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THE ANALYTIC TORSION OF THE CONE OVER AN ODD
DIMENSIONAL MANIFOLD

L. HARTMANN
M. SPREAFICO

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THE ANALYTIC TORSION OF THE CONE OVER AN ODD DIMENSIONAL MANIFOLD

L. HARTMANN AND M. SPREAFICO

ABSTRACT. We study the analytic torsion of the cone over an orientable odd dimensional compact connected Riemannian manifold W . We prove that the logarithm of the analytic torsion of the cone decomposes as the sum of the logarithm of the root of the analytic torsion of the boundary of the cone, plus a topological term, plus a further term that is a rational linear combination of local Riemannian invariants of the boundary. We also prove that this last term coincides with the anomaly boundary term appearing in the Cheeger Müller theorem [7] [26] for a manifold with boundary, according to Brüning and Ma [3]. We also prove Poincaré duality for the analytic torsion of a cone.

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1. INTRODUCTION AND STATEMENT OF THE RESULTS

Analytic torsion was originally introduced by Ray and Singer in [29], as an analytic counter part of the Reidemeister torsion of Reidemeister, Franz and de Rham [31] [13] [12]. Since then, analytic torsion became an important invariant of Riemannian manifolds, and has been intensively studied, several generalizations have been introduced and significative results obtained. Concerning the original invariant, the first important result was achieved by W. Müller [26] and J. Cheeger [7] who proved that for a compact connected Riemannian manifold without boundary, the analytic torsion and the Reidemeister torsion coincide, result conjectured by Ray and Singer in [29], because of the several similar properties shared by the two torsions. This result is nowadays known as the celebrated Cheeger-Müller theorem. The next natural question along this line of investigation was

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to answer the same problem for manifolds with boundary. It was soon realized that the answer to such a question was an highly non trivial one, since the boundary introduces some wild terms. The first case to be analyzed was the case of a product metric near the boundary. W. Lück proved in [22] that in this case the boundary term is topological, and depends only upon the Euler characteristic of the boundary. The answer to the general case required 20 more years of work, and has eventually been given in a recent paper of Brüning and Ma [3]. The new contribution of the boundary, beside the topological one given by Lück, is called anomaly boundary term and we denote it by $A_{BM,abs}$. The term $A_{BM,abs}$ has a quite complicate expression, but only depends on some local quantities constructed from the metric tensor near the boundary (see Section 2.6 for details). The next natural step is to study the analytic torsion for spaces with singularities. A first, natural type of space with singularity is the cone over a manifold, CW . Cones and spaces with conical singularities have been deeply investigated by J. Cheeger in a series of works [7] [9] [10] (see also [27]). Due to this investigation, all information on L^2 -forms, Hodge theory, and Laplace operator on forms on CW are available. Further information on the class of regular singular operators, that contains the Laplace operator on CW , are given in works of Brüning and Seely (see in particular [6]). As a result it is not difficult to obtain a complete description of the eigenvalues of the Laplace operator on CW in terms of the eigenvalues of the Laplace operator on W . With all these tools available, namely on one side the formula for the boundary term, and on the other some representation of the eigenvalues of the Laplace operator on the cone, it is natural to tackle the problem of investigating the analytic torsion of CW . What one expects in this case in fact is some relation between the torsion of the cone and the torsion of the section. A possible extension of the Cheeger Müller theorem could follow, or not. Indeed, it is general belief that in case of conical singularity such an extension would require intersection R torsion more than classical R torsion (see [11]). However, if the section is a rational homology manifold, then the two torsion coincide (see [7], end of Section 2), and the classical Cheeger Müller theorem is expected to extend.

If $\mathcal{C}(W)$ is the chain complex associated to some cell decomposition of W , then the algebraic mapping cone $Cone(\mathcal{C}(W))$ gives the chain complex for a cell decomposition of CW . It is then easy to see that the R torsion of CW only depends on the choice of a base for the zero dimensional homology. Even if Poincaré duality does not hold, it does hold between top and bottom dimension, and therefore we can use the method of Ray and Singer in order to fix the base for the zero homology using the Riemannian structure and harmonic forms (see [29] Section 3, see also [19]). The result for the R torsion is

$$\tau(CW) = \sqrt{\text{Vol}(CW)}.$$

On the other side, one wants the analytic torsion. The analytic tools necessary to deal with the zeta functions appearing in the definition of the analytic torsion, constructed with the eigenvalues of the Laplace operator on CW , are available by works of M. Spreafico [36] [37] [39]. In these works, the zeta function associated to a general class of double sequences is investigated. In particular, a decomposition result is presented and formulas for the zeta invariants of a decomposable sequence are given. This technique applies to the case of the zeta function on CW , and permits to obtain some results on the analytic torsion that we will describe here below. Before, we note that this approach has been also followed in [19], [40] and [18]. The main results of these papers are that in the case of W an odd low dimensional sphere, then the classical Cheeger Müller theorem with the anomalous boundary term of Brüning and Ma holds for CW , while if W is an even dimensional sphere, it does not hold. Explicit formulas for W the sphere of dimensions one, two and three are also given, and in [18] it is conjectured that the Cheeger Müller theorem with the anomalous boundary term of Brüning and Ma holds for W any odd dimensional sphere. This is proved to

be true in Theorem 1.2 below. In Theorem 1.3 the same result is proved for the cone over a generic orientable compact connected Riemannian manifold of odd dimension. We split the proof of Theorem 1.3 in two parts. We first prove in Section 9 that the result is true if the dimension of the manifold is smaller than six. A basic ingredient in this proof are results of P. Gilkey on local invariants of Riemannian manifolds [14]. Next, in Section 10, we prove the general statement. The reason for giving a different explicit proof for the low dimensional cases is due to the fact that the proof of the general case is based on a result that has not been published yet, namely a formula for the anomaly boundary term of Brüning and Ma with mixed boundary conditions. This result is contained in a preprint of the same authors [4], and we thanks the author for making available part of their work.

We are now ready to state the main results of this paper, for we fix some notation. Let (W, g) be an orientable compact connected Riemannian manifold of finite dimension m without boundary and with Riemannian structure g . We denote by $C_l W$ the cone over W with the Riemannian structure

$$dx \otimes dx + x^2 g,$$

on $CW - \{pt\}$, where pt denotes the tip of the cone and $0 < x \leq l$ (see Section 4.1 for details). The formal Laplace operator on forms on $CW - \{pt\}$ has a suitable L^2 -self adjoint extension $\Delta_{\text{abs/rel}}$ on $C_l W$ with absolute or relative boundary conditions on the boundary $\partial C_l W$ (see Section 4.3 for details), with pure discrete spectrum $\text{Sp} \Delta_{\text{abs/rel}}$. This permit to define the associated zeta function

$$\zeta(s, \Delta_{\text{abs/rel}}) = \sum_{\lambda \in \text{Sp}_+ \Delta_{\text{abs/rel}}} \lambda^{-s},$$

for $\text{Re}(s) > \frac{m+1}{2}$. This zeta function has a meromorphic analytic continuation to the whole complex s -plane with at most isolated poles (see Section 5 for details). It is then possible to define the analytic torsion of the cone

$$\log T_{\text{abs/rel}}(C_l W) = \frac{1}{2} \sum_{q=0}^{m+1} (-1)^q q \zeta'(0, \Delta_{\text{abs/rel}}^{(q)}).$$

In this setting, we have the following results (analogous results with relative boundary conditions also follow by Poincaré duality on the cone, proved in Theorem 5.1 below).

Theorem 1.1. *The analytic torsion on the cone $C_l W$ on an orientable compact connected Riemannian manifold (W, g) of odd dimension $2p - 1$ is*

$$\log T_{\text{abs}}(C_l W) = \frac{1}{2} \sum_{q=0}^{p-1} (-1)^{q+1} \text{rk} H_q(W; \mathbb{Q}) \log \frac{2(p-q)}{l} + \frac{1}{2} \log T(W, l^2 g) + S(\partial C_l W),$$

where the singular term $S(\partial C_l W)$ only depends on the boundary of the cone:

$$S(\partial C_l W) = \frac{1}{2} \sum_{q=0}^{p-1} \sum_{j=0}^{p-1} \sum_{k=0}^j \text{Res}_0 \Phi_{2k+1, q}(s) \binom{-\frac{1}{2} - k}{j - k} \sum_{h=0}^q (-1)^h \text{Res}_1 \zeta \left(s, \tilde{\Delta}^{(h)} \right) (q - p + 1)^{2(j-k)},$$

where the functions $\Phi_{2k+1, q}(s)$ are some universal functions, explicitly known by some recursive relations, and $\tilde{\Delta}$ is the Laplace operator on forms on the section of the cone.

It is important to observe that the singular term $S(\partial C_l W)$ is a universal linear combination of local Riemannian invariants of the boundary, for the residues of the zeta function of the section are such linear combination (see Section 9.3 for details).

Theorem 1.2. *When W is the odd dimensional sphere (of radius a), with the standard induced Euclidean metric, then the singular term of the analytic torsion of the cone $C_l W$ appearing in Theorem 1.1 coincides with the anomaly boundary term of Brüning and Ma, namely $S(\partial C_l S_a^{2p-1}) = A_{\text{BM,abs}}(\partial C_l S_a^{2p-1})$. In this case, the formula for the analytic torsion reads*

$$\log T_{\text{abs}}(C_l S_a^{2p-1}) = \frac{1}{2} \log \text{Vol}(C_l S_a^{2p-1}) + A_{\text{BM,abs}}(\partial C_l S_a^{2p-1}),$$

where

$$A_{\text{BM,abs}}(\partial C_l S_a^{2p-1}) = \frac{(2p-1)!}{4^p(p-1)!} \sum_{k=0}^{p-1} \frac{1}{(p-1-k)!(2k+1)} \sum_{j=0}^k \frac{(-1)^{k-j} 2^{j+1}}{(k-j)!(2j+1)!} a^{2k+1}.$$

Corollary 1.1. *The natural extension of Cheeger Müller theorem for manifold with boundary is valid for the cone over an odd dimensional sphere, namely*

$$\log T_{\text{abs}}(C_l S^{2p-1}) = \log \tau(C_l S^{2p-1}) + A_{\text{BM,abs}}(\partial C_l S^{2p-1}).$$

The result in the corollary should be understood as a particular case of the still unproved general result that the analytic torsion and the intersection R-torsion of a cone coincides up to the boundary term, for the intersection R-torsion is the classical R-torsion for the cone over a sphere.

