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GLOBAL ATTRACTOR FOR A MODEL OF EXTENSIBLE BEAM WITH NONLINEAR DAMPING AND SOURCE TERMS

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Resumo

Neste artigo estudamos a existência de um atrator global para a equação de viga com termos de dissipação e força não lineares,

$$u_{tt} + \Delta^2 u - M(\int_\Omega |\nabla u|^2 dx)\Delta u + f(u) + g(u_t) = h \quad \text{em} \quad \Omega \times \mathbb{R}^+,$$

onde Ω é um domínio limitado de $\mathbb{R}^N$, M é uma função real não negativa e $h \in L^2(\Omega)$. As não linearidades $f(u)$ e $g(u_t)$ são essencialmente $|u|^\rho u - |u|^\sigma u$ e $|u_t|^r u_t$ respectivamente, com $\rho, \sigma, r > 0$ e $\sigma < \rho$. Esta classe de problemas modela vibrações de vigas e placas extensíveis.

Global Attractor for a Model of Extensible Beam with Nonlinear Damping and Source Terms

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Abstract

This paper is concerned with the existence of a global attractor for the nonlinear beam equation, with nonlinear damping and source terms,

$$u_{tt} + \Delta^2 u - M(\textstyle\int_\Omega |\nabla u|^2 dx)\Delta u + f(u) + g(u_t) = h \quad \text{in} \quad \Omega \times \mathbb{R}^+,$$

where Ω is a bounded domain of $\mathbb{R}^N$, M is a nonnegative real function and $h \in L^2(\Omega)$. The nonlinearities $f(u)$ and $g(u_t)$ are essentially $|u|^\rho u - |u|^\sigma u$ and $|u_t|^r u_t$ respectively, with $\rho, \sigma, r > 0$ and $\sigma < \rho$. This kind of problem models vibrations of extensible beams and plates.

Keywords: Global attractor, Berger's plate, Kirchhoff's beam.

MSC: 35L75, 35B41, 35B40.

1 Introduction

In this paper we study the existence of a global attractor for a nonlinear evolution equation of the form

$$\begin{cases} u_{tt} + \Delta^2 u - M(\int_\Omega |\nabla u|^2 dx)\Delta u + f(u) + g(u_t) = h & \text{in} \quad \Omega \times \mathbb{R}^+, \\ u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}^+, \\ u(\,\cdot\,, 0) = u_0 \quad \text{and} \quad u_t(\,\cdot\,, 0) = u_1 \quad \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\Gamma = \partial\Omega$. The source and damping terms are of the form

$$f(u) \approx |u|^\rho u - |u|^\sigma u \quad (\sigma < \rho) \quad \text{and} \quad g(u_t) \approx |u_t|^r u_t,$$

with

$$0 < \rho, r \leq \frac{2}{N-2} \text{ if } N \geq 3 \quad \text{and} \quad \rho, r > 0 \text{ if } N = 1, 2. \tag{1.2}$$

This kind of wave equation is derived from the extensible beam equation of Woinowsky-Krieger [1],

$$\frac{\partial^2 u}{\partial t^2} + \frac{EI}{\rho}\frac{\partial^4 u}{\partial x^4} - \left(\frac{H}{\rho} + \frac{EA}{2\rho L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.3}$$

where L, E, I, ρ, H and A denote, respectively, the length of the beam in the rest position, the Young's modulus, the cross-sectional moment of inertia, the mass density, the tension in the rest position, and the cross-sectional area. Its modeling aspects were also discussed and extended in Berger [2] and Eisley [3]. Rigorous mathematical analysis for global existence and asymptotic behavior of these extensible beams goes back to the 1970's, for instance, in the papers by Ball [4, 5], Dickey [6] and Medeiros [7]. Later, several interesting results were published focusing different qualitative properties of the problem. We refer the reader to, for instance, [8, 9, 10, 11, 12, 13, 14] and the references therein.

In the case where the fourth order term is dropped in (1.1) or (1.3), we then obtain the Kirchhoff's wave equation [15], which was extensively studied by many authors. See for instance, [16, 17, 18, 19, 20, 21] and the references therein.

We are concerned with the long-time behavior of dynamical systems given by such extensible beam equations. More precisely, we seek the existence of global attractors. In this context, Eden and Milani [22] considered the problem (1.1) with a linear weak damping $g(u_t) = u_t$, $f(u) = 0$ and M being a linear function. Biazutti and Crippa [8] considered problem (1.1) in an abstract setting, with $f(u) = 0$ and a linear damping of fractionary order $g(u_t) = (-\Delta)^\beta u_t$, where $0 < \beta \leq 1$.

To our best knowledge, attractors for problem (1.1) with nonlinear damping and source terms were not previously considered. In fact, the existence and stability of global solutions for (1.1), with nonlinear terms f, g and M, were only recently proved by Cavalcanti et al [9]. Therefore our study complements the results in [9], by considering the existence of attractors, and extends the results in [8, 22] by considering nonlinear damping and source terms.

On the other hand, the existence of attractors for a related problem, with the boundary condition $u = \frac{\partial u}{\partial \nu} = 0$ of (1.1) replaced by $u = \Delta u = 0$, was considered with nonlinear terms f and g by Chueshov and Lasiecka [10]. Eden and Milani [22] also considered this kind of boundary condition, but with a linear damping. Some other related results can be found in, e.g. [23, 24, 25, 26].

This paper is organized as follows. In Section 2 we introduce the main assumptions and discuss the existence of a global weak solution to the problem (1.1). In Section 3, we establish our main result, Theorem 3.2, on the existence of bounded absorbing sets and global attractors for the system (1.1), in the phase space $H_0^2(\Omega) \times L^2(\Omega)$. Our approach is based on the one in Nakao [27], where it was considered a second order wave equation.

2 Assumptions and Global Existence

Let us begin with the precise hypotheses on the functions M, f, g and recall a result on the global existence and asymptotic stability for (1.1).

