UNIVERSIDADE DE SÃO PAULO
Instituto de Ciências Matemáticas e de Computação
ISSN 0103-2577

LÊ NUMBERS OF PHAM-BRIESKORN ARRANGEMENTS

M. F. Z. MORGADO
M. J. SAIA

Nº 330

NOTAS DO ICMC

SÉRIE MATEMÁTICA

São Carlos – SP
Maio/2010

LÊ NUMBERS OF PHAM-BRIESKORN ARRANGEMENTS

M. F. Z. MORGADO and M. J. SAIA

Resumo

Neste artigo é mostrado como calcular os números de Lê de polinômios que são produtos de polinômios de Pham-Brieskorn de mesmo tipo, somente usando os pesos e graus de quase homogeneidade do polinômio.

Abstract

In this article we show how to compute the Lê numbers of polynomials that are products of polynomials Pham-Brieskorn of the same type, we call Pham-Brieskorn arrangements, only using the weights, the degree of homogeneity and the number of variables.

Keywords: Arrangements, Pham-Brieskorn polynomial, Lê numbers, polar ratio, prepolar coordinates and Plücker formula.

Mathematics Subject Classification: 2000: 32C18, 32B10, 58K05, 58K65

1 Introduction

In the study of germs of hypersurfaces in $\mathbb{C}^{n+1}$ with isolated singularities, John Milnor introduced in [9] an important invariant, called the Milnor number, which is interpreted in terms of algebra, topology, differential geometry and so on. This invariant plays an important role in the modern theory of singularities, and there have been several generalizations of it in many directions. One important generalization of the Milnor number was done by David Massey in [8], where he introduced the Lê numbers and the Lê cycles associated to holomorphic applications germs with non-isolated critical points, or equivalently, to non-isolated singularities in germs of complex analytic hypersurfaces. The definition of these numbers is analytical but they can also be interpreted in terms of topology ([8], Theorem 3.3), in algebraic geometry ([3], 2.1), among others.

Under the analytical aspect, Massey describes in [8], Theorem 4.5, the ones called generalized Lê-Iomdine formulae that are a technique that reduces a hypersurface with singular set of arbitrary dimension s in a hypersurface with singular set of dimension $s-1$ and it relates the Lê numbers's first hypersurface to the Lê numbers of this one obtained. Massey uses these relations to get a Plücker type formula, which relates the Lê numbers of a homogeneous polynomial of degree d using only the degree and the number of variables ([8], Corollary 4.7). With the help of this formula, Massey characterizes each Lê number of an arrangement of hyperplanes, ie, the product of linear functions.

In this article we investigate a class of polynomials which are the product of Pham-Brieskorn's polynomials of the same type, this type of polynomial we call Pham-Brieskorn arrangements. Using the technique developed by Lê-Iomdine and Massey, and also other tools of Singularity Theory, we obtain a similar relationship to that done by Massey, as any Pham-Brieskorn's polynomial is weighted homogenous, we show a Plücker type formula for such polynomials in terms of its weights and degrees. Then, we characterize each Lê number of such arrangement also in terms of the weights and degrees.

2 Definitions and basic properties about Lê numbers

To define and describe the Lê numbers we follow the notation used by Massey in [8]. We assume that the reader is familiar with the notion of coherent sheaves, gap sheaves (references [11] and [8], Definition 1.1), schemes (references [4] and [1]) and cycles (reference [2]). In order to fix the notation, for a sheaf α and an analytic subset W in some affine space, we denote by α/W the corresponding gap sheaf and by $V(\alpha)/W$ the scheme associated with the sheaf α/W, where $V(\alpha)$ denotes the analytic space defined by the vanishing of α. We shall at times enclose cycles in square brackets, $[\,\cdot\,]$.

For the definitions in this section let $h : (U,0) \to (\mathbb{C},0)$ be an analytic function, U an open subset of $\mathbb{C}^{n+1}$ containing the origin and $z = (z_0, \dots, z_n)$ a linear choice of coordinates in $\mathbb{C}^{n+1}$.

Definition 2.1. *For $0 \le k \le n$, the k^{th} (relative) polar variety, $\Gamma^k_{h,z}$, of h with respect to z is the scheme $V\left(\frac{\partial h}{\partial z_k}, \dots, \frac{\partial h}{\partial z_n}\right) / \Sigma(h)$.*

Remark 2.2. *On the level of ideals, $\Gamma^k_{h,z}$ consists of the components of $V\left(\frac{\partial h}{\partial z_k}, \dots, \frac{\partial h}{\partial z_n}\right)$ which are not contained in $\Sigma(h)$.*

We denote by $[\Gamma^k_{h,z}]$ the cycle associated with this scheme.

Definition 2.3. *For $0 \le k \le n$, the k^{th} Lê cycle, $\Lambda^k_{h,z}$, of h with respect to z is the cycle*

$$\left[\Gamma^{k+1}_{h,z} \cap V\left(\frac{\partial h}{\partial z_k}\right)\right] - [\,\Gamma^k_{h,z}\,].$$

Remark 2.4. *In general we shall write simply $\Lambda^k_{h,z}$, and not $[\Lambda^k_{h,z}]$, because unlike the polar varieties, which are defined as schemes and we have to consider the associated cycle, this definition is given in terms of cycles.*

If the intersection of $\Lambda^k_{h,z}$ with the cycle of $V(z_0 - p_0, \ldots, z_{k-1} - p_{k-1})$ is purely 0-dimensional at a point $p = (p_0, \ldots, p_n)$, ie, either p is an isolated point of the intersection or p is not in the intersection, it is possible to define the Lê numbers as follows:

Definition 2.5. *For $0 \leq k \leq n$, the k^{th} Lê number, $\lambda^k_{h,z}(p)$, of h with respect to z at p, is defined as the intersection number*

$$(\Lambda^k_{h,z} \cdot V(z_0 - p_0, \ldots, z_{k-1} - p_{k-1})\ )_p\ .$$

Among the various properties involving Lê cycles and Lê numbers, we highlight two that are directly necessary for this work. The first describes how to calculate inductively the Lê cycles and the second shows the behavior of the Lê numbers taking hyperplane sections.

Proposition 2.6. ([*8*], Proposition 1.18) *If, for all j with $0 \leq j \leq k$, $\Lambda^j_{h,z}$ is purely j-dimensional at p, then the cycles $\left[\Gamma^{k+1}_{h,z} \cap V\left(\frac{\partial h}{\partial z_k}\right)\right]$ and $[\Gamma^{k+1}_{h,z}] \cdot \left[V\left(\frac{\partial h}{\partial z_k}\right)\right]$ are equal at p. Consequently, if s is the dimension of the critical locus of h, we have*

$$[\Gamma^{s+1}_{h,z}] = \left[V\left(\frac{\partial h}{\partial z_{s+1}}, \cdots, \frac{\partial h}{\partial z_n}\right)\right] \quad \textit{and}$$

$$[\Gamma^{i+1}_{h,z}] \cdot \left[V\left(\frac{\partial h}{\partial z_i}\right)\right] = [\Gamma^i_{h,z}] + [\Lambda^i_{h,z}],\ 0 \leq i \leq s.$$

Proposition 2.7. ([*8*], Proposition 1.21) *Suppose $\Sigma(h) \cap V(z_0 - p_0) = \Sigma(h_{|_{V(z_0-p_0)}})$ for $p \in \Sigma(h)$ and consider the coordinates $\tilde{z} = (z_1, \ldots, z_n)$ for $V(z_0 - p_0)$. If $\lambda^i_{h,z}(p)$ is defined for all i and $p \notin \Gamma^1_{h,z}$, then*

$$\lambda^i_{h_{|_{V(z_0-p_0)}}, \tilde{z}}(p) = \lambda^{i+1}_{h,z}(p).$$

2.1 The Polar Ratio and the Generalized Lê-Iomdine Formulae

The generalized Lê-Iomdine formulae describe how the Lê numbers of a singular hypersurface with citical set of dimension s are related to the Lê numbers of a certain sequence of singular hypersurfaces $X_1, \ldots, X_s$ that approximates the original singularity so that the critic sets of its terms X_i have dimension $s - i$. The polar radius is a number associated to the polar curve which ensures the existence of this sequence of hypersurfaces.

Suppose that the polar curve $\Gamma^1_{h,z}$ is one-dimensional at the origin. Let η be an irreducible component of $\Gamma^1_{h,z}$ (with its reduced structure).

Definition 2.8. *If $\eta \cap V(z_0)$ is zero-dimensional at the origin, we define the polar ratio of η (for h at 0 with respect to z) as* $\dfrac{(\ \eta \cdot V(h)\ )_0}{(\ \eta \cdot V(z_0)\ )_0}$. *Otherwise, we say that the polar ratio of η equals 1.*

Theorem 2.9. ([8], Theorem 4.5) **(Generalized Lê–Iomdine Formulae)** *Let $h : (U, 0) \to (\mathbb{C}, 0)$ be a germ of analytic function with $U \subseteq \mathbb{C}^{n+1}$ open and $s = dim_0\ \Sigma(h)$ with $s \geq 1$. Consider $z = (z_0, \ldots, z_n)$ a linear choice of coordinates such that $\lambda^i_{h,z}(0)$ is defined for all $i \leq s$. Let a be a complex number different from zero and consider the coordinates $\tilde{z} = (z_1, \ldots, z_n, z_0)$ for $h + az_0^j$, where $j \geq 2$. If j is greater or equal than the maximum polar ratio of h (at 0 with respect to z) then for all, except a finite number of complex numbers a, we have*

i) $dim_0\ \Sigma(h + az_0^j) = s - 1$;

ii) $\lambda^0_{h+az_0^j,\tilde{z}}(0) = \lambda^0_{h,z}(0) + (j-1)\lambda^1_{h,z}(0)$

iii) $\lambda^i_{h+az_0^j,\tilde{z}}(0) = (j-1)\lambda^{i+1}_{h,z}(0)$, *for* $1 \leq i \leq s-1$.

Moreover, if j is bigger than the maximum polar ratio of h, the inequalities over are valid for all $a \neq 0$. In particular, this is the case if $j \geq 2 + \lambda^0_{h,z}(0)$.

We remark here that applying the theorem inductively we obtain, for some complex numbers $a_0, \ldots, a_s$, the sequence of singular hypersurfaces $X_1, \ldots, X_s$ as

$$X_1 = V(h + a_0 z_0^{j_0}), \ldots, X_s = V(h + a_0 z_0^{j_0} + \ldots + a_s z_s^{j_s})$$

where j_i is greater or equal than the maximum polar ratio of $h + a_0 z_0^{j_0} + \cdots + z_{i-1}^{j_{i-1}}$ (at 0 with respect to $\tilde{z} = (z_i, \ldots, z_n, z_0, \ldots, z_{i-1})$).

This method was first considered by Iomdin in [5] and by Lê in [6] for the particular case that the hypersurfaces with critical set $\Sigma(h)$ being one dimensional.

2.2 Prepolar coordinate systems

The existence of the Lê numbers depends on the choice of coordinate system. Massey shows in [8] (Theorem 1.28) a coordinate system that is generic with respect to a certain kind of stratification of the hypersurface $V(h)$ in which the Lê numbers are always defined.

Definition 2.10. *A good stratification for h at a point $p \in V(h)$ is an analytic stratification $\mathcal{B} = \{S_\alpha\}$ of the hypersurface $V(h)$ in a neighborhood U of p such that the smooth part of $V(h)$ is a stratum and so that the stratification satisfies Thom's a_h condition with respect to $U - V(h)$, ie, if q_i is a sequence of points in $U - V(h)$ such that $q_i \to q \in S \in \mathcal{B}$ and $T_{q_i}V(h - h(q_i))$ converges to some hyperplane T, then $T_qS \subseteq T$.*

Definition 2.11. *Let $z = (z_0, \dots, z_n)$ be a choice of linear coordinates for $\mathbb{C}^{n+1}$ and $p \in V(h)$, with $s = \dim_p \Sigma(h)$. The system $z = (z_0, \dots, z_n)$ is prepolar for h in p if there exists a good stratification of h at the point p such that $V(z_0 - p_0)$ intercepts transversally all strata S_α, with possible exception the stratum $\{p\}$ and for all j such that $1 \leq j \leq s-1$, $V(z_j - p_j)$ intercepts transversally all $S_\alpha \cap V(z_0 - p_0, \dots, z_j - p_j)$, again with possible exception the stratum $\{p\}$.*

Theorem 2.12. ([8], Theorem 1.28) *Let $p \in V(h)$ with $s = \dim_p \Sigma(h)$ and $z = (z_0, \dots, z_n)$ be a prepolar system for h at p. Then, for all i with $1 \leq i \leq s$, $\lambda^i_{h,z}(p)$ is defined. Moreover, $\Gamma^i_{h,z} \cap V(z_0 - p_0, \dots, z_{i-1} - p_{i-1})$ is empty or 0-dimensional at p with $1 \leq i \leq s$.*

An important property of the prepolar coordinates is that they are also prepolar for each hypersurface of the sequence obtained by the generalized Lê-Iomdine formulae.

Corollary 2.13. *Let $z = (z_0, \dots, z_n)$ be a prepolar system of coordinates for h at the origin and j bigger or equal than the polar ratio of h (at 0 with respect to z), then $\tilde{z} = (z_1, \dots, z_n, z_0)$ is a prepolar system of coordinates for $h + az_0^j$ at the origin, except possible for a finite number of $a \in \mathbb{C}$.*

This result is obtained by applying Lemma 4.3 and Corollary 4.18 of [8].

3 Pham-Brieskorn arrangements

From now on we shall restrict ourselves to the class of polynomials that we are interested and describe its polar radius with respect to a specific coordinate system that is prepolar.

Definition 3.1. *A polynomial $f \in \mathbb{C}[x_0, \dots, x_n]$ is called a Pham-Brieskorn polynomial if*

$$f = \gamma_0 x_0^{a_0} + \cdots + \gamma_n x_n^{a_n} \text{ with integers } \gamma_i \text{ and } a_i \geq 2.$$

Note that f is a weighted homogeneous polynomial of type $(r_0, \dots, r_n; D)$, where $D = m.c.m.(r_0, \dots, r_n)$.

