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ON ACYCLIC AND SIMPLY CONNECTED OPEN MANIFOLDS

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In this paper we prove that an acyclic and simply connected open n-manifold is homeomorphic to the space $\mathbb{R}^n$ if and only if its one-point compactification is again a manifold. Furthermore, we prove that this statement is equivalent to the Generalized Poincaré Conjecture. Also, we connect these results with the existence of so-called compact spine of open manifolds.

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1. INTRODUCTION

Let X be a connected open n-manifold and let $\bar{X}$ its one-point compactification. If $\bar{X}$ is again a manifold, then it is a closed n-manifold. However, $\bar{X}$ is not necessarily a manifold. For example, the open cylinder $\mathcal{C} = S^1 \times (0,1)$ is a connected open 2-manifold, but its one-point compactification $\bar{\mathcal{C}}$ is not a manifold, in fact, $\bar{\mathcal{C}}$ is the "pinched" torus obtained from the torus by collapsing one meridian into a single point. Other more complicated example is the Whitehead Manifold $W = S^3 \setminus Wh$ which is contractible but is not simply connected at infinity, that is, its one-point compactification $W \cup \{\infty\}$ is such that the point ∞ has no simply connected neighborhood (see [4]).

We present now the statement of the first main theorem of this paper.

THEOREM 1.1 (Main Theorem A). *An acyclic and simply connected open n-manifold is homeomorphic to $\mathbb{R}^n$ if and only if its one-point compactification is again a manifold.*

We will prove this using the Generalized Poincaré Conjecture. Indeed, we will prove that the Main Theorem A is actually equivalent to this conjecture.

THEOREM 1.2 (Main Theorem B). *The Main Theorem A in dimension n is equivalent to the Generalized Poincaré Conjecture in dimension n.*

Currently, it is well known that the Generalized Poincaré Conjecture is true in every dimensions (on the category Top). In dimensions 1 and 2, this Conjecture is trivial, because of the classification of manifolds of these dimensions. In dimensions greater than 4, the proof of the Conjecture is a consequence of the famous h-Cobordism Theorem, whose proof was presented in 1961 by Smale [7]. The proof in dimension 4 was presented a little later in 1982 by Freedman [1]. Only very recently, a Russian mathematician called Grigori Perelman proved the Poincaré Conjecture in its original version, in dimension 3. (See [6]).

THEOREM 1.3 (Generalized Poincaré Conjecture). *Every homotopy n-sphere is homeomorphic to the sphere S^n.*

Throughout the text, in particular in the statement of the Generalized Poincaré Conjecture above, we use the following definition:

DEFINITION 1.1. An n-manifold X is said a *homotopy n-sphere* if it is homotopy equivalent to S^n. More simply, an n-manifold X is called a *homology n-sphere* if it have the same homology groups of the sphere S^n.

A homotopy or homology n-sphere is necessarily closed and orientable.

It can not be easy to check if the one-point compactification of an open manifold is a manifold. We define a new concept to help it.

DEFINITION 1.2. We say that an open n-manifold X has a compact spine if there is a compact n-manifold $\dot{X} \subset X$ (so-called the *compact spine* of X) with boundary $\partial\dot{X}$ such that X is homeomorphic to $\dot{X}$ with an open collar attached.

In this definition, we speak $\dot{X}$ is "the" compact spine and not "a" compact spine because $\dot{X}$, if there exists, is obviously uniquely determined up to homeomorphism.