We assume that $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a function of class C^1, satisfying

$$M(z)z \geq \widehat{M}(z) \quad \text{where} \quad \widehat{M}(z) = \int_0^z M(s)ds. \tag{2.4}$$

This condition is easily verified if M is monotone nondecreasing.

The functions $f, g : \mathbb{R} \to \mathbb{R}$ are also of class C^1, with $f(0) = g(0) = 0$, $|f'(u)| \leq k_0(1+|u|^\rho)$ and $g'(v) \geq 0$, for all $u, v \in \mathbb{R}$, where ρ satisfies (1.2). In addition, there exist constants $k_1, L_0, L_1 > 0$ such that

$$|f(u) - f(v)| \leq k_1(1 + |u|^\rho + |v|^\rho)|u - v|, \quad \forall\, u, v \in \mathbb{R} \tag{2.5}$$

and

$$-L_0 \leq \widehat{f}(u) \leq \frac{1}{2} f(u)u + L_1, \quad \forall\, u \in \mathbb{R}, \tag{2.6}$$

where $\widehat{f}(z) = \int_0^z f(s)ds$. There exist also constants $k_2, k_3 > 0$ such that

$$|g(u) - g(v)| \leq k_2(1 + |u|^r + |v|^r)|u - v|, \quad \forall\, u, v \in \mathbb{R} \tag{2.7}$$

and

$$(g(u) - g(v))(u - v) \geq k_3|u - v|^{r+2}, \quad \forall\, u, v \in \mathbb{R}, \tag{2.8}$$

where $r > 0$ satisfies (1.2).

Our analysis is given in the Sobolev space

$$\mathcal{H} = H_0^2(\Omega) \times L^2(\Omega), \tag{2.9}$$

equipped with the norm

$$\|(u, v)\|_{\mathcal{H}}^2 = \|\Delta u\|_2^2 + \|v\|_2^2,$$

where $\|\cdot\|_p$ denotes standard L^p norms. Note that assumption (1.2) implies that $H_0^2(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^{2(p+1)}(\Omega)$, with $p = \rho$ or $p = r$. The energy of the system (1.1) is defined by

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\Delta u(t)\|_2^2 + \frac{1}{2}\widehat{M}\left(\|\nabla u(t)\|_2^2\right) + \int_\Omega \widehat{f}(u(t))dx - \int_\Omega hu(t)dx, \tag{2.10}$$

where $u(t) = u(\cdot, t)$.

When $h = 0$, problem (1.1) was studied by Cavalcanti et al in [9], where global existence and polynomial decay rate for the energy were obtained. They used Faedo-Galerkin approximations on a simplified problem, by putting $M(\|\nabla u(t)\|_2^2) = \mu(t)$, and then constructed a local solution with fixed-point arguments. By using Zorn's Lemma and uniform estimates, the local solution is finally extended to a global solution.

Under our hypotheses on f, g, h, M, we can derive an existence result, by following the arguments of [9] almost step by step.

Theorem 2.1 *Assume that conditions (2.4)-(2.8) hold and that $h \in L^2(\Omega)$. Then, if the initial data (u_0, u_1) are given in*

$$\mathcal{H}_1 = (H^4(\Omega) \cap H_0^2(\Omega)) \times H_0^2(\Omega), \tag{2.11}$$

problem (1.1) has a unique strong solution $u = u(x,t)$ in the class

$$u \in L^\infty(\mathbb{R}^+; H^4(\Omega) \cap H_0^2(\Omega)), \quad u_t \in L^\infty(\mathbb{R}^+; H_0^2(\Omega)), \quad u_{tt} \in L^\infty(\mathbb{R}^+; L^2(\Omega)).$$

On the other hand, if the initial data (u_0, u_1) are given in

$$\mathcal{H} = H_0^2(\Omega) \times L^2(\Omega),$$

problem (1.1) has a unique weak solution u in the class

$$u \in C^0([0,\infty); H_0^2(\Omega)) \cap C^1([0,\infty); L^2(\Omega)). \tag{2.12}$$

In both cases,

$$\|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2 + \|u(t)\|_{\rho+2}^{\rho+2} \le C, \quad \forall t \ge 0, \tag{2.13}$$

where $C > 0$ is a constant depending on the initial data.

Remark 2.1 *If $h = 0$, it is proved in Cavalcanti et al [9] that the energy $E(t)$, for both strong and weak solutions, has the polynomial decay rate*

$$E(t) \le \left(\theta t + E(0)^{-\frac{r}{2}}\right)^{-\frac{2}{r}}, \quad \forall t \ge 0, \tag{2.14}$$

where $\theta > 0$ is a positive constant.

Remark 2.2 *The existence of a weak solution in $\mathcal{H}$ is obtained by using density arguments. Given initial data $(u_0, u_1) \in \mathcal{H}$, there exists a sequence $\{(u_0^\nu, u_1^\nu)\} \subset \mathcal{H}_1$ such that*

$$u_0^\nu \to u_0 \ \ in \ \ H_0^2(\Omega) \quad and \quad u_1^\nu \to u_1 \ \ in \ \ L^2(\Omega). \tag{2.15}$$

Then uniform estimates on the corresponding regular solutions u^ν show the existence of a subsequence $\{u^{\nu_k}\}$ that converges to a unique weak solution u of problem (1.1), satisfying

$$u_{tt} + \Delta^2 u - M(\textstyle\int_\Omega |\nabla u|^2 dx)\Delta u + f(u) + g(u_t) = h \quad in \quad L^2_{loc}(\mathbb{R}^+; H^{-2}(\Omega) + L^{\frac{r+2}{r+1}}(\Omega)).$$

Remark 2.3 *In view of Theorem 2.1 we can define on $\mathcal{H}$ a one-parameter operator*

$$S(t) : (u_0, u_1) \mapsto (u(t), u_t(t)), \quad t \ge 0,$$

where u is the unique solution of (1.1). It follows that $S(t)$ is a nonlinear C_0-semigroup defined on the phase space $\mathcal{H}$.

