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**INVARIANTS OF RELATIVE RIGHT AND CONTACT
EQUIVALENCES**

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INVARIANTS OF RELATIVE RIGHT AND CONTACT EQUIVALENCES

IMRAN AHMED¹ AND MARIA APARECIDA SOARES RUAS²

1. INTRODUCTION

Let $\mathcal{O}_n$ be the ring of germs of analytic functions $h : \mathbb{C}^n, 0 \rightarrow \mathbb{C}$. Consider the analytic variety $V = \{x : f_1(x) = \dots = f_r(x) = 0\} \subset \mathbb{C}^n, 0$, where $f_1, \dots, f_r$ are germs of analytic functions. In this note we study function germs $h : \mathbb{C}^n, 0 \rightarrow \mathbb{C}, 0$ under the equivalence relation that preserves the analytic variety $V, 0$.

We say that two function germs h_1 and $h_2 : \mathbb{C}^n, 0 \rightarrow \mathbb{C}, 0$ are $\mathcal{R}_V$ -equivalent if there exists a germ of a diffeomorphism $\psi : \mathbb{C}^n, 0 \rightarrow \mathbb{C}^n, 0$ with $\psi(V) = V$ and $h_1 \circ \psi = h_2$. That is,

$$\mathcal{R}_V = \{\psi \in \mathcal{R} : \psi(V) = V\}$$

where $\mathcal{R}$ is the group of germs of diffeomorphisms of $\mathbb{C}^n, 0$.

Two function germs h_1 and $h_2 : \mathbb{C}^n, 0 \rightarrow \mathbb{C}, 0$ are $\mathcal{K}_V$ -equivalent if there exists a germ of a diffeomorphism $\psi : \mathbb{C}^n, 0 \rightarrow \mathbb{C}^n, 0$ and a unit $u \in \mathcal{O}_n^*$ such that $\psi(V) = V$ and $h_1 = u \cdot (h_2 \circ \psi)$.

We denote by θ_n the set of germs of tangent vector fields in $\mathbb{C}^n, 0$; θ_n is a free $\mathcal{O}_n$ module of rank n . Let $I(V)$ be the ideal in $\mathcal{O}_n$ consisting of germs of analytic functions vanishing on V . We denote by $\Theta_V = \{\eta \in \theta_n : \eta(I(V)) \subseteq I(V)\}$, the submodule of germs of vector fields tangent to V .

The tangent space to the action of the group $\mathcal{R}_V$ is $T\mathcal{R}_V(h) = dh(\Theta_V^0) = \mathcal{J}_h(\Theta_V^0)$, where Θ_V^0 is the submodule of Θ_V given by the vector fields that are zero at zero. When the point $x = 0$ is a stratum in the logarithmic stratification of the analytic variety, this is the case when V has an isolated singularity at the origin, see [2] for details, both spaces Θ_V and Θ_V^0 coincide.

The tangent space to the action of the group $\mathcal{K}_V$ is $T\mathcal{K}_V(h) = \langle h, dh(\Theta_V^0) \rangle = \langle h, \mathcal{J}_h(\Theta_V^0) \rangle$.

We fix a system of local coordinates x of $\mathbb{C}^n$. Due to the identification between $\mathcal{O}_n$ and the ring of convergent power series $\mathbb{C}\{x_1, \dots, x_n\}$ we identify a germ $f \in \mathcal{O}_n$ with its power series $f(x) = \sum a_\alpha x^\alpha$, where $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

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The relative Milnor and Tjurina algebras, $M_V(h)$ and $T_V(h)$, of h are defined respectively, by

$$M_V(h) = \frac{\mathbb{C}\{x_1, \dots, x_n\}}{\mathcal{J}_h(\Theta_V^0)} \text{ and } T_V(h) = \frac{\mathbb{C}\{x_1, \dots, x_n\}}{\langle h, \mathcal{J}_h(\Theta_V^0) \rangle}.$$

When V is a weighted homogeneous variety, we can always choose weighted homogeneous generators for Θ_V . Moreover, it is finitely generated [3], Lemma 3.2, p.41.

We recall first Mather's Lemma 4.1 providing effective necessary and sufficient conditions for a connected submanifold to be contained in an orbit. In Theorem 4.3 we show that two arbitrary (i.e. not necessary with isolated singularities) quasihomogeneous polynomials f and g having isomorphic relative Milnor algebras $M_V(f)$ and $M_V(g)$ are $\mathcal{R}_V$ -equivalent.

In Theorem 4.5 we prove that two arbitrary complex-analytic hypersurfaces (i.e. not necessary with isolated singularities), one is quasihomogeneous and other is arbitrary, are determined by isomorphism of Jacobean ideals.

The Example of Gaffney and Hauser, in [5], suggests us that we can not extend our results for arbitrary analytic germs.