Definition 3.2. *We call a polynomial $h \in \mathbb{C}[x_0, \dots, x_n]$ a Pham-Brieskorn arrangement if it is written in the form*

$$h = c\, f_1 \cdots f_p f_{p+1}^{\alpha_{p+1}} \cdots f_l^{\alpha_l},$$

where each f_j is a Pham-Brieskorn polynomial of type $(r_0, \dots, r_n; D)$. We remark that in this case the polynomial h is weighted homogeneous of type $(r_0, \dots, r_n; d)$ where $d = \left(p + \sum_{i=p+1}^{l} \alpha_i\right) D$. Therefore, we call such polynomial h a Pham-Brieskorn arrangement of type $(r_0, \dots, r_n; d)$.

Suppose now that r_i and r_j are coprime for all $i \neq j$. Thus $D = r_0 \cdots r_n$ and $a_i = k_i r_0 \cdots \hat{r}_i \cdots r_n$, with k_i integers. The notation $\hat{r}_i$ means that there is not the term r_i with $i \in \{0, \ldots, n\}$. In this case, a Pham-Brieskorn arrangement of type $(r_0, \ldots, r_n; d)$ is written in the form

$$h = c\, f_1 \cdots f_p f_{p+1}^{\alpha_{p+1}} \cdots f_l^{\alpha_l},$$

where $f_j = \gamma_{0,j} x_0^{k_0(\hat{r}_0 r_1 \cdots r_n)} + \cdots + \gamma_{n,j} x_n^{k_n(r_0 r_1 \cdots \hat{r}_n)}$, with r_i and r_j coprime for all $i \neq j$, $\alpha_i > 1$ and $d = \left(p + \sum_{i=p+1}^{l} \alpha_i\right) r_0 \cdots r_n$.

To facilitate the calculations we study a particular case, when $k_i = 1$ with $i = 1, \ldots, n$. We emphasize that the results shown below are also valid for any integer $k_i \geq 1$ and the calculations are exactly the same.

3.1 Prepolar coordinates

To describe the Lê numbers of such arrangements, the first step is to find a convenient prepolar coordinate system. In order to do this we need to obtain a stra-tification of $V(h)$ which satisfies the Whitney conditions since such stratification is always a good stratification for h. Lê and Teissier in [7] showed that there always exists such a stratification of the variety $V(h)$, called by Lê and Teissier minimal Whitney stratification of $V(h)$.

Here use the idea given by Massey in [8], page 61, to obtain a Whitney stratification for arrangements given by hyperplanes, to give an explicit description of a stratification for Pham-Brieskorn arrangements. We show that this stratification is the minimal Whitney stratification of $V(h)$ as defined by Lê and Teissier.

We recover here this construction.

According to Lê and Teissier in [7] there exists a sequence of algebraic subvarieties of $V(h)$, $F_0 \supset F_1 \supset \cdots \supset F_k \cdots$, where $F_0 = V(h)$ and F_1 denotes the singular set of $V(h)$. To obtain the other F_j, let us denote $(F_{1,j_1})_{j_1 \in J_1}$ the irreducible components of F_1, then F_2 is the union of the singular set of F_1 with a finite number of closed algebraic sets given by the points in F_{1,j_1} such that the sequence of polar multiplicities of Lê-Teissier is different of the one computed on a generic point of F_{1,j_1}.

Here we are considering the k^{th} polar multiplicity in a point, as given by Lê-Teissier. This means the multiplicity of the polar variety $P_k(V(h), D_{n+1-k})$ in this point, defined by Lê and Teissier as,

$$P_k(V(h), D_{n+1-k}) = \overline{\{p \in V(h)_{reg} \mid \dim\left(T_p V(h)_{reg} \cap D_{n+1-k}\right) \geq k\}},$$

D_{n+1-k} is a generic hyperplane and $V(h)_{reg}$ denotes the regular part of $V(h)$.

We remark that this definition of polar variety is different from the one given by Massey in the Definition 2.1.

Now denote $(F_{2,j_2})_{j_2 \in J_2}$, the irreducible components of F_2, in the general case, let F_k be the union of the singular set of F_{k-1} with a finite number of closed algebraic sets given by the points such that the sequence of polar multiplicities of Lê-Teissier is different of the one computed on a generic point of $F_{k-1,j_{k-1}}$.

In order to construct the Whitney stratification of the set $V(h)$, we fix the following notation: For any i with $1 \leq i \leq l$ define $H_i = V(f_i^{m_i})$, with $m_i = 1$ if $1 \leq i \leq p$ and $m_i = \alpha_i$ if $p+1 \leq i \leq l$.

Call I the set of indexes $\{1, \dots, l\}$ and for any subset J of I, define

$$w_J := \bigcap_{i \in J} H_i \quad \text{and} \quad S_J := w_J - \bigcup_{J \subsetneq K} w_K.$$

Lemma 3.3. *$\{S_J\}_{J \subseteq I}$ is the minimal Whitney stratification of $V(h)$.*

Proof. We show, for each k, that the irreducible components F_{k,j_k} are

$$F_{k,j_k} = w_J := \bigcap_{i \in J} H_i \quad \text{with} \quad \#J = k+1.$$

As each function f_i has isolated singularity at the origin, the components of singular set of $V(h)$ are the intersections $H_i \cap H_j$ with $i \neq j$ and $i, j \in I$.

Hence we have that F_1 is given by the union of the intersections $H_i \cap H_j$ with $i \neq j$ and $i, j \in I$, or in other words, each F_1, j_1 is of the form $F_1, j_1 = H_i \cap H_j$.
By the same reason the singular set of F_1 has as components the intersections $H_i \cap H_j \cap H_k$ with $i, j, k \in I$, $i \neq j \neq k$.

Now if $p \in V(h)_{reg}$, then $p \in H_i - \bigcup_{j \neq i} H_j$ for some i, therefore $\dim\left(T_p V(h)_{reg} \cap D_{n+1-k}\right) \geq k \Rightarrow D_{n+1-k} \subset T_p V(h)_{reg} \Rightarrow \dfrac{\partial f_i}{\partial z_m}(p) = 0$ if $m \geq n+1-k$.
Then,

$$\begin{aligned}
P_k(V(h), D_{n+1-k}) &= \overline{\bigcup_{i=1}^{l} \left\{ p \in H_i - \bigcup_{j \neq i} H_j \mid \frac{\partial f_i}{\partial z_m}(p) = 0,\ m \geq n+1-k \right\}} \\
&= \bigcup_{i=1}^{l} \overline{\left\{ p \in H_i - \bigcup_{j \neq i} H_j \mid \frac{\partial f_i}{\partial z_m}(p) = 0,\ m \geq n+1-k \right\}} \\
&= \bigcup_{i=1}^{l} \left\{ p \in H_i \mid \frac{\partial f_i}{\partial z_m}(p) = 0,\ m \geq n+1-k \right\}.
\end{aligned}$$

Now, let $B_i = \left\{ p \in H_i \mid \frac{\partial f_i}{\partial z_m}(p) = 0 \text{ if } m \geq n+1-k \right\}$ and we obtain, for any $q \in P_k(V(h), D_{n+1-k})$,

$$mult_q\, P_k(V(h), D_{n+1-k}) = \sum_{i,\ q \in H_i} mult_q\, B_i,$$

where $mult_q$ denotes the multiplicity at q.

Therefore, the set of points $p \in F_{1,j_1}$ which has a different sequence of Lê-Teissier polar multiplicities of the sequence in a generic point is in the set of the intersections $H_i \cap H_j \cap H_k$ for $i, j, k \in I$, $i \neq j \neq k$. Then, we conclude that the components of F_2, F_{2,j_2} are the intersections $H_i \cap H_j \cap H_k$ for $i, j, k \in I$, $i \neq j \neq k$.

Proceeding inductively we obtain that the components of F_k, denoted F_{k,j_k}, are $w_J := \bigcap_{i \in J} H_i$ with $\# J = k+1$.

Then, by the Corollary 6.1.7 of [7], we obtain $S_J := w_J - \bigcup_{J \subsetneq K} w_k$, and $\{S_J\}$ is the minimal Whitney stratification of $V(h)$. □

Now we fix $r_0 < r_1 < \cdots < r_n$ to consider the system $z = (x_0, \ldots, x_n)$. The next proposition shows that this system is prepolar.

Proposition 3.4. *The coordinate system $z = (x_0, \ldots, x_n)$ is prepolar for h with respect to the stratification $\{S_J\}_{J \subseteq I}$.*

Proof. First we remark that the stratification $\{S_J\}$ is a good stratification for h at 0 since the a_h condition of Thom is a necessary condition for the Whitney stratification $\{S_J\}_{J \subseteq I}$.

Now we show that $V(x_0)$ intercepts transversally all strata S_J with $S_J \neq \{0\}$ and for all j with $1 \leq j \leq n-1$, $V(x_j)$ intercepts transversally all strata $S_J \cap V(x_0, \ldots, x_{j-1})$ with $S_J \cap V(x_0, \ldots, x_{j-1}) \neq \{0\}$.

But, to say that $V(x_0)$ intercepts transversally all such S_J means that for all $p \in V(x_0) \cap S_J$, $T_pV(x_0) + T_pS_J = \mathbb{C}^{n+1}$, or equivalently, the normal vector to $V(x_0)$ at p is not in the normal space to S_J at p.

Let $J = \{i\}$ and considering $p \in V(x_0) \cap S_J$, we see that the normal vector to S_J at p is the normal vector to $V(f_i)$ at p. Here we see that $v = (1, 0, \ldots, 0)$ is the normal vector of $V(x_0)$ and

$$w_i = (\ 0,\ \gamma_{1,i}(r_0\hat{r}_1 \cdots r_n)\tilde{x}_1^{(r_0\hat{r}_1 \cdots r_n)-1}, \ldots, \gamma_{n,i}(r_0 \cdots r_{n-1}\hat{r}_n)\tilde{x}_n^{(r_0 \cdots r_{n-1}\hat{r}_n)-1}\ )$$

is the normal vector of $V(f_i)$ at the point $(0, \tilde{x}_1, \ldots, \tilde{x}_n) \neq (0, \ldots, 0)$ of $V(x_0) \cap S_J$.

Clearly the vectors v and w_i are not parallels, therefore $V(x_0)$ intercepts transversally S_J for $J = \{i\}$ with $i = 1, \ldots, l$.

Now let $J = \{i, j\}$ with $i \neq j$, $i, j \in \{1, \ldots, l\}$ and $S_J \neq \{0\}$. The normal space to S_J at a point $(0, \ldots, 0) \neq (0, \tilde{x}_1, \ldots, \tilde{x}_n) \in V(x_0) \cap S_J$ is generated by the vectors

$$w_k = (\ 0,\ \gamma_{1,k}(r_0\hat{r}_1 \cdots r_n)\tilde{x}_1^{(r_0\hat{r}_1 \cdots r_n)-1}, \ldots, \gamma_{n,k}(r_0 \cdots r_{n-1}\hat{r}_n)\tilde{x}_n^{(r_0 \cdots r_{n-1}\hat{r}_n)-1}\ )$$

with $k = i, j$.

Since $v \notin [w_i, w_j]$, we conclude that $V(x_0)$ intercepts transversally S_J.

Following the same idea we can show that $V(x_0)$ intercepts transversally all strata S_J with $S_J \neq \{0\}$ and for all j with $1 \leq j \leq n-1$, $V(x_j)$ intercepts transversally all $S_J \cap V(x_0, \dots, x_{j-1})$ with $S_J \cap V(x_0, \dots, x_{j-1}) \neq \{0\}$. $\square$

3.2 Polar ratio

From now on we shall fix $h \in \mathbb{C}[x_0, \dots, x_n]$ as a Pham-Brieskorn arrangement of type $(r_0, \dots, r_n; d)$ with $r_0 < \dots < r_n$. We shall show that the polar ratio of h at the origin with respect to the system of coordinates $z = (x_0, \dots, x_n)$ depends only of the weights $r_0, \dots, r_n$ and the degree d.

Proposition 3.5. *The maximum polar ratio of h at the origin with respect to the coordinate system $z = (x_0, \dots, x_n)$ is smaller or equal than $\frac{d}{r_0}$.*

As we have seen before, we can write h as

$$h = c\, f_1 \cdots f_p . f_{p+1}^{\alpha_{p+1}} \cdots f_l^{\alpha_l}, \quad \text{with} \quad d = \left(p + \sum_{i=p+1}^{l} \alpha_i \right) r_0 \cdots r_n.$$

Call $g_{i,j} = f_1 \cdots \frac{\partial f_i}{\partial x_j} \cdots f_l$ and $g_j = g_{1,j} + \dots + g_{p,j} + \alpha_{p+1} g_{p+1,j} + \dots + \alpha_l g_{l,j}$, with $0 \leq j \leq n$.

In order to compute the polar radio of the Pham-Brieskorn arrangements, we use the following.

Lemma 3.6. *For all $k \in \{0, \dots, n\}$, the sets $V(g_n, g_{n-1}, \dots, g_k)$ have no components in the critical set of h.*

Proof. We note that

$$g_j = (r_0 \cdots \hat{r}_j \cdots r_n) x_j^{r_0 \cdots \hat{r}_j \cdots r_n - 1} \left((m_1 \gamma_{j,1})\, \hat{f}_1 f_2 \cdots f_l + \dots + (m_l \gamma_{j,l})\, f_1 \cdots \hat{f}_l \right),$$

where $\hat{f}_k$ means that there is no function f_k with $k \in \{1, \dots, l\}$.