3 Global Attractors

Let $S(t)$ be a semigroup defined on a metric space (H, d). Then a bounded closed subset $\mathcal{A} \subset H$ is a global attractor for $S(t)$ if, it is positive invariant, that is, $S(t)\mathcal{A} = \mathcal{A}$, $\forall\, t \geq 0$, and uniformly attracting, that is, for any bounded set $B \subset H$,

$$dist(S(t)B, \mathcal{A}) = \sup_{x \in S(t)B} \inf_{y \in \mathcal{A}} d(x, y) \to 0 \quad \text{as} \quad t \to \infty.$$

Since in many cases these conditions are hard to be verified, there are some equivalent statements.

A bounded set $\mathcal{B} \subset H$ is an absorbing set for $S(t)$ if for any bounded set $B \subset H$, there exists $t_B = t_B(B) \geq 0$ such that

$$S(t)B \subset \mathcal{B}, \quad \forall\, t \geq t_B,$$

which characterizes $S(t)$ as a dissipative semigroup.

A semigroup $S(t)$ is asymptotically smooth in H if for any bounded positive invariant set $B \subset H$, there exists a compact set $K \subset \overline{B}$, such that

$$dist(S(t)B, K) \to 0 \quad \text{as} \quad t \to \infty.$$

Then it is well-known the following theorem (see e.g. [10, 28, 29, 30]).

Theorem 3.1 *Let $S(t)$ be a dissipative semigroup defined on a metric space H. Then $S(t)$ has a compact global attractor in H if and only if it is asymptotically smooth in H.*

Our main result reads as follows.

Theorem 3.2 (Main Result) *Under the hypotheses of Theorem 2.1, the associate semigroup $S(t)$ of problem (1.1) has a global attractor $\mathcal{A}$ in $\mathcal{H}$. More precisely, there exists a constant $C_h > 0$ such that*

$$\mathcal{A} \subset \mathcal{B} = \{(u, v) \in \mathcal{H} \,|\, \|\Delta u\|_2^2 + \|v\|_2^2 \leq C_h(\|h\|_2^2 + L_0 + L_1)\},$$

and for each bounded subset $B \subset \mathcal{H}$, there exists a constant $C(B)$ such that

$$dist(S(t)B, \mathcal{B}) \leq C(B)(1+t)^{-\frac{1}{r}}. \tag{3.16}$$

Proof. The proof is based on two lemmas stated below. Lemma 3.3 guarantees that $S(t)$ has an absorbing set in $\mathcal{H}$ and Lemma 3.4 shows that $S(t)$ has the asymptotic smoothness property. Then the existence of a global attractor follows from Theorem 3.1. The absorbing rate (3.16) is a consequence of Lemma 3.3 as noticed in Remark 3.1. ■

In the remaining of this paper we deal with Lemmas 3.3 and 3.4. To this end we need the following Nakao's inequality.

Lemma 3.1 **([27], Lemma 2.1)** *Let $\phi(t)$ be a nonnegative continuous function defined on $[0,T)$, $1 < T \leq \infty$, which satisfies*

$$\sup_{t\leq s\leq t+1} \phi(s)^{1+\gamma} \leq A_0\left(\phi(t)-\phi(t+1)\right)+A_1, \quad 0\leq t\leq T-1, \tag{3.17}$$

where A_0, A_1, γ are positive constants. Then we have

$$\phi(t) \leq \left(A_0^{-1}\gamma(t-1)^+ + \left(\sup_{0\leq s\leq 1}\phi(s)\right)^{-\gamma}\right)^{-\frac{1}{\gamma}} + A_1^{\frac{1}{\gamma+1}}, \quad 0\leq t\leq T.$$

To prove the asymptotic smoothness property we follow an approach by Khanmamedov [24] and Chueshov and Lasiecka [10].

Lemma 3.2 **([10], Proposition 2.10)** *Assume that for any bounded positive invariant set $B \subset H$, and for any $\varepsilon > 0$, there exists $T = T(\varepsilon, B)$ such that*

$$d(S(T)x, S(T)y) \leq \varepsilon + \psi_T(x,y), \quad \forall\, x,y \in B, \tag{3.18}$$

where $\psi_T : H\times H \to \mathbb{R}$ satisfies, for any sequence $\{z_n\} \subset B$,

$$\liminf_{m\to\infty}\liminf_{n\to\infty} \psi_T(z_n, z_m) = 0. \tag{3.19}$$

Then $S(t)$ is asymptotically smooth.

3.1 Absorbing Set

Lemma 3.3 *Under the hypotheses of Theorem 3.2, $S(t)$ has an absorbing set $\mathcal{B} \subset \mathcal{H}$.*

Proof. Let us begin by fixing an arbitrary bounded set $B \subset \mathcal{H}$ and consider the solutions of problem (1.1) given by $(u(t), u_t(t)) = S(t)(u_0, u_1)$ with $(u_0, u_1) \in B$.

Our analysis is based on the modified energy functional

$$\begin{aligned}\widetilde{E}(t) \;&=\; \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\Delta u(t)\|_2^2 + \frac{1}{2}\widehat{M}\left(\|\nabla u(t)\|_2^2\right) + \int_\Omega \widehat{f}(u(t))dx \\ &\quad - \int_\Omega hu(t)dx + L_0|\Omega| + \frac{1}{\lambda_1}\|h\|_2^2, \end{aligned}\tag{3.20}$$

where $\lambda_1 > 0$ is the first eigenvalue of the bi-harmonic operator Δ^2 in $H_0^2(\Omega)$. Thus, $\widetilde{E}(t) = E(t) + L_0|\Omega| + \frac{1}{\lambda_1}\|h\|_2^2$ and $\widetilde{E}(t)$ dominates $\|(u(t),u_t(t))\|_{\mathcal{H}}^2$. Indeed, since λ_1 satisfies

$$\|u\|_2^2 \leq \lambda_1^{-1}\|\Delta u\|_2^2, \quad \forall\, u \in H_0^2(\Omega), \tag{3.21}$$

one has

$$\int_\Omega h(x)u(x,t)dx \leq \frac{1}{\lambda_1}\|h\|_2^2 + \frac{1}{4}\|\Delta u(t)\|_2^2,$$

and because $\widehat{f}(u) \geq -L_0$ (see 2.6), we get

$$\widetilde{E}(t) \geq \frac{1}{4}\left(\|\Delta u(t)\|_2^2 + \|u_t(t)\|_2^2\right). \tag{3.22}$$

To deal with this modified energy, we use density arguments as noticed in the Remark 2.2. That is, we may assume formally that u is regular.

