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### STABILITY OF GRADIENT SEMIGROUPS UNDER PERTURBATIONS

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# STABILITY OF GRADIENT SEMIGROUPS UNDER PERTURBATIONS

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ABSTRACT. This paper is concerned with the problem of establishing conditions under which gradient semigroups in a general metric space are stable under perturbation. This is done by proving that gradient-like semigroups in the sense defined in [2] are gradient semigroups (possess a Lyapunov function). The results are inspired by the works of [2] and Morse decomposition of invariant sets in [3, 11].

## 1. INTRODUCTION

The analysis of qualitative properties of semigroups in general phase spaces (infinite-dimensional Banach spaces or general metric spaces) has received very much attention throughout the last four decades (see, for instance, [4], [7], [12] or [1]). In particular, the study of compact attracting invariant sets has developed a long and profound area of research, providing crucial information for an increasing number of models for phenomena from different areas of Science as Physics, Biology, Economics, Engineering and others.

When a system is shown to possess a global attractor, all its asymptotic behaviour can be described by a detailed analysis of the internal dynamics on this compact invariant set. To this aim, a careful study of the geometrical structure -and its stability under perturbations- of the global attractor arises as a crucial fact. Probably the most general result in this line is what is now known as the *Fundamental Theorem of Dynamical Systems*, suggested in [8] from the results of [3], which describes any flow on a compact metric space as a decomposition of chain recurrent isolated invariant sets and connections between them. In the terminology of [3], this is called a Morse decomposition of a compact invariant set (see Definition 2.10), and has been considered in different frameworks, as in the case of flows ([3]) and semiflows on compact spaces ([11]), or even compact and non-compact topological spaces ([6, 9, 10]).

On the other hand, very recently, it has been introduced in [2] the so-called *gradient-like semigroups* (see Definition 2.8) in Banach spaces as an intermediate concept between

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gradient semigroups (i.e., those possessing a Lyapunov function) and semigroup possessing a gradient-like attractor; that is, an attractor that is characterized as the union of unstable sets of associated isolated invariant sets.

In this paper, given a gradient-like nonlinear semigroup in a general metric space, we construct a differentiable Lyapunov function for it proving that gradient-like nonlinear semigroups are in fact gradient semigroups. Our result avoids any hypotheses on compactness of the associated group or semigroup and any additional hypotheses on the phase space in which it is defined. Our proofs, though inspired in classical works as [3, 11], are different -and so new- from the results there, as we adopt essentially a dynamical approach from the results on gradient-like systems.

For the construction of the Lyapunov function we will firstly prove that the disjoint family of isolated invariant sets of a gradient-like semigroup on a general metric space can be reordered in such a way that it becomes a Morse decomposition for the global attractor. Again, the proofs are intuitive, focused on the dynamics of the semigroup and, for instance, not based on chain recurrence and related more classical notions in this theory. A refinement of the results from [3] would lead us to define a generalized Lyapunov function, not only on the attractor but on the whole phase space. Indeed, we will say that a semigroup $\{T(t) : t \geq 0\}$ with a global attractor $\mathcal{A}$ and a disjoint family of isolated invariant sets $\Xi = \{\Xi_1, \cdots, \Xi_n\}$ is a *generalized gradient semigroup* with respect to $\Xi$ if there exists a continuous function $V : X \to \mathbb{R}$ such that $[0, \infty) \ni t \mapsto V(T(t)x) \in \mathbb{R}$ is decreasing for each $x \in X$ and $V(T(t)x) = V(x)$ for all $t \geq 0$ if and only if $x \in \bigcup_{i=1}^{n} \Xi_i$.

Our main result can now be stated as follows

**Theorem 1.1.** *$\{T(t) : t \geq 0\}$ is a generalized gradient semigroup with respect to $\Xi$ if and only if it is a generalized gradient-like semigroup with respect to $\Xi$. In addition the Lyapunov function $V : X \to \mathbb{R}$ of a generalized gradient-like semigroup may be chosen in such a way that $V(\Xi_j) = j - 1$.*

One of the straightforward consequences of the previous result is that, if the disjoint family of isolated invariant sets $\mathcal{E} = \{\xi_1, \cdots, \xi_n\}$ is made of stationary points, then a nonlinear semigroup is gradient in the sense of [4] (see Definition 3.8.1) if and only if it is gradient-like (see Definition 1.3 in [2]). Then, as gradient-like nonlinear semigroups are stable under perturbation (see [2]), we conclude that gradient semigroups are stable under perturbation; that is, the existence of a continuous Lyapunov function is robust under perturbation.

Observe that any Morse decomposition $\Xi = (\Xi_1, \cdots, \Xi_n)$ of a compact invariant set $\mathcal{A}$ leads to a partial order among the isolated invariant sets $\Xi_i$; that is, we can define an order between two isolated invariant sets $\Xi_i$ and $\Xi_j$ if there is a chain of global solutions $\{\xi_\ell,\ 1 \leq \ell \leq j-i\}$, with $\lim_{t\to\infty} \xi_\ell(t) = \Xi_{i+\ell-1}$ and $\lim_{t\to-\infty} \xi_\ell(t) = \Xi_{i+\ell}$, $1 \leq \ell \leq j-i$. This defines a partial order and some of the isolated invariant sets in $\Xi$ may not be comparable. In

Section 5 we introduce a new Morse decomposition of the attractor of a generalized gradient-like semigroup which improves the construction and dynamical properties of its associated Lyapunov function. Indeed, we prove that, given any generalized gradient-like semigroup with respect to the disjoint family of isolated invariant sets $\Xi = (\Xi_1, \cdots, \Xi_n)$, there exists another Morse decomposition given by the so-called energy levels $\mathbf{N} = (\mathcal{N}_1, \mathcal{N}_2, \cdots, \mathcal{N}_p)$ which guarantees a total order in $\mathcal{A}$. Each of the levels $\mathcal{N}_i$, $1 \leq i \leq p$ is made of a finite union of the isolated invariant sets in $\Xi$ and $\mathbf{N}$ is totally ordered. The associated Lyapunov function has different values in any two different sets of $\mathbf{N}$ and any two elements of $\Xi$ which are contained in the same element of $\mathbf{N}$ (same energy level) are not connected.

In a general metric space $X$, consider a generalized gradient-like semigroup $\{T(t) : t \geq 0\}$ with respect to the isolated invariant sets $\Xi = (\Xi_1, \cdots, \Xi_n)$ and with a global attractor $\mathcal{A}$. In Section 2 we prove that $\Xi$ can be reordered in such a way that it becomes a Morse decomposition for the $\mathcal{A}$. In Section 3 we construct a Lyapunov function in $X$ for $\{T(t) : t \geq 0\}$ showing that a generalized-gradient like semigroup is a generalized gradient semigroup. Section 4 is dedicated to a recollection of the results in [2] to conclude that the generalized gradient semigroups are stable under perturbation. In Section 5 we introduce the level grouping isolated invariant sets with same dynamical characteristics. Finally in Section 6 we make some final comments and present some of our future research on the subject.

## 2. Morse decomposition of global attractors for generalized gradient-like semigroups

In this section we present the notions of generalized gradient-like semigroups and of Morse decomposition for a global attractor as well as the relationship between them. In order to do that we will need to introduce some basic notions and results (see [4] for example).

Let $X$ be a metric space with metric $d : X \times X \to \mathbb{R}^+$, where $\mathbb{R}^+ = [0, \infty)$. Given a subset $A \subset X$, the $\epsilon$-neighborhood of $A$ is the set $\mathcal{O}_\epsilon(A) = \{x \in X : d(x, a) < \epsilon \text{ for some } a \in A\}$

Next we introduce the notion of semigroups in the metric space $X$

**Definition 2.1.** *A family of mappings $\{T(t) : t \geq 0\}$ is a semigroup in $X$ if*

- *$T(0) = I_X$, with $I_X$ being the identity map in $X$,*
- *$T(t+s) = T(t)T(s)$, for all $t, s \in \mathbb{R}^+$ and*
- *$\mathbb{R}^+ \times X \ni (t, x) \mapsto T(t)x \in X$ is continuous.*

The notion of invariance plays a fundamental role in the study of the asymptotic behavior of semigroups

**Definition 2.2.** *A subset $A$ of $X$ is said invariant under the action semigroup $\{T(t) : t \geq 0\}$ if $T(t)A = A$ for all $t \geq 0$.*

Now we will introduce the notions of attraction and absorption. For that we recall the definition of Hausdorff semi-distance. Given $A, B \subset X$, the Hausdorff semidistance between $A$ and $B$ is given by

$$d_H(A,B) = \sup_{a\in A}\inf_{b\in B} d(a,b).$$

**Definition 2.3.** *Given two subsets $A, B$ of $X$ we say that $A$ attracts $B$ under the action of the semigroup $\{T(t) : t \geq 0\}$ if $d_H(T(t)B, A) \overset{t\to\infty}{\longrightarrow} 0$ and we say that $A$ absorbs $B$ under the action of $\{T(t) : t \geq 0\}$ if there is a $t_B > 0$ such that $T(t)B \subset A$ for all $t \geq t_B$.*

With this we are in condition to define *global attractors*

**Definition 2.4.** *A subset $\mathcal{A}$ of $X$ is a* global attractor *for a semigroup $\{T(t) : t \geq 0\}$ if it is compact, invariant under the action of $\{T(t) : t \geq 0\}$ and for every bounded subset $B$ of $X$ we have that $\mathcal{A}$ attracts $B$ under the action of $\{T(t) : t \geq 0\}$.*

Next we seek to introduce the notion of generalized gradient-like semigroups (see [2]). To that end we first need the definition of isolated invariant set

**Definition 2.5.** *Let $\{T(t) : t \geq 0\}$ be a semigroup. We say that an invariant set $\Xi \subset X$ for the semigroup $\{T(t) : t \geq 0\}$ is an isolated invariant set if there is an $\epsilon > 0$ such that $\Xi$ is the maximal invariant subset of $\mathcal{O}_\epsilon(\Xi)$.*