Suppose that X is an open n-manifold whose one-point compactification $\bar{X}$ is a manifold. Then X has a compact spine. In fact, write $\bar{X} = X \cup \{\infty\}$. Then the point ∞ has an open neighborhood V in $\bar{X}$ homeomorphic to an open n-ball. Now, reducing V, if necessary, $\dot{X} \setminus V$ is an compact n-manifold with boundary $\partial(\dot{X} \setminus V) = \partial(\bar{V}) \cong S^{n-1}$. Here $\bar{V}$ denotes the closure of V in $\bar{X}$. Furthermore, $\bar{V} \setminus \{\infty\}$ is homeomorphic to the cylinder $S^{n-1} \times [0,1)$ and X is exactly the space obtained from $\dot{X} \setminus V$ by attaching $\bar{V} \setminus \{\infty\}$ by identifying their boundaries (which are both homeomorphic to S^{n-1}). This proves that X is homeomorphic to $\dot{X} \setminus V$ with an open collar attached. Therefore, $\dot{X} \setminus V$ is the compact spine of X.

The cylinder (without boundary) $S^1 \times (0,1)$ is an open manifold which has a compact spine, though its one-point compactification is not a manifold.

The following theorem presents a condition for an acyclic and simply connected open n-manifold to be homeomorphic to $\mathbb{R}^n$, through the existence of compact spine.

THEOREM 1.4 (Main Theorem C). *Let X be an acyclic and simply connected open n-manifold. Then X is homeomorphic to $\mathbb{R}^n$ if and only if*

(1) *X has a compact spine, if $n = 1$ or 2;*
(2) *X has a compact spine with simply connected boundary, if $n \geq 3$.*

It follows that the one-point compactification of an acyclic and simply connected open n-manifold X is again a manifold if and only if either (1) or (2) in the Main Theorem C is verified.

The paper is presented with the following structure: In Section 2 we present some preliminary results to development of the paper. In Section 3 we prove the Main Theorem B and, consequently, the Main Theorem A, as already observed. In Section 4 we present several corollaries of the Main Theorem A. Finally, in Section 5, we prove the Main Theorem C and some corollaries of it.

2. REVIEW AND PRELIMINARY RESULTS

We start this section by remembering that a topological space has an one-point compactification if and only if it is Hausdorff, locally compact and not compact. Moreover, the one-point compactification, if there exists, is uniquely determined up to homeomorphism. In special, every open manifold has an one-point compactification. However, as we already have seen, such compactification is not necessarily a manifold. Also, an open n-manifold is homeomorphic to $\mathbb{R}^n$ if and only if its one-point compactfication is homeomorphic to the sphere S^n.

In according to [8], we say that a topological space X is m-connected if $\pi_i(X) = 0$ for $i \leq m$. In particular, X is connected if it is 0-connected and X is simply connected if it is 1-connected. In [8] we found the following consequences of the Hurewicz Theorem:

THEOREM 2.1. *Let X be a simply connected space. Assume that $n \geq 2$.*

(1) *If X is $(n-1)$-connected, then $\widetilde{H}_i(X) = 0$ for every $i < n$ and $\pi_n(X) \approx H_n(X)$.*
(2) *If $\widetilde{H}_i(X) = 0$ for every $i < n$, then X is $(n-1)$-connected and $\pi_n(X) \approx H_n(X)$.*

Below, we present a version of the Whitehead Theorem in terms of the homology groups. This version corresponds to Corollary 4.33 of [3] and is very useful to check whether two spaces are homotopy equivalent, what in general can be difficult.

THEOREM 2.2. *A map $f : X \to Y$ between simply connected CW complexes is a homotopy equivalence if and only if $f_*\colon H_i(X) \to H_i(Y)$ is an isomorphism for each i.*

We use this theorem to prove that a (closed) n-manifold with the same algebraic type of the sphere S^n (that is, having the same homology and homotopy groups of the sphere S^n) is actually homotopy equivalent to S^n.