By multiplying the first equation in (1.1) by u_t and integrating over Ω, one obtains

$$\frac{d}{dt}E(t) = -\int_\Omega g(u_t(t))u_t(t)dx.$$

Then integrating from t to $t+1$,

$$E(t) - E(t+1) = \int_t^{t+1}\int_\Omega g(u_t(s))u_t(s)dxds \geq 0,$$

in view of (2.8). So we can define an auxiliary functional $D(t)$ by setting

$$D(t)^2 = \widetilde{E}(t) - \widetilde{E}(t+1) = E(t) - E(t+1). \tag{3.23}$$

The two inequalities involving $D(t)$ will be useful. Firstly, noting that $\frac{r}{r+2} + \frac{2}{r+2} = 1$ and (2.8), we have

$$\int_t^{t+1} \|u_t(s)\|_2^2\, ds \leq |\Omega|^{\frac{r}{r+2}}\left(\int_t^{t+1}\int_\Omega |u_t|^{r+2}dxds\right)^{\frac{2}{r+2}} \leq \frac{1}{k_3}|\Omega|^{\frac{r}{r+2}}D(t)^{\frac{4}{r+2}}. \tag{3.24}$$

Then. by the Mean Value Theorem for integrals, there exist two numbers $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|u_t(t_i)\|_2^2 \leq \frac{4}{k_3}|\Omega|^{\frac{r}{r+2}}D(t)^{\frac{4}{r+2}}, \quad i = 1, 2. \tag{3.25}$$

In what follows, we compare $\widetilde{E}(t)^{1+\frac{r}{2}}$ with $D(t)^2$, in order to apply Lemma 3.1 with $\phi = \widetilde{E}$ and $\gamma = \frac{r}{2}$.

By multiplying the first equation in (1.1) by u and integrating over Ω, we see that

$$\begin{aligned}\|\Delta u(t)\|_2^2 \quad = \quad & -M(\|\nabla u(t)\|_2^2)\|\nabla u(t)\|_2^2 - \int_\Omega f(u(t))u(t)\, dx \\ & + \|u_t(t)\|_2^2 - \frac{d}{dt}\int_\Omega u_t(t)u(t)dx - \int_\Omega g(u_t(t))u(t)\, dx \\ & + \int_\Omega hu(t)dx.\end{aligned} \tag{3.26}$$

Then combining (3.26) with (3.20) we get

$$\begin{aligned}\widetilde{E}(t) &= \frac{1}{2}\left(\widehat{M}(\|\nabla u(t)\|_2^2) - M(\|\nabla u(t)\|_2^2)\|\nabla u(t)\|_2^2\right)\\ &\quad + \int_\Omega \left(\widehat{f}(u(t)) - \frac{1}{2}f(u(t))u(t)\right)dx + \|u_t(t)\|_2^2\\ &\quad - \frac{1}{2}\frac{d}{dt}\int_\Omega u_t(t)u(t)dx - \frac{1}{2}\int_\Omega g(u_t(t))u(t)dx \qquad (3.27)\\ &\quad + L_0|\Omega| + \frac{1}{\lambda_1}\|h\|_2^2 - \frac{1}{2}\int_\Omega hu(t)dx.\end{aligned}$$

Then using (2.4), (2.6) and integrating from t_1 to t_2,

$$\begin{aligned}\int_{t_1}^{t_2}\widetilde{E}(s)ds &\leq \int_{t_1}^{t_2}\|u_t(s)\|_2^2 ds - \frac{1}{2}\left(\int_\Omega u_t(t_2)u(t_2)dx - \int_\Omega u_t(t_1)u(t_1)dx\right)\\ &\quad - \frac{1}{2}\int_{t_1}^{t_2}\int_\Omega h(x)u(x,s)dxds + \frac{1}{2}\int_{t_1}^{t_2}\int_\Omega g(u_t(x,s))u(x,s)\,dxds\\ &\quad + \frac{1}{\lambda_1}\|h\|_2^2 + (L_0+L_1)|\Omega|. \qquad (3.28)\end{aligned}$$

Let us estimate the right hand side of (3.28) in terms of $D(t)$. We set $C_0 > 0$ as a generic constant not depending on the initial data.

Combining (3.21) with (3.25) and then noting that $\|\Delta u(s)\|_2^2 \leq 2\widetilde{E}(s)$, we have

$$\begin{aligned}\left|\int_\Omega u_t(t_2)u(t_2)dx - \int_\Omega u_t(t_1)u(t_1)dx\right| &\leq \frac{4}{\lambda_1^{1/2}}|\Omega|^{\frac{r}{2(r+2)}}D(t)^{\frac{2}{r+2}}\sup_{t\leq s\leq t+1}\|\Delta u(s)\|_2\\ &\leq C_0 D(t)^{\frac{4}{r+2}} + \delta\sup_{t\leq s\leq t+1}\widetilde{E}(s), \qquad (3.29)\end{aligned}$$

where $0<\delta<1/8$ is a fixed constant. Using (2.7) we have

$$\int_{t_1}^{t_2}\int_\Omega g(u_t)u\,dxds \leq k_2\int_{t_1}^{t_2}\int_\Omega |u_t||u|\,dxds + k_2\int_{t_1}^{t_2}\int_\Omega |u_t|^{r+1}|u|\,dxds.$$