Theorem 1.3. *When (W, g) is an orientable compact connected Riemannian manifold of odd dimension, then the singular term of the analytic torsion of the cone $C_l W$ appearing in Theorem 1.1 coincides with the anomaly boundary term of Brüning and Ma, namely $S(\partial C_l W) = A_{\text{BM,abs}}(\partial C_l W)$.*

We conclude with a remark on the even dimensional case, namely when the dimension of the section W is even. It is quit clear that all the arguments used in the odd dimensional case go through also in the even dimensional case. So we obtain formulas for the analytic torsion as in the theorems above. However, in the even dimensional case some further term appears: this was described in some details for $W = S^2$ in [19]. Since we do not have a clear understanding of this new term yet, we prefer to omit the non particularly illuminating formulas for the even dimensional case here.

2. PRELIMINARIES AND NOTATION

In this section we will recall some basic results in Riemannian geometry, Hodge de Rham theory and global analysis, and the definitions of the main objects we will deal with in this work. All the results are contained either in classical literature or in [29], [26], [9], [3], [14] [1]. This section can be skipped at first reading, and it is added exclusively for the reader's benefit. From one side, it permits to the interested reader to find out all the necessary tools to verify the arguments developed in the following sections, with precise reference to definitions and formulas, avoiding the fuss of searching in several different papers and books, from the other, it provides a unified notation, whereas notations of different authors are too often quite different in the available literature.

2.1. $\mathbb{Z}/2$ -graded algebras and Berezin integrals. Let V, W be finite dimensional vector spaces over a field $\mathbb{F}$ of characteristic zero, with Euclidean inner products $\langle -, - \rangle$. Let $V^* = \text{Hom}(V, \mathbb{F})$ denotes the dual of V , and fix an isomorphism of V onto V^* by $v^*(u) = \langle v, u \rangle$. Then, we identify $\mathbb{F}$ -homomorphisms with tensor product of tensors by $\text{Hom}(V, W) = V^* \otimes W$. If V, W have dimensions m, n respectively, and $\{e_k\}_{k=1}^m, \{b_l\}_{l=1}^n$ are orthonormal bases of V and W respectively, then we

identify $t \in \text{Hom}(V, W)$ with the tensor $T \in V^* \otimes W$ with components $T_{kl} = \langle b_l, t(e_k) \rangle = b^*(t(e_k))$, namely

$$T = \sum_{k=1}^m \sum_{l=1}^n T_{kl} e_k^* \otimes b_l,$$

where we denote by $\{e_k^*\}$ the dual base. We use the *Euclidean norm* of a linear homomorphism $|t| = \sqrt{\text{tr}(t^*t)}$. In the orthonormal base, this gives for the associated tensor

$$|T|^2 = \sum_{k=1}^m \sum_{l=1}^n T_{kl}^2.$$

We denote by ΛV the exterior algebra of V (the universal algebra with unit generated by V with the relation $v^2 = 0$). A $\mathbb{Z}/2$ -graded algebra A is a vector space with an involution such that $A = A_+ \oplus A_-$, and a product $\cdot$ that preserves the involution, i.e. such that $A_j A_k \subset A_{jk}$. The exterior algebra is a first example. Let A and B be two $\mathbb{Z}/2$ -graded algebras. The vector spaces tensor product $A \otimes B$ has a natural $\mathbb{Z}/2$ -grading, $A \otimes B = (A \otimes B)_+ \oplus (A \otimes B)_-$, where $(A \otimes B)_+ = A_+ \otimes B_+ \oplus A_- \otimes B_-$, and $(A \otimes B)_- = A_+ \otimes B_- \oplus A_- \otimes B_+$. This becomes a $\mathbb{Z}/2$ -graded algebra, that we denote by $A \hat{\otimes} B$, with the product defined by (we will omit the dot in the following)

$$(a \hat{\otimes} b) \cdot (a' \hat{\otimes} b') = (-1)^{|b||a'|} a \cdot a' \hat{\otimes} b \cdot b'.$$

There are two natural immersions of A in $A \hat{\otimes} A$ as an algebra: we identify A with $A \otimes 1$ and we denote by $\hat{A} = 1 \otimes A$. Since $A \otimes 1 \otimes 1 \otimes B = A \otimes B$, we have that $A \hat{\otimes} B = A \otimes \hat{B}$. Let $A = B = \Lambda V$, for some vector space V , where we denote the product by $\wedge$, as usual. Then, $V \hat{\otimes} V \in \Lambda V \hat{\otimes} \Lambda V$. If $v \in V$, then $v = v \hat{\otimes} 1 \in V = V \hat{\otimes} 1$, $\hat{v} = 1 \hat{\otimes} v \in \hat{V} = 1 \hat{\otimes} V$, and $v \wedge \hat{v} \in V \wedge \hat{V}$. Note that, $v \wedge \hat{v} = (v \hat{\otimes} 1) \wedge (1 \hat{\otimes} v)$, while $(1 \hat{\otimes} u) \wedge (v \hat{\otimes} 1) = -v \wedge \hat{u}$, and

$$\hat{u} \wedge \hat{v} = (1 \hat{\otimes} u) \wedge (1 \hat{\otimes} v) = 1 \hat{\otimes} (u \wedge v) = \widehat{u \wedge v} \in \widehat{V \wedge V}.$$

This permits to identify an antisymmetric endomorphism ϕ of V with the element

$$\hat{\phi} = \frac{1}{2} \sum_{j,k=1}^m \langle \phi(v_j), v_k \rangle \hat{v}_j \wedge \hat{v}_k,$$

of $\widehat{\Lambda^2 V}$. For the elements $\langle \phi(v_j), v_k \rangle$ are the entries of the tensor representing ϕ in the base $\{v_k\}$, and this is an antisymmetric matrix. Now assume that r is an antisymmetric endomorphism of $\Lambda^2 V$. Then, $(R_{jk} = \langle r(v_j), v_k \rangle)$ is a tensor of two forms in $\Lambda^2 V$. We extend the above construction identifying R with the element

$$(2.1) \quad \hat{R} = \frac{1}{2} \sum_{j,k=1}^m \langle r(v_j), v_k \rangle \hat{v}_j \wedge \hat{v}_k,$$

of $\Lambda^2 V \wedge \widehat{\Lambda^2 V}$. This can be generalized to higher dimensions. In particular, all the construction can be done taking the dual V^* instead of V . We conclude with the definition of the Berezin integral. Assume V to be oriented. We define the *Berezin integral* be

$$\begin{aligned} \int^B : \Lambda W \hat{\otimes} \Lambda V &\rightarrow \Lambda W, \\ \int^B : \alpha \otimes \beta &\mapsto c_B \beta(e_1, \dots, e_m) \alpha, \end{aligned}$$

where $c_B = \frac{(-1)^{\frac{m(m+1)}{2}}}{\pi^{\frac{m}{2}}}$. In particular, note that

$$\int^B e^{-\frac{A}{2}} = Pf\left(\frac{A}{2\pi}\right),$$

and this vanishes if $\dim E = m$ is odd.

2.2. Some Riemannian geometry. Let (W, g) be an orientable connected Riemannian manifold of dimension m without boundary, where g denotes the Riemannian structure. We denote by TW the (total space) of the tangent bundle over W , and by T^*W the dual bundle. We denote by ΛT^*W the exterior algebra of T^*W . We denote by $\Gamma(W, TW)$ and $\Gamma(W, T^*W)$ the corresponding spaces of smooth sections, and by $\Gamma_0(W, TW)$ and $\Gamma_0(W, T^*W)$ the spaces of smooth sections with compact support. Let $x = (x_1, \dots, x_m)$ be a local coordinate system on W . We denote by $\{\partial_j\}_{j=1}^m$ the local coordinate base on TW , and by $\{dx_j\}_{j=1}^m$ the dual base of one forms on T^*W : $dx_j(\partial_k) = \delta_{jk}$. We denote by $\{e_j\}_{j=1}^m$ and $\{e_j^*\}_{j=1}^m$ local orthonormal bases of TW and T^*W , $e_j^*(e_k) = \delta_{jk}$. Then,

$$g = \sum_{j,k=1}^m g_{jk} dx_j \otimes dx_k = \sum_{j,k=1}^m \delta_{jk} e_j^* \otimes e_k^*,$$

and the *volume element* on W is $dvol_g = e_1^* \wedge \dots \wedge e_m^* = \sqrt{|\det g|} dx_1 \wedge \dots \wedge dx_m$.