*A disjoint family of isolated invariant sets is a family $\{\Xi_1, \cdots, \Xi_n\}$ of isolated invariant sets with the property that, for some $\epsilon > 0$,*

$$\mathcal{O}_\epsilon(\Xi_i) \cap \mathcal{O}_\epsilon(\Xi_j) = \varnothing, \ 1 \leq i < j \leq n.$$

**Definition 2.6.** *A global solution for a semigroup $\{T(t) : t \geq 0\}$ is a continuous function $\xi : \mathbb{R} \to X$ with the property that $T(t)\xi(s) = \xi(t+s)$ for all $s \in \mathbb{R}$ and for all $t \in \mathbb{R}^+$. We say that $\xi : \mathbb{R} \to X$ is a global solution through $x \in X$ if it is a global solution and $\xi(0) = x$.*

**Definition 2.7.** *Let $\{T(t) : t \geq 0\}$ be a semigroup which has a disjoint family of isolated invariant sets $\Xi = \{\Xi_1, \cdots, \Xi_n\}$. A homoclinic structure associated to $\Xi$ is a subset $\{\Xi_{k_1}, \cdots, \Xi_{k_p}\}$ of $\Xi$ $(p \leq n)$ together with a set of global solutions $\{\phi_1, \cdots, \phi_p\}$ such that*

$$\Xi_{k_j} \overset{t\to-\infty}{\longleftarrow} \phi_j(t) \overset{t\to\infty}{\longrightarrow} \Xi_{k_{j+1}}, \ 1 \leq j \leq p$$

*where $\Xi_{k_{p+1}} := \Xi_{k_1}$.*

We are now ready to define generalized gradient-like semigroups

**Definition 2.8.** *Let $\{T(t) : t \geq 0\}$ be a semigroup with a global attractor $\mathcal{A}$ and a disjoint family of isolated invariant sets $\Xi = \{\Xi_1, \cdots, \Xi_n\}$. We say that $\{T(t) : t \geq 0\}$ is a generalized gradient-like semigroup relative to $\Xi$ if*

- *For any global solution $\xi : \mathbb{R} \to \mathcal{A}$ there are $1 \leq i, j \leq n$ such that*

$$\Xi_i \overset{t\to-\infty}{\longleftarrow} \xi(t) \overset{t\to\infty}{\longrightarrow} \Xi_j.$$

- *There is no homoclinic structure associated to* $\Xi$.

Now we will introduce the notion of a Morse decomposition for an attractor $\mathcal{A}$ of a gradient-like semigroup $\{T(t) : t \geq 0\}$. We start with the notion of attractor-repellers pairs.

**Definition 2.9.** *Let* $\{T(t) : t \geq 0\}$ *be a semigroup with a global attractor* $\mathcal{A}$. *We say that a non-empty subset* $\Xi$ *of* $\mathcal{A}$ *is a local attractor if there is an* $\epsilon > 0$ *such that* $\omega(\mathcal{O}_\epsilon(\Xi)) = \Xi$. *The repeller* $\Xi^*$ *associated to a local attractor* $\Xi$ *is the set defined by*

$$\Xi^* = \{x \in \mathcal{A} : \omega(x) \cap \Xi = \varnothing\}.$$

*The pair* $(\Xi, \Xi^*)$ *is called attractor-repeller pair for* $\{T(t) : t \geq 0\}$.

Note that if $\Xi$ is a local attractor, then $\Xi^*$ is closed and invariant.

**Definition 2.10.** *Given an increasing family* $\varnothing = A_0 \subset A_1 \subset \cdots \subset A_n = \mathcal{A}$, *of* $n+1$ *local attractors, define* $\Xi_j := A_j \cap A_{j-1}^*$. *The ordered n-upla* $\mathbf{\Xi} := (\Xi_1, \Xi_2, \cdots, \Xi_n)$ *is called a Morse decomposition on* $\mathcal{A}$.

Observe that $\Xi$ is a local attractor if and only if it is compact invariant and attracts $\mathcal{O}_\epsilon(\Xi)$ for some $\epsilon > 0$. We observe that the above definition differs slightly from the usual definition since the local attractor is required to attract a neighborhood of $\Xi$ in $X$ and not in $\mathcal{A}$ as in [3, 11]. We will prove next that both definitions coincide.

**Lemma 2.11.** *Let* $\{T(t) : t \geq 0\}$ *be a semigroup in* $X$ *with a global attractor* $\mathcal{A}$. *If* $\Xi$ *is a compact invariant set for* $\{T(t) : t \geq 0\}$ *and there is an* $\epsilon > 0$ *such that* $\Xi$ *attracts* $\mathcal{O}_\epsilon(\Xi) \cap \mathcal{A}$ *then, given* $\delta > 0$ *there is a* $\delta' > 0$ *such that* $\gamma^+(\mathcal{O}_{\delta'}(\Xi)) \subset \mathcal{O}_\delta(\Xi)$, *where* $\gamma^+(\mathcal{O}_{\delta'}(\Xi)) = \bigcup_{x \in \mathcal{O}_{\delta'}(\Xi)} \bigcup_{t \geq 0} T(t)x$.

*Proof.* Given $0 < \delta < \epsilon$ if there is no $\delta' > 0$ such that $\gamma^+(\mathcal{O}_{\delta'}(\Xi)) \subset \mathcal{O}_\delta(\Xi)$, there are $x \in \Xi$, $X \ni x_n \stackrel{n\to\infty}{\longrightarrow} x$ and $\mathbb{R} \ni t_n \stackrel{n\to\infty}{\longrightarrow} \infty$ such that $d(T(t_n)x_n, \Xi) = \delta$ and $T(t)x_n \in \mathcal{O}_\delta(\Xi)$, $t \in [0, t_n]$. Since the $\{T(t) : t \geq 0\}$ has a global attractor it is not difficult to see that there is a global solution $\xi : \mathbb{R} \to X$ such that $\xi_n : [-t_n, \infty) \to X$ given by $\xi_n(t) = T(t_n + t)x_n$ satisfies $\xi_n(t) \stackrel{n\to\infty}{\longrightarrow} \xi(t)$ for each $t \in \mathbb{R}$. Clearly $\xi(t) \in \overline{\mathcal{O}_\delta(\Xi)} \bigcap \mathcal{A} \subset \mathcal{O}_\epsilon(\Xi) \bigcap \mathcal{A}$ for all $t \leq 0$, and $d(\xi(0), \Xi) = \delta$, and consequently $\Xi$ cannot attract $\mathcal{O}_\epsilon(\Xi)$. □

**Lemma 2.12.** *If* $\{T(t) : t \geq 0\}$ *is a semigroup in* $X$ *with a global attractor* $\mathcal{A}$ *and* $S(t) := T(t)|_{\mathcal{A}}$, *clearly* $\{S(t) : t \geq 0\}$ *is a semigroup in the metric space* $\mathcal{A}$. *If* $\Xi$ *is a local attractor for* $\{S(t) : t \geq 0\}$ *in the metric space* $\mathcal{A}$ *and* $K$ *is a compact subset of* $\mathcal{A}$ *such that* $K \cap \Xi^* = \varnothing$, *then* $\Xi$ *attracts* $K$. *Furthermore* $\Xi$ *is a local attractor for* $\{T(t) : t \geq 0\}$ *in* $X$.

*Proof.* Let $K$ be a compact subset of $\mathcal{A}$ such that $K \cap \Xi^* = \varnothing$. If $\Xi$ does not attract $K$, there is a $\delta > 0$, $t_n \stackrel{n\to\infty}{\longrightarrow} \infty$, $x \in K$ and $K \ni x_n \stackrel{n\to\infty}{\longrightarrow} x$ such that $d(T(t_n)x_n, \Xi) \geq \delta$. From Lemma 2.11 there exists $0 < \delta' < \delta$ such that $d(T(t)x_n, \Xi) \geq \delta'$ for all $t \in [0, t_n]$. This implies that

$d(T(t)x, \Xi) \geq \delta'$ for all $t \geq 0$ and, consequently, $\omega(x) \cap \Xi = \varnothing$. That is a contradiction and proves the first part of the result.

For the remaining part of the result note that, from Lemma 2.11, there exists $\delta > 0$ such that $\omega(\mathcal{O}_{\delta'}(\Xi)) \cap \Xi^* = \varnothing$ for all $\delta' < \delta$. From the invariance of $\omega(\mathcal{O}_{\delta'}(\Xi))$ and the property that $\Xi$ attracts compact subsets of $\mathcal{A}$ that do not intersect $\Xi^*$, we must have that $\omega(\mathcal{O}_{\delta'}(\Xi)) \subset \Xi$. Since $\omega(\mathcal{O}_{\delta'}(\Xi))$ attracts $\mathcal{O}_{\delta'}(\Xi)$ we have the result. □

**Lemma 2.13.** *Let $\{T(t) : t \geq 0\}$ be a semigroup in $X$ with a global attractor $\mathcal{A}$ and an attractor-repeller $(\Xi, \Xi^*)$. A global solution $\xi : \mathbb{R} \to X$ of $\{T(t) : t \geq 0\}$ with the property that $\xi(t) \in \mathcal{O}_\delta(\Xi^*)$ for all $t \leq 0$ for some $\delta > 0$ such that $\mathcal{O}_\delta(\Xi^*) \cap \Xi = \varnothing$ must satisfy $d(\xi(t), \Xi^*) \overset{t\to-\infty}{\longrightarrow} 0$.*

*Proof.* If the conclusion is false there exists a $\delta' > 0$ and a sequence $t_n \overset{n\to\infty}{\longrightarrow} \infty$ such that $d(\xi(-t_n), \Xi^*) \geq \delta'$. This leads to a contradiction with the fact that $\Xi$ must attract $K = \{z \in \mathcal{A} : d(z, \Xi^*) \geq \delta'\}$. □

**Lemma 2.14.** *Let $\{T(t) : t \geq 0\}$ be a semigroup in $X$ with a global attractor $\mathcal{A}$ and $(\Xi, \Xi^*)$ an attractor-repeller for $\{T(t) : t \geq 0\}$. If $\xi : \mathbb{R} \to X$ is a global bounded solution for $\{T(t) : t \geq 0\}$ through $x \notin \Xi \cup \Xi^*$, then $\xi(t) \overset{t\to\infty}{\longrightarrow} \Xi$ and $\xi(t) \overset{t\to-\infty}{\longrightarrow} \Xi^*$. Furthermore, if $x \in X \backslash \mathcal{A}$ then, $T(t)x \overset{t\to\infty}{\longrightarrow} \Xi \cup \Xi^*$.*

*Proof.* Since $x \notin \Xi^*$ we have that $\omega(x) \cap \Xi$ is non-empty and from the fact that $\Xi$ is a local attractor we have that $\xi(t) \overset{t\to\infty}{\longrightarrow} \Xi$.