To get the critical set of h, as $\dfrac{\partial h}{\partial x_j} = (f_{p+1}^{\alpha_{p+1}-1} \cdots f_l^{\alpha_l - 1}) g_j$, then for all $i \neq j \in \{0, \dots, n\}$, we consider the system

$$\begin{cases} m_1 \gamma_{i,1}\, \hat{f}_1 f_2 \cdots f_l + & \cdots & + m_l \gamma_{i,l}\, f_1 f_2 \cdots \hat{f}_l = 0 \\ m_1 \gamma_{j,1}\, \hat{f}_1 f_2 \cdots f_l + & \cdots & + m_l \gamma_{j,l}\, f_1 f_2 \cdots \hat{f}_l = 0 \end{cases}$$

which is equivalent to the following system:

$$\begin{cases} m_1 \gamma_{i,1}\, \hat{f}_1 f_2 \cdots f_l & + & m_2 \gamma_{i,2}\, f_1 \hat{f}_2 \cdots f_l + \dots + m_l \gamma_{i,l}\, f_1 f_2 \cdots \hat{f}_l = 0 \\ 0 & + & m_2 k_2\, f_1 \hat{f}_2 \cdots f_l + \dots + m_l k_l\, f_1 f_2 \cdots \hat{f}_l = 0 \end{cases},$$

with $k_m = \gamma_{j,1} \gamma_{i,m} - \gamma_{i,1} \gamma_{j,m}$, com $2 \leq k \leq l$. From the second equation we have

$$f_1 = 0 \quad \text{or} \quad f_3 \cdots f_l = -\left(\frac{m_3 k_3}{m_2 k_2}\, f_2 \hat{f}_3 \cdots f_l + \dots + \frac{m_l k_l}{m_2 k_2}\, f_2 \cdots \hat{f}_l \right).$$

Then

$$(m_2\gamma_{i,2}\, f_1 + m_1\gamma_{i,1}\, f_2)(f_3\cdots f_l) =$$
$$= -(m_2\gamma_{i,2}\, f_1 + m_1\gamma_{i,1}\, f_2)\left(\frac{m_3k_3}{m_2k_2}\, f_2\hat{f}_3\cdots f_l + \cdots + \frac{m_lk_l}{m_2k_2}\, f_2\cdots\hat{f}_l\right). \quad (1)$$

From the first equation we see that

$$(m_2\gamma_{i,2}\, f_1 + m_1\gamma_{i,1}\, f_2)(f_3\cdots f_l) =$$
$$= f_1\left(m_3\gamma_{i,3}\, f_2\hat{f}_3\cdots f_l + \cdots + m_l\gamma_{i,l}\, f_2\cdots\hat{f}_l\right). \quad (2)$$

From the fact that the decomposition of a polynomial in irreducible factors is unique and from the equalities (1) and (2) we obtain $f_3\cdots f_l = 0$.

Considering again the first system we see that the solutions are the intersections $V(f_m)\cap V(f_n)$ with $m\neq n$ in $\{1,\ldots,l\}$. As i and j are arbitrary, the critical set of h (as a set) is the union

$$\bigcup_{i=p+1}^{l} V(f_i) \cup \bigcup_{i\neq j\in\{1,\ldots,p\}} V(f_i)\cap V(f_j).$$

Hence, since any f_i is irreducible we obtain from the description of the polynomials g_i's that $V(g_n, g_{n-1},\ldots,g_k)$ has no components in the critical set of h for all $k\in\{0,\ldots,n\}$. □

From the Proposition 3.4 and the Theorem 2.12, for the coordinate system $z=(x_0,\ldots,x_n)$, we conclude that $\Lambda^i_{h,z}$ is purely i-dimensional at 0 and the Lê numbers $\lambda^i_{h,z}(0)$ are defined for all $0\leq i\leq n$.

To compute the polar ratio, we need to get the polar curve.

First we see from the Proposition 2.6 that for all $0\leq i\leq n$,

$$[\Gamma^{i+1}_{h,z}]\cdot\left[V\left(\frac{\partial h}{\partial x_i}\right)\right] = [\Lambda^i_{h,z}] + [\Gamma^i_{h,z}].$$

In special for $i=n$ we have

$$\sum_{j=p+1}^{l}\left[V\left(f_j^{\alpha_j-1}\right)\right] + \left[V\left(\frac{\frac{\partial h}{\partial x_n}}{\prod_{j=p+1}^{l} f_j^{\alpha_j-1}}\right)\right] = \left[V\left(\frac{\partial h}{\partial x_n}\right)\right] = \Lambda^n_{h,z} + \Gamma^n_{h,z}.$$

By definition $\Lambda^n_{h,z}\subseteq\Sigma(h)$ and $\Lambda^n_{h,z}$ is disjoint from $\Gamma^n_{h,z}$.

Since $V(g_n) = V\left(\frac{\frac{\partial h}{\partial x_n}}{\prod f_j^{\alpha_j-1}}\right)$ is not in the critical set of h, from the Lemma 3.6, we obtain $[\Gamma^n_{h,z}] = [V(g_n)]$.

By induction we have

$$[\Gamma^n_{h,z}]\cdot\left[V\left(\frac{\partial h}{\partial x_{n-1}}\right)\right]=\sum_{j=p+1}^{l}\left(\left[V\left(f_j^{\alpha_j-1}\right)\right]\cdot[V(g_n)]\right)+([V(g_{n-1})]\cdot[V(g_n)])\,.$$

Since $\left[\Gamma^n_{h,z}\right]\cdot\left[V\left(\frac{\partial h}{\partial x_{n-1}}\right)\right]=\Lambda^{n-1}_{h,z}+\Gamma^{n-1}_{h,z}$, using the same argument we have $[\Gamma^{n-1}_{h,z}]=[V(g_{n-1},g_n)]$ and proceeding in the same way we conclude that

$$[\Gamma^1_{h,z}]=[V(g_1,\ldots,g_n)]\,.$$

Using again the Theorem 2.12 we have that $\Gamma^1_{h,z}$ is 1-dimensional, then $\Gamma^1_{h,z}$ is a complex algebraic curve determined by n independent polynomials functions with $n+1$ variables with coefficients in $\mathbb{C}$. Using the resultant we eliminate all less two of the variables and reduce the curve to an equivalent birrational plane curve $\hat{g}(x,y)=0$.

Hence each component of $\Gamma^1_{h,z}$ has a local parametrization. Now let η be an irreducible component of $\Gamma^1_{h,z}$ and consider $\phi(t)=(\phi_0(t),\ldots,\phi_n(t))$ such a parametrization of η, where $\phi_j(t)=c_jt^{k_j}+$ higher order terms.

We shall also need the following result to compute the polar ratio.

Lemma 3.7. ([8], page 10) *Let $p\in U$ with U an open set in $\mathbb{C}^{n+1}$. If W is an irreducible curve in U at p and $V(f)\subseteq U$ is an hypersurface which intercepts W properly at p (or $codim_pV(f)\cap W=codim_p\,V(f)+codim_p\,W$), consider a local parametrization $\phi(t)$ of W which takes 0 at p. Then $([W]\cdot[V(f)])_p=mult_t\,f(\phi(t))=$ the smallest degree of all non zero terms in the Taylor series of $f(\phi(t))$.*

Proof of the Proposition 3.5. We first suppose that η intercepts transversally $V(z_0)$ since in the other case the polar ratio is 1.

Now, by the Lemma 3.7,

$$(\eta\cdot V(f_i))_0=mult_t\,f_i(\phi(t))=\min_q\{k_q(r_0\cdots\hat{r}_q\cdots r_n)\}$$

and $(\eta\cdot V(z_0))_0=mult_t\,z_0(\phi(t))=k_0$. Therefore

$$\frac{(\eta\cdot V(h)\,)_0}{(\eta\cdot V(z_0)\,)_0}\leq\frac{\frac{d}{r_0\cdots r_n}(k_0(\hat{r}_0r_1\cdots r_n))}{k_0}=\frac{d}{r_0},$$

and the maximum polar ratio of h is smaller or equal than $\frac{d}{r_0}$. $\square$

We shall show now that we also can compute the polar ratio of the function $h+a_0x_0^{\frac{d}{r_0}}$ in terms of the weights and the degree.

Proposition 3.8. *The maximum polar ratio of* $h + a_0 x_0^{\frac{d}{r_0}}$ *at* 0 *with respect to the variables* $(x_1, \ldots, x_n, x_0)$ *is smaller or equal to* $\frac{d}{r_1}$.

The maximum polar ratio of $h + a_0 x_0^{\frac{d}{r_0}} + \cdots + a_{i-1} x_{i-1}^{\frac{d}{r_{i-1}}}$ *(at* 0 *with respect to* $(x_i, \ldots, x_n, x_0, \ldots, x_{i-1})$*) is smaller or equal to* $\frac{d}{r_i}$ *for all* $i = 1, \ldots, n-1$.

Proof. Since $z = (x_0, \ldots, x_n)$ is prepolar for h at 0 from the Proposition 2.13, $\tilde{z} = (x_1, \ldots, x_n, x_0)$ is prepolar for $h + a_0 x_0^{\frac{d}{r_0}}$ at 0, except for a finite number of a_0. Consider a_0 satisfying this condition, then the polar curve of $h + a_0 x_0^{\frac{d}{r_0}}$ in $\tilde{z}$ is purely 1-dimensional at 0.

In an analogous way as in the Proposition 2.7, we conclude that it is determined by n independent functions in $n+1$ variables with coefficients in $\mathbb{C}$. Hence any irreducible component of this polar curve has a parametrization. Let η be an irreducible component of the polar curve of $h + a_0 x_0^{\frac{d}{r_0}}$ in $\tilde{z}$ which intercepts transversally $V(x_1)$ and $\phi(t) = (\phi_0(t), \ldots, \phi_n(t))$ is a parametrization of η, with $\phi_q(t) = c_q t^{k_q} +$ higher order terms.

We know that

$$(\eta \cdot V(h + a_0 x_0^{\frac{d}{r_0}}))_0 = mult_t\ (h + a_0 x_0^{\frac{d}{r_0}})(\phi(t)) =$$

$$= mult_t \left(h(\phi(t)) + a_0 x_0^{\frac{d}{r_0}} (\phi(t)) \right) = mult_t\ h(\phi(t)) = min_q\ \{k_q(r_0 \cdots \hat{r}_q \cdots r_n)\}.$$

Therefore

$$\frac{(\ \eta \cdot V(h + x_0^{\frac{d}{r_0}}))_0}{(\ \eta \cdot V(x_1)\ )_0} \leq \frac{\frac{d}{r_0 \cdots r_n}\,(k_1(r_0 \hat{r}_1 \cdots r_n))}{k_1} = \frac{d}{r_1}.$$

Then the maximum polar ratio of $h + a_0 x_0^{\frac{d}{r_0}}$ at 0 with respect to the variables $(x_1, \ldots, x_n, x_0)$ is smaller or equal to $\frac{d}{r_1}$.

Proceeding in an analogous way, we obtain that the maximum polar ratio of $h + a_0 x_0^{\frac{d}{r_0}} + \cdots + a_{i-1} x_{i-1}^{\frac{d}{r_{i-1}}}$ (at 0 with respect to $(x_1, \ldots, x_n, x_0, \ldots, x_{i-1})$) is smaller or equal to $\frac{d}{r_i}$ for all $i = 1, \ldots, n-1$. □

4 Characterization of the Lê numbers

Massey shows in [8], Corollary 4.7, the following Plücker type formula for homogeneous polynomials of degree d in $n+1$ variables:

If $\lambda^i_{h,z}(0)$ *exists for all* $i \leq s$, *where* $s = dim\ \Sigma(h)$, *then*

$$\sum_{i=0}^{s} (d-1)^i \lambda^i_{h,z}(0) = (d-1)^{n+1}.$$

With the help of this formula he characterizes the Lê numbers of hyperplane arrangements.

We shall show here how to describe the Lê numbers at the origin of any Pham-Brieskorn arrangement $h \in \mathbb{C}[x_0, \ldots, x_n]$ of type $(r_0, \ldots, r_n; d)$ with $r_0 < \cdots < r_n$ in terms of the weights and the degree.

The key tool to show this description is to obtain a Plücker type formula for this case, which is shown in the next

Proposition 4.1. *Let h be a Pham-Brieskorn arrangement of type $(r_0, \ldots, r_n; d)$ at the origin with respect to the coordinate system $z = (x_0, \ldots, x_n)$. Then*

$$\sum_{i=0}^{n} \left(\prod_{k=0}^{i-1} \left(\frac{d}{r_k} - 1 \right) \right) \lambda_{h,z}^{i}(0) = \left(\frac{d}{r_0} - 1 \right) \cdots \left(\frac{d}{r_n} - 1 \right).$$

Proof. From the Proposition 3.5, we have that the maximum polar ratio of h at the origin with respect to the coordinate system $z = (x_0, \ldots, x_n)$ is smaller or equal than $\frac{d}{r_0}$. Then, we consider $j_0 = \frac{d}{r_0}$ to apply the Theorem 2.9 and obtain that there exists $a_0 \in \mathbb{C}^*$ such that $h + a_0 x_0^{\frac{d}{r_0}}$ has critical set of dimension $n-1$.

Applying the Proposition 3.8, for all except a finite number of $a_i \in \mathbb{C}^*$, the maximum polar ratio of $h + a_0 x_0^{\frac{d}{r_0}} + \cdots + a_{i-1} x_{i-1}^{\frac{d}{r_{i-1}}}$, at the origin with respect to the coordinate system $\tilde{z} = (x_i, \ldots, x_n, x_0, \ldots, x_{i-1})$ is smaller or equal than $\frac{d}{r_i}$ for all $1 \leq i \leq n-1$, then we apply the Theorem 2.9 inductively with $j_i = \frac{d}{r_i}$, $1 \leq i \leq n-1$ to obtain that there exist $a_1, \ldots, a_{n-1} \in \mathbb{C}^*$ such that $f := h + a_0 x_0^{j_0} + a_1 x_1^{j_1} + \cdots + a_{n-1} x_{n-1}^{j_{n-1}}$ has isolated singularity at the origin and

$$\mu(f) = \sum_{i=0}^{n} \left(\prod_{k=0}^{i-1} \left(\frac{d}{r_k} - 1 \right) \right) \lambda_{h,z}^{i}(0).$$

Since f is a weighted homogenous polynomial of type $(r_0, \ldots, r_n; d)$, we conclude from the Theorem 1 of [10] that $\mu(f) = \left(\frac{d}{r_0} - 1 \right) \cdots \left(\frac{d}{r_n} - 1 \right)$.

Therefore

$$\sum_{i=0}^{s} \left(\prod_{k=0}^{i-1} \left(\frac{d}{r_k} - 1 \right) \right) \lambda_{h,z}^{i}(0) = \left(\frac{d}{r_0} - 1 \right) \cdots \left(\frac{d}{r_n} - 1 \right).$$

□

We shall apply this formula to characterize each Lê number at the origin of any Pham-Brieskorn arrangement.