In Theorem 4.9 we show that two arbitrary (i.e. not necessary with isolated singularities) function germs f and g having isomorphic relative Tjurina algebras $T_V(f)$ and $T_V(g)$ are $\mathcal{K}_V$ -equivalent, where $V = \Phi^{-1}(0)$ be a variety in $\mathbb{C}^n, 0$ such that $\Phi \in \mathbb{C}\{x_1, \dots, x_n\}$ and Θ_V is finitely generated.

This is the relative version of the celebrated theorem by Mather and Yau [9], saying that the isolated hypersurface singularities are determined by their Tjurina algebras. For arbitrary hypersurface singularities, the Mather-Yau Theorem (even a more general version) has been proved by Greuel, Lossen and Shustin [6].

2. QUASIHOMOGENEOUS FUNCTIONS AND FILTRATIONS

We recall first some basic facts on quasihomogeneous functions and filtrations in the ring A of formal power series. We introduce, in the next section, their analogues for quasihomogeneous diffeomorphisms and vector fields. For a more complete introduction see [1], Chap. 1, §3.

A holomorphic function $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ (defined on the complex space $\mathbb{C}^n$) is a quasihomogeneous function of degree d with weights $w_1, \dots, w_n$ if

$$f(\lambda^{w_1}x_1, \dots, \lambda^{w_n}x_n) = \lambda^d f(x_1, \dots, x_n) \forall \lambda > 0.$$

In terms of the Taylor series $\sum f_k x^k$ of f , the quasihomogeneity condition means that the exponents of the nonzero terms of the series lie in the hyperplane

$$L = \{\underline{k} : w_1 k_1 + \dots + w_n k_n = d\}.$$

Any quasihomogeneous function f of degree d satisfies Euler's identity

$$(2.1) \quad \sum_{i=1}^n w_i x_i \frac{\partial f}{\partial x_i} = d.f$$

It implies that a quasihomogeneous function f belongs to its Jacobean ideal $\mathcal{J}_f$. The following is the well known result of Saito [10].

Theorem 2.1. *A function-germ $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ is equivalent to a quasihomogeneous function-germ if and only if $f \in \mathcal{J}_f$.*

Consider $\mathbb{C}^n$ with a fixed coordinate system $x_1, \dots, x_n$. The algebra of formal power series in the coordinates will be denoted by $A = \mathbb{C}[[x_1, \dots, x_n]]$. We assume that a quasihomogeneity type $\underline{w} = (w_1, \dots, w_n)$ is fixed. With each such $\underline{w}$ there is associated a filtration of the ring A , defined as follows.

The monomial $\mathbf{x}^{\underline{k}}$ is said to have degree d if $\langle \underline{w}, \underline{k} \rangle = w_1 k_1 + \dots + w_n k_n = d$.

The order d of a series (resp. polynomial) is the smallest of the degrees of the monomials that appear in that series (resp. polynomial).

The series of order larger than or equal to d form a subspace $A_d \subset A$. The order of a product is equal to the sum of the orders of the factors. Consequently, A_d is an ideal in the ring A . The family of ideals A_d constitutes a decreasing filtration of A : $A_d \subset A_{d'}$ whenever $d' > d$. We let A_{d+} denote the ideal in A formed by the series of order higher than d .

The quotient algebra A/A_{d+} is called the algebra of d -quasijets, and its elements are called d -quasijets.

3. QUASIHOMOGENEOUS DIFFEOMORPHISMS AND VECTOR FIELDS

Several Lie groups and algebras are associated with the filtration defined in the ring A of power series by the type of quasihomogeneity $\underline{w}$. In the case of ordinary homogeneity these are the general linear group, the group of k -jets of diffeomorphisms, its subgroup of k -jets with $(k-1)$ -jet equal to the identity, and their quotient groups. Their analogues for the case of a quasihomogeneous filtration are defined as follows.

A formal diffeomorphism $g : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ is a set of n power series $g_i \in A$ without constant terms for which the map $g^* : A \rightarrow A$ given by the rule $g^* f = f \circ g$ is an algebra isomorphism.

The diffeomorphism g is said to have order d if for every s

$$(g^* - 1)A_s \subset A_{s+d}.$$

The set of all diffeomorphisms of order $d \geq 0$ is a group G_d . The family of groups G_d yields a decreasing filtration of the group G of formal diffeomorphisms; indeed, for $d' > d \geq 0$, $G_{d'} \subset G_d$ and is a normal subgroup in G_d .

The group G_0 plays the role in the quasihomogeneous case that the full group of formal diffeomorphisms plays in the homogeneous case. We should emphasize that in the quasihomogeneous case $G_0 \neq G$, since certain diffeomorphisms have negative orders and do not belong to G_0 .

The group of d -quasijets of type $\underline{w}$ is the quotient group of the group of diffeomorphisms G_0 by the subgroup G_{d+} of diffeomorphisms of order higher than d : $J_d = G_0/G_{d+}$.