Let $\nabla : \Gamma(W, TW) \rightarrow \Gamma(W, \text{End}(TW))$ denotes the *covariant derivative* associated to the Levi Civita connection of the metric g . This is completely determined by the tensor $\Gamma = \sum_{\alpha,\beta,\gamma=1}^m \Gamma_{\alpha\beta\gamma} e_\alpha^* \otimes e_\beta^* \otimes e_\gamma \in \Gamma(W, T^*W \otimes T^*W \otimes TW)$, with components the *Christoffel symbols*:

$$\Gamma_{\alpha\beta\gamma} = \Gamma(e_\alpha, e_\beta, e_\gamma^*) = e_\gamma^*(\Gamma(e_\alpha, e_\beta, -)) = e_\gamma^*(\nabla_{e_\alpha} e_\beta) = g(e_\gamma, \nabla_{e_\alpha} e_\beta) = -\Gamma_{\alpha\gamma\beta},$$

that can be computed using the formula:

$$(2.2) \quad \Gamma_{\alpha\beta\gamma} = \frac{1}{2} (c_{\alpha\beta\gamma} + c_{\gamma\alpha\beta} + c_{\gamma\beta\alpha}),$$

where the *Cartan structure constants* $c_{\alpha\beta\gamma} = -c_{\beta\alpha\gamma}$ are defined by

$$[e_\alpha, e_\beta] = \sum_{\gamma=1}^m c_{\alpha\beta\gamma} e_\gamma.$$

The *connection one form* associated to ∇ is the matrix valued one form $\omega \in \mathfrak{so}(m, \Gamma(W, \Lambda T^*W))$, with components

$$\omega_{\beta\gamma} = \sum_{\alpha=1}^m \Gamma(e_\alpha, e_\gamma, e_\beta^*) e_\alpha^*.$$

The *curvature* associated to the Riemannian connection ∇ (of the metric g) is the linear map $r : \Gamma(W, TW \otimes TW) \rightarrow \Gamma(W, \text{End}(TW))$, defined by:

$$r(x, y, z) = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z,$$

where $x, y, z, r(x, y, z) \in \Gamma(W, TW)$, and corresponds to the tensor $R = \sum_{\alpha,\beta,\gamma,\delta=1}^m R_{\alpha\beta\gamma\delta} e_\alpha^* \otimes e_\beta^* \otimes e_\gamma^* \otimes e_\delta \in \Gamma(W, T^*W \otimes T^*W \otimes T^*W \otimes TW)$, with components

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= -R_{\beta\alpha\gamma\delta} = R(e_\alpha, e_\beta, e_\gamma, e_\delta^*) = e_\delta^*(r(e_\alpha, e_\beta, e_\gamma)) \\ &= e_\delta^*(\nabla_{e_\alpha} \nabla_{e_\beta} e_\gamma - \nabla_{e_\beta} \nabla_{e_\alpha} e_\gamma - \nabla_{[e_\alpha, e_\beta]} e_\gamma). \end{aligned}$$

The *curvature two* form associated to R is the matrix valued two form $\Omega \in \mathfrak{so}(m, \Gamma(W, \Lambda^2 T^*W))$, with components

$$\Omega_{\gamma\delta} = \sum_{\alpha, \beta=1, \alpha < \beta}^m R(e_\alpha, e_\beta, e_\delta, e_\gamma) e_\alpha^* \wedge e_\beta^*,$$

and can be computed by the following formula:

$$(2.3) \quad \Omega_{\alpha\beta} = d\omega_{\alpha\beta} + \sum_{\gamma=1}^m \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta}.$$

We introduce two more tensor fields. The Ricci tensor $Ric = \sum_{\alpha, \beta=1}^m Ric_{\alpha\beta} e_\alpha^* \otimes e_\beta^* \in \Gamma(W, T^*W \otimes T^*W)$, defined by

$$Ric(x, y) = \sum_{k=1}^m e_k^* (\nabla_{e_k} \nabla_x y - \nabla_x \nabla_{e_k} y - \nabla_{[e_k, x]} y),$$

and the scalar curvature tensor, defined by

$$\tau = \sum_{k=1}^m Ric(e_k, e_k).$$

The components of these tensors in terms of the curvature tensor are:

$$Ric_{\alpha\beta} = \sum_{k=1}^m R(e_k, e_\alpha, e_\beta, e_k^*),$$

$$\tau = \sum_{k, h=1}^m R(e_k, e_h, e_h, e_k^*).$$

2.3. Hodge theory and de Rham complex. We recall some results on the de Rham complex (see for example [25]) and some results from [29] and [26]. From now on we will assume that W is compact. In this section we also assume that W has no boundary.

Let denote by Ω^q the space of sections $\Gamma(W, \Lambda^{(q)} T^*W)$. The exterior differential d defines the *de Rham complex*

$$\mathcal{C}_{DR} : \dots \longrightarrow \Omega^{(q)}(W) \xrightarrow{d} \Omega^{(q+1)}(W) \longrightarrow \dots,$$

whose homology coincides with the rational homology of W . The Hodge star $\star : \Lambda^{(q)} T^*W \rightarrow \Lambda^{(m-q)} T^*W$, defines an isometry $\star : \Omega^q(W) \rightarrow \Omega^{m-q}(W)$, and an inner product on $\Omega^q(W)$

$$(\alpha, \beta) = \int_W \alpha \wedge \star \beta = \int_W \langle \alpha, \beta \rangle dvol_g.$$

The closure of Ω^q with respect to this inner product is the Hilbert space the L^2 q -forms on W . The de Rham complex with this product is an elliptic complex. The dual of the exterior derivative $d^\dagger$, defined by $(\alpha, d\beta) = (d^\dagger \alpha, \beta)$, satisfies $d^\dagger = (-1)^{mq+m+1} \star d \star$. The Laplace operator is $\Delta = (d + d^\dagger)^2$. It satisfies: 1) $\star \Delta = \Delta \star$, 2) Δ is self adjoint, and 3) $\Delta \omega = 0$ if and only if $d\omega = d^\dagger \omega = 0$. Let $\mathcal{H}^q(W) = \{\omega \in \Omega^{(q)}(W) \mid \Delta \omega = 0\}$, be the space of the q -harmonic forms. Then, we have the *Hodge decomposition*

$$\Omega^q(W) = \mathcal{H}^q(W) \oplus d\Omega^{q-1}(W) \oplus d^\dagger \Omega^{q+1}(W).$$

All this generalizes considering a bundle over W . In particular, we are interested in the following situation. Let $\rho : \pi_1(W) \rightarrow O(k, \mathbb{F})$ be a representation of the fundamental group of W , and let

E_ρ be the associated vector bundle over W with fibre $\mathbb{F}^k$ and group $O(k, \mathbb{F})$, $E_\rho = \widetilde{W} \times_\rho \mathbb{F}^k$. Then, we denote by $\Omega(W, E_\rho)$ be the graded linear space of smooth forms on W with values in E_ρ , namely $\Omega(W, E_\rho) = \Omega(W) \otimes E_\rho$. The exterior differential on W defines the exterior differential on $\Omega^q(W, E_\rho)$, $d : \Omega^q(W, E_\rho) \rightarrow \Omega^{q+1}(W, E_\rho)$. The metric g defines an Hodge operator on W and hence on $\Omega^q(W, E_\rho)$, $\star : \Omega^q(W, E_\rho) \rightarrow \Omega^{m-q}(W, E_\rho)$, and, using the inner product $\langle \cdot, \cdot \rangle$ in E_ρ , an inner product on $\Omega^q(W, E_\rho)$ is defined by

$$(2.4) \quad (\omega, \eta) = \int_W \langle \omega \wedge \star \eta \rangle.$$

It is clear that the adjoint $d^\dagger$ and the Laplacian $\Delta = (d + d^\dagger)^2$ are also defined on the spaces of sections with values in E_ρ . Setting $\mathcal{H}^q(W, E_\rho) = \{\omega \in \Omega^q(W, E_\rho) \mid \Delta\omega = 0\}$, be the space of the q -harmonic forms, we have the Hodge decomposition

$$(2.5) \quad \Omega^q(W, E_\rho) = \mathcal{H}^q(W, E_\rho) \oplus d\Omega^{q-1}(W, E_\rho) \oplus d^\dagger\Omega^{q+1}(W, E_\rho).$$

This induces a decomposition of the eigenspace of a given eigenvalue $\lambda \neq 0$ of $\Delta^{(q)}$ into the spaces of *closed forms* and *coclosed forms*: $\mathcal{E}_\lambda^{(q)} = \mathcal{E}_{\lambda, \text{cl}}^{(q)} \oplus \mathcal{E}_{\lambda, \text{ccl}}^{(q)}$, where

$$\begin{aligned} \mathcal{E}_{\lambda, \text{cl}}^{(q)} &= \{\omega \in \Omega^q(W, E_\rho) \mid \Delta\omega = \lambda\omega, d\omega = 0\}, \\ \mathcal{E}_{\lambda, \text{ccl}}^{(q)} &= \{\omega \in \Omega^q(W, E_\rho) \mid \Delta\omega = \lambda\omega, d^\dagger\omega = 0\}. \end{aligned}$$

Defining *exact forms* and *coexact forms* by

$$\begin{aligned} \mathcal{E}_{\lambda, \text{ex}}^{(q)} &= \{\omega \in \Omega^q(W, E_\rho) \mid \Delta\omega = \lambda\omega, \omega = d\alpha\}, \\ \mathcal{E}_{\lambda, \text{cex}}^{(q)} &= \{\omega \in \Omega^q(W, E_\rho) \mid \Delta\omega = \lambda\omega, \omega = d^\dagger\alpha\}. \end{aligned}$$

Note that, if $\lambda \neq 0$, then $\mathcal{E}_{\lambda, \text{cl}}^{(q)} = \mathcal{E}_{\lambda, \text{ex}}^{(q)}$, and $\mathcal{E}_{\lambda, \text{ccl}}^{(q)} = \mathcal{E}_{\lambda, \text{cex}}^{(q)}$, and we have an isometry

$$(2.6) \quad \begin{aligned} \phi : \mathcal{E}_{\lambda, \text{cl}}^{(q)} &\rightarrow \mathcal{E}_{\lambda, \text{cex}}^{(q-1)}, \\ \phi : \omega &\mapsto \frac{1}{\sqrt{\lambda}} d^\dagger \omega, \end{aligned}$$

whose inverse is $\frac{1}{\sqrt{\lambda}} d$. Also, the restriction of the Hodge star defines an isometry

$$\star : d^\dagger \Omega^{(q+1)}(W) \rightarrow d \Omega^{(m-q-1)}(W),$$

and that composed with the previous one gives the isometries:

$$(2.7) \quad \begin{aligned} \frac{1}{\sqrt{\lambda}} d\star : \mathcal{E}_{\lambda, \text{cl}}^{(q)} &\rightarrow \mathcal{E}_{\lambda, \text{cex}}^{(m-q+1)}, \\ \frac{1}{\sqrt{\lambda}} d^\dagger\star : \mathcal{E}_{\lambda, \text{ccl}}^{(q)} &\rightarrow \mathcal{E}_{\lambda, \text{ex}}^{(m-q-1)}. \end{aligned}$$