Lemma 4.2. $\Lambda^k_{h,z} = \sum_{\dim S_J = k} a_J[w_J]$, *with* $a_J = \lambda^k_{h_{|N},z}(p)$ *for some* $p \in S_J$ *with* $p = (p_0, \ldots, p_n)$ *and* $N = V(x_0 - p_0, \ldots, x_{k-1} - p_{k-1})$.

Proof. First we shall show that, as sets, the Lê cycles of h with respect to the coordinate system $z = (x_0, \ldots, x_n)$ are given as the union of the w_J with correct dimension.

We shall describe these cycles inductively. By definition $\Lambda^n_{h,z}$ is formed by the components of $V(\frac{\partial h}{\partial x_n})$ which are in the critical set of h.

As we saw before, $\frac{\partial h}{\partial x_n} = (f_{p+1}^{\alpha_{p+1}-1} \cdots f_l^{\alpha_l - 1}) g_n$ and $V(g_n)$ does not have components in the critical set, therefore $\Lambda^n_{h,z}$, as a set, is the union of the $V(f_i)$ with $i = p+1, \ldots, l$.

Now, consider the intersection of $V(\frac{\partial h}{\partial x_{n-1}})$ with the components of $V(\frac{\partial h}{\partial x_n})$ which are not in the critical set of h. We have that $\Lambda^{n-1}_{h,z}$ is formed by the components of this intersection which are in the critical set of h.

But $\frac{\partial h}{\partial x_{n-1}} = (f_{p+1}^{\alpha_{p+1}-1} \cdots f_l^{\alpha_l-1}) g_{n-1}$ and $V(g_{n-1}, g_n)$ has no components in the critical set, therefore $\Lambda^{n-1}_{h,z}$ is, as a set, the union of all intersections $V(f_i) \cap V(f_j)$ with $i \neq j$, $i = p+1, \ldots, l$ and $j = 1, \ldots, l$. Then the result follows inductively.

Hence, as cycles, for all k, $\Lambda^k_{h,z} = \sum_{\dim S_J = k} a_J \, [w_J]$, for some a_J.

Now, let $\tilde{J} \subseteq I$ such that the dimension of $S_{\tilde{J}}$ is k.

Consider $p = (p_0, \ldots, p_n) \in S_{\tilde{J}}$. As sets we have

$$\begin{aligned} \Sigma(h_{|\, V(x_0 - p_0, \ldots,\, x_{k-1} - p_{k-1})}) &= V\left(x_0 - p_0, \ldots,\, x_{k-1} - p_{k-1}, \frac{\partial h}{\partial z_k}, \ldots, \frac{\partial h}{\partial z_n}\right) \\ &= V(x_0 - p_0, \ldots,\, x_{k-1} - p_{k-1}) \cap (\Sigma(h) \cup \Gamma^k_{h,z}). \end{aligned}$$

As $\Lambda^k_{h,z}$ and $\Gamma^k_{h,z}$ are disjoint we have $p \notin \Gamma^k_{h,z}$, hence, by the Theorem 2.12, $V(x_0 - p_0, \ldots,\, x_{k-1} - p_{k-1}) \cap \Gamma^k_{h,z}$ is empty at p. Therefore, considering $N = V(x_0 - p_0, \ldots, x_{k-1} - p_{k-1})$,

$$\Sigma(h_{|\, N}) = \Sigma(h) \cap N \text{ in } p.$$

Using the Proposition 2.7, we obtain $\lambda^k_{h_{|N},\tilde{z}(p)} = \lambda^k_{h,z}(p)$, with the system of coordinates $\tilde{z} = (x_k, \ldots, x_n, x_0, \ldots, x_{k-1})$.

On the other side, the cycle $\Lambda^k_{h,z} \cdot V(x_0 - p_0, \ldots, x_{k-1} - p_{k-1})$ is of the form

$$\sum_{\dim S_J = k} a_J \; ([w_J] \cdot V(x_0 - p_0, \ldots, x_{k-1} - p_{k-1}))\,.$$

Now, $[w_J] \cdot V(x_0 - p_0, \ldots, x_{k-1} - p_{k-1}) = [p]$ if $J = \tilde{J}$ and we get that $\left(\Lambda^k_{h,z} \cdot V(x_0 - p_0, \ldots, x_{k-1} - p_{k-1})\right)_p = a_{\tilde{J}}$, hence $a_{\tilde{J}} = \lambda^k_{h_{|N},z}(p)$. $\square$

Remark 4.3. If we consider the translation $(x_0,\ldots,x_n)\xrightarrow{\phi}(X_0,\ldots,X_n)$ where $X_i=x_i-p_i$ takes p at the origin, we have $h_{|_N}$ at p in this new coordinate system can be seen in a neighbourhood of the origin as being $\prod_{j=1}^{l}\tilde{f}_j^{\,m_j}$ with $\tilde{f}_j=\phi(f_{i|_{x_0=\cdots=x_{k-1}=0}})$.

Since $p\notin V(f_j)$ if $j\notin J$ then $0\notin V(\tilde{f}_j)$ if $j\notin J$. Then, in a neighbourhood of the origin, $\prod_{j\notin J}\tilde{f}_j^{\,m_j}$ is a unity and we obtain that $h_{|_N}$ can be seen , in a neighbourhood of the origin, as $\prod_{j\in J}\tilde{f}_j^{\,m_j}$.

Now, for each hypersurface H_i we call *weighted multiplicity* of H_i, $\mathrm{mult}_P\, H_i$, the number $m_i(r_0\cdots r_n)$. To simplify the notation we write H for the defining hypersurfaces of $V(h)$ and w or v the possible intersections with them and $\mathcal{A}$ denotes the collection of all these w.

We denote $e(w):=\sum_{w\subseteq H}\mathrm{mult}_P\, H$ and $k(w):=\dfrac{\left(\frac{e(w)}{r_0}-1\right)\cdots\left(\frac{e(w)}{r_n}-1\right)}{\prod_{i=0}^{\dim w-1}\left(\frac{e(w)}{r_i}-1\right)}$.

Definition 4.4. *We define inductively on the dimension of w,*

$$\eta(w):=k(w)-\sum_{v\supsetneq w}\eta(v)\left(\prod_{i=1}^{\mathrm{codim}_v w}\left(\frac{e(w)}{r_{n-i}}-1\right)\right).$$

Theorem 4.5. *Let h be a Pham-Brieskorn arrangement, then for some generic coordinate system z,*

$$\lambda^i_{h,z}(0)=\sum_{w\in\mathcal{A},\ \dim w=i}\eta(w)$$

Proof. From the Lemma 4.2 and the Remark 4.3, $\Lambda^k_{h,z}=\sum_{\dim S_J=k}a_J\,[w_J]$, with $a_J=\lambda^0_{g_J,z}(0)$ where g is the product of Pham-Brieskorn type polynomials in $n+1-k$ variables, keeping the same weights of the variables and weighted homogeneous degree $e(w_J)$.

Therefore $\lambda^n_{h,z}(0)=\sum_{\dim S_J=n}a_J=\sum_{\dim S_J=n}\lambda^0_{g_J,z}(0)$, where each g_J is a polynomial in 1 variable with critical set 0-dimensional, $J=\{i\}$, $1,\ldots,l$ and degree $e(w_{\{i\}})$.

Applying the Proposition 4.1 for g_i at the origin with respect to the coordinate system $z=(X_n)$, we have $\lambda^0_{g_{\{i\}},z}(0)=\dfrac{e(w_{\{i\}})}{r_n}-1=\eta(w_{\{i\}})$. Therefore

$$\lambda^n_{h,z}(0)=\sum_{i=1}^{l}\eta(w_{\{i\}})=\sum_{\dim v=n}\eta(v).$$

Analogously, $\lambda^{n-1}_{h,z}(0) = \sum\limits_{\dim S_J=n-1} a_J = \sum\limits_{\dim S_J=n-1} \lambda^0_{g_J,z}(0)$, where each g_J is a polynomial in two variables with critical set 1-dimensional and degree $e(w_J)$.

Now we apply again the proposition 4.1 for the g_J at the origin, but with respect to the coordinate system $z = (X_{n-1}, X_n)$, then we obtain

$$\lambda^0_{g_{\{J\}},z}(0) + \left(\frac{e(w_J)}{r_{n-1}} - 1\right)\lambda^1_{g_{\{J\}},z}(0) = \left(\frac{e(w_J)}{r_{n-1}} - 1\right)\left(\frac{e(w_J)}{r_n} - 1\right).$$

From the Proposition 2.7, $\lambda^1_{g_{\{i\}},z}(0) = \lambda^0_{g_{\{i\}}|_{V(X_{n-1})},z}(0)$. On the other side,

$$\lambda^0_{g_{\{i\}}|_{V(X_{n-1})},z}(0) = \sum_{w_{\tilde{J}} \supsetneq w_J,\ \dim S_{\tilde{J}}=n} \lambda^0_{h_{|N_{\tilde{J}}},z}(p_{\tilde{J}}),$$

com $p_{\tilde{J}} = (p_{\tilde{J},1}, \ldots, p_{\tilde{J},n})$ e $N_{\tilde{J}} = (x_0 - p_{\tilde{J},0}, \ldots, x_{\tilde{J},n-1} - p_{\tilde{J},n-1})$.

We apply again the Remark 4.3 to conclude that $h_{|N_{\tilde{J}}}$ in a neighborhood of $p_{\tilde{J}}$ can be seen, by exchange of coordinates as a polynomial in a neighborhood of the origin of one variable with 0-dimensional critical set, $\tilde{J} = \{i\}, 1, \ldots, l$ and degree $e(w_{\{i\}})$. From the case before we obtain $\lambda^0_{h_{|N_{\tilde{J}}},z}(p_{\tilde{J}}) = \eta(w_{\tilde{J}})$.

Therefore $\lambda^1_{g_{\{i\}},z}(0) = \sum_{w_J \supsetneq w,\ \dim S_J=n} \eta(w)$ and

$$\lambda^0_{g_{\{J\}},z}(0) = \left(\frac{e(w_J)}{r_{n-1}} - 1\right)\left(\frac{e(w_J)}{r_n} - 1\right) - \sum_{w_J \supsetneq w,\ \dim S_J=n} \left(\frac{e(w_J)}{r_{n-1}} - 1\right)\eta(w).$$

Hence $\lambda^{n-1}_{h,z}(0) = \sum\limits_{\dim v=n-1} \eta(v)$.

Proceeding inductively we obtain $\lambda^i_{h,z}(0) = \sum\limits_{\dim v=i} \eta(v)$ with $i = 1, \ldots, n$. To get $\lambda^0_{h,z}(0)$ it is enough to exchange each $\lambda^i_{h,z}(0) = \sum\limits_{\dim v=i} \eta(v)$ with $i = 1, \ldots, n$ in the formula for $w = \bigcap\limits_{i=1}^{l} H_i$. □

Acknowledgments

This work is part of the Ph.D. Thesis of the first named author, done under supervision of the second author at ICMC-USP, São Carlos. The second named author is partially supported by FAPESP and CNPq and the first named author was su-pported by CAPES during her Ph.D. Thesis. The authors thank these agencies for this financial support.

References

[1] D. Eisenbud and J. Harris, Schemes: The Language of Modern Algebraic Geometry, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA. *The Wadsworth & Brooks/Cole Mathematics Series.* (1992), 157 pages.

[2] W. Fulton, Intersection theory, *Ergebnisse der Math.*, Springer-Verlag, (1984).

[3] T. Gaffney and R. Gassler, Segre numbers and hypersurface singularities, *J. Algebraic Geometry* **8**, (1996) 695–736.

[4] P. Griffiths and J. Harris, Principles of algebraic geometry, New York : Wiley, (1978).

[5] I. N. Iomdin, Varits complexes avec singularits de dimension un, *Sibirsk. Mat. Z.* **15**, (1974) 1061–1082.

[6] Ensembles analytiques complexes avec lieu singulier de dimension un (d'aprs I.N. Iomdin), *Sminaire sur les Singularits* (Paris, 1976-1977) *Publ. Math. Univ. Paris* **VII**, (1980) 87–95.

[7] Lê D. T. and B. Teissier, Variétés polaires locales et classes de Chern de variétés singulères, *Annals of Math.*, **114** (1981) 457–491.

[8] D. Massey, Lê cycles an hypersurface singularities, *Springer Lecture Notes in Mathematics* **1615**, (1995).

[9] J. Milnor, Singular points of complex hypersurfaces, *Annals of Mathematics Studies*, **61**, Princeton University Press, Princeton, N.J., (1968).

[10] Milnor, J. and Orlik, P., Isolated singularities defined by weighted homogeneous polynomials, *Topology* **9** (1970) 385–393.

[11] Y. T. Siu and G. Trautmann, Gap-sheaves and extension of coherent analytic subsheaves, *Springer Lecture Notes in Mathematics* **172** (1971).

Universidade de São Paulo, Instituto de Ciências Matemáticas e Computação de São Carlos, Departamento de Matemática, São Carlos, SP - Brasil
E-mails: ferreira@icmc.usp.br, mjsaia@icmc.usp.br

NOTAS DO ICMC

SÉRIE MATEMÁTICA

329/10	GRULHA JÚNIOR, N.G. – Stability of the Euler obstruction of f on free divisors.
328/10	AHMED, I.; RUAS, M.A.S. – Invariants of relative right and contact equivalences.
327/10	CARO, N.; LEVCOVITZ, D. – On theorem of Stafford.
326/10	FEDERSON, M.; GODOY, J.B. – Averaging for retarded functional differential equations.
325/10	FERNANDES, A. – Fast loops on semi-weighted homogeneous hypersurface singularities.
324/10	FEDERSON, M.; GODOY, J.B. – A new continuous dependence result for impulsive retarded functional differential equations.
323/10	AFONSO, S.; BONOTTO, E.; FEDERSON, M.; SCHWABIK, S. – Discontinuous local semiflows for Kurzweil equations leading to Lasalle's invariance principle for differential systems with impulses at variable times.
322/09	COSTA, J. C. F.; SAIA, M. J. ; SOARES JR., C. H. – Bi-Lipschitz, $\mathcal{G} = \mathcal{R}, \mathcal{C}, \mathcal{K}, \mathcal{R}_V, \mathcal{C}_V, \mathcal{K}_V$
321/09	FENILLE, M. C. – On acyclic and simply connected open manifolds.
320/09	FENILLE, M. C. – Closed injective systems and its fundamental limit spaces .