If $\xi(t)$ does not converges to $\Xi^*$ as $t \to -\infty$ we divide the proof of the result in two cases: If $\overline{\xi(\mathbb{R})} \cap \Xi^* = \varnothing$, then $\xi(\mathbb{R})$ is invariant, contains a point which is not in $\Xi^*$ and is attracted by $\Xi$ from Lemma 2.12 which is a contradiction. On the other hand, if $\overline{\xi(\mathbb{R})} \cap \Xi^*$ is non-empty, there is a $\delta > 0$ and sequences $t_n, \tau_n \overset{n\to\infty}{\longrightarrow} \infty$ such that $d(\xi(-t_n, \Xi^*) = \delta$ and $\xi(-t_n + \tau_n) \overset{n\to\infty}{\longrightarrow} z \in \Xi^*$ and $d(\xi(-t_n + t), \Xi) \leq \delta$ for al $0 \leq t \leq \tau_n$. From this we obtain a global solution $\zeta : \mathbb{R} \to \mathcal{A}$ with the property that $d(\zeta(t), \Xi^*) \leq \delta$ for all $t \geq 0$. In this case we have that $\omega(\zeta(0)) \notin \Xi$ which implies that $\zeta(0) \in \Xi^*$ leading to a contradiction.

For $x \in X \backslash \mathcal{A}$ let us prove that $T(t)x \overset{t\to\infty}{\longrightarrow} \Xi \cup \Xi^*$. If $\overline{\gamma^+(x)} \cap \Xi \neq \varnothing$ we have that $T(t)x \overset{t\to\infty}{\longrightarrow} \Xi$. Otherwise, there exists $\delta > 0$ with $\gamma^+(z) \cap \mathcal{O}_\delta(\Xi) = \varnothing$, and in this case we claim that $T(t)x \overset{t\to\infty}{\longrightarrow} \Xi^*$. If the claim is false, there are $\epsilon > 0$ and a sequence $t_n \overset{n\to\infty}{\longrightarrow} \infty$ such that $d(T(t_n)x, \Xi^*) \geq \epsilon$. Considering the sequence of functions $\xi_n : [-t_n, \infty) \to X$ defined by $\xi_n(t) = T(t + t_n)x$, $t \geq -t_n$, we construct a global solution $\xi : \mathbb{R} \to \mathcal{A}$ such that $d(\xi(0), \Xi^*) \geq \epsilon$ and $d(\xi(t), \Xi) \geq \delta$ for all $t \in \mathbb{R}$. Hence $\omega(\xi(0)) \cap \Xi^* = \varnothing$ and $\xi(0) \notin \Xi^*$ which is a contradiction. □

**Corollary 2.15.** *If $\{T(t) : t \geq 0\}$ is a semigroup in $X$ with a global attractor $\mathcal{A}$ and $(\Xi, \Xi^*)$ is an attractor-repeller pair for $\{T(t) : t \geq 0\}$, then $\{T(t) : t \geq 0\}$ is a generalized gradient-like semigroup associated to the disjoint family of isolated invariant sets $\{\Xi, \Xi^*\}$.*

Next we describe the construction of a Morse decomposition of the attractor of a gradient-like semigroup relative to the disjoint family of isolated invariant sets $\{\Xi_1,\cdots,\Xi_n\}$ and of the associated collection of increasing local attractors starting from the collection of isolated invariant sets $\{\Xi_1,\cdots,\Xi_n\}$. The following result plays a fundamental role on that

**Lemma 2.16.** *Let $\{T(t):t\geq 0\}$ be a generalized gradient-like semigroup with associated family of isolated invariant sets $\Xi=\{\Xi_1,\cdots,\Xi_n\}$. Then, there is a $1\leq k\leq n$ such that $\Xi_k$ is a local attractor for $\{T(t):t\geq 0\}$.*

*Proof.* It is sufficient to show that there is a $1\leq k\leq n$ such $\Xi_k$ is a local attractor for $\{T(t)\big|_{\mathcal{A}}:t\geq 0\}$. If there is no local attractor in $\Xi$ we have that for each $1\leq i\leq n$ there is a global solution $\xi_i:\mathbb{R}\to\mathcal{A}$ such that $\xi_i(t)\overset{t\to-\infty}{\longrightarrow}\Xi_i$. Since $\xi_i(t)$ must converge to one element of $\Xi$, this produces a homoclinic structure and gives us a contradiction. $\square$

Let $\{T(t):t\geq 0\}$ be a generalized gradient-like semigroup with associated family of isolated invariant sets $\Xi=\{\Xi_1,\cdots,\Xi_n\}$. If (after possible reordering) $\Xi_1$ is a local attractor for $\{T(t):t\geq 0\}$ and

$$\Xi_1^*=\{a\in\mathcal{A}:\omega(a)\cap\Xi_1=\varnothing\}$$

each $\Xi_i$, $i>1$ is contained in $\Xi_1^*$ and that for any $a\notin\mathcal{A}\backslash\{\Xi_1\cup\Xi_1^*\}$ and global solution $\phi:\mathbb{R}\to\mathcal{A}$ with $\phi(0)=a$ we have that

$$\Xi_1^*\overset{t\to-\infty}{\longleftarrow}\phi_j(t)\overset{t\to\infty}{\longrightarrow}\Xi_1.$$

Considering the restriction $T_1(t)$ of $T(t)$ to $\Xi_1^*$ we have that $T_1(t)$ is a generalized gradient-like semigroup in $\Xi_1^*$ with isolated invariant sets $\{\Xi_2,\cdots,\Xi_n\}$ and we may assume without loss of generality that $\Xi_2$ is a local attractor for the semigroup $\{T_1(t):t\geq 0\}$ in $\Xi_1^*$. If $\Xi_{2,1}^*$ is the repeller associated to the isolated invariant set $\Xi_2$ for $\{T_1(t):t\geq 0\}$ in $\Xi_1^*$ we may proceed and consider the restriction $\{T_2(t):t\geq 0\}$ of the semigroup $\{T_1(t):t\geq 0\}$ to $\Xi_{2,1}^*$ and $\{T_2(t):t\geq 0\}$ is a generalized gradient-like semigroup in $\Xi_{2,1}^*$ with associated isolated invariant sets $\{\Xi_3,\cdots,\Xi_n\}$.

Proceeding with this until all isolated invariant sets are exhausted we obtain a reordering of $\{\Xi_1,\cdots,\Xi_n\}$ in such a way that $\Xi_j$ is a local attractor for the restriction of $\{T(t):t\geq 0\}$ to $\Xi_{j-1,j-2}^*$ ($\Xi_{0,-1}^*=\mathcal{A}$).

With the construction above, if a global solution $\xi:\mathbb{R}\to\mathcal{A}$ satisfies

$$\Xi_\ell\overset{t\to-\infty}{\longleftarrow}\xi(t)\overset{t\to\infty}{\longrightarrow}\Xi_k \tag{2.1}$$

then $\ell\geq k$. To see that we first observe that if $(\Xi,\Xi^*)$ is an attractor-repeller pair, any global solution $\zeta:\mathbb{R}\to X$ with $\zeta(0)\in\Xi^*$ satisfies $\zeta(t)\in\Xi^*$ for all $t\in\mathbb{R}$. This is immediate from the fact that $\omega(\zeta(t))=\omega(\zeta(0))$ for all $t\in\mathbb{R}$. Now for $k=1$ the result is trivial. For $k=2$ we necessarily have that $\xi(0)\in\Xi_1^*$. This implies that $\xi(t)\in\Xi_1^*$ for all $t\in\mathbb{R}$. Since $\Xi_j\subset\Xi_1^*$ for all $j\geq 2$, we necessarily have that $\ell\geq 2=k$.

Before we proceed to the general case, let us prove that, if $3 \leq \ell \leq k$, (2.1) holds and $\xi(0) \in \Xi^*_{\ell-2,\ell-3}$ then $\xi(0) \in \Xi^*_{\ell-1,\ell-2}$. The case $\ell = 3$ is obvious. If $\ell > 3$ and $\xi(0) \notin \Xi^*_{\ell-1,\ell-2}$, since $\xi(0) \in \Xi^*_{\ell-2,\ell-3}$ and $\Xi_{\ell-1}$ is an attractor in $\Xi^*_{\ell-2,\ell-3}$, the fact that $\xi(0) \notin \Xi^*_{\ell-1,\ell-2}$ implies $\omega(\xi(0)) \subset \Xi_{\ell-1}$, forcing that $\Xi_{\ell-1} = \Xi_k$; that is, $\ell - 1 = k$ contradicting $\ell \leq k$.

From this implication we have that $k \leq 3$, $3 \leq \ell \leq k$, (2.1) holds and $\xi(0) \in \Xi^*_{\ell-2,\ell-3}$ implies $\xi(0) \in \Xi^*_{k-1,k-2}$ and since the elements of $\Xi = \{\Xi_1, \cdots, \Xi_n\}$ in $\Xi^*_{k-1,k-2}$ are those $\Xi_i$'s with $i \geq k$, we have that $\ell \geq k$ proving the results.