2.4. Manifolds with boundary. Next consider a manifold with boundary. If W has a boundary ∂W , then there is a natural splitting near the boundary, of ΛW as direct sum of vector bundles $\Lambda T^* \partial W \oplus N^* W$, where $N^* W$ is the dual to the normal bundle to the boundary. Locally, this reads as follows. Let ∂_x denotes the outward pointing unit normal vector to the boundary, and dx the corresponding one form. Near the boundary we have the collar decomposition $\text{Coll}(\partial W) = (-\epsilon, 0] \times \partial W$, and if y is a system of local coordinates on the boundary, then (x, y) is a local system

of coordinates in $\text{Coll}(\partial W)$. The metric tensor decomposes near the boundary in this local system as

$$g = dx \otimes dx + g_\partial(x),$$

where $g_\partial(x)$ is a family of metric structure on ∂W such that $g_\partial(0) = i^*g$, where $i : \partial W \rightarrow W$ denotes the inclusion. The smooth forms on W near the boundary decompose as $\omega = \omega_{\text{tan}} + \omega_{\text{norm}}$, where ω_{norm} is the orthogonal projection on the subspace generated by dx , and ω_{tan} is in $C^\infty(W) \otimes \Lambda(\partial W)$. We write $\omega = \omega_1 + dx \wedge \omega_2$, where $\omega_j \in C^\infty(W) \otimes \Lambda(\partial W)$, and

$$(2.8) \quad \star \omega_2 = -dx \wedge \star \omega.$$

Define absolute boundary conditions by

$$B_{\text{abs}}(\omega) = \omega_{\text{norm}}|_{\partial W} = \omega_2|_{\partial W} = 0,$$

and relative boundary conditions by

$$B_{\text{rel}}(\omega) = \omega_{\text{tan}}|_{\partial W} = \omega_1|_{\partial W} = 0.$$

Note that, if $\omega \in \Omega^q(W)$, then $B_{\text{abs}}(\omega) = 0$ if and only if $B_{\text{rel}}(\star \omega) = 0$, $B_{\text{rel}}(\omega) = 0$ implies $B_{\text{rel}}(d\omega) = 0$, and $B_{\text{abs}}(\omega) = 0$ implies $B_{\text{abs}}(d^\dagger \omega) = 0$. Let $\mathcal{B}(\omega) = B(\omega) \oplus B((d + d^\dagger)(\omega))$. Then the operator $\Delta = (d + d^\dagger)^2$ with boundary conditions $\mathcal{B}(\omega) = 0$ is self adjoint, and if $\mathcal{B}(\omega) = 0$, then $\Delta \omega = 0$ if and only if $(d + d^\dagger)\omega = 0$. Note that $\mathcal{B}$ correspond to

$$(2.9) \quad B_{\text{abs}}(\omega) = 0 \quad \text{if and only if} \quad \begin{cases} \omega_{\text{norm}}|_{\partial W} = 0, \\ (d\omega)_{\text{norm}}|_{\partial W} = 0, \end{cases}$$

$$(2.10) \quad \mathcal{B}_{\text{rel}}(\omega) = 0 \quad \text{if and only if} \quad \begin{cases} \omega_{\text{tan}}|_{\partial W} = 0, \\ (d^\dagger \omega)_{\text{tan}}|_{\partial W} = 0, \end{cases}$$

Let

$$\begin{aligned} \mathcal{H}^q(W, E_\rho) &= \{\omega \in \Omega^q(W, E_\rho) \mid \Delta^{(q)}\omega = 0\}, \\ \mathcal{H}_{\text{abs}}^q(W, E_\rho) &= \{\omega \in \Omega^q(W, E_\rho) \mid \Delta^{(q)}\omega = 0, B_{\text{abs}}(\omega) = 0\}, \\ \mathcal{H}_{\text{rel}}^q(W, E_\rho) &= \{\omega \in \Omega^q(W, E_\rho) \mid \Delta^{(q)}\omega = 0, B_{\text{rel}}(\omega) = 0\}, \end{aligned}$$

be the spaces of harmonic forms with boundary conditions, then the Hodge decomposition reads

$$\begin{aligned} \Omega_{\text{abs}}^q(W, E_\rho) &= \mathcal{H}_{\text{abs}}^q(W, E_\rho) \oplus d\Omega_{\text{abs}}^{q-1}(W, E_\rho) \oplus d^\dagger \Omega_{\text{abs}}^{q+1}(W, E_\rho), \\ \Omega_{\text{rel}}^q(W, E_\rho) &= \mathcal{H}_{\text{rel}}^q(W, E_\rho) \oplus d\Omega_{\text{rel}}^{q-1}(W, E_\rho) \oplus d^\dagger \Omega_{\text{rel}}^{q+1}(W, E_\rho). \end{aligned}$$

2.5. The form valued zeta functions and the analytic torsion. By the results of the previous sections, the Laplace operator $\Delta^{(q)}$, with boundary conditions $\mathcal{B}_{\text{abs/rel}}$ has a pure point spectrum $\text{Sp}\Delta_{\text{abs/rel}}^{(q)}$ consisting of real non negative eigenvalues. The sequence $\text{Sp}_+ \Delta_{\text{abs/rel}}^{(q)}$ is a totally regular sequence of spectral type accordingly to Section 3.1, and the *forms valued zeta function* is the associated zeta function, defined by

$$\zeta(s, \Delta_{\text{abs/rel}}^{(q)}) = \zeta(s, \text{Sp}_+ \Delta_{\text{abs/rel}}^{(q)}) = \sum_{\lambda \in \text{Sp}_+ \Delta_{\text{abs/rel}}^{(q)}} \lambda^{-s},$$

when $\operatorname{Re}(s) > \frac{m}{2}$ (see Propositions 3.2 and 3.3). The *analytic torsion* $T_{\text{abs/rel}}((W, g); \rho)$ of (W, g) with respect to the representation $\rho : \pi_1(W) \rightarrow O(k, \mathbb{R})$ is defined by

$$\log T_{\text{abs/rel}}((W, g); \rho) = \frac{1}{2} \sum_{q=1}^m (-1)^q q \zeta'(0, \Delta_{\text{abs/rel}}^{(q)}).$$

We will omit the representation in the notation, whenever we mean the trivial representation.

Theorem 2.1. Poincaré duality for analytic torsion [22]. *Let (W, g) be an orientable compact connected Riemannian manifold of dimension m , with possible boundary. Then, for any representation ρ ,*

$$\log T_{\text{abs}}((W, g); \rho) = (-1)^{m+1} \log T_{\text{rel}}((W, g); \rho).$$

We now use results of section 2.3 to define *closed*, *coclosed*, *exact* and *coexact zeta functions*. We again restrict ourselves to the case of a manifold without boundary (see [29] for the case of manifold with boundary). By the very definition, we have

$$\zeta(s, \Delta^{(q)}) = \sum_{\lambda \in \operatorname{Sp}_+ \Delta^{(q)}} \dim \mathcal{E}_\lambda^{(q)} \lambda^{-s} = \zeta_{\text{cl}}(s, \Delta^{(q)}) + \zeta_{\text{ccl}}(s, \Delta^{(q)}),$$

where

$$\begin{aligned} \zeta_{\text{cl}}(s, \Delta^{(q)}) &= \sum_{\lambda \in \operatorname{Sp}_+ \Delta^{(q)}} \dim \mathcal{E}_{\lambda, \text{cl}}^{(q)} \lambda^{-s}, \\ \zeta_{\text{ccl}}(s, \Delta^{(q)}) &= \sum_{\lambda \in \operatorname{Sp}_+ \Delta^{(q)}} \dim \mathcal{E}_{\lambda, \text{ccl}}^{(q)} \lambda^{-s}. \end{aligned}$$

Since, by (2.6), $\zeta_{\text{cl}}(s, \Delta^{(q)}) = \zeta_{\text{ccl}}(s, \Delta^{(q-1)})$, we obtain from the above relations the following formulas for the torsion of a closed m dimensional manifold W :

$$\begin{aligned} \log T((W, g); \rho) &= \frac{1}{2} \sum_{q=1}^m (-1)^q q \zeta'(0, \Delta^{(q)}) = \frac{1}{2} \sum_{q=1}^m (-1)^q \zeta'_{\text{cl}}(0, \Delta^{(q)}) \\ &= -\frac{1}{2} \sum_{q=0}^{m-1} (-1)^q \zeta'_{\text{ccl}}(0, \Delta^{(q)}). \end{aligned}$$

In particular, using again duality, for an odd dimensional manifold W of dimension $m = 2p - 1$,

$$\begin{aligned} \log T((W, g); \rho) &= \sum_{q=1}^{p-1} (-1)^q \zeta'_{\text{cl}}(0, \Delta^{(q)}) + \frac{(-1)^p}{2} \zeta'_{\text{cl}}(0, \Delta^{(p)}) \\ (2.11) \quad &= -\sum_{q=0}^{p-2} (-1)^q \zeta'_{\text{ccl}}(0, \Delta^{(q)}) + \frac{(-1)^p}{2} \zeta'_{\text{ccl}}(0, \Delta^{(p-1)}). \end{aligned}$$

2.6. The Cheeger Müller theorem for manifolds with boundary, and the anomaly boundary term of Brüning and Ma. In case of a smooth orientable compact connect Riemannian manifold (W, g) with boundary ∂W , for any representation ρ of the fundamental group (for simplicity assume $\operatorname{rk}(\rho) = 1$), the analytic torsion is given by the Reidemeister torsion plus

some further contributions. It was shown by J. Cheeger in [7], that this further contribution only depends on the boundary, namely that

$$\log T_{\text{abs}}((W, g); \rho) = \log \tau(W; \rho) + c(\partial W).$$

In the case of a product metric near the boundary, the following formula for this boundary contribution was given by W. Lück [22], where $\chi(X)$ denotes the Euler characteristic of X ,

$$\log T_{\text{abs}}((W, g); \rho) = \log \tau(W; \rho) + \frac{1}{4} \chi(\partial W) \log 2.$$