THEOREM 2.3. *Let X be a simply connected n-manifold, $n \geq 2$. Then X is a homotopy n-sphere if and only if it is a homology n-sphere.*

Proof. The "only if" part is trivial. In order to prove the "if" part, suppose that X is a homology n-sphere. If $n = 1$, the result is obvious, indeed, in this case, X is actually homeomorphic to the sphere S^1. Then, assume $n \geq 2$. By Theorem 2.1, X is $(n-1)$-connected and $\pi_n(X) \approx H_n(X) \approx \mathbb{Z}$. Thus, there is a map $f : S^n \to X$ representing a generator of the group $\pi_n(X)$. Since $\pi_n(X) \approx \mathbb{Z} \approx \pi_n(S^n)$, the map f induces an isomorphism $f_\# : \pi_n(S^n) \to \pi_n(X)$. By composing $f_\#$ and $f_* : H_n(S^n) \to H_n(X)$ with the appropriated Hurewicz isomorphisms we conclude that $f_* : H_n(S^n) \to H_n(X)$ is an isomorphism. By Theorem 2.2, f is a homotopy equivalence and so X is a homotopy n-sphere. ∎

The following result can be considered as a consequence of the Whitehead Theorem, by using the fact that every compact manifold is dominated by a CW complex.

THEOREM 2.4. *A map $f : X \to Y$ between compact and simply connected manifolds is a homotopy equivalence if and only if $f_* : H_i(X) \to H_i(Y)$ is an isomorphism for each i. A compact, acyclic and simply connected manifold is contractible.*

Proof. Since each compact manifold is homotopy equivalent to a CW complex (see [3]), the first part of the theorem is a consequence of Theorem 2.2. Now, if X is a compact, acyclic and simply connected manifold, let $x_0 \in X$ be a point and let $l : \{x_0\} \hookrightarrow X$ be the natural inclusion. Obviously, $l_* : H_i(\{x_0\}) \to H_i(X)$ is an isomorphism for each i. By the first part of the theorem, the inclusion l is a homotopy equivalence. Therefore, X is contractible. ∎

Despite the equivalence of the previous theorem, apparently is more restrictive to assume that a compact manifold is contractive than to assume that it is acyclic and simply connected. The assumption *acyclic and simply connected* appear in most of the statements of our theorems.

3. PROOFS OF THE MAIN THEOREMS A AND B

In order to prove the Main Theorems A and B, we start by proving that if the one-point compactification $\bar{X}$ of an acyclic and simply connected open n-manifold is again a manifold, then $\bar{X}$ is a homotopy n-sphere.

THEOREM 3.1. *Let X be an acyclic and simply connected open n-manifold and let $\bar{X}$ be its one-point compactification. If $\bar{X}$ is a manifold, then it is a homotopy n-sphere.*

Proof. Let $\bar{X} = X \cup \{\infty\}$ be the one-point compactification of X. Suppose that $\bar{X}$ is a manifold. Then, it is clear that $\bar{X}$ is a closed n-manifold. Let V be an open neighborhood of ∞ homeomorphic to an open n-ball of $\mathbb{R}^n$. Then $\bar{X} = X \cup V$ with X and V open in $\bar{X}$. Now, the intersection $X \cap V$ is exactly the subspace $V \setminus \{\infty\}$ of $\bar{X}$, which is homeomorphic to the cylinder $S^{n-1} \times (0,1)$ which is homotopy equivalent to the sphere S^{n-1}. Thus, we have the following Mayer-Vietoris exact sequence related to space $\bar{X} = X \cup V$,

$$\to H_i(S^{n-1}) \to H_i(V) \oplus H_i(X) \to H_i(\bar{X}) \to H_{i-1}(S^{n-1}) \to$$

Now, by assumption, $\widetilde{H}_i(X) = 0$ for each i. Moreover, it is clear that $H_i(V) = 0$ for each $i \geq 1$, since V is contractible. Then, the exactness of the sequence implies that $H_i(\bar{X}) \approx \widetilde{H}_{i-1}(S^{n-1})$ for each $i \geq 1$. Therefore $H_i(\bar{X}) \approx H_i(S^n)$ for each i. It follows that $\bar{X}$ is a homology n-sphere.

Now, since $\bar{X} = X \cup V$, with X and V open and simply connected, and moreover $X \cap Y$ is path connected if $n \geq 2$ (the case $n = 1$ is trivial), it follows by the van Kampen Theorem that $\bar{X}$ is also simply connected.

