Then for every $\eta > 0$ the set N is singularly strongly admissible with respect to η and the family $(\tilde{\pi}_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$, where $\tilde{\pi}_0 = \pi_0$.* □

We can now prove the following Conley index continuation principle for singular families of abstract parabolic equations:

Theorem 6.3. *Let N be a closed and bounded isolating neighborhood of an invariant set K_0 relative to π_0. For $\varepsilon \in \,]0,\varepsilon_0]$ and for every $\eta \in \,]0,\infty[$ set*

$$N_{\varepsilon,\eta} := \{\, u \in H_1^\varepsilon \mid Q_\varepsilon u \in N \text{ and } |(I - Q_\varepsilon)u|_{H_1^\varepsilon} \le \eta \,\}$$

and $K_{\varepsilon,\eta} := \mathrm{Inv}_{\pi_\varepsilon}(N_{\varepsilon,\eta})$ i.e. $K_{\varepsilon,\eta}$ is the largest π_ε-invariant set in $N_{\varepsilon,\eta}$. Then for every $\eta \in \,]0,\infty[$ there exists an $\varepsilon^c = \varepsilon^c(\eta) \in \,]0,\varepsilon_0]$ such that for every $\varepsilon \in \,]0,\varepsilon^c]$ the set $N_{\varepsilon,\eta}$ is a strongly admissible isolating neighborhood of $K_{\varepsilon,\eta}$ relative to π_ε and

$$h(\pi_\varepsilon, K_{\varepsilon,\eta}) = h(\pi_0, K_0).$$

Furthermore, for every $\eta > 0$, the family $(K_{\varepsilon,\eta})_{\varepsilon\in[0,\varepsilon^c(\eta)]}$ of invariant sets, where $K_{0,\eta} = K_0$, is upper semicontinuous at $\varepsilon = 0$ with respect to the family $|\cdot|_{H_1^\varepsilon}$ of norms i.e.

$$\lim_{\varepsilon\to 0^+} \sup_{w\in K_{\varepsilon,\eta}} \inf_{u\in K_0} |w-u|_{H_1^\varepsilon} = 0.$$

Proof. The isomorphism Ψ_ε conjugates the local semiflow π_ε to the local semiflow $\tilde{\pi}_\varepsilon$. Thus whenever S is a strongly admissible isolating neighborhood with respect to π_ε, then $\Psi_\varepsilon(S)$ is a strongly admissible isolating neighborhood with respect to $\tilde{\pi}_\varepsilon$ and

$$h(\pi_\varepsilon, S) = h(\tilde{\pi}_\varepsilon, \Psi_\varepsilon(S)).$$

Corollaries 6.1 and 6.2 imply that the family of semiflows $(\tilde{\pi}_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ and the set N satisfy the hypotheses of [3, Theorem 4.1]. Notice also that any closed ball in Y_ε is contractible. Hence [3, Theorem 4.1] and [3, Corollary 4.11] completes the proof. □

Remark 6.4. The family $(K_{\varepsilon,\eta})_{\varepsilon\in]0,\varepsilon^c(\eta)]}$ is asymptotically independent of η i.e. whenever η_1 and $\eta_2 \in\,]0,\infty[$ then there is an $\varepsilon' \in\,]0, \min(\varepsilon^c(\eta_1), \varepsilon^c(\eta_2))]$ such that $K_{\varepsilon,\eta_1} = K_{\varepsilon,\eta_2}$ for $\varepsilon \in\,]0,\varepsilon']$.

We can now give a

Proof of Theorem 1.4. Proposition 3.3, Corollary 4.6, Theorem 6.3 and Remark 6.4 imply the theorem. □

We also prove the following (co)homology index continuation principle:

Theorem 6.5. *Assume the hypotheses of Theorem 6.3 and for every $\eta \in\,]0,\infty[$ let $\varepsilon^c(\eta) \in\,]0,\varepsilon_0]$ be as in that theorem. Let $(M_{p,0})_{p\in P}$ be a $\prec$-ordered Morse decomposition of K_0 relative to π_0. For each $p \in P$, let $V_p \subset N$ be closed in X_0 and such that $M_{p,0} = \mathrm{Inv}_{\pi_0}(V_p) \subset \mathrm{Int}_{H_1^0}(V_p)$. (Such sets V_p, $p\in P$, exist.) For $\varepsilon \in\,]0,\varepsilon_0]$, for every $\eta \in\,]0,\infty[$ and $p \in P$ set $M_{p,\varepsilon,\eta} := \mathrm{Inv}_{\pi_\varepsilon}(V_{p,\varepsilon,\eta})$, where*

$$V_{p,\varepsilon,\eta} := \{\, u \in H_1^\varepsilon \mid Q_\varepsilon u \in V_p \text{ and } |(I-Q_\varepsilon)u|_{H_1^\varepsilon} \le \eta \,\}.$$

Then for every $\eta \in\,]0,\infty[$ there is an $\tilde{\varepsilon} = \tilde{\varepsilon}(\eta) \in\,]0,\varepsilon^c(\eta)]$ such that for every $\varepsilon \in\,]0,\tilde{\varepsilon}]$ and $p \in P$, $M_{p,\varepsilon,\eta} \subset \mathrm{Int}_{H_1^\varepsilon}(V_{p,\varepsilon,\eta})$ and the family $(M_{p,\varepsilon,\eta})_{p\in P}$ is a $\prec$-ordered Morse decomposition of $K_{\varepsilon,\eta}$ relative to π_ε and the (co)homology index braids of $(\pi_0, K_0, (M_{p,0})_{p\in P})$ and $(\pi_\varepsilon, K_{\varepsilon,\eta}, (M_{p,\varepsilon,\eta})_{p\in P}))$, $\varepsilon \in\,]0,\tilde{\varepsilon}]$, are isomorphic and so they determine the same collection of C-connection matrices.