In the general case a further contribution appears, that measures how the metric is far from a product metric. A formula for this new anomaly boundary contribution is contained in some recent result of Brüning and Ma [3]. More precisely, in [3] (equation (0.6)) is given a formula for the ratio of the analytic torsion of two metrics, g_0 and g_1 ,

$$(2.12) \quad \log \frac{T_{\text{abs}}((W, g_1); \rho)}{T_{\text{abs}}((W, g_0); \rho)} = \frac{1}{2} \int_{\partial W} (B(\nabla_1) - B(\nabla_0)),$$

where ∇_j is the covariant derivative of the metric g_j , the forms $B(\nabla_j)$ are defined as follows, according to Section 1.3 of [3] (observe, however, that we take the opposite sign with respect to the definition in [3], since we are considering left actions instead of right actions, and also that we use the formulas of [3] in the particular case of a flat trivial bundle F). Using the notation of Section 2.1 (in particular see equation (2.1)), we define the following forms

$$(2.13) \quad \begin{aligned} \mathcal{S}_j &= \frac{1}{2} \sum_{k=1}^{m-1} (i^* \omega_j - i^* \omega_0)_{0k} \wedge \hat{e}_k^*, \\ i^* \widehat{\Omega}_j &= \frac{1}{2} \sum_{k,l=1}^{m-1} i^* \Omega_{j,kl} \wedge \hat{e}_k^* \wedge \hat{e}_l^* \\ \hat{\Theta} &= \frac{1}{2} \sum_{k,l=1}^{m-1} \Theta_{kl} \wedge \hat{e}_k^* \wedge \hat{e}_l^*. \end{aligned}$$

Here, ω_j are the connection one forms, and Ω_j , $j = 0, 1$, the curvature two forms associated to the metrics g_0 and g_1 , respectively, while Θ is the curvature two form of the boundary (with the metric induced by the inclusion), and $\{e_k\}_{k=0}^{m-1}$ is an orthonormal base of TW (with respect to the metric g). Then, set

$$(2.14) \quad B(\nabla_j) = \frac{1}{2} \int_0^1 \int^B e^{-\frac{1}{2} \hat{\Theta} - u^2 \mathcal{S}_j^2} \sum_{k=1}^{\infty} \frac{1}{\Gamma(\frac{k}{2} + 1)} u^{k-1} \mathcal{S}_j^k du.$$

Taking $g_1 = g$, and g_0 an opportune deformation of g , that is a product metric near the boundary, it is easy to see that (see equation (4.8) of Section 4.2)

$$\log \frac{T_{\text{abs}}((W, g_1); \rho)}{T_{\text{abs}}((W, g_0); \rho)} = \frac{1}{2} \int_{\partial W} B(\nabla_1).$$

Note that the right end side of this equation is (as expected) a local quantity, and is well defined if there exists a regular collar neighborhood of the boundary. If this is the case, we define the

Brüning and Ma *anomaly boundary term* by

$$(2.15) \quad A_{\text{BM,abs}}(\partial W) = \frac{1}{2} \int_{\partial W} B(\nabla_1),$$

and we have

$$(2.16) \quad \log T_{\text{abs}}((W, g); \rho) = \log \tau(W; \rho) + \frac{1}{4} \chi(\partial W) \log 2 + A_{\text{BM,abs}}(\partial W).$$

3. ZETA DETERMINANTS

In this section we collect some results on the theory of the zeta function associated to a sequence of spectral type introduced in works of M. Spreafico [35] [36] [37] and [39]. This will give the analytic tools necessary in order to evaluate the zeta determinants appearing in the calculation of the analytic torsion in the following sections. We give the basic definition in Section 3.1, some results concerning simple sequences in Section 3.2, the main results for double sequences in Section 3.3, and we specialize to the zeta functions associated to the Laplace operator on Riemannian manifolds in Section 3.4. We present here a simplified version of the theory, that is sufficient for our purpose here; we refer to the mentioned papers for further details (see in particular the general formulation in Theorem 3.9 of [39] or the Spectral Decomposition Lemma of [37]).

3.1. Zeta functions for sequences of spectral type. Let $S = \{a_n\}_{n=1}^{\infty}$ be a sequence of non vanishing complex numbers, ordered by increasing modules, with the unique point of accumulation at infinite. The positive real number (possibly infinite)

$$s_0 = \limsup_{n \rightarrow \infty} \frac{\log n}{\log |a_n|},$$

is called the *exponent of convergence* of S , and denoted by $e(S)$. We are only interested in sequences with $e(S) = s_0 < \infty$. If this is the case, then there exists a least integer p such that the series $\sum_{n=1}^{\infty} a_n^{-p-1}$ converges absolutely. We assume $s_0 - 1 < p \leq s_0$, we call the integer p the *genus* of the sequence S , and we write $p = g(S)$. We define the *zeta function* associated to S by the uniformly convergent series

$$\zeta(s, S) = \sum_{n=1}^{\infty} a_n^{-s},$$

when $\text{Re}(s) > e(S)$, and by analytic continuation otherwise. We call the open subset $\rho(S) = \mathbb{C} - S$ of the complex plane the *resolvent set* of S . For all $\lambda \in \rho(S)$, we define the *Gamma function* associated to S by the canonical product

$$(3.1) \quad \frac{1}{\Gamma(-\lambda, S)} = \prod_{n=1}^{\infty} \left(1 + \frac{-\lambda}{a_n} \right) e^{\sum_{j=1}^{g(S)} \frac{(-1)^j}{j} \frac{(-\lambda)^j}{a_n^j}}.$$

When necessary in order to define the meromorphic branch of an analytic function, the domain for λ will be the open subset $\mathbb{C} - [0, \infty)$ of the complex plane. We use the notation $\Sigma_{\theta, c} = \{z \in \mathbb{C} \mid |\arg(z - c)| \leq \frac{\theta}{2}\}$, with $c \geq \delta > 0$, $0 < \theta < \pi$. We use $D_{\theta, c} = \mathbb{C} - \Sigma_{\theta, c}$, for the complementary (open) domain and $\Lambda_{\theta, c} = \partial \Sigma_{\theta, c} = \{z \in \mathbb{C} \mid |\arg(z - c)| = \frac{\theta}{2}\}$, oriented counter clockwise, for the boundary. With this notation, we define now a particular subclass of sequences. Let S be as above, and assume that $e(S) < \infty$, and that there exist $c > 0$ and $0 < \theta < \pi$, such that S is contained in the interior of the sector $\Sigma_{\theta, c}$. Furthermore, assume that the logarithm of the

associated Gamma function has a uniform asymptotic expansion for large $\lambda \in D_{\theta,c}(S) = \mathbb{C} - \Sigma_{\theta,c}$ of the following form

$$\log \Gamma(-\lambda, S) \sim \sum_{j=0}^{\infty} a_{\alpha_j,0}(-\lambda)^{\alpha_j} + \sum_{k=0}^{g(S)} a_{k,1}(-\lambda)^k \log(-\lambda),$$

where $\{\alpha_j\}$ is a decreasing sequence of real numbers. Then, we say that S is a *totally regular sequence of spectral type with infinite order*. We call the open set $D_{\theta,c}(S)$ the *asymptotic domain* of S .

3.2. The zeta determinant of some simple sequences. The results of this section are known to specialists, and can be found in different places. We will use the formulation of [34]. For positive real numbers l and q , define the *non homogeneous quadratic Bessel zeta function* by

$$z(s, \nu, q, l) = \sum_{k=1}^{\infty} \left(\frac{j_{\nu,k}^2}{l^2} + q^2 \right)^{-s},$$

for $\operatorname{Re}(s) > \frac{1}{2}$. Then, $z(s, \nu, q, l)$ extends analytically to a meromorphic function in the complex plane with simple poles at $s = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \dots$. The point $s = 0$ is a regular point and

$$(3.2) \quad \begin{aligned} z(0, \nu, q, l) &= -\frac{1}{2} \left(\nu + \frac{1}{2} \right), \\ z'(0, \nu, q, l) &= -\log \sqrt{2\pi l} \frac{I_{\nu}(lq)}{q^{\nu}}. \end{aligned}$$

In particular, taking the limit for $q \rightarrow 0$,

$$z'(0, \nu, 0, l) = -\log \frac{\sqrt{\pi} l^{\nu+\frac{1}{2}}}{2^{\nu-\frac{1}{2}} \Gamma(\nu+1)}.$$

3.3. Zeta determinant for a class of double sequences. Let $S = \{\lambda_{n,k}\}_{n,k=1}^{\infty}$ be a double sequence of non vanishing complex numbers with unique accumulation point at the infinity, finite exponent $s_0 = e(S)$ and genus $p = g(S)$. Assume if necessary that the elements of S are ordered as $0 < |\lambda_{1,1}| \leq |\lambda_{1,2}| \leq |\lambda_{2,1}| \leq \dots$. We use the notation S_n (S_k) to denote the simple sequence with fixed n (k). We call the exponents of S_n and S_k the *relative exponents* of S , and we use the notation $(s_0 = e(S), s_1 = e(S_k), s_2 = e(S_n))$. We define *relative genus* accordingly.

Definition 3.1. Let $S = \{\lambda_{n,k}\}_{n,k=1}^{\infty}$ be a double sequence with finite exponents (s_0, s_1, s_2) , genus (p_0, p_1, p_2) , and positive spectral sector Σ_{θ_0, c_0} . Let $U = \{u_n\}_{n=1}^{\infty}$ be a totally regular sequence of spectral type of infinite order with exponent r_0 , genus q , domain $D_{\phi, d}$. We say that S is *spectrally decomposable over U with power κ , length ℓ and asymptotic domain $D_{\theta, c}$* , with $c = \min(c_0, d, c')$, $\theta = \max(\theta_0, \phi, \theta')$, if there exist positive real numbers κ , ℓ (integer), c' , and θ' , with $0 < \theta' < \pi$, such that:

- (1) the sequence $u_n^{-\kappa} S_n = \left\{ \frac{\lambda_{n,k}}{u_n^{\kappa}} \right\}_{k=1}^{\infty}$ has spectral sector $\Sigma_{\theta', c'}$, and is a totally regular sequence of spectral type of infinite order for each n ;
- (2) the logarithmic Γ -function associated to S_n/u_n^{κ} has an asymptotic expansion for large n uniformly in λ for λ in $D_{\theta, c}$, of the following form

$$(3.3) \quad \log \Gamma(-\lambda, u_n^{-\kappa} S_n) = \sum_{h=0}^{\ell} \phi_{\sigma_h}(\lambda) u_n^{-\sigma_h} + \sum_{l=0}^L P_{\rho_l}(\lambda) u_n^{-\rho_l} \log u_n + o(u_n^{-r_0}),$$

where σ_h and ρ_l are real numbers with $\sigma_0 < \dots < \sigma_\ell$, $\rho_0 < \dots < \rho_L$, the $P_{\rho_l}(\lambda)$ are polynomials in λ satisfying the condition $P_{\rho_l}(0) = 0$, ℓ and L are the larger integers such that $\sigma_\ell \leq r_0$ and $\rho_L \leq r_0$.