Finally, by Theorem 2.3, $\bar{X}$ is a homotopy n-sphere. ∎

Now, we will prove the Main Theorem B. As we already observed, the Main Theorem A is a consequence of it. At first, we will prove that the Generalized Poincaré Conjecture implies the Main Theorem A and next we will prove that the Main Theorem A implies the Generalized Poincaré Conjecture.

Proof (Proof of the Main Theorem B). Suppose the Generalized Poincaré Conjecture is true in dimension n. We will prove that the Main Theorem A is true in dimension n. Let X be an acyclic and simply connected open n-manifold and let $\bar{X} = X \cup \{\infty\}$ be its one-point compactification. If X is homeomorphic to $\mathbb{R}^n$, then it is clear that $\bar{X}$ is a manifold; indeed, in this case, $\bar{X}$ is homeomorphic to S^n. Now, suppose that $\bar{X}$ is a manifold. Then, by Theorem 3.1, $\bar{X}$ is a homotopy n-sphere. By the Generalized Poincaré Conjecture in dimension n, $\bar{X}$ is homeomorphic to the sphere S^n and so X is homeomorphic to $\mathbb{R}^n$.

The proof of the reciprocal is more complicated and requires specific arguments for some dimensions. For $n = 1$, it is no necessary a proof. Assume that $n \geq 2$. Let Y be a homotopy n-sphere. Then Y is closed, orientable, connected and simply connected. In special $H_n(Y) \approx \mathbb{Z}$. Let $y_0 \in Y$ be an arbitrary point in Y and let V be an open neighborhood of y_0 in Y, homeomorphic to an open n-ball. Let $Y_0 = Y \setminus \{y_0\}$. Then Y_0 is a connected open n-manifold whose one-point compactification is Y. We will prove that Y_0 is acyclic and simply connected, concluding that Y is homeomorphic to S^n, by the Main Theorem A and, therefore, that the Generalized Poincaré Conjecture is true in dimension n. At first, note that V, Y_0 and $V \cap Y_0$ are path connected open subsets of Y. Moreover $V \cap Y_0$ is exactly the subset $V \setminus \{y_0\}$, which we denote simply by V_0. Note that V_0 is homeomorphic to the cylinder $S^{n-1} \times (0,1)$ and so it is homotopy equivalent to S^{n-1}. Finally, we note that $Y = Y_0 \cup V$ and we apply the van Kampen Theory.

Since V is contractible, the fundamental group $\pi_1(Y)$ is isomorphic to the quotient $\pi_1(Y_0)/N(j_\#)$ of the group $\pi_1(Y_0)$ by its normal subgroup $N(j_\#)$ generated by the image of the homomorphism $j_\#$ induced on fundamental groups by the natural inclusion $j : V_0 \hookrightarrow Y_0$. Now, we have two cases:

Case 1: If $n \geq 3$, then Y and V_0 are simply connected. Hence Y_0 is simply connected.

Case 2: If $n = 2$, then V_0 has the homotopy type of the sphere S^1 and so $\pi_1(V_0) \approx \mathbb{Z}$. Since Y is simply connected and $\pi_1(Y) \approx \pi_1(Y_0)/N(j_\#)$, either Y_0 is simply connected or $\pi_1(Y_0) \approx \mathbb{Z}$ and the homomorphisms $j_\# : \pi_1(V_0) \to \pi_1(Y_0)$ and $j_* : H_1(V_0) \to H_1(Y_0)$ are isomorphisms. By Exercise (22.45) of [2] (which is true!), the inclusion $\partial V \hookrightarrow Y_0$ induces trivial homomorphisms on homology. But it is clear that such homomorphisms are identical to homomorphisms induced by the inclusion $j : V_0 \hookrightarrow Y_0$. Therefore, $j_* : H_1(V_0) \to H_1(Y_0)$ can not be an isomorphism. Hence Y_0 is simply connected.