From (3.24),

$$\begin{aligned}k_2\int_{t_1}^{t_2}\int_\Omega |u_t(s)||u(s)|\,dxds &\leq \frac{k_2^2}{2\delta\lambda_1}\int_t^{t+1}\int_\Omega |u_t(s)|^2dxds + \frac{\delta\lambda_1}{2}\int_t^{t+1}\int_\Omega |u(s)|^2dxds\\ &\leq C_0 D(t)^{\frac{4}{r+2}} + \delta\sup_{t\leq s\leq t+1}\widetilde{E}(s).\end{aligned}$$

Also, since $\frac{r+1}{r+2}+\frac{1}{r+2}=1$, from (3.24),

$$\begin{aligned}k_2\int_{t_1}^{t_2}\int_\Omega |u_t|^{r+1}|u|\,dxds &\leq \left(\int_t^{t+1}\int_\Omega |u_t|^{r+2}dxds\right)^{\frac{r+1}{r+2}}\left(\int_t^{t+1}\int_\Omega |u|^{r+2}dxds\right)^{\frac{1}{r+2}}\\ &\leq D(t)^{\frac{2(r+1)}{r+2}}\sup_{t\leq s\leq t+1}\|u(s)\|_{r+2}\end{aligned}$$

$$
\begin{aligned}
&\leq D(t)^{\frac{2(r+1)}{r+2}}\mu \sup_{t\leq s\leq t+1}\|\Delta u(s)\|_2,\\
&\leq C_0D(t)^{\frac{4(r+1)}{r+2}}+\delta \sup_{t\leq s\leq t+1}\widetilde{E}(s).
\end{aligned}
$$

where μ is the embedding constant of $H_0^2(\Omega)$ in $L^{r+2}(\Omega)$. Then we infer that

$$\int_{t_1}^{t_2}\int_\Omega g(u_t)u\,dxds \leq C_0(D(t)^{\frac{4}{r+2}}+D(t)^{\frac{4(r+1)}{r+2}})+2\delta \sup_{t\leq s\leq t+1}\widetilde{E}(s). \tag{3.30}$$

Finally, we note that

$$\int_{t_1}^{t_2}\int_\Omega h(x)u(x,s)\,dxds \leq \frac{1}{2\delta\lambda_1}\|h\|_2^2+\delta \sup_{t\leq s\leq t+1}\widetilde{E}(s). \tag{3.31}$$

Therefore, inserting (3.24), (3.29), (3.30) and (3.31) into (3.28), we obtain

$$\int_{t_1}^{t_2}\widetilde{E}(s)ds \leq C_0\left(D(t)^{\frac{4}{r+2}}+D(t)^{\frac{4(r+1)}{r+2}}\right)+C_1(\|h\|_2^2+L_0+L_1)+4\delta \sup_{t\leq s\leq t+1}\widetilde{E}(s), \tag{3.32}$$

where $C_1>0$ is a generic constant not depending on t and B. Using again the Mean Value Theorem, there exists $\tau\in[t_1,t_2]$ such that

$$\int_{t_1}^{t_2}\widetilde{E}(s)ds=\widetilde{E}(\tau)(t_2-t_1)\geq \frac{1}{2}\widetilde{E}(t+1),$$

because $\widetilde{E}(t)$ is decreasing as $E(t)$. Then, from $\widetilde{E}(t+1)=\widetilde{E}(t)-D(t)^2$, we conclude that

$$\widetilde{E}(t)\leq D(t)^2+2\int_{t_1}^{t_2}\widetilde{E}(s)ds. \tag{3.33}$$

Combining (3.32) and (3.33) we get

$$\widetilde{E}(t)\leq D(t)^2+C_0\left(D(t)^{\frac{4}{r+2}}+D(t)^{\frac{4(r+1)}{r+2}}\right)+C_1(\|h\|_2^2+L_0+L_1)+8\delta\widetilde{E}(t). \tag{3.34}$$

Since $\frac{4}{r+2}<2<\frac{4(r+1)}{r+2}$ and $0<8\delta<1$, inequality (3.34) can be written as

$$\widetilde{E}(t)\leq C_0D(t)^{\frac{4}{r+2}}\left(1+D(t)^{\frac{2r}{r+2}}+D(t)^{\frac{4r}{r+2}}\right)+C_1(\|h\|_2^2+L_0+L_1). \tag{3.35}$$

In addition, from the definition of $D(t)$ and estimate (2.13), we infer that $(1+D(t)^{\frac{24}{r+2}}+D(t)^{\frac{4r}{r+2}})$ is bounded by a constant $C_0'>0$, which depends on B. Thus, (3.35) can be rewritten as

$$\widetilde{E}(t)\leq C_0C_0'D(t)^{\frac{4}{r+2}}+C_1(\|h\|_2^2+L_0+L_1), \tag{3.36}$$

which implies that

$$\widetilde{E}(t)^{1+\frac{r}{2}}\leq C_B(\widetilde{E}(t)-\widetilde{E}(t+1))+(C_1(\|h\|_2^2+L_0+L_1))^{\frac{r+2}{2}}, \tag{3.37}$$

where $C_B > 0$ is now a constant depending on B. Hence we can finally apply Nakao's lemma, with $\phi = \widetilde{E}$ and $\gamma = \frac{r}{2}$, to conclude that

$$\widetilde{E}(t) \leq \left(C_B^{-1} \frac{r}{2} (t-1)^+ + \widetilde{E}(0)^{-\frac{r}{2}} \right)^{-\frac{2}{r}} + C_1(\|h\|_2^2 + L_0 + L_1). \tag{3.38}$$