When a double sequence S is spectrally decomposable over a simple sequence U , Theorem 3.9 of [39] gives a formula for the derivative of the associated zeta function at zero. In order to understand such a formula, we need to introduce some other quantities. First, we define the functions

$$(3.4) \quad \Phi_{\sigma_h}(s) = \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_{\theta,c}} \frac{e^{-\lambda t}}{-\lambda} \phi_{\sigma_h}(\lambda) d\lambda dt.$$

Next, by Lemma 3.3 of [39], for all n , we have the expansions:

$$(3.5) \quad \begin{aligned} \log \Gamma(-\lambda, S_n/u_n^\kappa) &\sim \sum_{j=0}^\infty a_{\alpha_j,0,n}(-\lambda)^{\alpha_j} + \sum_{k=0}^{p_2} a_{k,1,n}(-\lambda)^k \log(-\lambda), \\ \phi_{\sigma_h}(\lambda) &\sim \sum_{j=0}^\infty b_{\sigma_h,\alpha_j,0}(-\lambda)^{\alpha_j} + \sum_{k=0}^{p_2} b_{\sigma_h,k,1}(-\lambda)^k \log(-\lambda), \end{aligned}$$

for large λ in $D_{\theta,c}$. We set (see Lemma 3.5 of [39])

$$(3.6) \quad \begin{aligned} A_{0,0}(s) &= \sum_{n=1}^\infty \left(a_{0,0,n} - \sum_{h=0}^\ell b_{\sigma_h,0,0} u_n^{-\sigma_h} \right) u_n^{-\kappa s}, \\ A_{j,1}(s) &= \sum_{n=1}^\infty \left(a_{j,1,n} - \sum_{h=0}^\ell b_{\sigma_h,j,1} u_n^{-\sigma_h} \right) u_n^{-\kappa s}, \quad 0 \leq j \leq p_2. \end{aligned}$$

We can now state the formula for the derivative at zero of the double zeta function. We give here a modified version of Theorem 3.9 of [39], more suitable for our purpose here. This is based on the following fact. The key point in the proof of Theorem 3.9 of [39] is the decomposition given in Lemma 3.5 of that paper of the sum

$$\mathcal{T}(s, \lambda, S, U) = \sum_{n=1}^\infty u_n^{-\kappa s} \log \Gamma(-\lambda, u_n^{-\kappa} S_n),$$

in two terms: the regular part $\mathcal{P}(s, \lambda, S, U)$ and the remaining singular part. The regular part is obtained subtracting from $\mathcal{T}$ some terms constructed starting from the expansion of the logarithmic Gamma function given in equation (3.3), namely

$$\mathcal{P}(s, \lambda, S, u) = \mathcal{T}(s, \lambda, S, U) - \sum_{h=0}^\ell \phi_{\sigma_h}(\lambda) u_n^{-\sigma_h} - \sum_{l=0}^L P_{\rho_l}(\lambda) u_n^{-\rho_l} \log u_n.$$

Now, assume instead we subtract only the terms such that the zeta function $\zeta(s, U)$ has a pole at $s = \sigma_h$ or at $s = \rho_l$. Let $\hat{\mathcal{P}}(s, \lambda, S, U)$ be the resulting function. Then the same argument as the one used in Section 3 of [39] in order to prove Theorem 3.9 applies, and we obtain similar formulas for the values of the residue, and of the finite part of the zeta function $\zeta(s, S)$ and of its derivative at zero, with just two differences: first, in all the sums, all the terms with index σ_h such that $s = \sigma_h$ is not a pole of $\zeta(s, U)$ must be omitted; and second, we must substitute the terms $A_{0,0}(0)$ and $A'_{0,1}(0)$, with the finite parts of the analytic continuation of $A_{0,0}(s)$, and $A'_{0,1}(s)$. The first modification is an obvious consequence of the substitution of the function $\mathcal{P}$ by the function $\hat{\mathcal{P}}$. The

second modification, follows by the same reason noting that the function $A_{\alpha_j, k}(s)$ defined in Lemma 3.5 of [39] are no longer regular at $s = 0$ themselves. However, they both admits a meromorphic extension regular at $s = 0$, using the extension of the zeta function $\zeta(s, U)$, and the expansion of the coefficients $a_{\alpha_j, k, n}$ for large n . Thus we have the following result.

Theorem 3.1. *The formulas of Theorem 3.9 of [39] hold if all the quantities with index σ_h such that the zeta function $\zeta(s, U)$ has not a pole at $s = \sigma_h$ are omitted. In such a case, the result must be read by means of the analytic extension of the zeta function $\zeta(s, U)$.*

Next, assuming some simplified pole structure for the zeta function $\zeta(s, U)$, sufficient for the present analysis, we state the main result of this section.

Theorem 3.2. *Let S be spectrally decomposable over U as in Definition 3.1. Assume that the functions $\Phi_{\sigma_h}(s)$ have at most simple poles for $s = 0$. Then, $\zeta(s, S)$ is regular at $s = 0$, and*

$$\begin{aligned} \zeta(0, S) &= -A_{0,1}(0) + \frac{1}{\kappa} \sum_{h=0}^{\ell} \operatorname{Res}_1 \Phi_{\sigma_h}(s) \operatorname{Res}_1 \zeta(s, U), \\ \zeta'(0, S) &= -A_{0,0}(0) - A'_{0,1}(0) + \frac{\gamma}{\kappa} \sum_{h=0}^{\ell} \operatorname{Res}_1 \Phi_{\sigma_h}(s) \operatorname{Res}_1 \zeta(s, U) \\ &\quad + \frac{1}{\kappa} \sum_{h=0}^{\ell} \operatorname{Res}_0 \Phi_{\sigma_h}(s) \operatorname{Res}_1 \zeta(s, U) + \sum_{h=0}^{\ell'} \operatorname{Res}_1 \Phi_{\sigma_h}(s) \operatorname{Res}_0 \zeta(s, U), \end{aligned}$$

where the notation $\sum'$ means that only the terms such that $\zeta(s, U)$ has a pole at $s = \sigma_h$ appear in the sum.

This result should be compared with the Spectral Decomposition Lemma of [37] and Proposition 1 of [38].

Remark 3.1. *We call regular part of $\zeta(0, S)$ the first term appearing in the formula given in the theorem, and regular part of $\zeta'(0, S)$ the first two terms. The other terms gives what we call singular part.*

Corollary 3.1. *Let $S_{(j)} = \{\lambda_{(j), n, k}\}_{n, k=1}^{\infty}$, $j = 1, \dots, J$, be a finite set of double sequences that satisfy all the requirements of Definition 3.1 of spectral decomposability over a common sequence U , with the same parameters κ , ℓ , etc., except that the polynomials $P_{(j), \rho}(\lambda)$ appearing in condition (2) do not vanish for $\lambda = 0$. Assume that some linear combination $\sum_{j=1}^J c_j P_{(j), \rho}(\lambda)$, with complex coefficients, of such polynomials does satisfy this condition, namely that $\sum_{j=1}^J c_j P_{(j), \rho}(\lambda) = 0$. Then, the linear combination of the zeta function $\sum_{j=1}^J c_j \zeta(s, S_{(j)})$ is regular at $s = 0$ and satisfies the linear combination of the formulas given in Theorem 3.2.*

3.4. Zeta invariants of compact Riemannian manifolds. We recall in this section some known facts about zeta invariants of a compact manifold. We will rewrite such results in the terminology of zeta functions associated to sequences of spectral type just introduced. Our main reference are the works of P. Gilkey, in particular we refer to the book [14].

Let (W, g) be a compact connected Riemannian manifold of dimension m , with metric g . Let $\Delta^{(q)}$ denote the metric Laplacian on forms on W , and $\operatorname{Sp} \Delta^{(q)} = \{\lambda_n\}_{n=0}^{\infty}$ ($\lambda_0 = 0$) its spectrum.

Then, there exists a full asymptotic expansion for the trace of the heat kernel of $\Delta^{(q)}$ for small t ,

$$(3.7) \quad \text{Tr}_{L^2} e^{-t\Delta^{(q)}} = t^{-\frac{m}{2}} \sum_{j=0}^{\infty} e_{q,j} t^{\frac{j}{2}},$$

where the coefficients depend only on local invariants constructed from the metric tensor, and are in principle calculable from it.