Now, we continue with valid arguments for every $n \geq 2$. We will prove now that Y_0 is acyclic. Since V_0 is homotopy equivalent to S^{n-1}, the Mayer-Vietoris exact sequence of the space $Y = Y_0 \cup V$ can be write as

$$\to H_i(S^{n-1}) \to H_i(V) \oplus H_i(Y_0) \to H_i(Y) \to H_{i-1}(S^{n-1}) \to$$

It is obvious that the homomorphism $H_0(S^{n-1}) \to H_0(V) \oplus H_0(Y_0)$ is injective. Moreover, since V is contractible, it follows that $H_i(Y_0) \approx H_i(Y)$ for each $i < n-1$. By Theorem 2.1, $\widetilde{H}_i(Y_0) = 0$ for each $i < n-1$. Also, since Y_0 is open, we have $H_n(Y_0) = 0$. Thus, we have the short exact sequence

$$0 \to H_n(Y) \to H_{n-1}(S^{n-1}) \xrightarrow{\tau} H_{n-1}(Y_0) \to 0$$

where the homomorphism τ is identical to that induced by the natural inclusion $\partial V \hookrightarrow Y_0$, which is trivial. Therefore, $H_{n-1}(Y_0) = 0$. Thus, we conclude that $\widetilde{H}_i(Y_0) = 0$ for every i, what shows that Y_0 is acyclic. Moreover, since we have proved that Y_0 is simply connected and, by assumption, its one-point compactification, Y, is a manifold, the Main Theorem A implies that Y_0 is homeomorphic to $\mathbb{R}^n$ and so Y is homeomorphic to S^n. ∎

4. CONSEQUENCES OF THE MAIN THEOREM A

In this section, we prove some corollaries of the Main Theorem A. By the Main Theorem B, such corollaries are consequences also of the Generalized Poincaré Conjecture. However, we use essentially the Main Theorem A to prove them.

COROLLARY 4.1. *A $(n-1)$-connected open n-manifold is homeomorphic to $\mathbb{R}^n$ if and only if its one-point-compactification is a manifold.*

Proof. For $n = 1$ the result is trivial. For $n \geq 2$, let X be a $(n-1)$-connected open n-manifold. Then X is connected and simply connected and, moreover, $\widetilde{H}_i(X) = 0$ for each $i < n$, by Theorem 2.1. Hence X is acyclic. Now, we apply the Main Theorem A. ∎

This corollary applies, in particular, if X is a contractible open n-manifold, what proves that *a contractible open n-manifold is homeomorphic to $\mathbb{R}^n$ if and only if its one-point compactification is a manifold.*

COROLLARY 4.2. *Let Y be a closed n-manifold not homeomorphic to the sphere S^n and let $y_0 \in Y$ be a point. If $Y \setminus \{y_0\}$ is acyclic, then $\pi_1(Y \setminus \{y_0\})$ is a nontrivial perfect group.*

Proof. Certainly, $Y \setminus \{y_0\}$ is not homeomorphic to $\mathbb{R}^n$. Moreover, Y is the one-point compactification of the open n-manifold $Y \setminus \{y_0\}$. By the Main Theorem A, $Y \setminus \{y_0\}$ either is not acyclic or is not simply connected. Now, if $Y\setminus\{y_0\}$ is acyclic, then $H_1(Y\setminus\{y_0\})=0$, but $\pi_1(Y\setminus\{y_0\})$ is nontrivial. Since H_1 is the abelianization of π_1 it follows that $\pi_1(Y\setminus\{y_0\})$ is a nontrivial perfect group. ▮

COROLLARY 4.3. *Let Y be a closed n-manifold and $y_0 \in Y$ be a point. If $Y \setminus \{y_0\}$ is acyclic and simply connected, then Y is homeomorphic to S^n and so $Y \setminus \{y_0\}$ is homeomorphic to $\mathbb{R}^n$.*

Proof. The statement of this corollary can be considered a restatement of the previous corollary. ▮

We say that an open n-manifold W is a *model Whitehead Manifold* if it is contractible but is not homeomorphic to $\mathbb{R}^n$.