We will prove that this reordering of $\{\Xi_1, \cdots, \Xi_n\}$ (which we denote the same) is a Morse decomposition to $\mathcal{A}$ for a suitably chosen sequence $A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_n$ of local attractors. To that end we need the definition of unstable set of an invariant set which is given next.

**Definition 2.17.** *Let $\{T(t) : t \geq 0\}$ be a semigroup. The* unstable set *of an invariant set $\Xi$ is defined by*

$$W^{\mathrm{u}}(\Xi) = \{\, z \in X : \text{ there is a global solution } \xi : \mathbb{R} \to X \\ \text{such that } \xi(0) = z \text{ and } \lim_{t \to -\infty} \operatorname{dist}(\xi(t), \Xi) = 0\}.$$

Define $A_0 = \varnothing$, $A_1 = \Xi_1$ and for $j = 2, 3, \cdots, n$

$$A_j = A_{j-1} \cup W^{\mathrm{u}}(\Xi_j). \tag{2.2}$$

It is clear that $A_n = \mathcal{A}$.

**Theorem 2.18.** *Let $\{T(t) : t \geq 0\}$ be a generalized gradient-like semigroup with associated family of isolated invariant sets $\Xi = \{\Xi_1, \cdots, \Xi_n\}$ reordered in such a way that $\Xi_j$ is an attractor for the restriction of $\{T(t) : t \geq 0\}$ to $\Xi^*_{j-1,j-2}$. Then $A_j$ defined in* (2.2) *is a local attractor for $\{T(t) : t \geq 0\}$ in $X$, and*

$$\Xi_j = A_j \cap A^*_{j-1}.$$

*As a consequence, $\Xi$ defines a Morse decomposition on $\mathcal{A}$.*

*Proof.* From Lemma 2.12, it is sufficient to prove that $A_j = A_{j-1} \cup W^{\mathrm{u}}(\Xi_j)$ is a local attractor for $\{T(t) : t \geq 0\}$ restricted to $\mathcal{A}$.

Choose $d > 0$ such that $\mathcal{O}_d(\cup_{i=1}^{j} \Xi_i) \cap (\cup_{i=j+1}^{n} \Xi_i) = \varnothing$. If there are $\delta < d$ and $\delta' < \delta$ such that $\gamma^+(\mathcal{O}_{\delta'}(A_j)) \subset \mathcal{O}_\delta(A_j)$, then $\omega(\mathcal{O}_{\delta'}(A_j))$ attracts $\mathcal{O}_{\delta'}(A_j)$ and (as $\omega(\mathcal{O}_{\delta'}(A_j))$ is invariant) is contained in $A_j$ proving that $A_j$ is a local attractor. If that is not the case, there is a sequence $\{x_k\}$ in $\mathcal{A} \backslash A_j$ with $d(x_k, A_j) \overset{k\to\infty}{\longrightarrow} 0$, for each $x_k$ a global solution $\xi_k : \mathbb{R} \to \mathcal{A}$ through $x_k$ which converges to $\bigcup_{i=j+1}^{n} \Xi_i$ and a sequence $t_k \overset{k\to\infty}{\longrightarrow} \infty$, such that $d(\xi(t), A_j) \leq \delta$ for all $t \in [0, t_k]$ and $d(\xi(t_k), A_j) = \delta$. In this way, we construct a global solution $\xi : \mathbb{R} \to \mathcal{A}$ such that $d(\xi(t), A_j) \leq \delta$ for all $t \leq 0$. This is a contradiction with the fact that $\xi(0) \notin A_j$.

To prove that $\Xi_j = A_j \cap A^*_{j-1}$ note that

$$A_j = \bigcup_{i=1}^{j} W^{\mathrm{u}}(\Xi_i)$$

and $A^*_{j-1} = \{z \in \mathcal{A} : \omega(z) \cap A_{j-1} = \varnothing\}$. Hence, given $z \in A_j \cap A^*_{j-1}$ we have that the global solution $\xi : \mathbb{R} \to \mathcal{A}$ through $z$ must satisfy that

$$\cup_{i=1}^{j} \Xi_i \overset{t\to-\infty}{\longleftarrow} \xi(t) \overset{t\to\infty}{\longrightarrow} \cup_{i=j}^{n} \Xi_i.$$

As a consequence of that and of the fact that $\{T(t) : t \geq 0\}$ is a generalized gradient-like semigroup with isolated invariant sets $\{\Xi_1, \cdots, \Xi_n\}$ for which any global solution $\xi : \mathbb{R} \to \mathcal{A}$ satisfies $\Xi_\ell \overset{t\to-\infty}{\longleftarrow} \xi(t) \overset{t\to-\infty}{\longrightarrow} \Xi_k$ with $\ell \geq k$, we obtain that $z \in \Xi_j$. This shows that $A_j \cap A^*_{j-1} \subset \Xi_j$. The other inclusion is immediate from the definition of $A_j$ and $A^*_{j-1}$. $\square$

**Proposition 2.19.** *Let $\{T(t) : t \geq 0\}$ be a generalized gradient-like semigroup with associated family of isolated invariant sets $\Xi = \{\Xi_1, \cdots, \Xi_n\}$ reordered in such a way that it is a Morse decomposition. Then,*

$$\bigcap_{j=0}^{n} (A_j \cup A_j^*) = \bigcup_{j=1}^{n} \Xi_j.$$

*Proof.* If $z \in \bigcup_{j=1}^{n} \Xi_j$, let $k \in \{1, 2, \cdots, n\}$ be such that $z \in \Xi_k = A_k \cap A^*_{k-1}$. Hence $z \in A_k \subset A_{k+1} \subset \cdots \subset A_n$ and $z \in A^*_{k-1} \subset A^*_{k-2} \subset \cdots \subset A^*_0$. Thus

$$z \in (\bigcap_{j=k}^{n} A_j) \cap (\bigcap_{j=1}^{k-1} A_j^*) \subset \left[\bigcap_{j=k}^{n} (A_j \cup A_j^*)\right] \cap \left[\bigcap_{j=0}^{k-1} (A_j \cup A_j^*)\right] = \bigcap_{j=0}^{n} (A_j \cup A_j^*),$$

proving the inclusion $\bigcup_{j=1}^{n} \Xi_j \subset \bigcap_{j=0}^{n} A_j \cup A_j^*$.

Now, let $z \in \bigcap_{j=0}^{n} (A_j \cup A_j^*)$ and $I := \{i_1, i_2, \cdots, i_k\}$ $J := \{j_1, j_2, \cdots, j_l\}$ such that $I \cup J = \{0, 1, \cdots, n\}$ with $I \cap J = \varnothing$ and $z \in A_i$ for all $i \in I$ and $z \in A_j^*$ for all $j \in J$. Clearly, if $i := \min I$, necessarily $I = \{i, i+1, i+2, \cdots, n\}$ and $J = \{0, 1, \cdots, i-1\}$, consequently $z \in A_i$ and $z \in A^*_{i-1}$. So, $z \in A_i \cap A^*_{i-1} = \Xi_i$, from which we conclude that $\bigcap_{j=0}^{n} (A_j \cup A_j^*) \subset \bigcup_{j=1}^{n} \Xi_j$ and the proof is completed. $\square$

## 3. A Lyapunov function for a generalized gradient-like semigroup

Inspired in the work of [3] (see also [11]) we will prove in this section the equivalence between generalized gradient systems and generalized gradient-like nonlinear semigroups. The generalized gradient-like semigroups have been defined in Definition 2.8 and a generalized gradient semigroup is defined as follows

Before we proceed let us fix that, if $I, J$ are subsets of $\mathbb{R}$, a function $w : I \to J$ is said decreasing (increasing) if $w(s) \leq w(t)$ ($w(s) \geq w(t)$) whenever $s \geq t$. If in addition, $w(s) < w(t)$ ($w(s) > w(t)$) whenever $s < t$ we will say that $w$ is strictly decreasing (increasing).

**Definition 3.1.** *We say that a semigroup $\{T(t) : t \geq 0\}$ with a global attractor $\mathcal{A}$ and a disjoint family of isolated invariant sets $\Xi = \{\Xi_1, \cdots, \Xi_n\}$ is a generalized gradient semigroup with respect to $\Xi$ if there is a continuous function $V : X \to \mathbb{R}$ such that $[0, \infty) \ni t \mapsto V(T(t)x) \in \mathbb{R}$ is decreasing for each $x \in X$ and $V(T(t)x) = V(x)$ for all $t \geq 0$ if and only if $x \in \bigcup_{i=1}^n \Xi_i$. A function $V$ with the properties above is called a Lyapunov function for the generalized gradient semigroup $\{T(t) : t \geq 0\}$ with respect to $\Xi$.*

The following lemmas will play a crucial role in the proof of such equivalence:

**Lemma 3.2.** *If $\{T(t) : t \geq 0\}$ is a semigroup with global attractor $\mathcal{A}$, the map $h : X \to \mathbb{R}$ defined by*

$$h(z) := \sup_{t \geq 0} d(T(t)z, \mathcal{A}), z \in X,$$

*is well defined, continuous, decreasing along solutions of $\{T(t) : t \geq 0\}$ and $h^{-1}(0) = \mathcal{A}$.*

*Proof.* Indeed, by Lemma 2.12, given $\varepsilon > 0$ let $0 < \varepsilon' < \varepsilon$ such that $\gamma^+(\mathcal{O}_{\varepsilon'}(\mathcal{A})) \subset \mathcal{O}_\varepsilon(\mathcal{A})$, showing the continuity of $h$ on $\mathcal{A}$. Let $z_0 \in X \backslash \mathcal{A}$ be given, so that $h(z_0) > 0$. Consider $\mathcal{O}_\mu(\mathcal{A})$ for some $0 < \mu < h(z_0)$. From the continuity of the function $X \ni x \mapsto d(x, \mathcal{A}) \in [0, \infty)$ let $V$ be a bounded neighborhood of $z_0$ such that $d(z, \mathcal{A}) > \mu$ if $z \in V$. Finally, let $\tau > 0$ such that $\gamma^+(T(\tau)V) \subset \mathcal{O}_\mu(\mathcal{A})$ so that it follows the continuity of $h$ en $z_0$, as for $z \in V$ it holds that $h(z) = \sup_{0 \leq s \leq \tau} d(T(s)z, \mathcal{A})$ and, from the continuity properties of the semigroup $\{T(t) : t \geq 0\}$, it follows that $h|_V : V \to \mathbb{R}$ is continuous.