Proposition 3.1.

$$\begin{aligned} e_{q,0} &= \frac{1}{(4\pi)^{\frac{m}{2}}} \binom{m}{q} \int_W d\text{vol}_g, \\ e_{q,2} &= \frac{1}{6(4\pi)^{\frac{m}{2}}} \left(\binom{m}{q} - 6 \binom{m-2}{q-1} \right) \int_W \tau d\text{vol}_g, \\ e_{q,4} &= \frac{1}{360(4\pi)^{\frac{m}{2}}} \left(5 \binom{m}{q} - 60 \binom{m-2}{q-1} + 180 \binom{m-4}{p-2} \right) \int_W \tau^2 d\text{vol}_g \\ &\quad + \frac{1}{360(4\pi)^{\frac{m}{2}}} \left(-2 \binom{m}{q} + 180 \binom{m-2}{q-1} - 720 \binom{m-4}{p-2} \right) \int_W |\text{Ric}|^2 d\text{vol}_g \\ &\quad + \frac{1}{360(4\pi)^{\frac{m}{2}}} \left(2 \binom{m}{q} - 30 \binom{m-2}{q-1} + 180 \binom{m-4}{p-2} \right) \int_W |R|^2 d\text{vol}_g. \end{aligned}$$

Proposition 3.2. *The sequence $\text{Sp}_+ \Delta^{(q)}$ of the positive eigenvalues of the metric Laplacian on forms on a compact connected Riemannian manifold of dimension m , is a totally regular sequence of spectral type, with finite exponent $\mathbf{e} = \frac{m}{2}$, genus $\mathbf{g} = [\mathbf{e}]$, spectral sector $\Sigma_{\theta,c}$ with some $0 < c < \lambda_1$, $\epsilon < \theta < \frac{\pi}{2}$, asymptotic domain $D_{\theta,c} = \mathbb{C} - \Sigma_{\theta,c}$, and infinite order.*

Proposition 3.3. *The zeta function $\zeta(s, \text{Sp}_+ \Delta^{(q)})$ has a meromorphic continuation to the whole complex plane up to simple poles at the values of $s = \frac{m-h}{2}$, $h = 0, 1, 2, \dots$, that are not negative integers nor zero, with residues*

$$\text{Res}_{s=\frac{m-h}{2}} \zeta(s, \text{Sp}_+ \Delta^{(q)}) = \frac{e_{q,h}}{\Gamma(\frac{m-h}{2})},$$

the point $s = -k = 0, -1, -2, \dots$ are regular points and

$$\begin{aligned} \zeta(0, \text{Sp}_+ \Delta^{(q)}) &= e_{q,m} - \dim \ker \Delta^{(q)}, \\ \zeta(-k, \text{Sp}_+ \Delta^{(q)}) &= (-1)^k k! e_{q,m+2k}. \end{aligned}$$

4. GEOMETRIC SETTING AND LAPLACE OPERATOR

4.1. The finite metric cone. Let (W, g) be an orientable compact connected Riemannian manifold of finite dimension m without boundary and with Riemannian structure g . We denote by CW the cone over W , namely the mapping cone of the constant map $: W \rightarrow \{p\}$. Then, CW is compact connected separable Hausdorff space, but in general is not a topological manifold. However, if we remove the tip of the cone p , then $CW - \{p\}$ is an open differentiable manifold, with the obvious differentiable structure. Embedding W in the opportune Euclidean space $\mathbb{R}^k$, and $\mathbb{R}^k$ in some hyperplane of $\mathbb{R}^{k+h}$, with opportune h , disconnected from the origin, a geometric realization of CW is the given by the set of the finite length l line segments joining the origin to the embedded

copy of W . Let x the euclidean geodesic distance from the origin, if we equip $CW - \{p\}$ with the Riemannian structure

$$(4.1) \quad dx \otimes dx + x^2 g,$$

this coincides with the metric structure induced by the described embedding. We denote by $C_{(0,l]}W$ the space $(0, l] \times W$ with the metric in equation (4.1). We denote by $C_l W$ the compact space $\overline{C_{(0,l]}W} = C_{(0,l]}W \cup \{p\}$. We call the space $C_l W$ the *(completed finite metric) cone* over W . We call the subspace $\{l\} \times W$ of $C_l W$, the *boundary of the cone*, and we denote it by $\partial C_l W$. This is of course diffeomorphic to W , and isometric to $(W, l^2 g)$. For the global coordinate x corresponds to the local coordinate $x' = l - x$, where x' is the geodesic distance from the boundary. Therefore, $g_\partial(x') = (l - x')^2 g$, and if $i : W \rightarrow C_{(0,l]}W$ denotes the inclusion, $i^*(dx \otimes dx + x^2 g) = g_\partial(0) = l^2 g$. Following common notation, we will call (W, g) the *section* of the cone. Also following usual notation, a tilde will denotes operations on the section (of course $\tilde{g} = g$), and not on the boundary. All the results of Section 2.4 are valid. In particular, given a local coordinate system y on W , then (x, y) is a local coordinate system on the cone.

We now give the explicit form of $\star$, $d^\dagger$ and Δ . See [8] [9] and [27] Section 5 for details. If $\omega \in \Omega^q(C_{(0,l]}W)$, set

$$\omega(x, y) = f_1(x)\omega_1(y) + f_2(x)dx \wedge \omega_2(y),$$

with smooth functions f_1 and f_2 , and $\omega_j \in \Omega(W)$. Then a straightforward calculation gives

$$(4.2) \quad \star \omega(x, y) = x^{m-2q+2} f_2(x) \tilde{\star} \omega_2(y) + (-1)^q x^{m-2q} f_1(x) dx \wedge \tilde{\star} \omega_1(y),$$

$$(4.3) \quad \begin{aligned} d\omega(x, y) &= f_1(x) \tilde{d}\omega_1(y) + \partial_x f_1(x) dx \wedge \omega_1(y) - f_2(x) dx \wedge d\omega_2(y), \\ d^\dagger \omega(x, y) &= x^{-2} f_1(x) \tilde{d}^\dagger \omega_1(y) - ((m-2q+2)x^{-1} f_2(x) + \partial_x f_2(x)) \omega_2(y) \\ &\quad - x^{-2} f_2(x) dx \wedge \tilde{d}^\dagger \omega_2(y), \end{aligned}$$

$$(4.4) \quad \begin{aligned} \Delta \omega(x, y) &= (-\partial_x^2 f_1(x) - (m-2q)x^{-1} \partial_x f_1(x)) \omega_1(y) + x^{-2} f_1(x) \tilde{\Delta} \omega_1(y) - 2x^{-1} f_2(x) \tilde{d}\omega_2(y) \\ &\quad + dx \wedge (x^{-2} f_2(x) \tilde{\Delta} \omega_2(y) + \omega_2(y) (-\partial_x^2 f_2(x) - (m-2q+2)x^{-1} \partial_x f_2(x) \\ &\quad + (m-2q+2)x^{-2} f_2(x)) - 2x^{-3} f_1(x) \tilde{d}^\dagger \omega_1(y)). \end{aligned}$$

4.2. Riemannian tensors on the cone. We give here the explicit form of the main Riemannian quantities on the cone. All calculation are based on the formulas given in Sections 2.2 and 2.6. Recall that a tilde denotes quantities relative to the section, that we have local coordinate $(x, y_1, \dots, y_m)$ on $C_l W$, and that the metric is

$$g_1 = dx \otimes dx + x^2 g.$$

Let $\{b_k\}_{k=1}^m$ be a local orthonormal base of TW , and $\{b_k^*\}_{k=1}^m$ the associated dual base. Then,

$$\begin{aligned} e_0 &= \partial_x & e_0^* &= dx, \\ e_k &= \frac{1}{x} b_k, & e_k^* &= x b_k^*, \end{aligned} \quad 1 \leq k \leq m.$$

Direct calculations give Cartan structure constants

$$\begin{aligned} c_{jk0} &= 0, & 1 \leq j, k \leq m, \\ c_{0kl} &= -c_{k0l} = -\frac{\delta_{kl}}{x}, & 1 \leq k, l \leq m, \\ c_{jkl} &= \frac{1}{x} \tilde{c}_{jkl}, & 1 \leq j, k, l \leq m. \end{aligned}$$

The Christoffel symbols are

$$\begin{aligned} \Gamma_{0kl} &= 0, & 1 \leq k, l \leq m, \\ \Gamma_{j0k} &= -\Gamma_{jk0} = \frac{\delta_{jk}}{x}, & 1 \leq j, k \leq m, \\ \Gamma_{jkl} &= \frac{1}{x} \tilde{\Gamma}_{jkl}, & 1 \leq j, k, l \leq m. \end{aligned}$$

The connection one form matrix relatively to the metric g_1 has components

$$\begin{aligned} \omega_{1,00} &= 0, \\ \omega_{1,0j} &= -\omega_{1,j0} = -\frac{1}{x} e_j^* = -b_j^*, & 1 \leq j \leq m, \\ \omega_{1,jk} &= \sum_{h=1}^m \Gamma_{hjk} e_h^* = \frac{1}{x} \sum_{h=1}^m \tilde{\Gamma}_{hjk} e_h^* = \sum_{h=1}^m \tilde{\Gamma}_{hjk} b_h^* = \tilde{\omega}_{jk}, & 1 \leq j, k \leq m. \end{aligned} \quad (4.5)$$

To compute the curvature we calculate

$$d\omega_{1,0j} = -\sum_{l=1}^m (\partial_l b_j^*) \wedge dy_l = -\sum_{l,k=1}^m (\partial_l b_{kj}^*) dy_k \wedge dy_l,$$

where $b_j^* = \sum_{k=1}^m b_{kj}^* dy_k$, and, for $1 \leq j, k \leq m$,

$$d\omega_{1,jk} = \tilde{d}\tilde{\omega}_{jk};$$

while

$$\begin{aligned} -(\omega_1 \wedge \omega_1)_{k0} &= (\omega_1 \wedge \omega_1)_{0k} = \sum_{l=0}^m \omega_{1,0l} \wedge \omega_{1,lk} = \sum_{l=1}^m \omega_{1,0l} \wedge \omega_{1,lk} = -\sum_{l=1}^m b_l^* \wedge \tilde{\omega}_{lk}, \\ (\omega_1 \wedge \omega_1)_{jk} &= \sum_{l=0}^m \omega_{1,jl} \wedge \omega_{1,lk} = \omega_{1,j0} \wedge \omega_{1,0k} + \sum_{l=1}^m \omega_{1,jl} \wedge \omega_{1,lk} = -b_j^* \wedge b_k^* + (\tilde{\omega} \wedge \tilde{\omega})_{jk}, \end{aligned}$$

for $1 \leq j, k \leq m$. The curvature two form has components

$$\begin{aligned} \Omega_{1,00} &= 0, \\ \Omega_{1,0j} &= -\sum_{l,k=1}^m (\partial_l b_{kj}^*) dy_k \wedge dy_l - \sum_{l=1}^m b_l^* \wedge \tilde{\omega}_{lk}, & 1 \leq j \leq m, \\ \Omega_{1,jk} &= \tilde{d}\tilde{\omega}_{jk} - b_j^* \wedge b_k^* + (\tilde{\omega} \wedge \tilde{\omega})_{jk} = \tilde{\Omega}_{jk} - b_j^* \wedge b_k^*, & 1 \leq j, k \leq m. \end{aligned}$$