Note that in the ordinary homogeneous case our numbering differs from the standard one by 1: for us J_0 is the group of 1-jets and so on.

J_d acts as a group of linear transformations on the space A/A_{d+} of d -quasijets of functions. A special importance is attached to the group J_0 , which is the quasihomogeneous generalization of the general linear group.

A diffeomorphism $g \in G_0$ is said to be quasihomogeneous of type $\underline{w}$ if each of the spaces of quasihomogeneous functions of degree d (and type $\underline{w}$) is invariant under the action of g^* .

The set of all quasihomogeneous diffeomorphisms is a subgroup of G_0 . This subgroup is canonically isomorphic to J_0 , the isomorphism being provided by the restriction of the canonical projection $G_0 \rightarrow J_0$.

The infinitesimal analogues of the concepts introduced above look as follows.

A formal vector field $v = \sum v_i \partial_i$, where $\partial_i = \partial/\partial x_i$, is said to have order d if differentiation in the direction of v raises the degree of any function by at least d : $L_v A_s \subset A_{s+d}$.

We let $\mathfrak{g}_d$ denote the set of all vector fields of order d . The filtration arising in this way in the Lie algebra $\mathfrak{g}$ of vector fields (i.e., of derivations of the algebra A) is compatible with the filtrations in A and in the group of diffeomorphisms G :

1. $f \in A_d, v \in \mathfrak{g}_s \Rightarrow fv \in \mathfrak{g}_{d+s}, L_v f \in A_{d+s}$
2. The module $\mathfrak{g}_d$, $d \geq 0$, is a Lie algebra w.r.t. the Poisson bracket of vector fields.
3. The Lie algebra $\mathfrak{g}_d$ is an ideal in the Lie algebra $\mathfrak{g}_0$.
4. The Lie algebra $\mathfrak{j}_d$ of the Lie group J_d of d -quasijets of diffeomorphisms is equal to the quotient algebra $\mathfrak{g}_0/\mathfrak{g}_{d+}$.
5. The quasihomogeneous vector fields of degree 0 form a finite-dimensional Lie subalgebra of the Lie algebra $\mathfrak{g}_0$; this subalgebra is canonically isomorphic to the Lie algebra $\mathfrak{j}_0$ of the group of 0-jets of diffeomorphisms.

The support of a quasihomogeneous function of degree d and type $\underline{w}$ is the set of all points $\underline{k}$ with nonnegative integer coordinates on the diagonal

$$L = \{\underline{k} : \langle \underline{k}, \underline{w} \rangle = d\}.$$

Quasihomogeneous functions can be regarded as functions given on their supports: $\sum f_{\underline{k}} x^{\underline{k}}$ assumes at $\underline{k}$ the value $f_{\underline{k}}$. The set of all such functions is a linear space $\mathbb{C}^r$,

where r is the number of points in the support. Both the group of quasihomogeneous diffeomorphisms (of type $\underline{w}$) and its Lie algebra $\mathfrak{a}$ act on this space.

The Lie algebra $\mathfrak{a}$ of a quasihomogeneous vector field of degree 0 is spanned, as a $\mathbb{C}$ -linear space, by all monomial fields $x^{\underline{P}}\partial_i$ for which $\langle \underline{P}, \underline{w} \rangle = w_i$. For example, the n fields $x_i\partial_i$ belong to $\mathfrak{a}$ for any $\underline{w}$.

Example 3.1. Consider the quasihomogeneous polynomial $f = x^2y + z^2$ of degree $d = 6$ w.r.t. weights $(2, 2, 3)$. Note that the Lie algebra of quasihomogeneous vector fields of degree 0 is spanned by

$$\mathfrak{a} = \langle x^{\underline{P}}\partial_i : \langle \underline{P}, \underline{w} \rangle = w_i, i = 1, 2, 3 \rangle = \langle x\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, z\frac{\partial}{\partial z} \rangle$$

4. FUNCTION GERMS WITH ISOMORPHIC LOCAL ALGEBRAS

We recall first Mather's lemma providing effective necessary and sufficient conditions for a connected submanifold (in our case the path P) to be contained in an orbit.

Mather's Lemma 4.1. ([8]) *Let $m : G \times M \rightarrow M$ be a smooth action and $P \subset M$ a connected smooth submanifold. Then P is contained in a single G -orbit if and only if the following conditions are fulfilled:*

- (a) $T_x(G.x) \supset T_xP$, for any $x \in P$.
- (b) $\dim T_x(G.x)$ is constant for $x \in P$.

For arbitrary (i.e. not necessary with isolated singularities) quasihomogeneous polynomials we establish the following results.