To see that $h$ is non-increasing along solutions note that, if $z \in X$ and $t_1 > 0$, then

$$h(T(t_1)z) = \sup_{t \geq 0} d(T(t)T(t_1)z, \mathcal{A}) = \sup_{t \geq 0} d(T(t + t_1)z, \mathcal{A}) =$$

$$\sup_{t \geq t_1} d(T(t)z, \mathcal{A}) \leq \sup_{t \geq 0} d(T(t)z, \mathcal{A}) = h(z).$$

□

**Proposition 3.3.** *Let $\{T(t) : t \geq 0\}$ be a nonlinear semigroup in a metric space $(X, d)$ with global attractor $\mathcal{A}$, and let $(\Xi, \Xi^*)$ an attractor-repeller pair in $\mathcal{A}$. Then, there exists a function $f : X \to \mathbb{R}$ satisfying the following:*

*(i)* $f : X \to \mathbb{R}$ *is continuous in* $X$.
*(ii)* $f : X \to \mathbb{R}$ *is non-increasing along solutions.*
*(iii)* $f^{-1}(0) = \Xi$ *and* $f^{-1}(1) \cap \mathcal{A} = \Xi^*$.
*(iv) Given* $z \in X$, *if* $f(T(t)z) = f(z)$ *for all* $t \geq 0$, *then* $z \in (\Xi \cup \Xi^*)$.

*Proof.* Firstly, observe that $\Xi$ and $\Xi^*$ are disjoint closed subsets of $\mathcal{A}$ and, since $\mathcal{A}$ is a compact subset of $X$, $\Xi$ and $\Xi^*$ are disjoint closed subsets of $X$. With the convention that

$d(z, \varnothing) = 1$ for each $z \in X$, define the function (the canonical Urysohn function if $\Xi$ and $\Xi^*$ are non-empty) $l : X \to [0,1]$ associated to $(\Xi, \Xi^*)$ by

$$l(z) := \frac{d(z,\Xi)}{d(z,\Xi) + d(z,\Xi^*)}, \; z \in X.$$

Clearly $l$ is well defined, uniformly continuous in $X$ (since, for $d_0 := d(\Xi, \Xi^*) > 0$, it holds that $|l(z) - l(w)| \leq \frac{2}{d_0}|d(z,\Xi) - d(w,\Xi)| \leq \frac{2}{d_0} d(z,w)$, for any $z$ y $w$ in $X$). Moreover, $l^{-1}(0) = \Xi$ and $l^{-1}(1) = \Xi^*$.

If we define $k : X \to \mathbb{R}$ by

$$k(z) := \sup_{t \geq 0} \; l(T(t)z),$$

we now show that $k : X \to \mathbb{R}$ is continuous and decreasing along solutions of $\{T(t) : t \geq 0\}$, $k(X) \subset [0,1]$ (with equality when $X$ is connected and $\Xi$ and $\Xi^*$ are non-empty), $k^{-1}(0) = \Xi$ and $k^{-1}(1) \cap \mathcal{A} = \Xi^*$.

The fact that $k(X) \subset [0,1]$ follows because $l(T(t)z) \in [0,1]$ for all $z \in X$ and $t \geq 0$. In the case in which $X$ is connected, and $\Xi$ and $\Xi^*$ are non-empty, due to the fact that $\mathcal{A}$ is connected we have that $k(X) = [0,1]$.

To prove that $[0,\infty) \ni t \mapsto k(T(t)z) \in [0,1]$ is decreasing for each $z \in X$ note that, if $0 \leq t_1 \leq t_2$ we have

$$\begin{aligned} k(T(t_1)z) &= \sup_{t \geq 0} \; l(T(t)T(t_1)z) = \sup_{t \geq 0} \; l(T(t+t_1)z) = \sup_{t \geq t_1} \; l(T(t)z) \\ &\geq \sup_{t \geq t_2} \; l(T(t)z) = \sup_{t \geq 0} \; l(T(t+t_2)z) = k(T(t_2)z). \end{aligned}$$

It is clear from the definition of $k$ and from the invariance of $\Xi$ and $\Xi^*$ that $k(\Xi) = \{0\}$ and $k(\Xi^*) = \{1\}$. Now, if $z \in X$ is such that $k(z) = 0$, then $l(T(t)z) = 0$ for all $t \geq 0$. In particular, $0 = l(T(0)z) = l(z)$, and so, $z \in \Xi$, that is, $k^{-1}(0) \subset \Xi$ which shows that $k^{-1}(0) = \Xi$. On the other hand, if $z \in \mathcal{A}$ is such that $k(z) = 1$ and $z \notin \Xi^*$, then $\omega(z) \subset \Xi$. From the continuity of $l$ and the fact that $\omega(z)$ attracts $z$, we obtain that $\lim_{t \to \infty} l(T(t)z) = 0$. So, there exists a $t_0 > 0$ such that $1 = k(z) = \sup_{0 \leq t \leq t_0} l(T(t)z)$. This implies the existence of a $t' \in [0, t_0]$ such that $l(T(t')z) = 1$; that is, $T(t')z \in \Xi^*$. Consequently $\omega(z) = \omega(T(t')z) \subset \Xi^*$, which contradicts the fact that $\omega(z) \subset \Xi$ and so, if $k(z) = 1$ for some $z \in \mathcal{A}$ we must have that $z \in \Xi^*$. From this we conclude that $k^{-1}(1) \cap \mathcal{A} \subset \Xi^*$ and so $k^{-1}(1) \cap \mathcal{A} = \Xi^*$.

We now prove that, if $z \in \mathcal{A}$ and $k(T(t)z) = k(z)$ for all $t \geq 0$ then $z \in \Xi \cup \Xi^*$. If $z \notin \Xi \cup \Xi^*$, $\omega(z) \subset \Xi$ (note that $z \in \mathcal{A}$) and from the definition of $k$ and the fact that $\omega(z)$ attracts $z$ we have that $k(z) = \lim_{t \to \infty} k(T(t)z) = 0$. Since $k^{-1}(0) = \Xi$, $z$ must belong to $\Xi$ which is a contradiction.

Next we prove the continuity of $k : X \to \mathbb{R}$. We split the proof into three cases:

*Case* 1) Continuity of $k : X \to \mathbb{R}$ in $\Xi^*$.

Since $l(z) \leq k(z) \leq 1$, for all $z \in X$, given $z_0 \in \Xi^*$ and $z \in X$ we have that

$$|k(z) - k(z_0)| = 1 - k(z) \leq 1 - l(z).$$

This and the continuity of $l : X \to \mathbb{R}$ in $z_0$ imply the continuity of $k : X \to \mathbb{R}$ in $z_0$.

*Case* 2) Continuity of $k : X \to \mathbb{R}$ in $\Xi$.

From the continuity of $l : X \to \mathbb{R}$ in $\Xi$, given $\varepsilon > 0$, there is a $\delta > 0$ such that $l(\mathcal{O}_\delta(\Xi)) \subset [0, \varepsilon)$. Lemma 2.11 implies that there exists $\delta' \in (0, \delta)$ such that $\gamma^+(\mathcal{O}_{\delta'}(\Xi)) \subset \mathcal{O}_\delta(\Xi)$, from which we conclude that $k(\mathcal{O}_{\delta'}(\Xi)) \subset [0, \varepsilon]$.

*Case* 3) Continuity of $k : X \to \mathbb{R}$ in $X \backslash (\Xi \cup \Xi^*)$.

Given $z_0 \in X \backslash (\Xi \cup \Xi^*)$, from Lemma 2.14 we have that, either $\lim_{t\to\infty} d(T(t)z_0, \Xi) = 0$, or $\lim_{t\to\infty} d(T(t)z_0, \Xi^*) = 0$.

If $\lim_{t\to\infty} d(T(t)z_0, \Xi^*) = 0$ let us prove the continuity of $k$ in $z_0$. First note that $k(z_0) = 1$. Now, given $\varepsilon > 0$, from the continuity of $l : X \to \mathbb{R}$ in $\Xi^*$ there is an open neighborhood $V$ of $\Xi^*$ in $X$ such that $l(V) \subset (1-\varepsilon, 1]$. If $t_0 > 0$ is such that $T(t_0)z_0 \in V$, from the continuity of $T(t_0) : X \to X$, let $U$ be a neighborhood of $z_0$ such that $T(t_0)U \subset V$, from which it follows that $k(z) > 1 - \varepsilon$ for all $z \in U$ (for $T(t_0)z \in V$ and then $1 - \varepsilon < l(T(t_0)z) \leq k(z)$). This proves the continuity of $k$ in points $z_0$ of $X \backslash (\Xi \cup \Xi^*)$ for which $\lim_{t\to\infty} d(T(t)z_0, \Xi^*) = 0$.

If $z_0 \in X \backslash (\Xi \cup \Xi^*)$ and $\lim_{t\to\infty} d(T(t)z_0, \Xi) = 0$, it holds that $l(z_0) > 0$. Choose $\delta > 0$ such that $l(\mathcal{O}_\delta(\Xi)) \subset [0, \frac{l(z_0)}{2})$ and, from Lemma 2.11, there is a $\delta' \in (0, \delta)$ such that $\gamma^+(\mathcal{O}_{\delta'}(\Xi)) \subset \mathcal{O}_\delta(\Xi)$. From this, there is a $t_0 > 0$ with the property that $T(t)z_0 \in \mathcal{O}_\delta(\Xi)$ for all $t \geq t_0$. From the continuity of $T(t_0) : X \to X$, there is a neighborhood $U_1$ of $z_0$ in $X$ such that $T(t_0)U_1 \subset \mathcal{O}_{\delta'}(\Xi)$. Then, for all $z \in U_1$ we have that $T(t_0)z \in \mathcal{O}_{\delta'}(\Xi)$ so that $T(t)z \in \mathcal{O}_\delta(\Xi)$ for all $t \geq t_0$. Finally, from the continuity of $l$, let $U_2$ be a neighborhood of $z_0$ in $X$ such that $l(z) > \frac{l(z_0)}{2}$ for all $z \in U_2$ and write $U := U_1 \cap U_2$, so that for all $z \in U$ it holds that $k(z) = \sup_{0\leq t\leq t_0} l(T(t)z)$. Reasoning as before we obtain the continuity of $k$ in points $z_0$ of $X \backslash (\Xi \cup \Xi^*)$ for which $\lim_{t\to\infty} d(T(t)z_0, \Xi) = 0$.