UNIVERSIDADE DE SÃO PAULO

Instituto de Ciências Matemáticas e de Computação

ISSN 0103-2577

LÊ NUMBERS OF PHAM-BRIESKORN ARRANGEMENTS

M. F. Z. MORGADO
M. J. SAIA

Nº 330

NOTAS DO ICMC

SÉRIE MATEMÁTICA

São Carlos – SP
Maio/2010

LÊ NUMBERS OF PHAM-BRIESKORN ARRANGEMENTS

M. F. Z. MORGADO and M. J. SAIA

Resumo

Neste artigo é mostrado como calcular os números de Lê de polinômios que são produtos de polinômios de Pham-Brieskorn de mesmo tipo, somente usando os pesos e graus de quase homogeneidade do polinômio.

Abstract

In this article we show how to compute the Lê numbers of polynomials that are products of polynomials Pham-Brieskorn of the same type, we call Pham-Brieskorn arrangements, only using the weights, the degree of homogeneity and the number of variables.

Keywords: Arrangements, Pham-Brieskorn polynomial, Lê numbers, polar ratio, prepolar coordinates and Plücker formula.

Mathematics Subject Classification: 2000: 32C18, 32B10, 58K05, 58K65

1 Introduction

In the study of germs of hypersurfaces in $\mathbb{C}^{n+1}$ with isolated singularities, John Milnor introduced in [9] an important invariant, called the Milnor number, which is interpreted in terms of algebra, topology, differential geometry and so on. This invariant plays an important role in the modern theory of singularities, and there have been several generalizations of it in many directions. One important generalization of the Milnor number was done by David Massey in [8], where he introduced the Lê numbers and the Lê cycles associated to holomorphic applications germs with non-isolated critical points, or equivalently, to non-isolated singularities in germs of complex analytic hypersurfaces. The definition of these numbers is analytical but they can also be interpreted in terms of topology ([8], Theorem 3.3), in algebraic geometry ([3], 2.1), among others.

Under the analytical aspect, Massey describes in [8], Theorem 4.5, the ones called generalized Lê-Iomdine formulae that are a technique that reduces a hypersurface with singular set of arbitrary dimension s in a hypersurface with singular set of dimension $s-1$ and it relates the Lê numbers's first hypersurface to the Lê numbers of this one obtained. Massey uses these relations to get a Plücker type formula, which relates the Lê numbers of a homogeneous polynomial of degree d using only the degree and the number of variables ([8], Corollary 4.7). With the help of this formula, Massey characterizes each Lê number of an arrangement of hyperplanes, ie, the product of linear functions.

In this article we investigate a class of polynomials which are the product of Pham-Brieskorn's polynomials of the same type, this type of polynomial we call Pham-Brieskorn arrangements. Using the technique developed by Lê-Iomdine and Massey, and also other tools of Singularity Theory, we obtain a similar relationship to that done by Massey, as any Pham-Brieskorn's polynomial is weighted homogenous, we show a Plücker type formula for such polynomials in terms of its weights and degrees. Then, we characterize each Lê number of such arrangement also in terms of the weights and degrees.

2 Definitions and basic properties about Lê numbers

To define and describe the Lê numbers we follow the notation used by Massey in [8]. We assume that the reader is familiar with the notion of coherent sheaves, gap sheaves (references [11] and [8], Definition 1.1), schemes (references [4] and [1]) and cycles (reference [2]). In order to fix the notation, for a sheaf α and an analytic subset W in some affine space, we denote by α/W the corresponding gap sheaf and by $V(\alpha)/W$ the scheme associated with the sheaf α/W, where $V(\alpha)$ denotes the analytic space defined by the vanishing of α. We shall at times enclose cycles in square brackets, $[\,\cdot\,]$.

For the definitions in this section let $h : (U, 0) \to (\mathbb{C}, 0)$ be an analytic function, U an open subset of $\mathbb{C}^{n+1}$ containing the origin and $z = (z_0, \dots, z_n)$ a linear choice of coordinates in $\mathbb{C}^{n+1}$.

Definition 2.1. *For $0 \leq k \leq n$, the k^{th} (relative) polar variety, $\Gamma^k_{h,z}$, of h with respect to z is the scheme $V\left(\frac{\partial h}{\partial z_k}, \dots, \frac{\partial h}{\partial z_n}\right) / \Sigma(h)$.*

Remark 2.2. *On the level of ideals, $\Gamma^k_{h,z}$ consists of the components of $V\left(\frac{\partial h}{\partial z_k}, \dots, \frac{\partial h}{\partial z_n}\right)$ which are not contained in $\Sigma(h)$.*

We denote by $[\Gamma^k_{h,z}]$ the cycle associated with this scheme.

Definition 2.3. *For $0 \leq k \leq n$, the k^{th} Lê cycle, $\Lambda^k_{h,z}$, of h with respect to z is the cycle*

$$\left[\Gamma^{k+1}_{h,z} \cap V\left(\frac{\partial h}{\partial z_k}\right)\right] - [\,\Gamma^k_{h,z}\,].$$

Remark 2.4. *In general we shall write simply $\Lambda^k_{h,z}$, and not $[\Lambda^k_{h,z}]$, because unlike the polar varieties, which are defined as schemes and we have to consider the associated cycle, this definition is given in terms of cycles.*

If the intersection of $\Lambda^k_{h,z}$ with the cycle of $V(z_0 - p_0, \dots, z_{k-1} - p_{k-1})$ is purely 0-dimensional at a point $p = (p_0, \dots, p_n)$, ie, either p is an isolated point of the intersection or p is not in the intersection, it is possible to define the Lê numbers as follows:

Definition 2.5. *For $0 \leq k \leq n$, the k^{th} Lê number, $\lambda^k_{h,z}(p)$, of h with respect to z at p, is defined as the intersection number*

$$(\Lambda^k_{h,z} \cdot V(z_0 - p_0, \dots, z_{k-1} - p_{k-1})\ )_p\ .$$

Among the various properties involving Lê cycles and Lê numbers, we highlight two that are directly necessary for this work. The first describes how to calculate inductively the Lê cycles and the second shows the behavior of the Lê numbers taking hyperplane sections.

Proposition 2.6. ([*8*], Proposition 1.18) *If, for all j with $0 \leq j \leq k$, $\Lambda^j_{h,z}$ is purely j-dimensional at p, then the cycles $\left[\Gamma^{k+1}_{h,z} \cap V\left(\frac{\partial h}{\partial z_k}\right)\right]$ and $[\Gamma^{k+1}_{h,z}] \cdot \left[V\left(\frac{\partial h}{\partial z_k}\right)\right]$ are equal at p. Consequently, if s is the dimension of the critical locus of h, we have*

$$[\Gamma^{s+1}_{h,z}] = \left[V\left(\frac{\partial h}{\partial z_{s+1}}, \dots, \frac{\partial h}{\partial z_n}\right)\right] \quad \textit{and}$$

$$[\Gamma^{i+1}_{h,z}] \cdot \left[V\left(\frac{\partial h}{\partial z_i}\right)\right] = [\Gamma^i_{h,z}] + [\Lambda^i_{h,z}],\ 0 \leq i \leq s.$$

Proposition 2.7. ([*8*], Proposition 1.21) *Suppose $\Sigma(h) \cap V(z_0 - p_0) = \Sigma(h_{|V(z_0-p_0)})$ for $p \in \Sigma(h)$ and consider the coordinates $\tilde{z} = (z_1, \dots, z_n)$ for $V(z_0 - p_0)$. If $\lambda^i_{h,z}(p)$ is defined for all i and $p \notin \Gamma^1_{h,z}$, then*

$$\lambda^i_{h_{|V(z_0-p_0)}, \tilde{z}}(p) = \lambda^{i+1}_{h,z}(p).$$

2.1 The Polar Ratio and the Generalized Lê-Iomdine Formulae

The generalized Lê-Iomdine formulae describe how the Lê numbers of a singular hypersurface with citical set of dimension s are related to the Lê numbers of a certain sequence of singular hypersurfaces $X_1, \dots, X_s$ that approximates the original singularity so that the critic sets of its terms X_i have dimension $s - i$. The polar radius is a number associated to the polar curve which ensures the existence of this sequence of hypersurfaces.

Suppose that the polar curve $\Gamma^1_{h,z}$ is one-dimensional at the origin. Let η be an irreducible component of $\Gamma^1_{h,z}$ (with its reduced structure).

Definition 2.8. *If $\eta \cap V(z_0)$ is zero-dimensional at the origin, we define the polar ratio of η (for h at 0 with respect to z) as* $\dfrac{(\,\eta \cdot V(h)\,)_0}{(\,\eta \cdot V(z_0)\,)_0}$. *Otherwise, we say that the polar ratio of η equals 1.*

Theorem 2.9. ([8], Theorem 4.5) **(Generalized Lê–Iomdine Formulae)** *Let $h : (U, 0) \to (\mathbb{C}, 0)$ be a germ of analytic function with $U \subseteq \mathbb{C}^{n+1}$ open and $s = \dim_0 \Sigma(h)$ with $s \geq 1$. Consider $z = (z_0, \ldots, z_n)$ a linear choice of coordinates such that $\lambda^i_{h,z}(0)$ is defined for all $i \leq s$. Let a be a complex number different from zero and consider the coordinates $\tilde{z} = (z_1, \ldots, z_n, z_0)$ for $h + az_0^j$, where $j \geq 2$. If j is greater or equal than the maximum polar ratio of h (at 0 with respect to z) then for all, except a finite number of complex numbers a, we have*

i) $\dim_0 \Sigma(h + az_0^j) = s - 1$;

ii) $\lambda^0_{h+az_0^j,\tilde{z}}(0) = \lambda^0_{h,z}(0) + (j-1)\lambda^1_{h,z}(0)$

iii) $\lambda^i_{h+az_0^j,\tilde{z}}(0) = (j-1)\lambda^{i+1}_{h,z}(0)$, *for* $1 \leq i \leq s-1$.

Moreover, if j is bigger than the maximum polar ratio of h, the inequalities over are valid for all $a \neq 0$. In particular, this is the case if $j \geq 2 + \lambda^0_{h,z}(0)$.

We remark here that applying the theorem inductively we obtain, for some complex numbers $a_0, \ldots, a_s$, the sequence of singular hypersurfaces $X_1, \ldots, X_s$ as

$$X_1 = V(h + a_0 z_0^{j_0}), \ldots, X_s = V(h + a_0 z_0^{j_0} + \ldots + a_s z_s^{j_s})$$

where j_i is greater or equal than the maximum polar ratio of $h + a_0 z_0^{j_0} + \cdots + z_{i-1}^{j_{i-1}}$ (at 0 with respect to $\tilde{z} = (z_i, \ldots, z_n, z_0, \ldots, z_{i-1})$).

This method was first considered by Iomdin in [5] and by Lê in [6] for the particular case that the hypersurfaces with critical set $\Sigma(h)$ being one dimensional.

2.2 Prepolar coordinate systems

The existence of the Lê numbers depends on the choice of coordinate system. Massey shows in [8] (Theorem 1.28) a coordinate system that is generic with respect to a certain kind of stratification of the hypersurface $V(h)$ in which the Lê numbers are always defined.

Definition 2.10. *A good stratification for h at a point $p \in V(h)$ is an analytic stratification $\mathcal{B} = \{S_\alpha\}$ of the hypersurface $V(h)$ in a neighborhood U of p such that the smooth part of $V(h)$ is a stratum and so that the stratification satisfies Thom's a_h condition with respect to $U - V(h)$, ie, if q_i is a sequence of points in $U - V(h)$ such that $q_i \to q \in S \in \mathcal{B}$ and $T_{q_i}V(h - h(q_i))$ converges to some hyperplane T, then $T_q S \subseteq T$.*

Definition 2.11. *Let $z = (z_0, \dots, z_n)$ be a choice of linear coordinates for $\mathbb{C}^{n+1}$ and $p \in V(h)$, with $s = \dim_p \Sigma(h)$. The system $z = (z_0, \dots, z_n)$ is prepolar for h in p if there exists a good stratification of h at the point p such that $V(z_0 - p_0)$ intercepts transversally all strata S_α, with possible exception the stratum $\{p\}$ and for all j such that $1 \leq j \leq s-1$, $V(z_j - p_j)$ intercepts transversally all $S_\alpha \cap V(z_0 - p_0, \dots, z_j - p_j)$, again with possible exception the stratum $\{p\}$.*

Theorem 2.12. ([8], Theorem 1.28) *Let $p \in V(h)$ with $s = \dim_p \Sigma(h)$ and $z = (z_0, \dots, z_n)$ be a prepolar system for h at p. Then, for all i with $1 \leq i \leq s$, $\lambda^i_{h,z}(p)$ is defined. Moreover, $\Gamma^i_{h,z} \cap V(z_0 - p_0, \dots, z_{i-1} - p_{i-1})$ is empty or 0-dimensional at p with $1 \leq i \leq s$.*

An important property of the prepolar coordinates is that they are also prepolar for each hypersurface of the sequence obtained by the generalized Lê-Iomdine formulae.

Corollary 2.13. *Let $z = (z_0, \dots, z_n)$ be a prepolar system of coordinates for h at the origin and j bigger or equal than the polar ratio of h (at 0 with respect to z), then $\tilde{z} = (z_1, \dots, z_n, z_0)$ is a prepolar system of coordinates for $h + az_0^j$ at the origin, except possible for a finite number of $a \in \mathbb{C}$.*

This result is obtained by applying Lemma 4.3 and Corollary 4.18 of [8].

3 Pham-Brieskorn arrangements

From now on we shall restrict ourselves to the class of polynomials that we are interested and describe its polar radius with respect to a specific coordinate system that is prepolar.

Definition 3.1. *A polynomial $f \in \mathbb{C}[x_0, \dots, x_n]$ is called a Pham-Brieskorn polynomial if*

$$f = \gamma_0 x_0^{a_0} + \cdots + \gamma_n x_n^{a_n} \text{ with integers } \gamma_i \text{ and } a_i \geq 2.$$

Note that f is a weighted homogeneous polynomial of type $(r_0, \dots, r_n; D)$, where $D = m.c.m.(r_0, \dots, r_n)$.