Since the first term in the right hand side of (3.38) goes to zero as $t \to \infty$, there exists $t_B > 0$, depending on B, such that

$$\widetilde{E}(t) \leq C_1(\|h\|_2^2 + L_0 + L_1), \quad \forall t > t_B.$$

Hence, using (3.22) we conclude that

$$\mathcal{B} = \left\{ (u,v) \in \mathcal{H} \mid \|\Delta u\|_2^2 + \|v\|_2^2 \leq C_h(\|h\|_2^2 + L_0 + L_1) \right\} \tag{3.39}$$

is an absorbing set for $S(t)$, where $C_h > 4C_1$. ■

Remark 3.1 *The energy estimate* (3.38) *shows that, when* $h = 0$, *the energy* $E(t)$ *has the same polynomial decay rate* (2.14), *as obtained by Cavalcanti et al [9]. It combined with (3.39) also shows that the absorbing rate* (3.16) *holds.*

3.2 Asymptotic Smoothness

Lemma 3.4 *Under hypotheses of Theorem 3.2, the semigrupo $S(t)$ is asymptotically smooth in $\mathcal{H}$.*

Proof. We apply Lemmas 3.1 and 3.2. Let $B \subset \mathcal{H}$ be a bounded positive invariant set for $S(t)$. Given initial data $(u_0, u_1), (v_0, v_1) \in B$, let u, v be the corresponding solutions of (1.1). Then from Theorem 2.1, $w = u - v$ is a weak solution of

$$\begin{cases} w_{tt} + \Delta^2 w + (g(u_t) - g(v_t)) \\ \qquad = M(\|\nabla u\|_2^2)\Delta u - M(\|\nabla v\|_2^2)\Delta v - (f(u) - f(v)) \quad \text{in} \quad \Omega \times \mathbb{R}^+, \\ w = \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}^+, \\ w(\cdot, 0) = u_0 - v_0, \quad w_t(\cdot, 0) = u_1 - v_1 \quad \text{in} \quad \Omega. \end{cases} \tag{3.40}$$

Let us define the functional

$$E_w(t) = \|w_t(t)\|_2^2 + \|\Delta w(t)\|_2^2 + M(\|\nabla u(t)\|_2^2)\|\nabla w(t)\|_2^2.$$

We claim that, from Nakao's lemma, there exists positive constants C_B and C_T, with C_B not depending on T, such that

$$E_w(t) \leq C_B(1+t)^{-\frac{2}{r}} + C_T \left(\sup_{0 \leq \sigma \leq T} \int_\sigma^{\sigma+1} \|\nabla w(s)\|_2 ds \right)^{\frac{2}{r+2}}, \quad 0 \leq t \leq T. \tag{3.41}$$

Indeed, the proof is similar to the one of Lemma 3.3. In order to estimate $E_w(t)$, by density arguments, we may assume that $w = u - v$ is a regular solution of (3.40).

By multiplying the equation of (3.40) by w_t and integrating over Ω, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}E_w(t) &+ \int_\Omega (g(u_t(t)) - g(v_t(t)))w_t(t)dx \\
&= - M'(\|\nabla u(t)\|_2^2)\,\|\nabla w(t)\|_2^2 \int_\Omega \Delta u(t) u_t(t)dx \\
&+ (M(\|\nabla u(t)\|_2^2) - M(\|\nabla v(t)\|_2^2)) \int_\Omega \Delta v(t) w_t(t)dx \\
&- \int_\Omega (f(u(t)) - f(v(t)))\, w_t(t)dx.
\end{aligned} \tag{3.42}$$

To estimate the right hand side of (3.42), we set $C_2 > 0$ as a generic constant depending on B but not on t.

The continuity of M' and estimate (2.13) show that

$$M'(\|\nabla u(t)\|_2^2)\,\|\nabla w(t)\|_2^2 \int_\Omega u_t(t) - \Delta u(t)dx \le C_2\|\nabla w(t)\|_2^2. \tag{3.43}$$

Noting that $M(a^2) - M(b^2) \le M'(\sup\{a^2, b^2\})|a+b||a-b|$, we see that

$$\begin{aligned}
\big(M(\|\nabla u(t)\|_2^2) &- M(\|\nabla v(t)\|_2^2)\big) \int_\Omega \Delta v(t) w_t(t)dx \\
&\le C_2\|\nabla w(t)\|_2\|w_t(t)\|_2 \\
&\le C_2\|\nabla w(t)\|_2^{\frac{r+2}{r+1}} + \frac{k_3}{4}\|w_t(t)\|_{r+2}^{r+2}.
\end{aligned} \tag{3.44}$$

Applying Hölder inequality with $\frac{\rho}{2(\rho+1)} + \frac{1}{2(\rho+1)} + \frac{1}{2} = 1$, we infer that

$$\begin{aligned}
\int_\Omega (f(u(t)) - f(v(t)))w_t(t)dx &\le k_1 \int_\Omega (1 + |u|^\rho + |v|^\rho)|w(t)||w_t(t)|dx \\
&\le k_1 \left(\int_\Omega (1 + |u|^\rho + |v|^\rho)^{\frac{2(\rho+1)}{\rho}} dx\right)^{\frac{\rho}{2(\rho+1)}} \|w(t)\|_{2(\rho+1)}\|w_t(t)\|_2 \\
&\le C_2\|\nabla w(t)\|_2\|w_t(t)\|_{r+2} \\
&\le C_2\|\nabla w(t)\|_2^{\frac{r+2}{r+1}} + \frac{k_3}{4}\|w_t(t)\|_{r+2}^{r+2}.
\end{aligned} \tag{3.45}$$

On the other hand, from (2.8),

$$\int_\Omega (g(u_t(t)) - g(v_t(t)))\,(u_t(t) - v_t(t))dx \ge k_3\|w_t(t)\|_{r+2}^{r+2}. \tag{3.46}$$

Therefore, by inserting (3.43)-(3.46) into (3.42) we obtain

$$\frac{1}{2}\frac{d}{dt}E_w(t) + \frac{k_3}{2}\|w_t(t)\|_{r+2}^{r+2} \le C_2\|\nabla w(t)\|_2\left(\|\nabla w(t)\|_2 + \|\nabla w(t)\|_2^{\frac{1}{r+1}}\right), \ \forall\, t \ge 0. \tag{3.47}$$