Let $h : X \to \mathbb{R}$ be the function defined in Lemma 3.2, that is, $h(z) = \sup_{t\geq 0} d(T(t)z, \mathcal{A})$, $z \in X$, and define $f : X \to \mathbb{R}$ by

$$f(z) := k(z) + h(z), \ z \in X.$$

The continuity of $f : X \to \mathbb{R}$ follows from the continuity of $k$ (proved above) and $h$ (proved in Lemma 3.2). Since $k$ and $h$ are decreasing along solutions of $\{T(t) : t \geq 0\}$ (see above and Lemma 3.2), $f$ also possesses this property.

Clearly $f(\Xi) = \{0\}$. If $f(z) = 0$ for some $z \in X$, then $h(z) = k(z) = 0$ and we must have that $z \in \Xi$. This shows that $f^{-1}(0) = \Xi$.

Also, since $f \mid_{\mathcal{A}} = k \mid_{\mathcal{A}}$ we have that $f^{-1}(1) \cap \mathcal{A} = k^{-1}(1) \cap \mathcal{A} = \Xi^*$.

Finally, $(iv)$ is proved in the following manner. Let $z \in X$ such that $f(T(t)z) = f(z)$ for all $t \geq 0$. If $z \in \mathcal{A}$, we have that $k(T(t)z) = k(z)$ and, consequently, $z \in \Xi \cup \Xi^*$ completing the proof. If $z \in X \backslash \mathcal{A}$, let us show that $\lim_{t \to \infty} d(T(t)z, \Xi^*) = 0$. If that is not the case, then $\lim_{t \to \infty} d(T(t)z, \Xi) = 0$, and

$$f(z) = \lim_{t \to \infty} f(T(t)z) = \lim_{t \to \infty} k(T(t)z) + \lim_{t \to \infty} h(T(t)z) = 0 + 0 = 0. \tag{3.1}$$

This implies that $z$ is in $\Xi \subset \mathcal{A}$ and that contradicts the fact that $z \in X \backslash \mathcal{A}$. This proves that $\lim_{t \to \infty} d(T(t)z, \Xi^*) = 0$.

Using the same reasoning as in (3.1) we conclude that

$$f(z) = \lim_{t \to \infty} f(T(t)z) = \lim_{t \to \infty} k(T(t)z) + \lim_{t \to \infty} h(T(t)z) = 1 + 0 = 1.$$

This is a contradiction, since $k(z) \geq k(T(t)z)$ for all $t \geq 0$, and $1 = \lim_{t \to \infty} k(T(t)) \leq k(z) \leq 1$, that is, $k(z) = 1$, but $f(z) = k(z) + h(z)$ and we must have that $h(z) = 0$ which is a contradiction with the fact that $z \notin \mathcal{A}$. Thus, $f(T(t)z) = f(z)$ for all $t \geq 0$ if and only if $z \in \Xi \cup \Xi^*$ and this completes the proof. $\square$

**Theorem 3.4.** *Let $\{T(t) : t \geq 0\}$ be a semigroup with global attractor $\mathcal{A}$ and a disjoint family of isolated invariant sets $\Xi = \{\Xi_1, \cdots, \Xi_n\}$. Then, $\{T(t) : t \geq 0\}$ is a generalized gradient semigroup with respect to $\Xi$ if and only if it is a generalized gradient-like semigroup with respect to $\Xi$. In addition, the Lyapunov function $V : X \to \mathbb{R}$ of a generalized gradient-like semigroup may be chosen in such a way that $V(\Xi_k) = k - 1$, $k = 1, \cdots, n$.*

*Proof.* It is clear that a generalized gradient semigroup with respect to $\Xi$ is a generalized gradient-like semigroup with respect to $\Xi$.

Suppose that $\{T(t) : t \geq 0\}$ is a generalized gradient-like semigroup with respect to $\Xi$ reordered in such a way that it is a Morse decomposition for $\mathcal{A}$. Let $\varnothing = A_0 \subset A_1 \subset \cdots \subset A_n = \mathcal{A}$ be the sequence of local attractors defined in (2.2) and $\varnothing = A_n^* \subset A_{n-1}^* \subset \cdots \subset A_0^* = \mathcal{A}$ their corresponding repellers such that for each $j = 1, 2, \cdots, n$, we have $\Xi_j = A_j \cap A_{j-1}^*$.

Let $h : X \to \mathbb{R}$ be the function defined in Lemma 3.2 and $k_j : X \to \mathbb{R}$ be the function constructed in Proposition 3.3 for the attractor-repeller pair $A_j, A_j^*$, $j = 1, \cdots, n$.

Define the continuous function $V : X \to \mathbb{R}$ by

$$V(z) := h(z) + \sum_{j=1}^{n} k_j(z), \ z \in X.$$

Then $V : X \to \mathbb{R}$ is a Lyapunov function for the generalized gradient semigroup $\{T(t) : t \geq 0\}$ with respect to $\Xi$.

Indeed, since $h : X \to \mathbb{R}$ and each $k_j : X \to \mathbb{R}$, $1 \leq j \leq n$, are decreasing along solutions of $\{T(t) : t \geq 0\}$, $V$ is also decreasing along solutions of $\{T(t) : t \geq 0\}$.

Now, if $z \in X$ is such that $V(T(t)z) = V(z)$ for all $t \geq 0$, then, using that $h : X \to \mathbb{R}$ and each $k_j$, $1 \leq j \leq n$, are decreasing along solutions of $\{T(t) : t \geq 0\}$, we conclude that $f_j(T(t)z) = k_j(T(t)z) + h(T(t)z) = k_j(z) + h(z) = f_j(z)$ for all $t \geq 0$, and for each $j = 1, \cdots, n$. From part *(iv)* of Proposition 3.3, we have that $z \in (A_j \cup A_j^*)$, for each $j = 0, 1, \cdots, n$; that is, $z \in \bigcap_{j=0}^n (A_j \cup A_j^*)$. From Lemma 2.19 we have that

$$\bigcap_{j=0}^{n} (A_j \cup A_j^*) = \bigcup_{j=1}^{n} \Xi_j,$$

and so $z \in \bigcup_{j=1}^n \Xi_j$.

Let $\varnothing = A_0 \subset A_1 \subset \cdots \subset A_n = \mathcal{A}$ be the chain of attractors defined in (2.2) and $\varnothing = A_n^* \subset A_{n-1}^* \subset \cdots \subset A_0^* = \mathcal{A}$ their repellers. Then, for each $j = 1, 2, \cdots, n$, $\Xi_j = A_j \cap A_{j-1}^*$. For the function $V : X \to \mathbb{R}$ constructed above we have that, for each $j = 0, 1, \cdots, n$, $k_j : \mathcal{A} \to [0,1]$, $k_j^{-1}(0) = A_j$ and $k_j^{-1}(1) \cap \mathcal{A} = A_j^*$.

If $k \in \{1, \cdots, n\}$ and $z \in \Xi_k = A_k \cap A_{k-1}^*$, it follows that $z \in A_k \subset A_{k+1} \subset \cdots \subset A_n = \mathcal{A}$ and $z \in A_{k-1}^* \subset A_{k-2}^* \subset \cdots \subset A_0^* = \mathcal{A}$. Hence $k_j(z) = 0$ if $k \leq j \leq n$ and $k_j(z) = 1$ if $1 \leq j \leq k-1$, and so

$$V(z) = \sum_{j=1}^{n} k_j(z) = \sum_{j=0}^{k-1} k_j(z) + \sum_{j=k}^{n} k_j(z) = \sum_{j=0}^{k-1} 1 + \sum_{j=k}^{n} 0 = k - 1.$$

□

Under some very mild additional assumptions it is possible to construct a Lyapunov function which is strictly decreasing outside the isolated invariant sets and that is differentiable. Next we present such a construction.

**Definition 3.5.** *We say that a semigroup $\{T(t) : t \geq 0\}$ in a metric spaces $X$ is locally bounded if, for each $z \in X$, there is an $\epsilon > 0$ such that $\gamma^+(\mathcal{O}_\epsilon(z))$ is bounded.*

We now give sufficient conditions ensuring that a generalized gradient-like semigroup has Lyapunov function with the same properties of those in Theorem 3.4 and with the additional properties that it is differentiable along solutions and strictly decreasing along solutions that do not originate in the isolated invariant sets. In fact we prove the following result

**Proposition 3.6.** *Let $\{T(t) : t \geq 0\}$ be a semigroup which possesses a global attractor $\mathcal{A}$. Assume that $\{T(t) : t \geq 0\}$ is a generalized gradient-like semigroup with respect to the disjoint family of isolated invariant sets $\Xi = \{\Xi_1, \cdots, \Xi_n\}$. Then, there is a function $W : X \to \mathbb{R}$*

*which is a Lyapunov function for the generalized gradient-like semigroup* $\{T(t) : t \geq 0\}$ *relative to* $\Xi$ *and such that*

(*i*) $[0,\infty) \ni t \mapsto W(T(t)z)$ *is differentiable for all* $z \in X$ *and*

(*ii*) $[0,\infty) \ni t \mapsto W(T(t)z)$ *is strictly decreasing whenever* $z \notin \bigcup_{i=1}^n \Xi_i$.