COROLLARY 4.4. *The one-point compactification of any model Whitehead Manifold is not a manifold.*

Proof. Let W be a model Whitehead Manifold of dimension n and let $\overline{W}$ its one-point compactification. Suppose that $\overline{W}$ is a manifold and so a closed n-manifold. Certainly, $\overline{W}$ is not homeomorphic to S^n. By Corollary 4.2, either W is not acyclic or $\pi_1(W)$ is nontrivial. This is a contradiction, since W is contractible. ▮

THEOREM 4.1. *Let Y be an orientable and connected closed 3-manifold. Then Y is homeomorphic to S^3 if and only if it is simply connected. If Y is not homeomorphic to S^3 but $H_1(Y)$ is trivial, then $\pi_1(Y)$ is a nontrivial perfect group.*

Proof. Certainly, $H_3(Y) \approx \mathbb{Z} \approx H_0(Y)$. Furthermore, by the Poincaré Duality (see Theorem 26.6 of [2]), we have $H_1(Y) \approx H_2(Y)$.

Suppose that Y is simply connected. Then both $H_1(Y)$ and $H_2(Y)$ are trivial. Let y_0 be a point in Y and let $Y_0 = Y \setminus \{y_0\}$. Then Y_0 is an open 3-manifold whose one-point compactification is Y. Moreover, the same arguments of the proof of the Main Theorem B show that Y_0 is acyclic and simply connected. By the Main Theorem A, Y_0 is homeomorphic to $\mathbb{R}^3$ and so Y is homeomorphic to S^3.

Now, assume that Y is not homeomorphic to S^3 but $H_1(Y)$ is trivial. By the first part of the theorem, $\pi_1(Y)$ is not trivial. Then, since $H_1(Y)$ is the abelianization of $\pi_1(Y)$, it follows that $\pi_1(Y)$ is a nontrivial perfect group. ▮

The previous theorem has a natural generalization in high dimensions.

THEOREM 4.2. *Let Y be an orientable and connected closed n-manifold, $n \geq 3$. Suppose that $H_i(Y)=0$ for each integer $1 < i \leq (n+1)/2$. Then Y is homeomorphic to S^n if and only if it is simply connected. Moreover, if Y is not homeomorphic to S^n but $H_1(Y)$ is trivial, then $\pi_1(Y)$ is a nontrivial perfect group.*

Proof. We use again the Poincaré Duality and repeat the arguments of the proof of the previous theorem. ∎

5. PROOF OF THE MAIN THEOREM C

Let Y be a compact n-manifold with boundary ∂Y. Then $\partial Y \times [0,1)$ is a n-manifold with boundary $\partial(\partial Y \times [0,1)) \cong \partial Y$. As in Proposition 3.42 of [3], we can consider the space $X = Y \cup_{\partial Y} (\partial Y \times [0,1))$ obtained from the disjoint union of Y and $\partial Y \times [0,1)$ by identifying each $x \in \partial Y$ with $(x,0) \in \partial Y \times [0,1)$. The space X constructed in this way is an open n-manifold, which is said to be obtained from Y by attaching an open collar. By the way, we say also that X is Y with an open collar attached.

In according to Definition 1.2, a compact manifold $\dot{X}$, which boundary $\partial\dot{X}$, contained in an open manifold X, is the compact spine of X if X is homeomorphic to $\dot{X}$ with an open collar attached. In particular, if Y is a compact manifold with boundary, then Y is the compact spine of Y with an open collar attached.