Lemma 4.2. *Let $f, g \in H_{\underline{w}}^d(n, 1; \mathbb{C}) = H_{\underline{w}}^d$ be two quasihomogeneous polynomials of degree d and $\Phi \in H_{\underline{w}}^r(n, 1; \mathbb{C})$ be a quasihomogeneous polynomial of degree r w.r.t. the same weights $\underline{w} = (w_1, \dots, w_n)$ such that $\mathcal{J}_f(\Theta_V^0) = \mathcal{J}_g(\Theta_V^0)$, where $\Phi^{-1}(0) = V$ is a hypersurface in $(\mathbb{C}^n, 0)$. Then $f \stackrel{\mathcal{R}_V}{\sim} g$, where $\stackrel{\mathcal{R}_V}{\sim}$ denotes the relative right equivalence.*

Proof. To prove this claim choose an appropriate submanifold of $H_{\underline{w}}^d(n, 1; \mathbb{C})$ containing f and g and then apply Mather's lemma to get the result.

Let $f, g \in H_{\underline{w}}^d(n, 1; \mathbb{C})$ such that $\mathcal{J}_f(\Theta_V^0) = \mathcal{J}_g(\Theta_V^0)$. Set $f_t = (1 - t)f + tg \in H_{\underline{w}}^d(n, 1; \mathbb{C})$. Consider the $\mathcal{R}_V$ -equivalence action on $H_{\underline{w}}^d(n, 1; \mathbb{C})$ under the group $\mathcal{R}_V^0 = \mathcal{R}_V \cap J_0$, we have

$$(4.1) \quad T_{f_t}(\mathcal{R}_V^0.f_t) = \mathcal{J}_{f_t}(\Theta_V^0) \cap H_{\underline{w}}^d = \langle df_t(\xi_i) : i = 1, \dots, p \rangle \cap H_{\underline{w}}^d \subset T_{f_t}(J_0.f_t)$$

where $df_t(\xi_i) = \sum_{j=1}^n a_{ij}x^{\underline{P}}\frac{\partial f_t}{\partial x_j} = \sum_{j=1}^n a_{ij}x^{\underline{P}}[(1 - t)\frac{\partial f}{\partial x_j} + t\frac{\partial g}{\partial x_j}]$, $\langle \underline{P}, \underline{w} \rangle = w_j$.

Note that we have the inclusion of finite dimensional $\mathbb{C}$ -vector spaces

$$(4.2) \quad T_{f_t}(\mathcal{R}_V^0.f_t) = \langle df_t(\xi_i) \rangle \cap H_{\underline{w}}^d \subset \mathcal{J}_f(\Theta_V^0) \cap H_{\underline{w}}^d$$

with equality for $t = 0$ and $t = 1$. The spaces $\mathcal{J}_{f_t}(\Theta_V^0) \cap H_{\underline{w}}^d$ and $\mathcal{J}_f(\Theta_V^0) \cap H_{\underline{w}}^d$ are not trivial by Euler identity 2.1.

Let's show that we have equality for all $t \in [0, 1]$ except finitely many values. Take $\dim(\mathcal{J}_{f_t}(\Theta_V^0) \cap H_{\underline{w}}^d) = \dim(\mathcal{J}_f(\Theta_V^0) \cap H_{\underline{w}}^d) = s$ (say). Let's fix $\{e_1, \dots, e_s\}$ a basis of $\mathcal{J}_f(\Theta_V^0) \cap H_{\underline{w}}^d$. Consider the s polynomials corresponding to the generators of the space (4.1):

$$\alpha_i(t) = df_t(\xi_i) = \sum_{j=1}^n a_{ij} x^P \frac{\partial f_t}{\partial x_j} = \sum_{j=1}^n a_{ij} x^P [(1-t) \frac{\partial f}{\partial x_j} + t \frac{\partial g}{\partial x_j}], \quad \langle P, \underline{w} \rangle = w_j$$

We can express each $\alpha_i(t)$, $i = 1, \dots, s$ in terms of above mentioned fixed basis as

$$(4.3) \quad \alpha_i(t) = \phi_{i1}(t)e_1 + \dots + \phi_{is}(t)e_s, \quad \forall i = 1, \dots, s$$

where each $\phi_{ij}(t)$ is linear in t . Consider the matrix of transformation corresponding to the eqs. (4.3)

$$(\phi_{ij}(t))_{s \times s} = \begin{pmatrix} \phi_{11}(t) & \phi_{12}(t) & \dots & \phi_{1s}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{s1}(t) & \phi_{s2}(t) & \dots & \phi_{ss}(t) \end{pmatrix}$$

having rank at most s . Note that the equality

$$\mathcal{J}_{f_t}(\Theta_V^0) \cap H_{\underline{w}}^d = \mathcal{J}_f(\Theta_V^0) \cap H_{\underline{w}}^d$$

holds for those values of t in $\mathbb{C}$ for which the rank of above matrix is precisely s . We have the $s \times s$ -matrix whose determinant is a polynomial of degree s in t and by the fundamental theorem of algebra it has at most s roots in $\mathbb{C}$ for which rank of the matrix of transformation will be less than s . Therefore, the above-mentioned equality does not hold for at most finitely many values, say $t_1, \dots, t_q$ where $1 \leq q \leq s$.