*Proof.* Let $V : X \to \mathbb{R}$ be the function defined in Theorem 3.4 and $W : X \to \mathbb{R}$ be the function defined by

$$W(z) := \int_0^\infty e^{-t} V(T(t)z)dt.$$

It is easy to see that the function is well defined. Next we prove that this function has the desired properties.

We start with the continuity of $W$. First note that $W(z) \leq n + V(z)$ for all $z \in X$. Now, from the fact that the semigroup $\{T(t) : t \geq 0\}$ is locally bounded it is easy to see that for each $z \in X$ there is an $\epsilon_z > 0$ and $t_1 > 0$ such that $V(\gamma^+(T(t_1)\mathcal{O}_{\epsilon_z}))$ is bounded. Hence, given $\varepsilon > 0$ and $z' \in X$ we choose $\bar{t} > 0$ and a neighborhood $B$ of $z'$ such that,

$$\int_{\bar{t}}^\infty e^{-t} dt < \frac{\varepsilon}{4(M_B+1)}, \tag{3.2}$$

where $M_B := \sup\{V(T(t)w) : w \in B,\ t \geq t_1\} > 0$.

Now, from the continuity of $V$ and of the mapping $[0,\infty) \times X \mapsto T(t)x \in X$, it is easy to see that there exists $\delta > 0$ such that, if $z \in X$ satisfies $d(z,z') < \delta$ then,

$$\int_0^{\bar{t}} e^{-s} \left| V(T(t)z) - V(T(t)z') \right| dt \leq \frac{\varepsilon}{2}.$$

This and (3.2) show that, for $z \in X$ with $d(z,z') < \delta$,

$$|W(z) - W(z')| \leq \int_0^{\bar{t}} e^{-s} |V(T(t)z) - V(T(t)z')|\, dt + 2M_B \int_{\bar{t}}^\infty e^{-t} dt \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Clearly, $W : X \to \mathbb{R}$ is decreasing along solutions of $\{T(t) : t \geq 0\}$. Now, if $z \in \bigcup_{i=1}^n \Xi_i$, we have that $T(t)z \in \bigcup_{i=1}^n \Xi_i$ for all $t \geq 0$, and $V(T(t)z)$ is constant for all $t \geq 0$, yielding that $W(T(t)z)$ is constant. Conversely, if $z \in X$ is such that $W(T(t)z) = \int_0^\infty e^{-s} V(T(t+s)z)dt$ is constant, then $V(T(t)z)$ is constant for all $t \geq 0$ and, consequently, $z \in \bigcup_{i=1}^n \Xi_i$ from the properties of $V$.

Next, given $z \in X \backslash \bigcup_{i=1}^n \Xi_i$ let us prove that $[0,\infty) \ni t \mapsto W(T(t)z)$ is strictly decreasing. Indeed, given $t > 0$, we have that

$$W(T(t)z) - W(z) = \int_0^\infty e^{-s}[V(T(s+t)z) - V(T(s)z)]ds.$$

From this we see that, if for some $t > 0$ we have that $W(T(t)z) - W(z) = 0$, then $V(T(s+t)z) - V(T(s)z) = 0$ for all $s \geq 0$. In particular, $V(T(t)z) = V(z)$ and, as a consequence of that, $V(T(t)z) = V(T(s)z) = V(z)$ for all $s \in [0,t]$. Repeating this reasoning we conclude that $V(T(s)z) = V(z)$ for all $s \geq 0$, which is in contradiction with the choice of $z$.

Now, given $z \in X$, $t \geq 0$ and $h \in \mathbb{R}$ we have that

$$\frac{g(T(t+h)z) - g(T(t)z)}{h} = \frac{e^t}{h}\left[(e^h - 1)\int_{t+h}^{\infty} e^{-s} f(T(s)z)ds - \int_t^{t+h} e^{-s} f(T(s)z)ds\right],$$

which converges to

$$e^t \int_t^{\infty} e^{-s} f(T(s)z)ds - f(T(t)z) \leq 0,$$

proving the differentiability of $[0, \infty) \ni t \mapsto V(T(t)z) \in \mathbb{R}$. □

## 4. Stability under perturbations of generalized gradient semigroups

The equivalence between generalized gradient semigroups and generalized gradient–like semigroups, proved in the previous section, together with the results in [2] prove that generalized gradient semigroups are stable under perturbations. In this section we briefly describe this fact. To that end, we first need to introduce a parameter dependent family of semigroups, as well as the notions of continuity and asymptotic compactness for a parameter dependent family. We start with the notion of continuity for a family of semigroups.

**Definition 4.1.** *We say that a family of semigroups* $\{T_\eta(t) : t \geq 0\}_{\eta \in [0,1]}$, *is continuous at* $\eta = 0$ *if* $T_\eta(t, x) \stackrel{\eta \to 0}{\longrightarrow} T_0(t, x)$ *uniformly for* $(t, x)$ *in compact subsets of* $\mathbb{R}^+ \times X$ *as* $\eta \to 0$.

The notion of collectively asymptotic compactness which is given next plays a fundamental role in the proof of the main result in [2].

**Definition 4.2.** *We say that a family of semigroups* $\{T_\eta(t) : t \geq 0\}_{\eta \in [0,1]}$ *is* collectively asymptotically compact *at* $\eta = 0$ *if, given a sequence* $\{\eta_k\}_{k \in \mathbb{N}}$ *with* $\eta_k \stackrel{k \to \infty}{\longrightarrow} 0$, *a bounded sequence* $\{x_k\}_{k \in \mathbb{N}}$ *in* $X$ *and a sequence* $\{t_k\}_{k \in \mathbb{N}}$ *in* $\mathbb{R}^+$ *with* $t_k \stackrel{k \to \infty}{\longrightarrow} \infty$, *then* $\{T_{\eta_k}(t_k)x_k\}$ *is relatively compact.*

We are now ready to state the following result from [2].

**Theorem 4.3** (Carvalho-Langa). *Let* $\{T_\eta(t) : t \geq 0\}_{\eta \in [0,1]}$, *be a collectively compact family of semigroups which is continuous at* $\eta = 0$. *Assume that*

a) $\{T_\eta(t) : t \geq 0\}$ *possesses a global attractor* $\mathcal{A}_\eta$ *for each* $\eta \in [0, 1]$ *and* $\cup_{\eta \in [0,1]} \mathcal{A}_\eta$ *is bounded.*

b) *There exists* $n \in \mathbb{N}$ *such that* $\mathcal{A}_\eta$ *has* $n$ *isolated invariant sets* $\mathcal{S}_\eta = \{\Xi_{1,\eta}, \cdots, \Xi_{n,\eta}\}$ *for all* $\eta \in [0, 1]$, *and* $\sup_{1 \leqslant i \leqslant n}[\mathrm{dist}_H(\Xi^*_{i,\eta}, \Xi^*_{i,0}) + \mathrm{dist}_H(\Xi^*_{i,0}, \Xi^*_{i,\eta})] \stackrel{\eta \to 0}{\longrightarrow} 0$.

c) $\{T_0(t) : t \geq 0\}$ *is a generalized gradient-like semigroup.*

*Then, there exists* $\eta_0 > 0$ *such that, for all* $\eta \leqslant \eta_0$, $\{T_\eta(t) : t \geq 0\}$ *is a generalized gradient-like semigroup and consequently*

$$\mathcal{A}_\eta = \cup_{i=1}^n W^u(\Xi^*_{i,\eta}), \ \forall \eta \in [0, \eta_0].$$

As an immediate consequence of this result and the ones in Section 3 we have the following result.

**Corollary 4.4.** *Under the assumption of Theorem 4.3, there exists $\eta_0 > 0$ such that, for all $\eta \leqslant \eta_0$, $\{T_\eta(t) : t \geq 0\}$ is a generalized gradient semigroup.*

**Corollary 4.5.** *Under the assumption of Theorem 4.3, suppose there exists $n \in \mathbb{N}$ such that $\mathcal{A}_\eta$ has $n$ stationary solutions $\mathcal{S}_\eta = \{\xi_{1,\eta}, \cdots, \xi_{n,\eta}\}$ for all $\eta \in [0,1]$ and $\sup_{1\leqslant i\leqslant n} \mathrm{dist}(\xi^*_{i,\eta}, \xi^*_{i,0}) \overset{\eta\to 0}{\longrightarrow} 0$. Then, there exists $\eta_0 > 0$ such that, for all $\eta \leqslant \eta_0$, $\{T_\eta(t) : t \geq 0\}$ is a gradient semigroup in the sense of* [4].

## 5. Energy level decomposition of a generalized gradient-like system

Let $\{T(t) : t \geq 0\}$ be a semigroup in $X$ with global attractor $\mathcal{A}$. Next we will give a dynamical description of a generalized gradient-like system by reordering and regrouping the corresponding isolated invariant subsets to obtain a *totally ordered* family of isolated invariant sets that we will refer to as *energy levels*. This new family of isolated invariant sets is a Morse decomposition of $\mathcal{A}$ with fewer invariant sets but in such a way that it still gives us a Lyapunov function that is constant only in the original isolated invariant sets. In a certain sense this decomposition is the coarsest decomposition which still gives us a Lyapunov function which is constant only in the original isolated invariant sets.

Assume that $\{T(t) : t \geq 0\}$ is a generalized gradient-like semigroup with respect to the disjoint family of isolated invariant sets $\Xi = \{\Xi_1, \Xi_2, \cdots, \Xi_n\}$.