The space $\mathbb{R}^n$ is homeomorphic to D^n with an open collar attached, where D^n is the unitary closed n-disc in $\mathbb{R}^n$. Therefore, the space $\mathbb{R}^n$ has a compact spine whose boundary is the sphere S^{n-1}. For an acyclic and simply connected open manifold with compact spine, we have the following general result:

THEOREM 5.1. *Let X be an acyclic and simply connected open n-manifold and suppose that $\dot{X}$ is the compact spine of X. Then the boundary $\partial\dot{X}$ is a homology $(n-1)$-sphere.*

Proof. Certainly, $\dot{X}$ is a strong deformation retract of X. Thus, also $\dot{X}$ is acyclic and simply connected. By Theorem 2.4, $\dot{X}$ is contractible. Since X is orientable (since X is simply connected), also $int(\dot{X})$ is an orientable n-manifold. By the Lefschetz Duality (see Theorem 28.18 of [2]), we have $H_i(\dot{X}, \partial\dot{X}) \approx H^{n-i}(\dot{X})$ for each $i \geq 0$. Now, since $\dot{X}$ is acyclic, it has the cohomology of a point. Hence, $H_i(\dot{X}, \partial\dot{X}) = 0$ for each $i \neq n$ and $H_n(\dot{X}, \partial\dot{X}) \approx \mathbb{Z}$. Considering the exact sequence of homology of the pair $(\dot{X}, \partial\dot{X})$, namely,

$$\rightarrow H_i(\partial\dot{X}) \rightarrow H_i(\dot{X}) \rightarrow H_i(\dot{X}, \partial\dot{X}) \rightarrow H_{i-1}(\partial\dot{X}) \rightarrow$$

we conclude that $\widetilde{H}_i(\partial\dot{X}) = 0$ for each $i \neq n-1$ and $H_{n-1}(\partial\dot{X}) \approx \mathbb{Z}$. ∎

Proof (Proof of the Main Theorem C). (1) Let $n = 1$ or 2 and let X be an acyclic and simply connected open n-manifold. If X is homeomorphic to $\mathbb{R}^n$, it is obvious that X has a compact spine, as we have seen. Now, suppose that $\dot{X}$ is the compact spine of X. Then, by Theorem 5.1 and the classification of the compact manifolds of dimensions 1 and 2, we have that $\partial\dot{X}$ is homeomorphic to S^{n-1}. Moreover, since $\dot{X}$ is contractible (due Theorem 2.4), $\dot{X}$ is homeomorphic to the closed disc D^n. Hence, X is homeomorphic to $\mathbb{R}^n$.

(2) Let $n \geq 3$ and let X be an acyclic and simply connected open n-manifold. If X is homeomorphic to $\mathbb{R}^n$, then it is obvious that X has compact spine with simply connected boundary, indeed, in this case, the boundary of a compact spine of X is homeomorphic

to the sphere S^{n-1}. Now, suppose that $\dot{X}$ is a compact spine of X and $\partial\dot{X}$ is simply connected. Then, by Theorems 2.3 and 5.1, $\partial\dot{X}$ is a homotopy $(n-1)$-sphere. By the Generalized Poincaré Conjecture, $\partial\dot{X}$ is homeomorphic to the sphere S^{n-1}. Now, by Theorem 2.4, $\dot{X}$ is contractible. Then, by the characterization of closed n-discs (see [5]), $\dot{X}$ is homeomorphic to the closed disc D^n. Therefore, X is homeomorphic to $\mathbb{R}^n$. ∎

As consequences of the Main Theorem A and C and Theorem 5.1 we obtain the following corollaries:

Corollary 5.1. *Let $n \geq 3$ and let X be an acyclic and simply connected open n-manifold not homeomorphic to $\mathbb{R}^n$. Suppose that $\dot{X}$ is the compact spine of X. Then $\pi_1(\partial\dot{X})$ is a nontrivial perfect group.*

Proof. By the Main Theorem C, $\pi_1(\partial\dot{X})$ is not trivial. However, by Theorem 5.1, $H_1(\partial\dot{X}) = 0$. Therefore, $\pi_1(\partial\dot{X})$ is a nontrivial perfect group. ∎

Corollary 5.2. *Any model Whitehead Manifold does not have compact spine with boundary simply connected.*

Proof. It follows from Main Theorem A and C and Corollary 4.4. ∎

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