It follows that the dimension of the space (4.1) is constant for all $t \in \mathbb{C}$ except finitely many values $\{t_1, \dots, t_q\}$.

For an arbitrary smooth path

$$\alpha : \mathbb{C} \longrightarrow \mathbb{C} \setminus \{t_1, \dots, t_q\}$$

with $\alpha(0) = 0$ and $\alpha(1) = 1$, we have the connected smooth submanifold

$$P = \{f_t = (1 - \alpha(t))f(x) + \alpha(t)g(x) : t \in \mathbb{C}\}$$

of $H_{\underline{w}}^d$. By the above, it follows $\dim T_{f_t}(\mathcal{R}_V^0.f_t)$ is constant for $f_t \in P$.

Now, to apply Mather's lemma, we need to show that the tangent space to the submanifold P is contained in that to the orbit $\mathcal{R}_V^0.f_t$ for any $f_t \in P$. One clearly has

$$T_{f_t}P = \{\dot{f}_t = -\dot{\alpha}(t)f(x) + \dot{\alpha}(t)g(x) : \forall t \in \mathbb{C}\}$$

Therefore, by Euler formula 2.1, we have

$$T_{f_t}P \subset T_{f_t}(\mathcal{R}_V^0.f_t)$$

By Mather's lemma the submanifold P is contained in a single orbit. Hence the result. $\square$

Theorem 4.3. *Let $f, g \in H_{\underline{w}}^d(n, 1; \mathbb{C}) = H_{\underline{w}}^d$ be two quasihomogeneous polynomials of degree d and $\Phi \in H_{\underline{w}}^r(n, 1; \mathbb{C})$ be a quasihomogeneous polynomial of degree r w.r.t. the same weights $\underline{w} = (w_1, \dots, w_n)$. If $M_V(f) \simeq M_V(g)$ (isomorphism of graded $\mathbb{C}$ -algebra) then $f \stackrel{\mathcal{R}_V}{\sim} g$, where $\Phi^{-1}(0) = V$ is a hypersurface in $(\mathbb{C}^n, 0)$.*

Proof. We show firstly that an isomorphism of graded $\mathbb{C}$ -algebras

$$\varphi : (M_V(g))_l = \left(\frac{\mathbb{C}[x_1, \dots, x_n]}{\mathcal{J}_g(\Theta_V^0)} \right)_l \xrightarrow{\simeq} (M_V(f))_l = \left(\frac{\mathbb{C}[x_1, \dots, x_n]}{\mathcal{J}_f(\Theta_V^0)} \right)_l$$

is induced by an isomorphism $u : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that $u^*(\mathcal{J}_g(\Theta_V^0)) = \mathcal{J}_f(\Theta_V^0)$.

Consider the following commutative diagram.

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 (\mathcal{J}_g(\Theta_V^0))_{d+l} & \xrightarrow{u^*} & (\mathcal{J}_f(\Theta_V^0))_{d+l} \\
 \downarrow i & & \downarrow j \\
 J_{\underline{w}}^{d+l} & \xrightarrow{u^*} & J_{\underline{w}}^{d+l} \\
 \downarrow p & & \downarrow q \\
 (M_V(g))_l & \xrightarrow[\simeq]{\varphi} & (M_V(f))_l \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array}$$

Define the morphism $u^* : J_{\underline{w}}^{d+l} \rightarrow J_{\underline{w}}^{d+l}$ by

$$(4.4) \quad u^*(x_i) = L_i(x_1, \dots, x_n) = \sum_{j=1}^n a_{ij} x_j^{\alpha_j} + \sum a_{ik_1 \dots k_n} x_{k_1}^{\beta_1} \dots x_{k_n}^{\beta_n}; \quad i = 1, \dots, n$$

where $k_m \in \{1, \dots, n\}$ & $w_{k_1}\beta_1 + \dots + w_{k_n}\beta_n = \deg_{\underline{w}}(x_i) = w_j\alpha_j$, which is well defined by commutativity of diagram below.

$$\begin{array}{ccc} x_i & \xrightarrow{u^*} & L_i \\ \downarrow p & & \downarrow q \\ \widehat{x}_i & \xrightarrow[\simeq]{\varphi} & \widehat{L}_i \end{array}$$

Note that the isomorphism φ is a degree preserving map and is also given by the same morphism u^* . Therefore, u^* is an isomorphism.