Proof. Since the isomorphism Ψ_ε conjugates the local semiflow π_ε to the local semiflow $\tilde{\pi}_\varepsilon$, using [5, Proposition 2.7], it follows that whenever S is a strongly admissible isolating neighborhood with respect to π_ε and $(M_p)_{p\in P}$ is a $\prec$-ordered Morse decomposition of S relative to π_ε, then $\Psi_\varepsilon(S)$ is a strongly admissible isolating neighborhood with respect to $\tilde{\pi}_\varepsilon$ and $(\Psi_\varepsilon(M_p))_{p\in P}$ is a $\prec$-ordered Morse decomposition of S relative to $\tilde{\pi}_\varepsilon$ and the (co)homology index braids of $(\pi_\varepsilon, S, (M_p)_{p\in P})$ and $(\tilde{\pi}_\varepsilon, \Psi_\varepsilon(S), (\Psi_\varepsilon(M_p))_{p\in P}))$, $\varepsilon \in\,]0,\varepsilon_0]$, are isomorphic.

Corollaries 6.1 and 6.2 imply that the family of semiflows $(\tilde{\pi}_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ and the set N satisfy the hypotheses of [4, Theorem 3.10]. Since any closed ball in Y_ε is contractible, an application of [4, Theorem 3.10] completes the proof. □

Remark 6.6. Again, for each $p \in P$, the family $(M_{p,\varepsilon,\eta})_{\varepsilon\in[0,\bar{\varepsilon}(\eta)]}$, where $M_{p,0,\eta} = M_{p,0}$ is upper semicontinuous at $\varepsilon = 0$ respect to the family $|\cdot|_{H_1^\varepsilon}$ of norms and the family $(M_{p,\varepsilon,\eta})_{\varepsilon\in]0,\bar{\varepsilon}(\eta)]}$ is asymptotically independent of η.

We can now give a

Proof of Theorem 1.5. Proposition 3.3, Corollary 4.6, Theorem 6.5 and Remark 6.6 imply the theorem. □

References

[1] J.M. Arrieta, A.N. Carvalho and A. Rodríguez-Bernal, Parabolic problems with nonlinear boundary conditions and critical nonlinearities, *Journal of Differential Equations*, **156** (1999), 376–406.

[2] M.C. Carbinatto and K.P. Rybakowski, On convergence, admissibility and attractors for damped wave equations on squeezed domains, *Proc. of the Royal Soc. of Edinburgh*, **132A** (2002), 765–791.

[3] M.C. Carbinatto and K.P. Rybakowski, On a general Conley index continuation principle for singular perturbation problems, *Ergod. Th. & Dynam. Sys.*, **22** (2002), 729–755.

[4] M.C. Carbinatto and K.P. Rybakowski Continuation of the connection matrix in singular perturbation problems, *Ergod. Th. & Dynam. Sys.*, **26** (2006), 1021–1059.

[5] M.C. Carbinatto and K.P. Rybakowski, Continuation of the connection matrix for singularly perturbed hyperbolic equations, *Fund. Math.*, **196** (2007), 253–273.

[6] D.G. de Figueiredo, *The Ekeland Variational Principle with Applications and Detours*, Springer-Verlag, Berlin, 1989.

[7] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, **840**, Springer-Verlag, NY, 1981.

[8] J.L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Springer-Verlag, Berlin, 1972.

[9] A. Rodriguez-Bernal, Localized spatial homogenization and large diffusion, *SIAM J. Math. Anal.*, **29** (1998), 1361–1380.

[10] K.P. Rybakowski, On the homotopy index for infinite-dimensional semiflows, *Trans. Amer. Math. Soc.*, **269** (1982), 351–382.

[11] K.P. Rybakowski *The Homotopy Index and Partial Differential Equations*, Springer-Verlag, Berlin, 1987.

[12] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland Publishing Company, Amsterdam, 1978.

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316/09 CARBINATTO, M. C.; RYBAKOWSKI, K. P. – Conley índex and homology index braids in singular perturbation problems without uniqueness of solutions.

315/09 HARTMANN, L.; SPREAFICO, M. – Analytic torsion for manifolds with totally geodesic boundary.

314/09 CALLEJAS-BEDREGAL, R.; JORGE PÉREZ, V.H. – Mixed multiplicities of arbitrary modules.

313/09 BRONZI, M.; TAHZIBI, A. – Homoclinic tangency and variation of entropy.

312/09 TAHZIBI, A. – Carlos Teobaldo Gutiérrez Vidalon 1944-2008.

311/09 GUTIERREZ, C.(1944-2008); BIASI, C.; SANTOS, E.L. – Global inverse mapping theorems.

310/09 BUOSI, M.; IZUMIYA, S.; RUAS, M.A. – Horo-tight spheres in hyperbolic space.

309/09 BIASI, C.; LIBARDI, A.K.M.; MATTOS, D.; SANTOS, E.L. – On codimensions k immersions of m-manifolds for $k = m - 3$, $k = m - 5$ and $k = m - 6$.

308/09 BARBOSA, G.F.; GRULHA JR., N.; SAIA, M.J. – Stable types and the minimal Whitney stratification of discriminants.

307/09 ARRIETA, J.M.; CARVALHO, A.N.; LANGA, J.A.; RODRIGUEZ-BERNAL, A. – Continuity of dynamical structures for non-autonomous evolution equations under singular perturbations.