Definition 3.2. *We call a polynomial $h \in \mathbb{C}[x_0, \dots, x_n]$ a Pham-Brieskorn arrangement if it is written in the form*

$$h = c\, f_1 \cdots f_p f_{p+1}^{\alpha_{p+1}} \cdots f_l^{\alpha_l},$$

where each f_j is a Pham-Brieskorn polynomial of type $(r_0, \dots, r_n; D)$. We remark that in this case the polynomial h is weighted homogeneous of type $(r_0, \dots, r_n; d)$ where $d = \left(p + \sum_{i=p+1}^{l} \alpha_i\right) D$. Therefore, we call such polynomial h a Pham-Brieskorn arrangement of type $(r_0, \dots, r_n; d)$.

Suppose now that r_i and r_j are coprime for all $i \neq j$. Thus $D = r_0 \cdots r_n$ and $a_i = k_i r_0 \cdots \hat{r}_i \cdots r_n$, with k_i integers. The notation $\hat{r}_i$ means that there is not the term r_i with $i \in \{0, \ldots, n\}$. In this case, a Pham-Brieskorn arrangement of type $(r_0, \ldots, r_n; d)$ is written in the form

$$h = c\, f_1 \cdots f_p f_{p+1}^{\alpha_{p+1}} \cdots f_l^{\alpha_l},$$

where $f_j = \gamma_{0,j} x_0^{k_0(\hat{r}_0 r_1 \cdots r_n)} + \cdots + \gamma_{n,j} x_n^{k_n(r_0 r_1 \cdots \hat{r}_n)}$, with r_i and r_j coprime for all $i \neq j$, $\alpha_i > 1$ and $d = \left(p + \sum_{i=p+1}^{l} \alpha_i\right) r_0 \cdots r_n$.

To facilitate the calculations we study a particular case, when $k_i = 1$ with $i = 1, \ldots, n$. We emphasize that the results shown below are also valid for any integer $k_i \geq 1$ and the calculations are exactly the same.

3.1 Prepolar coordinates

To describe the Lê numbers of such arrangements, the first step is to find a convenient prepolar coordinate system. In order to do this we need to obtain a stra-tification of $V(h)$ which satisfies the Whitney conditions since such stratification is always a good stratification for h. Lê and Teissier in [7] showed that there always exists such a stratification of the variety $V(h)$, called by Lê and Teissier minimal Whitney stratification of $V(h)$.

Here use the idea given by Massey in [8], page 61, to obtain a Whitney stratification for arrangements given by hyperplanes, to give an explicit description of a stratification for Pham-Brieskorn arrangements. We show that this stratification is the minimal Whitney stratification of $V(h)$ as defined by Lê and Teissier.

We recover here this construction.

According to Lê and Teissier in [7] there exists a sequence of algebraic subvarieties of $V(h)$, $F_0 \supset F_1 \supset \cdots \supset F_k \cdots$, where $F_0 = V(h)$ and F_1 denotes the singular set of $V(h)$. To obtain the other F_j, let us denote $(F_{1,j_1})_{j_1 \in J_1}$ the irreducible components of F_1, then F_2 is the union of the singular set of F_1 with a finite number of closed algebraic sets given by the points in F_{1,j_1} such that the sequence of polar multiplicities of Lê-Teissier is different of the one computed on a generic point of F_{1,j_1}.

Here we are considering the k^{th} polar multiplicity in a point, as given by Lê-Teissier. This means the multiplicity of the polar variety $P_k(V(h), D_{n+1-k})$ in this point, defined by Lê and Teissier as,

$$P_k(V(h), D_{n+1-k}) = \overline{\{p \in V(h)_{reg} \mid \dim\left(T_p V(h)_{reg} \cap D_{n+1-k}\right) \geq k\}},$$

D_{n+1-k} is a generic hyperplane and $V(h)_{reg}$ denotes the regular part of $V(h)$.

We remark that this definition of polar variety is different from the one given by Massey in the Definition 2.1.

Now denote $(F_{2,j_2})_{j_2\in J_2}$, the irreducible components of F_2, in the general case, let F_k be the union of the singular set of F_{k-1} with a finite number of closed algebraic sets given by the points such that the sequence of polar multiplicities of Lê-Teissier is different of the one computed on a generic point of $F_{k-1,j_{k-1}}$.

In order to construct the Whitney stratification of the set $V(h)$, we fix the following notation: For any i with $1 \le i \le l$ define $H_i = V\left(f_i^{m_i}\right)$, with $m_i = 1$ if $1 \le i \le p$ and $m_i = \alpha_i$ if $p+1 \le i \le l$.

Call I the set of indexes $\{1,\ldots,l\}$ and for any subset J of I, define

$$w_J := \bigcap_{i\in J} H_i \quad \text{and} \quad S_J := w_J - \bigcup_{J\subsetneq K} w_K.$$

Lemma 3.3. *$\{S_J\}_{J\subseteq I}$ is the minimal Whitney stratification of $V(h)$.*

Proof. We show, for each k, that the irreducible components F_{k,j_k} are

$$F_{k,j_k} = w_J := \bigcap_{i\in J} H_i \quad \text{with} \quad \#J = k+1.$$

As each function f_i has isolated singularity at the origin, the components of singular set of $V(h)$ are the intersections $H_i \cap H_j$ with $i \neq j$ and $i,j \in I$.

Hence we have that F_1 is given by the union of the intersections $H_i \cap H_j$ with $i \neq j$ and $i,j \in I$, or in other words, each F_1, j_1 is of the form $F_1, j_1 = H_i \cap H_j$.
By the same reason the singular set of F_1 has as components the intersections $H_i \cap H_j \cap H_k$ with $i,j,k \in I$, $i \neq j \neq k$.

Now if $p \in V(h)_{reg}$, then $p \in H_i - \bigcup_{j\neq i} H_j$ for some i, therefore $\dim\left(T_pV(h)_{reg} \cap D_{n+1-k}\right) \ge k \Rightarrow D_{n+1-k} \subset T_pV(h)_{reg} \Rightarrow \dfrac{\partial f_i}{\partial z_m}(p) = 0$ if $m \ge n+1-k$.
Then,

$$\begin{aligned}
P_k(V(h), D_{n+1-k}) &= \overline{\bigcup_{i=1}^{l}\left\{p \in H_i - \bigcup_{j\neq i} H_j \mid \frac{\partial f_i}{\partial z_m}(p) = 0,\ m \ge n+1-k\right\}} \\
&= \bigcup_{i=1}^{l}\overline{\left\{p \in H_i - \bigcup_{j\neq i} H_j \mid \frac{\partial f_i}{\partial z_m}(p) = 0,\ m \ge n+1-k\right\}} \\
&= \bigcup_{i=1}^{l}\left\{p \in H_i \mid \frac{\partial f_i}{\partial z_m}(p) = 0,\ m \ge n+1-k\right\}.
\end{aligned}$$

Now, let $B_i = \left\{p \in H_i \mid \frac{\partial f_i}{\partial z_m}(p) = 0 \text{ if } m \ge n+1-k\right\}$ and we obtain, for any $q \in P_k(V(h), D_{n+1-k})$,

$$mult_q\, P_k(V(h), D_{n+1-k}) = \sum_{i,\ q\in H_i} mult_q\, B_i,$$

where $mult_q$ denotes the multiplicity at q.

Therefore, the set of points $p \in F_{1,j_1}$ which has a different sequence of Lê-Teissier polar multiplicities of the sequence in a generic point is in the set of the intersections $H_i \cap H_j \cap H_k$ for $i, j, k \in I$, $i \neq j \neq k$. Then, we conclude that the components of F_2, F_{2,j_2} are the intersections $H_i \cap H_j \cap H_k$ for $i, j, k \in I$, $i \neq j \neq k$.

Proceeding inductively we obtain that the components of F_k, denoted F_{k,j_k}, are $w_J := \bigcap_{i \in J} H_i$ with $\# J = k + 1$.

Then, by the Corollary 6.1.7 of [7], we obtain $S_J := w_J - \bigcup_{J \subsetneq K} w_k$, and $\{S_J\}$ is the minimal Whitney stratification of $V(h)$. □

Now we fix $r_0 < r_1 < \cdots < r_n$ to consider the system $z = (x_0, \ldots, x_n)$. The next proposition shows that this system is prepolar.

Proposition 3.4. *The coordinate system $z = (x_0, \ldots, x_n)$ is prepolar for h with respect to the stratification $\{S_J\}_{J \subseteq I}$.*

Proof. First we remark that the stratification $\{S_J\}$ is a good stratification for h at 0 since the a_h condition of Thom is a necessary condition for the Whitney stratification $\{S_J\}_{J \subseteq I}$.

Now we show that $V(x_0)$ intercepts transversally all strata S_J with $S_J \neq \{0\}$ and for all j with $1 \leq j \leq n - 1$, $V(x_j)$ intercepts transversally all strata $S_J \cap V(x_0, \ldots, x_{j-1})$ with $S_J \cap V(x_0, \ldots, x_{j-1}) \neq \{0\}$.

But, to say that $V(x_0)$ intercepts transversally all such S_J means that for all $p \in V(x_0) \cap S_J$, $T_pV(x_0) + T_pS_J = \mathbb{C}^{n+1}$, or equivalently, the normal vector to $V(x_0)$ at p is not in the normal space to S_J at p.

Let $J = \{i\}$ and considering $p \in V(x_0) \cap S_J$, we see that the normal vector to S_J at p is the normal vector to $V(f_i)$ at p. Here we see that $v = (1, 0, \ldots, 0)$ is the normal vector of $V(x_0)$ and

$$w_i = (\ 0,\ \gamma_{1,i}(r_0\hat{r}_1 \cdots r_n)\tilde{x}_1^{(r_0\hat{r}_1\cdots r_n)-1}, \ldots, \gamma_{n,i}(r_0 \cdots r_{n-1}\hat{r}_n)\tilde{x}_n^{(r_0\cdots r_{n-1}\hat{r}_n)-1}\ )$$

is the normal vector of $V(f_i)$ at the point $(0, \tilde{x}_1, \ldots, \tilde{x}_n) \neq (0, \ldots, 0)$ of $V(x_0) \cap S_J$.

Clearly the vectors v and w_i are not parallels, therefore $V(x_0)$ intercepts transversally S_J for $J = \{i\}$ with $i = 1, \ldots, l$.

Now let $J = \{i, j\}$ with $i \neq j$, $i, j \in \{1, \ldots, l\}$ and $S_J \neq \{0\}$. The normal space to S_J at a point $(0, \ldots, 0) \neq (0, \tilde{x}_1, \ldots, \tilde{x}_n) \in V(x_0) \cap S_J$ is generated by the vectors

$$w_k = (\ 0,\ \gamma_{1,k}(r_0\hat{r}_1 \cdots r_n)\tilde{x}_1^{(r_0\hat{r}_1\cdots r_n)-1}, \ldots, \gamma_{n,k}(r_0 \cdots r_{n-1}\hat{r}_n)\tilde{x}_n^{(r_0\cdots r_{n-1}\hat{r}_n)-1}\ )$$

with $k = i, j$.

Since $v \notin [w_i, w_j]$, we conclude that $V(x_0)$ intercepts transversally S_J.

Following the same idea we can show that $V(x_0)$ intercepts transversally all strata S_J with $S_J \neq \{0\}$ and for all j with $1 \leq j \leq n-1$, $V(x_j)$ intercepts transversally all $S_J \cap V(x_0, \ldots, x_{j-1})$ with $S_J \cap V(x_0, \ldots, x_{j-1}) \neq \{0\}$. $\square$

3.2 Polar ratio

From now on we shall fix $h \in \mathbb{C}[x_0, \ldots, x_n]$ as a Pham-Brieskorn arrangement of type $(r_0, \ldots, r_n; d)$ with $r_0 < \cdots < r_n$. We shall show that the polar ratio of h at the origin with respect to the system of coordinates $z = (x_0, \ldots, x_n)$ depends only of the weights $r_0, \ldots, r_n$ and the degree d.

Proposition 3.5. *The maximum polar ratio of h at the origin with respect to the coordinate system $z = (x_0, \ldots, x_n)$ is smaller or equal than $\frac{d}{r_0}$.*

As we have seen before, we can write h as

$$h = c\, f_1 \cdots f_p . f_{p+1}^{\alpha_{p+1}} \cdots f_l^{\alpha_l}, \quad \text{with} \quad d = \left(p + \sum_{i=p+1}^{l} \alpha_i \right) r_0 \cdots r_n.$$

Call $g_{i,j} = f_1 \cdots \frac{\partial f_i}{\partial x_j} \cdots f_l$ and $g_j = g_{1,j} + \cdots + g_{p,j} + \alpha_{p+1} g_{p+1,j} + \cdots + \alpha_l g_{l,j}$, with $0 \leq j \leq n$.

In order to compute the polar radio of the Pham-Brieskorn arrangements, we use the following.

Lemma 3.6. *For all $k \in \{0, \ldots, n\}$, the sets $V(g_n, g_{n-1}, \ldots, g_k)$ have no components in the critical set of h.*

Proof. We note that

$$g_j = (r_0 \cdots \hat{r}_j \cdots r_n) x_j^{r_0 \cdots \hat{r}_j \cdots r_n - 1} \left((m_1 \gamma_{j,1})\, \hat{f}_1 f_2 \cdots f_l + \cdots + (m_l \gamma_{j,l})\, f_1 \cdots \hat{f}_l \right),$$

where $\hat{f}_k$ means that there is no function f_k with $k \in \{1, \ldots, l\}$.