Now, we show that $u^*(\mathcal{J}_g(\Theta_V^0)) = \mathcal{J}_f(\Theta_V^0)$. For every $G \in (\mathcal{J}_g(\Theta_V^0))_{d+l}$, we have $u^*(G) \in (\mathcal{J}_f(\Theta_V^0))_{d+l}$ by commutative diagram below.

$$\begin{array}{ccc} G & \xrightarrow{u^*} & F = u^*(G) \\ \downarrow p & & \downarrow q \\ \widehat{0} & \xrightarrow{\varphi} & \widehat{F} = \widehat{0} \end{array}$$

It implies that $u^*((\mathcal{J}_g(\Theta_V^0))_{d+l}) \subset (\mathcal{J}_f(\Theta_V^0))_{d+l}$. As u^* is an isomorphism, therefore it is invertible and by repeating the above argument for its inverse, we have $u^*((\mathcal{J}_g(\Theta_V^0))_{d+l}) \supset (\mathcal{J}_f(\Theta_V^0))_{d+l}$. Therefore, $u^*((\mathcal{J}_g(\Theta_V^0))_{d+l}) = (\mathcal{J}_f(\Theta_V^0))_{d+l}$. It follows that $u^*(\mathcal{J}_g(\Theta_V^0)) = \mathcal{J}_f(\Theta_V^0)$. Thus, u^* is an isomorphism with $u^*(\mathcal{J}_g(\Theta_V^0)) = \mathcal{J}_f(\Theta_V^0)$.

By eq. (4.4), the map $u : \mathbb{C}^n \rightarrow \mathbb{C}^n$ can be defined by

$$u(z_1, \dots, z_n) = (L_1(z_1, \dots, z_n), \dots, L_n(z_1, \dots, z_n))$$

where $L_i(z_1, \dots, z_n) = \sum_{j=1}^n a_{ij}x_j^{\alpha_j} + \sum a_{ik_1\dots k_n}x_{k_1}^{\beta_1}\dots x_{k_n}^{\beta_n}$; $i = 1, \dots, n$, $k_m \in \{1, \dots, n\}$ & $w_{k_1}\beta_1 + \dots + w_{k_n}\beta_n = \deg_{\underline{w}}(x_i) = w_j\alpha_j$. Note that u is an isomorphism by Prop. 3.16 [4], p.23.

In this way, we have shown that the isomorphism φ is induced by the isomorphism $u : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that $u^*(\mathcal{J}_g(\Theta_V^0)) = \mathcal{J}_f(\Theta_V^0)$.

Consider $u^*(\mathcal{J}_g(\Theta_V^0)) = \langle g_1 \circ u, \dots, g_n \circ u \rangle = \mathcal{J}_{g \circ u}(\Theta_V^0)$, where g_j are the generators of $\mathcal{J}_g(\Theta_V^0)$. Therefore, $\mathcal{J}_{g \circ u}(\Theta_V^0) = \mathcal{J}_f(\Theta_V^0) \Rightarrow g \circ u \stackrel{\mathcal{R}_V}{\sim} f$, by Lemma 4.2. Hence, by definition there exists an analytic isomorphism $h \in \mathcal{R}_V$ such that $g \circ u = f \circ h$. Since $\mathcal{R}_V$ is a group, therefore $h^{-1} \in \mathcal{R}_V$. Taking $u = h^{-1}$ we have $g \circ h \stackrel{\mathcal{R}_V}{\sim} g$. Thus, $f \stackrel{\mathcal{R}_V}{\sim} g$. $\square$

Remark 4.4. The converse implication, namely

$$f \stackrel{\mathcal{R}_V}{\sim} g \Rightarrow M_V(f) \simeq M_V(g)$$

always holds (even for analytic germs f, g defining IHS).

Proof. Let $f \stackrel{\mathcal{R}_V}{\sim} g$. Then, by definition, there exists an analytic isomorphism $h \in \mathcal{R}_V$ such that $f \circ h = g$. It follows that $\mathcal{J}_{f \circ h}(\Theta_V^0) = \mathcal{J}_g(\Theta_V^0) \Rightarrow h^*(\mathcal{J}_f(\Theta_V^0)) = \mathcal{J}_g(\Theta_V^0)$. By Prop. 3.16 [4], p.23 h^* is also analytic isomorphism. Thus, $M_V(f) \cong M_V(g)$ by the commutativity of the diagram below.

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 \mathcal{J}_g(\Theta_V^0) & \xrightarrow{h^*} & \mathcal{J}_f(\Theta_V^0) \\
 \downarrow i & & \downarrow j \\
 \mathbb{C}[x_1, \dots, x_n] & \xrightarrow{h^*} & \mathbb{C}[x_1, \dots, x_n] \\
 \downarrow p & & \downarrow q \\
 M_V(g) & \xrightarrow{\varphi} & M_V(f) \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array}$$

□

Theorem 4.5. Let $f \in H_{\underline{w}}^d(n, 1; \mathbb{C}) = H_{\underline{w}}^d$ be a quasihomogeneous polynomials of degree d and $\Phi \in H_{\underline{w}}^r(n, 1; \mathbb{C})$ be a quasihomogeneous polynomial of degree r w.r.t. the same weights $\underline{w} = (w_1, \dots, w_n)$. Let g be an arbitrary analytic germ such that $\mathcal{J}_f(\Theta_V^0) \cong \mathcal{J}_g(\Theta_V^0)$, where $\Phi^{-1}(0) = V$ is a hypersurface in $(\mathbb{C}^n, 0)$. Then $f \stackrel{\mathcal{R}_V}{\sim} g$.