Note that, since $w = u - v$, estimate (2.13) also implies that $\|(w(t), w_t(t))\|_{\mathcal{H}} \leq C_2$, $t \geq 0$. So, in (3.47), the last term in the brackets is also bounded. We can fix a constant $C_F > 0$ such that

$$\frac{d}{dt}E_w(t) + \|w_t(t)\|_{r+2}^{r+2} \leq C_F\|\nabla w(t)\|_2 \quad \forall t \geq 0. \tag{3.48}$$

Then, integrating (3.48) from t to $t+1$, we obtain

$$\int_t^{t+1} \|w_t(s)\|_{r+2}^{r+2} ds \leq E_w(t) - E_w(t+1) + C_F \int_t^{t+1} \|\nabla w(s)\|_2 \, ds \equiv F(t)^2. \tag{3.49}$$

The functional $F(t)$ will play a similar role to $D(t)$ defined in (3.23).

As in (3.24), (3.25) and (3.29), we infer that

$$\int_t^{t+1} \|w_t(s)\|_2^2 ds \leq C_2 F(t)^{\frac{4}{r+2}}, \tag{3.50}$$

and the existence of $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|w_t(t_i)\|_2^2 \leq C_2 F(t)^{\frac{4}{r+2}}, \quad i = 1, 2,$$

which implies that

$$\left| \int_\Omega w_t(t_2)w(t_2)dx - \int_\Omega w_t(t_1)w(t_1)dx \right| \leq C_2 F(t)^{\frac{4}{r+2}} + \frac{1}{8} \sup_{t\leq\sigma\leq t+1} E_w(\sigma). \tag{3.51}$$

Now, multiplying the equation in (3.40) by w and integrating over Ω, we obtain

$$\begin{aligned} &\|\Delta w(t)\|_2^2 + M(\|\nabla u(t)\|_2^2)\|\nabla w(t)\|_2^2 \\ &\quad = -\frac{d}{dt}\int_\Omega w_t(t)w(t)dx + \|w_t(t)\|_2^2 \\ &\qquad + (M(\|\nabla u(t)\|_2^2) - M(\|\nabla v(t)\|_2^2)) \int_\Omega \Delta v(t)w(t)dx \\ &\qquad - \int_\Omega (f(u(t)) - f(v(t))w(t)dx - \int_\Omega (g(u_t(t)) - g(v_t(t))w(t)dx. \end{aligned} \tag{3.52}$$

Integrating (3.52) from t_1 to t_2 and then using (3.50) and (3.51), we conclude that

$$\begin{aligned} &\int_{t_1}^{t_2} (\|\Delta w(s)\|_2^2 + M(\|\nabla u(s)\|_2^2)\|\nabla w(s)\|_2^2) \, ds \\ &\quad \leq C_2 F(t)^{\frac{4}{r+2}} + \frac{1}{8} \sup_{t\leq\sigma\leq t+1} E_w(\sigma) \\ &\qquad + \int_{t_1}^{t_2} (M(\|\nabla u(s)\|_2^2) - M(\|\nabla v(s)\|_2^2)) \int_\Omega \Delta v(s)w(s)dxds \\ &\qquad - \int_{t_1}^{t_2}\int_\Omega (f(u) - f(v))wdxds - \int_{t_1}^{t_2}\int_\Omega (g(u_t) - g(v_t))wdxds. \end{aligned} \tag{3.53}$$

A similar procedure to (3.45) yields

$$\begin{aligned}\int_\Omega (g(u_t(t)) - g(v_t(t))w(t)dx &\leq C_2\|w_t(t)\|_2\|w(t)\|_{2(r+1)} \\ &\leq C_2\|w_t(t)\|_{r+2}\|\Delta w(t)\|_2.\end{aligned}$$

Combining this with (3.50) we get

$$\int_{t_1}^{t_2}\int_\Omega (g(u_t(s)) - g(v_t(s))w(s)dxds \leq C_2F(t)^{\frac{4}{r+2}} + \frac{1}{8}\sup_{t\leq\sigma\leq t+1} E_w(\sigma). \tag{3.54}$$

Also, the estimate (2.13) implies that

$$\int_{t_1}^{t_2}\int_\Omega (f(u) - f(v))wdxds \leq C_2\int_{t_1}^{t_2} \|\nabla w(s)\|_2\, ds \tag{3.55}$$

and

$$\int_{t_1}^{t_2} (M(\|\nabla u(s)\|_2^2) - M(\|\nabla v(s)\|_2^2)) \int_\Omega \Delta v(s)w(s)\, dxds \leq C_2\int_{t_1}^{t_2} \|\nabla w(s)\|_2\, ds. \tag{3.56}$$

By inserting (3.54), (3.55) and (3.56) into (3.53) we obtain

$$\begin{aligned}&\int_{t_1}^{t_2} \left(\|\Delta w(s)\|_2^2 + M(\|\nabla u(s)\|_2^2)\|\nabla w(s)\|_2^2 \right) ds \\ &\qquad\leq C_2F(t)^{\frac{4}{r+2}} + \frac{1}{4}\sup_{t\leq\sigma\leq t+1} E_w(\sigma) + C_2\int_t^{t+1} \|\nabla w(s)\|_2\, ds.\end{aligned} \tag{3.57}$$

Then, from the definition of $E_w(t)$, (3.50) and (3.57) we have

$$\int_{t_1}^{t_2} E_w(s)ds \leq C_2F(t)^{\frac{4}{r+2}} + \frac{1}{4}\sup_{t\leq\sigma\leq t+1} E_w(\sigma) + C_2\int_t^{t+1} \|\nabla w(s)\|_2 ds.$$

Now, by Mean Value Theorem, there exists $t^* \in [t_1, t_2]$ such that

$$E_w(t^*) \leq C_2F(t)^{\frac{4}{r+2}} + \frac{1}{2}\sup_{t\leq\sigma\leq t+1} E_w(\sigma) + C_2\int_t^{t+1} \|\nabla w(s)\|_2^2 ds. \tag{3.58}$$