To get the critical set of h, as $\dfrac{\partial h}{\partial x_j} = (f_{p+1}^{\alpha_{p+1}-1} \cdots f_l^{\alpha_l - 1}) g_j$, then for all $i \neq j \in \{0, \ldots, n\}$, we consider the system

$$\begin{cases} m_1 \gamma_{i,1}\, \hat{f}_1 f_2 \cdots f_l + & \cdots & + m_l \gamma_{i,l}\, f_1 f_2 \cdots \hat{f}_l = 0 \\ m_1 \gamma_{j,1}\, \hat{f}_1 f_2 \cdots f_l + & \cdots & + m_l \gamma_{j,l}\, f_1 f_2 \cdots \hat{f}_l = 0 \end{cases}$$

which is equivalent to the following system:

$$\begin{cases} m_1 \gamma_{i,1}\, \hat{f}_1 f_2 \cdots f_l & + & m_2 \gamma_{i,2}\, f_1 \hat{f}_2 \cdots f_l + \cdots + m_l \gamma_{i,l}\, f_1 f_2 \cdots \hat{f}_l = 0 \\ 0 & + & m_2 k_2\, f_1 \hat{f}_2 \cdots f_l + \cdots + m_l k_l\, f_1 f_2 \cdots \hat{f}_l = 0 \end{cases},$$

with $k_m = \gamma_{j,1} \gamma_{i,m} - \gamma_{i,1} \gamma_{j,m}$, com $2 \leq k \leq l$. From the second equation we have

$$f_1 = 0 \quad \text{or} \quad f_3 \cdots f_l = -\left(\frac{m_3 k_3}{m_2 k_2}\, f_2 \hat{f}_3 \cdots f_l + \cdots + \frac{m_l k_l}{m_2 k_2}\, f_2 \cdots \hat{f}_l \right).$$

Then

$$(m_2\gamma_{i,2}\, f_1 + m_1\gamma_{i,1}\, f_2)(f_3\cdots f_l) =$$
$$= -(m_2\gamma_{i,2}\, f_1 + m_1\gamma_{i,1}\, f_2)\left(\frac{m_3k_3}{m_2k_2}\, f_2\hat{f}_3\cdots f_l + \cdots + \frac{m_lk_l}{m_2k_2}\, f_2\cdots\hat{f}_l\right). \quad (1)$$

From the first equation we see that

$$(m_2\gamma_{i,2}\, f_1 + m_1\gamma_{i,1}\, f_2)(f_3\cdots f_l) =$$
$$= f_1\left(m_3\gamma_{i,3}\, f_2\hat{f}_3\cdots f_l + \cdots + m_l\gamma_{i,l}\, f_2\cdots\hat{f}_l\right). \quad (2)$$

From the fact that the decomposition of a polynomial in irreducible factors is unique and from the equalities (1) and (2) we obtain $f_3\cdots f_l = 0$.

Considering again the first system we see that the solutions are the intersections $V(f_m)\cap V(f_n)$ with $m\neq n$ in $\{1,\dots,l\}$. As i and j are arbitrary, the critical set of h (as a set) is the union

$$\bigcup_{i=p+1}^{l} V(f_i) \cup \bigcup_{i\neq j\in\{1,\dots,p\}} V(f_i)\cap V(f_j).$$

Hence, since any f_i is irreducible we obtain from the description of the polynomials g_i's that $V(g_n, g_{n-1},\dots,g_k)$ has no components in the critical set of h for all $k\in\{0,\dots,n\}$. □

From the Proposition 3.4 and the Theorem 2.12, for the coordinate system $z=(x_0,\dots,x_n)$, we conclude that $\Lambda^i_{h,z}$ is purely i-dimensional at 0 and the Lê numbers $\lambda^i_{h,z}(0)$ are defined for all $0\le i\le n$.

To compute the polar ratio, we need to get the polar curve.

First we see from the Proposition 2.6 that for all $0\le i\le n$,

$$[\Gamma^{i+1}_{h,z}]\cdot\left[V\left(\frac{\partial h}{\partial x_i}\right)\right] = [\Lambda^i_{h,z}] + [\Gamma^i_{h,z}].$$

In special for $i=n$ we have

$$\sum_{j=p+1}^{l}\left[V\left(f_j^{\alpha_j-1}\right)\right] + \left[V\left(\frac{\frac{\partial h}{\partial x_n}}{\prod_{j=p+1}^{l} f_j^{\alpha_j-1}}\right)\right] = \left[V\left(\frac{\partial h}{\partial x_n}\right)\right] = \Lambda^n_{h,z} + \Gamma^n_{h,z}.$$

By definition $\Lambda^n_{h,z}\subseteq\Sigma(h)$ and $\Lambda^n_{h,z}$ is disjoint from $\Gamma^n_{h,z}$.

Since $V(g_n) = V\left(\frac{\frac{\partial h}{\partial x_n}}{\prod f_j^{\alpha_j-1}}\right)$ is not in the critical set of h, from the Lemma 3.6, we obtain $[\Gamma^n_{h,z}] = [V(g_n)]$.

By induction we have

$$[\Gamma^n_{h,z}]\cdot\left[V\left(\frac{\partial h}{\partial x_{n-1}}\right)\right]=\sum_{j=p+1}^{l}\left(\left[V\left(f_j^{\alpha_j-1}\right)\right]\cdot[V(g_n)]\right)+([V(g_{n-1})]\cdot[V(g_n)])\,.$$

Since $\left[\Gamma^n_{h,z}\right]\cdot\left[V\left(\frac{\partial h}{\partial x_{n-1}}\right)\right]=\Lambda^{n-1}_{h,z}+\Gamma^{n-1}_{h,z}$, using the same argument we have $[\Gamma^{n-1}_{h,z}]=[V(g_{n-1},g_n)]$ and proceeding in the same way we conclude that

$$[\Gamma^1_{h,z}]=[V\left(g_1,\ldots,g_n\right)]\,.$$

Using again the Theorem 2.12 we have that $\Gamma^1_{h,z}$ is 1-dimensional, then $\Gamma^1_{h,z}$ is a complex algebraic curve determined by n independent polynomials functions with $n+1$ variables with coefficients in $\mathbb{C}$. Using the resultant we eliminate all less two of the variables and reduce the curve to an equivalent birrational plane curve $\hat{g}(x,y)=0$.

Hence each component of $\Gamma^1_{h,z}$ has a local parametrization. Now let η be an irreducible component of $\Gamma^1_{h,z}$ and consider $\phi(t)=(\phi_0(t),\ldots,\phi_n(t))$ such a parametrization of η, where $\phi_j(t)=c_jt^{k_j}+$ higher order terms.

We shall also need the following result to compute the polar ratio.

Lemma 3.7. ([8], page 10) *Let $p\in U$ with U an open set in $\mathbb{C}^{n+1}$. If W is an irreducible curve in U at p and $V(f)\subseteq U$ is an hypersurface which intercepts W properly at p (or $codim_pV(f)\cap W=codim_p\,V(f)+codim_p\,W$), consider a local parametrization $\phi(t)$ of W which takes 0 at p. Then $([W]\cdot[V(f)])_p=$ $mult_t\,f(\phi(t))=$ the smallest degree of all non zero terms in the Taylor series of $f(\phi(t))$.*

Proof of the Proposition 3.5. We first suppose that η intercepts transversally $V(z_0)$ since in the other case the polar ratio is 1.
Now, by the Lemma 3.7,

$$(\eta\cdot V(f_i))_0=mult_t\,f_i(\phi(t))=\min_q\left\{k_q(r_0\cdots\hat{r}_q\cdots r_n)\right\}$$

and $(\eta\cdot V(z_0))_0=mult_t\,z_0(\phi(t))=k_0$. Therefore

$$\frac{(\eta\cdot V(h)\,)_0}{(\eta\cdot V(z_0)\,)_0}\leq\frac{\dfrac{d}{r_0\cdots r_n}(k_0(\hat{r}_0r_1\cdots r_n))}{k_0}=\frac{d}{r_0},$$

and the maximum polar ratio of h is smaller or equal than $\frac{d}{r_0}$. □

We shall show now that we also can compute the polar ratio of the function $h+a_0x_0^{\frac{d}{r_0}}$ in terms of the weights and the degree.

Proposition 3.8. *The maximum polar ratio of* $h + a_0 x_0^{\frac{d}{r_0}}$ *at* 0 *with respect to the variables* $(x_1, \ldots, x_n, x_0)$ *is smaller or equal to* $\frac{d}{r_1}$.

The maximum polar ratio of $h + a_0 x_0^{\frac{d}{r_0}} + \cdots + a_{i-1} x_{i-1}^{\frac{d}{r_{i-1}}}$ *(at* 0 *with respect to* $(x_i, \ldots, x_n, x_0, \ldots, x_{i-1})$*) is smaller or equal to* $\frac{d}{r_i}$ *for all* $i = 1, \ldots, n-1$.

Proof. Since $z = (x_0, \ldots, x_n)$ is prepolar for h at 0 from the Proposition 2.13, $\tilde{z} = (x_1, \ldots, x_n, x_0)$ is prepolar for $h + a_0 x_0^{\frac{d}{r_0}}$ at 0, except for a finite number of a_0. Consider a_0 satisfying this condition, then the polar curve of $h + a_0 x_0^{\frac{d}{r_0}}$ in $\tilde{z}$ is purely 1-dimensional at 0.

In an analogous way as in the Proposition 2.7, we conclude that it is determined by n independent functions in $n+1$ variables with coefficients in $\mathbb{C}$. Hence any irreducible component of this polar curve has a parametrization. Let η be an irreducible component of the polar curve of $h + a_0 x_0^{\frac{d}{r_0}}$ in $\tilde{z}$ which intercepts transversally $V(x_1)$ and $\phi(t) = (\phi_0(t), \ldots, \phi_n(t))$ is a parametrization of η, with $\phi_q(t) = c_q t^{k_q}+$ higher order terms.

We know that

$$(\eta \cdot V(h + a_0 x_0^{\frac{d}{r_0}}))_0 = mult_t\ (h + a_0 x_0^{\frac{d}{r_0}})(\phi(t)) =$$

$$= mult_t \left(h(\phi(t)) + a_0 x_0^{\frac{d}{r_0}} (\phi(t)) \right) = mult_t\ h(\phi(t)) = min_q\ \{k_q (r_0 \cdots \hat{r}_q \cdots r_n)\}.$$

Therefore

$$\frac{(\ \eta \cdot V(h + x_0^{\frac{d}{r_0}}))_0}{(\ \eta \cdot V(x_1)\ )_0} \leq \frac{\frac{d}{r_0 \cdots r_n} (k_1 (r_0 \hat{r}_1 \cdots r_n))}{k_1} = \frac{d}{r_1}.$$

Then the maximum polar ratio of $h + a_0 x_0^{\frac{d}{r_0}}$ at 0 with respect to the variables $(x_1, \ldots, x_n, x_0)$ is smaller or equal to $\frac{d}{r_1}$.

Proceeding in an analogous way, we obtain that the maximum polar ratio of $h + a_0 x_0^{\frac{d}{r_0}} + \cdots + a_{i-1} x_{i-1}^{\frac{d}{r_{i-1}}}$ (at 0 with respect to $(x_1, \ldots, x_n, x_0, \ldots, x_{i-1})$) is smaller or equal to $\frac{d}{r_i}$ for all $i = 1, \ldots, n-1$. □

4 Characterization of the Lê numbers

Massey shows in [8], Corollary 4.7, the following Plücker type formula for homogeneous polynomials of degree d in $n+1$ variables:

If $\lambda^i_{h,z}(0)$ *exists for all* $i \leq s$, *where* $s = dim\ \Sigma(h)$, *then*

$$\sum_{i=0}^{s} (d-1)^i \lambda^i_{h,z}(0) = (d-1)^{n+1}.$$

With the help of this formula he characterizes the Lê numbers of hyperplane arrangements.

We shall show here how to describe the Lê numbers at the origin of any Pham-Brieskorn arrangement $h \in \mathbb{C}[x_0, \ldots, x_n]$ of type $(r_0, \ldots, r_n; d)$ with $r_0 < \cdots < r_n$ in terms of the weights and the degree.

The key tool to show this description is to obtain a Plücker type formula for this case, which is shown in the next

Proposition 4.1. *Let h be a Pham-Brieskorn arrangement of type $(r_0, \ldots, r_n; d)$ at the origin with respect to the coordinate system $z = (x_0, \ldots, x_n)$. Then*

$$\sum_{i=0}^{n} \left(\prod_{k=0}^{i-1} \left(\frac{d}{r_k} - 1 \right) \right) \lambda^i_{h,z}(0) = \left(\frac{d}{r_0} - 1 \right) \cdots \left(\frac{d}{r_n} - 1 \right).$$

Proof. From the Proposition 3.5, we have that the maximum polar ratio of h at the origin with respect to the coordinate system $z = (x_0, \ldots, x_n)$ is smaller or equal than $\frac{d}{r_0}$. Then, we consider $j_0 = \frac{d}{r_0}$ to apply the Theorem 2.9 and obtain that there exists $a_0 \in \mathbb{C}^*$ such that $h + a_0 x_0^{\frac{d}{r_0}}$ has critical set of dimension $n-1$.

Applying the Proposition 3.8, for all except a finite number of $a_i \in \mathbb{C}^*$, the maximum polar ratio of $h + a_0 x_0^{\frac{d}{r_0}} + \cdots + a_{i-1} x_{i-1}^{\frac{d}{r_{i-1}}}$, at the origin with respect to the coordinate system $\tilde{z} = (x_i, \ldots, x_n, x_0, \ldots, x_{i-1})$ is smaller or equal than $\frac{d}{r_i}$ for all $1 \leq i \leq n-1$, then we apply the Theorem 2.9 inductively with $j_i = \frac{d}{r_i}$, $1 \leq i \leq n-1$ to obtain that there exist $a_1, \ldots, a_{n-1} \in \mathbb{C}^*$ such that $f := h + a_0 x_0^{j_0} + a_1 x_1^{j_1} + \cdots + a_{n-1} x_{n-1}^{j_{n-1}}$ has isolated singularity at the origin and

$$\mu(f) = \sum_{i=0}^{n} \left(\prod_{k=0}^{i-1} \left(\frac{d}{r_k} - 1 \right) \right) \lambda^i_{h,z}(0).$$

Since f is a weighted homogenous polynomial of type $(r_0, \ldots, r_n; d)$, we conclude from the Theorem 1 of [10] that $\mu(f) = \left(\frac{d}{r_0} - 1 \right) \cdots \left(\frac{d}{r_n} - 1 \right)$.

Therefore

$$\sum_{i=0}^{s} \left(\prod_{k=0}^{i-1} \left(\frac{d}{r_k} - 1 \right) \right) \lambda^i_{h,z}(0) = \left(\frac{d}{r_0} - 1 \right) \cdots \left(\frac{d}{r_n} - 1 \right).$$

□

We shall apply this formula to characterize each Lê number at the origin of any Pham-Brieskorn arrangement.