Proof. Let $\mathcal{J}_f(\Theta_V^0) \cong \mathcal{J}_g(\Theta_V^0)$. Then there exists an analytic isomorphism $h \in \mathcal{R}$ such that $h^*(\mathcal{J}_g(\Theta_V^0)) = \mathcal{J}_f(\Theta_V^0)$ by Prop. 3.16 [4], p.23. It follows that $\mathcal{J}_{g \circ h}(\Theta_V^0) = \mathcal{J}_f(\Theta_V^0)$, where $g \circ h$ is quasihomogeneous polynomial. It implies that $g \circ h \stackrel{\mathcal{R}_V}{\sim} f$ by Lemma 4.2. Hence, by definition there exists an analytic isomorphism $u \in \mathcal{R}_V$ such that $g \circ h = f \circ u$. Since $\mathcal{R}_V$ is a group, therefore $u^{-1} \in \mathcal{R}_V$. Taking $h = u^{-1}$ we have $g \circ h \stackrel{\mathcal{R}_V}{\sim} g$. Thus, $f \stackrel{\mathcal{R}_V}{\sim} g$. □

The following Example of Gaffney and Hauser, in [5], suggests us that we can not extend the Lemma 4.2 and Theorem 4.5 for arbitrary analytic germs.

Example 4.6. Let $h : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be any function satisfying $h \notin \mathcal{J}_h \subseteq \mathcal{O}_n$ i.e. $h \notin H_{\underline{w}}^d(n, 1; \mathbb{C})$. Define a family $f_t : (\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ by $f_t(x, y, z) = h(x) + (1 + z + t)h(y)$, and let $(X_t, 0) \subseteq (\mathbb{C}^{2n+1}, 0)$ be the hypersurface defined by f_t . Note that

$$\mathcal{J}_{f_t} = \left\langle \frac{\partial h}{\partial x_i}(x), \frac{\partial h}{\partial y_j}(y), h(y) \right\rangle, \quad t \in \mathbb{C}.$$

On the other hand, the family $\{(X_t, 0)\}_{t \in \mathbb{C}}$ is not trivial i.e. $(X_t, 0) \not\cong (X_0, 0)$: For, if $\{f_t\}_{t \in \mathbb{C}}$ were trivial, we would have by Prop. 2, §1, [5]

$$\frac{\partial f_t}{\partial t} = h(y) \in (f_t) + m_{2n+1}\mathcal{J}_{f_t} = (f_t) + m_{2n+1}\mathcal{J}_{h(x)} + m_{2n+1}\mathcal{J}_{h(y)} + m_{2n+1}(h(y))$$

Solving for $h(y)$ implies either $h(y) \in \mathcal{J}_{h(y)}$ or $h(x) \in \mathcal{J}_{h(x)}$ contradicting the assumption on h .

It follows that f_t is not $\mathcal{R}$ -equivalent to f_0 .

Before proceeding further, we state the lifting lemma.

Lemma 4.7. *Let φ be a morphism of analytic K -algebras*

$$\varphi : A = K\langle x_1, \dots, x_n \rangle / I \rightarrow B = K\langle y_1, \dots, y_m \rangle / J$$

Then φ has a lifting $\tilde{\varphi} : K\langle \mathbf{x} \rangle \rightarrow K\langle \mathbf{y} \rangle$ which can be chosen as an isomorphism in the case that φ is an isomorphism and $n = m$.

For arbitrary hypersurface singularities (i.e. not necessary with isolated singularities), the following result has been obtained by Greuel, Lossen and Shustin [6], 2007.

Theorem 4.8. *(Mather-Yau Theorem) Let $f, g \in \mathbb{C}\{x_1, \dots, x_n\}$ be two arbitrary hypersurface singularities having isomorphic Tjurina algebras $T(f) \simeq T(g)$. Then $f \stackrel{\mathcal{K}}{\sim} g$, where $\stackrel{\mathcal{K}}{\sim}$ denotes the contact equivalence.*

For arbitrary hypersurface singularities (i.e. not necessary with isolated singularities), we establish now the relative version of Mather-Yau Theorem under the hypothesis that $V = \Phi^{-1}(0)$ be a variety in $\mathbb{C}^n, 0$ such that $\Phi \in \mathbb{C}\{x_1, \dots, x_n\}$ and Θ_V is finitely generated. We remark that if $\Phi \in \mathbb{C}[[x_1, \dots, x_n]]$, the ring of formal power series, it is known that Θ_V is always finitely generated [7], Th. 5.4, p.771. However, we do not know whether the result holds for all $\Phi \in \mathbb{C}\{x_1, \dots, x_n\}$.