We claim that

$$\sup_{t\leq\sigma\leq t+1} E_w(\sigma) \leq E_w(t^*) + F(t)^2 + 2C_F\int_t^{t+1} \|\nabla w(s)\|_2 ds. \tag{3.59}$$

Indeed, we first note that from (3.49),

$$E_w(t) \leq E_w(t+1) + F(t)^2.$$

Next, let $\tau \in [t, t+1]$ such that $E_w(\tau) = \sup_{t\le\sigma\le t+1} E_w(\sigma)$. Then integrating (3.48) over $[t,\tau]$ and over $[t^*, t+1]$, we have

$$
\begin{aligned}
E_w(\tau) &\le E_w(t) + C_F \int_t^{t+1} \|\nabla w(s)\|_2 \, ds \\
&\le E_w(t+1) + F(t)^2 + C_F \int_t^{t+1} \|\nabla w(s)\|_2 \, ds \\
&\le E_w(t^*) + F(t)^2 + 2C_F \int_t^{t+1} \|\nabla w(s)\|_2 \, ds,
\end{aligned}
$$

which shows that (3.59) holds.

Inserting (3.58) into (3.59) we obtain

$$
\begin{aligned}
\sup_{t\le\sigma\le t+1} E_w(\sigma) &\le C_2 F(t)^{\frac{2r}{r+2}} \left(1 + F(t)^{2-\frac{4}{r+2}}\right) \\
&\quad + C_2 \int_t^{t+1} \|\nabla w(s)\|_2 \left(1 + \|\nabla w(s)\|_2\right) ds \\
&\le C_2 F(t)^{\frac{4}{r+2}} + C_2 \int_t^{t+1} \|\nabla w(s)\|_2 ds,
\end{aligned}
$$

since the terms in the brackets are bounded. Therefore we get

$$
\begin{aligned}
\sup_{t\le\sigma\le t+1} E_w(\sigma)^{\frac{r+2}{2}} &\le C_2 F(t)^2 + C_2 \left(\sup_{0\le\sigma\le T} \int_\sigma^{\sigma+1} \|\nabla w(s)\|_2 \, ds \right)^{\frac{r+2}{2}} \\
&\le C_2 F(t)^2 + C_3 \sup_{0\le\sigma\le T} \int_\sigma^{\sigma+1} \|\nabla w(s)\|_2 \, ds, \qquad (3.60)
\end{aligned}
$$

where

$$
C_3 = C_2 \left(\sup_{0\le\sigma\le T} \int_\sigma^{\sigma+1} \|\nabla w(s)\|_2 \, ds \right)^{\frac{r}{2}}
$$

is a constant which depends on T. Therefore, combining (3.49) with (3.60) we have

$$
\sup_{t\le\sigma\le t+1} E_w(\sigma)^{1+\frac{r}{2}} \le C_2(E_w(t) - E_w(t+1)) + (C_F + C_3) \sup_{0\le\sigma\le T} \int_\sigma^{\sigma+1} \|\nabla w(s)\|_2 \, ds.
$$

By applying Nakao's lemma we get estimate (3.41).

Now we apply Lemma 3.2. From definition of $E_w(t)$, there exist constants $C'_B, C'_T > 0$ such that

$$
\|(w(t), w_t(t))\|_{\mathcal{H}} \le C'_B(1+t)^{-\frac{1}{r}} + C'_T \sup_{0\le\sigma\le T} \left(\int_\sigma^{\sigma+1} \|\nabla w(s)\|_2 \, ds \right)^{\frac{1}{r+2}}, \quad 0 \le t \le T. \quad (3.61)
$$

Given $\varepsilon > 0$, we fix a sufficiently large T so that,

$$
C'_B(1+T)^{-\frac{1}{r}} < \varepsilon. \qquad (3.62)
$$

Then we define $\psi_T : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ by

$$\psi_T((u_0,u_1),(v_0,v_1)) = C'_T \sup_{0\le\sigma\le T} \left(\int_\sigma^{\sigma+1} \|\nabla u(s) - \nabla v(s)\|_2 \, ds \right)^{\frac{1}{r+2}}.$$

So, from (3.61) and (3.62) we have

$$\|S(T)(u_0,u_1) - S(T)(v_0,v_1)\|_{\mathcal{H}} \le \varepsilon + \psi_T((u_0,u_1),(v_0,v_1)), \tag{3.63}$$

for all $(u_0,u_1),(v_0,v_1) \in B$, and so (3.18) holds.

Let us verify that (3.19) also holds. In fact, given a sequence $\{(u_0^n,u_1^n)\} \subset B$, since B is bounded and positive invariant, the corresponding solutions $(u^n(t),u_t^n(t))$ of problem (1.1) are uniformly bounded in $\mathcal{H}$. Hence

$$\{u^n\} \quad \text{is bounded in} \quad C([0,\infty),H_0^2(\Omega)) \cap C^1([0,\infty),L^2(\Omega)).$$

Then, since $H_0^2(\Omega) \hookrightarrow H_0^1(\Omega)$ compactly, there exists a subsequence $\{u^{n_k}\}$ which converges strongly in $C([0,T+1],H_0^1(\Omega))$. (Simon [31]). Therefore

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{l\to\infty} \psi_T((u_0^{n_k},u_1^{n_k}),(u_0^{n_l},u_1^{n_l})) \\
&\qquad = \lim_{k\to\infty}\lim_{l\to\infty} C'_T \sup_{0\le\sigma\le T} \left(\int_\sigma^{\sigma+1} \|\nabla u^{n_k}(s) - \nabla u^{n_l}(s)\|_2 \, ds \right)^{\frac{1}{r+2}} = 0,
\end{aligned}$$

and consequently (3.19) holds. Then the asymptotic smoothness of $S(t)$ follows from Lemma 3.2. ■

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