(*a*) Given $\Xi_{l_1}$ and $\Xi_{l_2} \in \Xi$, we say that $\Xi_{l_1}$ precedes $\Xi_{l_2}$ (we write $\Xi_{l_1} \prec \Xi_{l_2}$), if there exists a global solution $\xi : \mathbb{R} \to X$ of $\{T(t) : t \geq 0\}$ such that $\xi(\mathbb{R}) \subsetneq \Xi_{l_1} \cup \Xi_{l_2}$ and

$$\lim_{t\to-\infty} d(\xi(t), \Xi_{l_2}) = 0 \text{ and } \lim_{t\to\infty} d(\xi(t), \Xi_{l_1}) = 0.$$

(*b*) Let us consider

$$\mathcal{M}_1 \ : \ = \{\Xi_\ell \in \Xi : \text{ there is no element } \Xi \in \Xi \text{ that preceeds } \Xi_\ell\}$$

and, for any integer $k \geq 2$

$$\mathcal{M}_k := \{\Xi^*_\ell \in \Xi : \text{ if } \Xi \in \Xi \text{ e } \Xi \prec \Xi_\ell \text{ then } \Xi \in \mathcal{M}_{k-1}\}.$$

Note that, by definition, $\mathcal{M}_k \subset \mathcal{M}_{k+1}$.

(*c*) We now define the sets

$$\mathcal{N}_1 := \bigcup_{\Xi\in\mathcal{M}_1} \Xi, \text{ and } \mathcal{N}_k := \bigcup_{\Xi\in\mathcal{M}_k\setminus\mathcal{M}_{k-1}} \Xi, \text{ for all } k \geq 2.$$

Since $\Xi$ is finite, there exists a positive integer $q$ such that $\mathcal{M}_k = \mathcal{M}_q$ for each $k > q$, so that, $\mathcal{N}_k = \varnothing$ for all $k > q$. Thus, we consider $\mathcal{N}_1, \mathcal{N}_2, \cdots, \mathcal{N}_p$, with $p := \min\{q \in \mathbb{N} : \mathcal{M}_k = \mathcal{M}_q \text{ for each } k > q\}$.

We have the following first result related to this family of sets:

**Lemma 5.1.** *Let $\{T(t) : t \geq 0\}$ be a semigroup with global attractor $\mathcal{A}$. Assume that $\{T(t) : t \geq 0\}$ is a generalized gradient-like semigroup with respect to the disjoint family of isolated invariant sets $\mathbf{\Xi} = \{\Xi_1, \Xi_2, \cdots, \Xi_n\}$. Then each element of $\mathbf{\Xi}$ belongs to $\mathcal{N}_k$, for some $k \leq p$.*

*Proof.* Suppose that $\mathcal{N}_1$ is empty and fix $\Xi \in \mathbf{\Xi}$. Then, since $\Xi \notin \mathcal{N}_1 = \varnothing$, by the definition of $\mathcal{N}_1$, there exists $\Xi^1 \in \mathbf{\Xi}$ with $\Xi^1 \prec \Xi$ and $\Xi^1 \neq \Xi$. Analogously, $\Xi^1 \notin \mathcal{N}_1$, from which we find $\Xi^2 \in \mathbf{\Xi}$ with $\Xi^2 \prec \Xi^2$ and $\Xi^2 \neq \Xi^1$. Also $\Xi^2 \neq \Xi$ since, otherwise, there would be a homoclinic structure $\Xi \prec \Xi^1 \prec \Xi^2 = \Xi$. Following in this way, as $\mathbf{\Xi}$ is finite we arrive at a contradiction in a finite number of steps.

If $\mathbf{\Xi} = \mathcal{N}_1$, we are done. If not, we must have that $\mathcal{N}_2$ is non-empty. Otherwise, given $\Xi \in \mathbf{\Xi} \backslash \mathcal{N}_1$ with $\Xi \notin \mathcal{N}_2$. Then, there exists $\Xi^1 \in \mathbf{\Xi} \backslash \mathcal{N}_1$ with $\Xi \prec \Xi^1$. As before, the fact that $\{T(t) : t \geq 0\}$ is a generalized gradient-like semigroup with respect to $\mathbf{\Xi}$ implies that $\Xi^1 \neq \Xi$. Since $\Xi^1 \notin \mathcal{N}_2$, there is a $\Xi^2 \in \mathbf{\Xi} \backslash \mathcal{N}_1$ with $\Xi^1 \prec \Xi^2$ and, again, $\Xi^2 \neq \Xi^1$ and $\Xi^2 \neq \Xi$. As before, we arrive at a contradiction in a finite number of steps, so that $\mathcal{N}_2 \neq \varnothing$.

If $\mathbf{\Xi} = \mathcal{N}_1 \cup \mathcal{N}_2 = \mathcal{M}_2$, we finish. If not, again $\mathcal{N}_3 \neq \varnothing$. As this argument has to finish in a finite number of steps completing the proof. □

The following result will show that $\mathcal{N} = (\mathcal{N}_1, \mathcal{N}_2, \cdots, \mathcal{N}_p)$ is a Morse decomposition for $\mathcal{A}$.

**Theorem 5.2.** *Let $\{T(t) : t \geq 0\}$ be a nonlinear semigroup with global attractor $\mathcal{A}$. If $\{T(t) : t \geq 0\}$ is a generalized gradient-like semigroup with respect to $\mathbf{\Xi} = \{\Xi_1, \Xi_2, \cdots, \Xi_n\}$, then $(\mathcal{N}_1, \mathcal{N}_2, \cdots, \mathcal{N}_p)$ is a Morse decomposition for $\mathcal{A}$.*

*Proof.* Clearly $\{T(t) : t \geq 0\}$ is a generalized gradient-like semigroup with respect to $\mathcal{N}$. The proof of the result now follows as the proof of Theorem 2.18. □

**Remark 5.3.** *Given a compact invariant set $\mathcal{A}$, we define the finest Morse decomposition on $\mathcal{A}$ (cf.* [10]*) as $(\Xi_1, \ldots, \Xi_n)$ in which, for each isolated compact invariant set $\Xi_i$ the unique Morse decomposition on it is the trivial one. Then, the energy level decomposition of the finest Morse decomposition can be considered as the optimal decomposition of $\mathcal{A}$, as it gives a total description of $\mathcal{A}$ by a Lyapunov (energy) function which is constant on each level and strictly decreasing on connecting global solutions among all the different levels.*

5.0.1. *Energy level decomposition for a gradient-like system.* All the concepts and results in the previous section can be written in the particular case in which we have a finite set of equilibria:

**Definition 5.4.** *Let $\{T(t) : t \geq 0\}$ be a gradient-like semigroup in $X$ with global attractor $\mathcal{A}$ with equilibrium points $\mathcal{E} = \{\zeta_1, \cdots, \zeta_n\}$.*

(a) *Given $\zeta_r$ and $\zeta_s \in \mathcal{E}$, we say that $\zeta_r$ precedes $\zeta_s$ (we write $\zeta_r \prec \zeta_s$), if there exists a non-constant global solution $\xi : \mathbb{R} \to X$ of $\{T(t) : t \geq 0\}$ such that*

$$\lim_{t\to-\infty} d(\xi(t), \zeta_s) = 0 \quad \text{and} \quad \lim_{t\to\infty} d(\xi(t), \zeta_r) = 0.$$

(b) *Let us consider*

$$\mathcal{M}_1 \ : \ = \{\zeta_\ell \in \mathcal{E} : \text{ there is no element } \zeta \in \mathcal{E} \text{ that preceeds } \zeta_\ell\}$$

*and, for any integer $k \geq 2$*

$$\mathcal{M}_k := \{\zeta_\ell \in \mathcal{E} : \text{ if } \zeta \in \mathcal{E} \text{ e } \zeta \prec \zeta_\ell \text{ then } \zeta \in \mathcal{M}_{k-1}\}.$$

*Note that, by definition, $\mathcal{M}_k \subset \mathcal{M}_{k+1}$.*

(c) *We now define the sets*

$$\mathcal{N}_1 := \mathcal{M}_1 \text{ and } \mathcal{N}_k := \mathcal{M}_k \backslash \mathcal{M}_{k-1}, \text{ for all } k \geq 2.$$

*Since $\mathcal{E}$ is finite, there exists a positive integer $q$ such that $\mathcal{M}_k = \mathcal{M}_q$ for each $k > q$, so that, $\mathcal{N}_k = \varnothing$ for all $k > q$. Thus, we consider $\mathcal{N}_1, \mathcal{N}_2, \cdots, \mathcal{N}_p$, with $p := \min\{q \in \mathbb{N} : \mathcal{M}_k = \mathcal{M}_q$ for each $k > q\}$.*

**Lemma 5.5.** *If $\{T(t) : t \geq 0\}$ is a gradient-like semigroup, then every $\zeta \in \mathcal{E}$ is in $\mathcal{N}_i$ for some $i = 1, \cdots, p$, where $p \geq 1$ is the maximum number of nonvoid energy levels in $\{T(t) : t \geq 0\}$.*

**Lemma 5.6.** *Let $\{T(t) : t \geq 0\}$ be a gradient-like semigroup with global attractor $\mathcal{A}$ and a finite set of equilibria $\mathcal{E} = \{\zeta_1, \cdots, \zeta_n\}$, and $(\mathcal{N}_1, \cdots, \mathcal{N}_p)$ the ordered n-upla of energy levels $\{T(t) : t \geq 0\}$. Then, $(\mathcal{N}_1, \cdots, \mathcal{N}_p)$ is a Morse decomposition in $\mathcal{A}$.*

## 6. Final comments and further research

We have proved that a generalized gradient-like nonlinear semigroup is a generalized gradient one. In particular, a gradient-like semigroup is gradient, concluding that it is stable under perturbation. Note that any global attractor for an infinite dimensional global attractor admits at least a Morse decomposition -the trivial one-, being very usual to get a better description of the internal dynamics on it in many non-trivial examples. In any case, our results apply to any Morse decomposition of a global attractor.

On the other hand, it is shown in [2] that a non-autonomous perturbation of a generalized gradient-like system is still gradient-like. Thus, there exists a very natural extension of the results on this paper to a non-autonomous framework.

Finally, we think that the concept of energy levels deserves being exploited, because it is giving a very good description of connections between isolated sets from any given decomposition. In particular, its stability under perturbation seems a proper non-trivial problem to be considered, connected, for instance, to Morse-Smale systems. We plan to present results in these last two lines of research in the near future.

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# *NOTAS DO ICMC*

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