Theorem 4.9. *Let $f, g \in \mathbb{C}\{x_1, \dots, x_n\}$ be two arbitrary hypersurface singularities having isomorphic relative Tjurina algebras $T_V(f) \simeq T_V(g)$, where $V = \Phi^{-1}(0)$ be a variety in $\mathbb{C}^n, 0$ such that $\Phi \in \mathbb{C}\{x_1, \dots, x_n\}$ and Θ_V is finitely generated. Then $f \stackrel{\mathcal{K}_V}{\sim} g$, where $\stackrel{\mathcal{K}_V}{\sim}$ denotes the relative contact equivalence.*

Proof. Consider the isomorphism of graded $\mathbb{C}$ -algebras

$$\varphi : T_V(f) = \frac{\mathbb{C}\{x_1, \dots, x_n\}}{\langle f, \mathcal{J}_f(\Theta_V^0) \rangle} \xrightarrow{\simeq} T_V(g) = \frac{\mathbb{C}\{x_1, \dots, x_n\}}{\langle g, \mathcal{J}_g(\Theta_V^0) \rangle}.$$

Note that φ lifts to an isomorphism $\tilde{\varphi} : \mathbb{C}\{x_1, \dots, x_n\} \rightarrow \mathbb{C}\{x_1, \dots, x_n\}$ with $\tilde{\varphi}(\langle f, \mathcal{J}_f(\Theta_V^0) \rangle) = \langle g, \mathcal{J}_g(\Theta_V^0) \rangle$ by Lifting Lemma 4.7. Since $\tilde{\varphi}(\langle f, \mathcal{J}_f(\Theta_V^0) \rangle) = \langle \tilde{\varphi}(f), \mathcal{J}_{\tilde{\varphi}(f)}(\Theta_V^0) \rangle$, we may assume that

$$(4.5) \quad \langle f, \mathcal{J}_f(\Theta_V^0) \rangle = \langle g, \mathcal{J}_g(\Theta_V^0) \rangle.$$

Let $f, g \in \mathbb{C}\{x_1, \dots, x_n\}$. Set $F(x, t) = f_t(x) = (1 - t)f(x) + tg(x)$ for $t \in \mathbb{C}$. Thus (f_t) is a 1-parameter family of germs with $f_0 = f$, $f_1 = g$. We intend to show that any two germs in this family satisfying Equation 4.5 are $\mathcal{K}_V$ equivalent.

Consider the family of ideals

$$I_{f_t} = \langle f_t, \mathcal{J}_{f_t}(\Theta_V^0) \rangle \subset \mathbb{C}\{x_1, \dots, x_n, t\}, t \in \mathbb{C}.$$

By Eq. 4.5, $I_{f_t} \subset I_f = \langle f, \mathcal{J}_f(\Theta_V^0) \rangle$ and $I_f = I_g$.

Now, represent f, g in a neighbourhood $W = W(\mathbf{0}) \subset \mathbb{C}^n$ by holomorphic functions and consider the coherent $\mathcal{O}_{W \times \mathbb{C}}$ -module

$$\mathfrak{F} = \langle f, \mathcal{J}_f(\Theta_V^0) \rangle / \langle f_t, \mathcal{J}_{f_t}(\Theta_V^0) \rangle,$$

whose support is a closed analytic set in $W \times \mathbb{C}$, see A.7 [6]. Moreover, note that

$$\text{supp}(\mathfrak{F}) \cap (\{0\} \times \mathbb{C}) = \{t \in \mathbb{C} | \mathfrak{F}_{(0,t)} \neq 0\} = \{t \in \mathbb{C} | I_f \neq I_{f_t}\},$$

which is a closed analytic, hence a discrete, set of points in $\mathbb{C} = \{0\} \times \mathbb{C}$. It follows that the set $U = \{t \in \mathbb{C} | I_{f_t} = I_f\}$ is open and connected and contains 0 and 1. Note that

$$\frac{\partial f_t}{\partial t} = g - f \in I_f = I_{f_t} = \langle f_t, \mathcal{J}_{f_t}(\Theta_V^0) \rangle$$

for all $t \in U$. By Theorem 2.22 [6], p.126, we get that $f_t \stackrel{\mathcal{K}_V}{\sim} f_{\hat{t}}$ for $t, \hat{t} \in U$ such that $|t - \hat{t}|$ is sufficiently small. Therefore, $f_t \stackrel{\mathcal{K}_V}{\sim} f$ for all $t \in U$, in particular, $f \stackrel{\mathcal{K}_V}{\sim} g$. $\square$

Note that the converse implication is just an application of the chain rule, as performed in the proof of Lemma 2.10 [6], p.119.

The Theorems 4.3 and 4.5 are particular cases of Theorem 4.9. We dealt with the quasihomogeneous part first to explore the ideas deeply.

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