Lemma 4.2. $\Lambda^k_{h,z} = \sum_{\dim S_J = k} a_J[w_J]$, *with* $a_J = \lambda^k_{h_{|N},z}(p)$ *for some* $p \in S_J$ *with* $p = (p_0, \ldots, p_n)$ *and* $N = V(x_0 - p_0, \ldots, x_{k-1} - p_{k-1})$.

Proof. First we shall show that, as sets, the Lê cycles of h with respect to the coordinate system $z = (x_0, \ldots, x_n)$ are given as the union of the w_J with correct dimension.

We shall describe these cycles inductively. By definition $\Lambda^n_{h,z}$ is formed by the components of $V(\frac{\partial h}{\partial x_n})$ which are in the critical set of h.

As we saw before, $\frac{\partial h}{\partial x_n} = (f_{p+1}^{\alpha_{p+1}-1} \cdots f_l^{\alpha_l - 1})g_n$ and $V(g_n)$ does not have components in the critical set, therefore $\Lambda^n_{h,z}$, as a set, is the union of the $V(f_i)$ with $i = p+1, \ldots, l$.

Now, consider the intersection of $V(\frac{\partial h}{\partial x_{n-1}})$ with the components of $V(\frac{\partial h}{\partial x_n})$ which are not in the critical set of h. We have that $\Lambda^{n-1}_{h,z}$ is formed by the components of this intersection which are in the critical set of h.

But $\frac{\partial h}{\partial x_{n-1}} = (f_{p+1}^{\alpha_{p+1}-1} \cdots f_l^{\alpha_l-1})g_{n-1}$ and $V(g_{n-1}, g_n)$ has no components in the critical set, therefore $\Lambda^{n-1}_{h,z}$ is, as a set, the union of all intersections $V(f_i) \cap V(f_j)$ with $i \neq j$, $i = p+1, \ldots, l$ and $j = 1, \ldots, l$. Then the result follows inductively.

Hence, as cycles, for all k, $\Lambda^k_{h,z} = \sum_{\dim S_J = k} a_J\, [w_J]$, for some a_J.

Now, let $\tilde{J} \subseteq I$ such that the dimension of $S_{\tilde{J}}$ is k.

Consider $p = (p_0, \ldots, p_n) \in S_{\tilde{J}}$. As sets we have

$$\begin{aligned}
\Sigma(h_{|\, V(x_0 - p_0, \ldots,\, x_{k-1} - p_{k-1})}) &= V\left(x_0 - p_0, \ldots,\, x_{k-1} - p_{k-1}, \frac{\partial h}{\partial z_k}, \ldots, \frac{\partial h}{\partial z_n}\right) \\
&= V(x_0 - p_0, \ldots,\, x_{k-1} - p_{k-1}) \cap (\Sigma(h) \cup \Gamma^k_{h,z}).
\end{aligned}$$

As $\Lambda^k_{h,z}$ and $\Gamma^k_{h,z}$ are disjoint we have $p \notin \Gamma^k_{h,z}$, hence, by the Theorem 2.12, $V(x_0 - p_0, \ldots,\, x_{k-1} - p_{k-1}) \cap \Gamma^k_{h,z}$ is empty at p. Therefore, considering $N = V(x_0 - p_0, \ldots, x_{k-1} - p_{k-1})$,

$$\Sigma(h_{|\, N}) = \Sigma(h) \cap N \text{ in } p.$$

Using the Proposition 2.7, we obtain $\lambda^k_{h_{|N},\tilde{z}}(p) = \lambda^k_{h,z}(p)$, with the system of coordinates $\tilde{z} = (x_k, \ldots, x_n, x_0, \ldots, x_{k-1})$.

On the other side, the cycle $\Lambda^k_{h,z} \cdot V(x_0 - p_0, \ldots, x_{k-1} - p_{k-1})$ is of the form

$$\sum_{\dim S_J = k} a_J\ ([w_J] \cdot V(x_0 - p_0, \ldots, x_{k-1} - p_{k-1}))\,.$$

Now, $[w_J] \cdot V(x_0 - p_0, \ldots, x_{k-1} - p_{k-1}) = [p]$ if $J = \tilde{J}$ and we get that $\left(\Lambda^k_{h,z} \cdot V(x_0 - p_0, \ldots, x_{k-1} - p_{k-1})\right)_p = a_{\tilde{J}}$, hence $a_{\tilde{J}} = \lambda^k_{h_{|N},z}(p)$. $\square$

Remark 4.3. If we consider the translation $(x_0, \ldots, x_n) \xrightarrow{\phi} (X_0, \ldots, X_n)$ where $X_i = x_i - p_i$ takes p at the origin, we have $h_{|_N}$ at p in this new coordinate system can be seen in a neighbourhood of the origin as being $\prod_{j=1}^{l} \tilde{f}_j^{m_j}$ with $\tilde{f}_j = \phi(f_{i|_{x_0=\cdots=x_{k-1}=0}})$.

Since $p \notin V(f_j)$ if $j \notin J$ then $0 \notin V(\tilde{f}_j)$ if $j \notin J$. Then, in a neighbourhood of the origin, $\prod_{j \notin J} \tilde{f}_j^{m_j}$ is a unity and we obtain that $h_{|_N}$ can be seen , in a neighbourhood of the origin, as $\prod_{j \in J} \tilde{f}_j^{m_j}$.

Now, for each hypersurface H_i we call *weighted multiplicity* of H_i, $\text{mult}_P\, H_i$, the number $m_i(r_0 \cdots r_n)$. To simplify the notation we write H for the defining hypersurfaces of $V(h)$ and w or v the possible intersections with them and $\mathcal{A}$ denotes the collection of all these w.

We denote $e(w) := \sum_{w \subseteq H} \text{mult}_P\, H$ and $k(w) := \dfrac{\left(\dfrac{e(w)}{r_0} - 1\right) \cdots \left(\dfrac{e(w)}{r_n} - 1\right)}{\prod_{i=0}^{\dim w - 1} \left(\dfrac{e(w)}{r_i} - 1\right)}$.

Definition 4.4. *We define inductively on the dimension of w,*

$$\eta(w) := k(w) - \sum_{v \supsetneq w} \eta(v) \left(\prod_{i=1}^{\text{codim}_v w} \left(\frac{e(w)}{r_{n-i}} - 1 \right) \right).$$

Theorem 4.5. *Let h be a Pham-Brieskorn arrangement, then for some generic coordinate system z,*

$$\lambda^i_{h,z}(0) = \sum_{w \in \mathcal{A},\ \dim w = i} \eta(w)$$

Proof. From the Lemma 4.2 and the Remark 4.3, $\Lambda^k_{h,z} = \sum_{\dim S_J = k} a_J\,[w_J]$, with $a_J = \lambda^0_{g_J,z}(0)$ where g is the product of Pham-Brieskorn type polynomials in $n + 1 - k$ variables, keeping the same weights of the variables and weighted homogeneous degree $e(w_J)$.

Therefore $\lambda^n_{h,z}(0) = \sum_{\dim S_J = n} a_J = \sum_{\dim S_J = n} \lambda^0_{g_J,z}(0)$, where each g_J is a polynomial in 1 variable with critical set 0-dimensional, $J = \{i\}$, $1, \ldots, l$ and degree $e(w_{\{i\}})$.

Applying the Proposition 4.1 for g_i at the origin with respect to the coordinate system $z = (X_n)$, we have $\lambda^0_{g_{\{i\}},z}(0) = \dfrac{e(w_{\{i\}})}{r_n} - 1 = \eta(w_{\{i\}})$. Therefore

$$\lambda^n_{h,z}(0) = \sum_{i=1}^{l} \eta(w_{\{i\}}) = \sum_{\dim v = n} \eta(v).$$

Analogously, $\lambda^{n-1}_{h,z}(0) = \sum\limits_{\dim S_J = n-1} a_J = \sum\limits_{\dim S_J = n-1} \lambda^0_{g_J,z}(0)$, where each g_J is a polynomial in two variables with critical set 1-dimensional and degree $e(w_J)$.

Now we apply again the proposition 4.1 for the g_J at the origin, but with respect to the coordinate system $z = (X_{n-1}, X_n)$, then we obtain

$$\lambda^0_{g_{\{J\}},z}(0) + \left(\frac{e(w_J)}{r_{n-1}} - 1\right)\lambda^1_{g_{\{J\}},z}(0) = \left(\frac{e(w_J)}{r_{n-1}} - 1\right)\left(\frac{e(w_J)}{r_n} - 1\right).$$

From the Proposition 2.7, $\lambda^1_{g_{\{i\}},z}(0) = \lambda^0_{g_{\{i\}|_{V(X_{n-1})}},z}(0)$. On the other side,

$$\lambda^0_{g_{\{i\}|_{V(X_{n-1})}},z}(0) = \sum_{w_{\tilde{J}} \supsetneq w_J,\ \dim S_{\tilde{J}} = n} \lambda^0_{h_{|N_{\tilde{J}}},z}(p_{\tilde{J}}),$$

com $p_{\tilde{J}} = (p_{\tilde{J},1}, \ldots, p_{\tilde{J},n})$ e $N_{\tilde{J}} = (x_0 - p_{\tilde{J},0}, \ldots, x_{\tilde{J},n-1} - p_{\tilde{J},n-1})$.

We apply again the Remark 4.3 to conclude that $h_{|N_{\tilde{J}}}$ in a neighborhood of $p_{\tilde{J}}$ can be seen, by exchange of coordinates as a polynomial in a neighborhood of the origin of one variable with 0-dimensional critical set, $\tilde{J} = \{i\}, 1, \ldots, l$ and degree $e(w_{\{i\}})$. From the case before we obtain $\lambda^0_{h_{|N_{\tilde{J}}},z}(p_{\tilde{J}}) = \eta(w_{\tilde{J}})$.

Therefore $\lambda^1_{g_{\{i\}},z}(0) = \sum_{w_J \supsetneq w,\ \dim S_J = n} \eta(w)$ and

$$\lambda^0_{g_{\{J\}},z}(0) = \left(\frac{e(w_J)}{r_{n-1}} - 1\right)\left(\frac{e(w_J)}{r_n} - 1\right) - \sum_{w_J \supsetneq w,\ \dim S_J = n} \left(\frac{e(w_J)}{r_{n-1}} - 1\right)\eta(w).$$

Hence $\lambda^{n-1}_{h,z}(0) = \sum\limits_{\dim v = n-1} \eta(v)$.

Proceeding inductively we obtain $\lambda^i_{h,z}(0) = \sum\limits_{\dim v = i} \eta(v)$ with $i = 1, \ldots, n$. To get $\lambda^0_{h,z}(0)$ it is enough to exchange each $\lambda^i_{h,z}(0) = \sum\limits_{\dim v = i} \eta(v)$ with $i = 1, \ldots, n$ in the formula for $w = \bigcap\limits_{i=1}^{l} H_i$. □

Acknowledgments

This work is part of the Ph.D. Thesis of the first named author, done under supervision of the second author at ICMC-USP, São Carlos. The second named author is partially supported by FAPESP and CNPq and the first named author was su-pported by CAPES during her Ph.D. Thesis. The authors thank these agencies for this financial support.

References

[1] D. Eisenbud and J. Harris, Schemes: The Language of Modern Algebraic Geometry, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA. *The Wadsworth & Brooks/Cole Mathematics Series.* (1992), 157 pages.

[2] W. Fulton, Intersection theory, *Ergebnisse der Math.*, Springer-Verlag, (1984).

[3] T. Gaffney and R. Gassler, Segre numbers and hypersurface singularities, *J. Algebraic Geometry* **8**, (1996) 695–736.

[4] P. Griffiths and J. Harris, Principles of algebraic geometry, New York : Wiley, (1978).

[5] I. N. Iomdin, Varits complexes avec singularits de dimension un, *Sibirsk. Mat. Z.* **15**, (1974) 1061–1082.

[6] Ensembles analytiques complexes avec lieu singulier de dimension un (d'aprs I.N. Iomdin), *Sminaire sur les Singularits* (Paris, 1976-1977) *Publ. Math. Univ. Paris* **VII**, (1980) 87–95.

[7] Lê D. T. and B. Teissier, Variétés polaires locales et classes de Chern de variétés singulères, *Annals of Math.*, **114** (1981) 457–491.

[8] D. Massey, Lê cycles an hypersurface singularities, *Springer Lecture Notes in Mathematics* **1615**, (1995).

[9] J. Milnor, Singular points of complex hypersurfaces, *Annals of Mathematics Studies*, **61**, Princeton University Press, Princeton, N.J., (1968).

[10] Milnor, J. and Orlik, P., Isolated singularities defined by weighted homogeneous polynomials, *Topology* **9** (1970) 385–393.

[11] Y. T. Siu and G. Trautmann, Gap-sheaves and extension of coherent analytic subsheaves, *Springer Lecture Notes in Mathematics* **172** (1971).

Universidade de São Paulo, Instituto de Ciências Matemáticas e Computação de São Carlos, Departamento de Matemática, São Carlos, SP - Brasil
E-mails: ferreira@icmc.usp.br, mjsaia@icmc.usp.br

NOTAS DO ICMC

SÉRIE MATEMÁTICA

329/10	GRULHA JÚNIOR, N.G. – Stability of the Euler obstruction of f on free divisors.
328/10	AHMED, I.; RUAS, M.A.S. – Invariants of relative right and contact equivalences.
327/10	CARO, N.; LEVCOVITZ, D. – On theorem of Stafford.
326/10	FEDERSON, M.; GODOY, J.B. – Averaging for retarded functional differential equations.
325/10	FERNANDES, A. – Fast loops on semi-weighted homogeneous hypersurface singularities.
324/10	FEDERSON, M.; GODOY, J.B. – A new continuous dependence result for impulsive retarded functional differential equations.
323/10	AFONSO, S.; BONOTTO, E.; FEDERSON, M.; SCHWABIK, S. – Discontinuous local semiflows for Kurzweil equations leading to Lasalle's invariance principle for differential systems with impulses at variable times.
322/09	COSTA, J. C. F.; SAIA, M. J. ; SOARES JR., C. H. – Bi-Lipschitz, $\mathcal{G} = \mathcal{R}, \mathcal{C}, \mathcal{K}, \mathcal{R}_V, \mathcal{C}_V, \mathcal{K}_V$
321/09	FENILLE, M. C. – On acyclic and simply connected open manifolds.
320/09	FENILLE, M. C. – Closed injective systems and its fundamental limit spaces .