Next, considering the metric $g_0 = dx \otimes dx + g$, similar calculations gives:

$$\begin{aligned} \omega_{0,0j} &= 0, & 0 \leq j \leq m, \\ \omega_{0,jk} &= \tilde{\omega}_{jk}, & 1 \leq j, k \leq m. \end{aligned} \quad (4.6)$$

By equations (4.5) and (4.6),

$$(4.7) \quad \mathcal{S}_1 = -\frac{1}{2l} \sum_{k=1}^m e_k^* \wedge e_k^* = -\frac{l}{2} \sum_{k=1}^m b_k^* \wedge b_k^* = -\frac{1}{2} \sum_{k=1}^m b_k^* \wedge e_k^*,$$

$$(4.8) \quad \mathcal{S}_0 = 0.$$

We also need the curvature two form Θ on the boundary $\partial C_l W$. A similar calculation gives

$$\Theta_{jk} = \tilde{\Omega}_{jk}.$$

Note in particular that it is easy to verify the equation (1.16) of [3]: $\hat{\Theta} = \widehat{i^* \Omega_1} - 2\mathcal{S}_1^2$. For

$$2\mathcal{S}_1^2 = -\frac{l^2}{2} \sum_{j,k=1}^m b_j^* \wedge b_k^* \wedge \hat{b}_j^* \wedge \hat{b}_k^*,$$

$$\hat{\Theta} = \frac{l^2}{2} \sum_{j,k=1}^m \tilde{\Omega}_{jk} \wedge \hat{b}_j^* \wedge \hat{b}_k^*,$$

while $(i^* \Omega)_{jk} = \tilde{\Omega}_{jk} - b_j^* \wedge b_k^*$, gives

$$\widehat{i^* \Omega_{1,jk}} = \frac{l^2}{2} \sum_{j,k=1}^m (\tilde{\Omega}_{jk} - b_j^* \wedge b_k^*) \wedge \hat{b}_j^* \wedge \hat{b}_k^*.$$

4.3. The Laplace operator on the cone and its spectrum. We study the Laplace operator on forms on the space $C_l W$. This is essentially based on [9] and [6]. Let denote by $\mathcal{L}$ the formal differential operator defined by equation (4.4) acting on smooth forms on $C_{(0,l]} W$, $\Gamma(C_{(0,l]} W, \Lambda T^* C_{(0,l]} W)$. We define in Lemma 4.1 a self adjoint operator Δ acting on $L^2(C_l W, \Lambda^{(q)} C_l W)$, and such that $\Delta \omega = \mathcal{L} \omega$, if $\omega \in \text{dom} \Delta$. Then, in Lemma 4.2, we list all the solutions of the eigenvalues equation for $\mathcal{L}$. Eventually, in Lemma 4.3, we give the spectrum of Δ .

Lemma 4.1. *The formal operator $\mathcal{L}$ in equation (4.4) with the absolute/relative boundary conditions given in equations (2.9)/(2.10) on the boundary $\partial C_l W$ defines a unique self adjoint semi bounded operator on $L^2(C_l W, \Lambda^{(q)} T^* C_l W)$, that we denote by the symbol $\Delta_{\text{abs}}/\Delta_{\text{rel}}$, respectively, with pure point spectrum.*

Proof. Let $L^{(q)}$ denote the minimal operator defined by the formal operator $\mathcal{L}^{(q)}$, with domain the q -forms with compact support in $C_{(0,l]} W$, namely $\text{dom} L^{(q)} = \Gamma_0(C_{(0,l]} W, \Lambda T^* C_{(0,l]} W)$. The boundary values problem at the boundary $x = l$, i.e. $\partial C_l W$, is trivial, and gives the self adjoint extensions stated. The point $x = 0$ requires more work. First, note that $L^{(q)}$ reduces by unitary transformation to an operator of the type

$$(4.9) \quad D^2 + \frac{A(x)}{x^2}, \quad D = -i \frac{d}{dx},$$

where $A(x)$ is smooth family of symmetric second order elliptic operators [6] pg. 370. More precisely, the map

$$\psi_q : C_0^\infty((0, l], \Lambda^{(q)} T^* W \times \Lambda^{(q-1)} T^* W) \rightarrow C^\infty(C_{(0,l]} W, \Lambda^{(q)} T^* C_{(0,l]} W),$$

$$\psi_q : (\omega^{(q)}, \omega^{(q-1)}) \mapsto x^{q-m/2} \pi^* \omega^{(q)}(x) + x^{p-1-m/2} \pi^* \omega^{(q-1)}(x) \wedge dx,$$

where $\pi : C_{(0,l]}W \rightarrow W$, is bijective onto forms with compact support. Moreover, ψ_q is unitary with respect to the usual L^2 structure on the function space $\mathcal{F}(C_l W, \Lambda^{(q)} T^* C_l W)$ and the Hilbert space structure on $\mathcal{F}([0, l], \Lambda^{(q)} T^* W \times \Lambda^{(q-1)} T^* W)$ given by

$$\int_0^l \left(\|\omega^{(q)}(x)\|_{\Lambda^{(q)} T^* W, g}^2 + \|\omega^{(q-1)}(x)\|_{\Lambda^{(q-1)} T^* W, g}^2 \right) dx.$$

Under the transformation ψ_q , $L^{(q)}$ has the form in equation (4.9), with $A(x)$ the constant smooth family of symmetric second order elliptic operators in $\Gamma(W, \Lambda^{(q)} T^* W \times \Lambda^{(q-1)} T^* W)$:

$$A(x) = A(0) = \begin{pmatrix} \tilde{\Delta}^{(q)} + \left(\frac{m}{2} - q\right) \left(\frac{m}{2} - q - 1\right) & 2(-1)^q \tilde{d} \\ \tilde{\Delta}^{(q-1)} + \left(\frac{m}{2} + 2 - q\right) \left(\frac{m}{2} + 1 - q\right) & \end{pmatrix}$$

Next, by its definition, $A(x)$ satisfies all the requirements at pg. 373 of [6], with $p = 1$ (in particular this follows from the fact that $A(x)$ is defined by the Laplacian on forms on a compact space). We can apply the results of Brüning and Seeley [5] [6], observing that in the present case we are in what they call “constant coefficient case” (Section 3 of [6]). By Theorem 5.1 of [6], the operator L extends to a unique self adjoint bounded operator $\Delta^{(q)}$. Note that this extension is the Friedrich extension by Theorem 6.1 of [6]. Note also that boundary condition at $x = 0$ are necessary in general in the definition of the domain of $\Delta^{(q)}$, see (L2) (c), pg. 410 of [6] for these conditions.

Eventually, by Theorem 5.2 of [6], the square (here $p = 1$, so $m = 2$) of the resolvent of $\Delta^{(q)}$ is of trace class. This means that the resolvent is Hilbert Schmidt, and consequently the spectrum of $\Delta^{(q)}$ is pure point, by the spectral theorem for compact operators. Note that we do not need the cut off function γ appearing in Theorem 5.2 of [6], since here $0 < x \leq l$.

□

Lemma 4.2. [9] *Let $\{\varphi_{\text{har},n}^{(q)}, \varphi_{\text{cex},n}^{(q)}, \varphi_{\text{ex},n}^{(q)}\}$ be an orthonormal base of $\Gamma(W, \Lambda^{(q)} T^* W)$ consisting of harmonic, coexact and exact eigenforms of $\tilde{\Delta}^{(q)}$ on W . Let $\lambda_{q,n}$ denotes the eigenvalue of $\varphi_{\text{cex},n}^{(q)}$ and $m_{\text{cex},q,n}$ its multiplicity (so that $m_{\text{cex},q,n} = \dim \mathcal{E}_{\text{cex},n}^{(q)} = \dim \mathcal{E}_{\text{ccl},n}^{(q)}$). Let J_ν be the Bessel function of index ν . Define*

$$\alpha_q = \frac{1}{2}(1 + 2q - m),$$

$$\mu_{q,n} = \sqrt{\lambda_{q,n} + \alpha_q^2}.$$

Then, assuming that $\mu_{q,n}$ is not an integer, all the solutions of the equation $\Delta u = \lambda^2 u$, with $\lambda \neq 0$, are convergent sums of forms of the following six types:

$$\begin{aligned} \psi_{\pm,1,n,\lambda}^{(q)} &= x^{\alpha_q} J_{\pm\mu_{q,n}}(\lambda x) \varphi_{\text{cex},n}^{(q)}, \\ \psi_{\pm,2,n,\lambda}^{(q)} &= x^{\alpha_{q-1}} J_{\pm\mu_{q-1,n}}(\lambda x) \tilde{d}\varphi_{\text{cex},n}^{(q-1)} + \partial_x(x^{\alpha_{q-1}} J_{\pm\mu_{q-1,n}}(\lambda x)) dx \wedge \varphi_{\text{cex},n}^{(q-1)} \\ \psi_{\pm,3,n,\lambda}^{(q)} &= x^{2\alpha_{q-1}+1} \partial_x(x^{-\alpha_{q-1}} J_{\pm\mu_{q-1,n}}(\lambda x)) \tilde{d}\varphi_{\text{cex},n}^{(q-1)} \\ &\quad + x^{\alpha_{q-1}-1} J_{\pm\mu_{q-1,n}}(\lambda x) dx \wedge \tilde{d}^\dagger \tilde{d}\varphi_{\text{cex},n}^{(q-1)} \\ \psi_{\pm,4,n,\lambda}^{(q)} &= x^{\alpha_{q-2}+1} J_{\pm\mu_{q-2,n}}(\lambda x) dx \wedge \tilde{d}\varphi_{\text{cex},n}^{(q-2)} \\ \psi_{\pm,E,\lambda}^{(q)} &= x^{\alpha_q} J_{\pm|\alpha_q|}(\lambda x) \varphi_{\text{har},n}^{(q)} \\ \psi_{\pm,O,\lambda}^{(q)} &= \partial_x(x^{\alpha_{q-1}} J_{\pm|\alpha_{q-1}|}(\lambda x)) dx \wedge \varphi_{\text{har},n}^{(q-1)}. \end{aligned}$$

When $\mu_{q,n}$ is an integer the $-$ solutions must be modified including some logarithmic term (see for example [41] for a set of linear independent solutions of the Bessel equation).

Proof. The proof is a direct verification of the assertion, using the definitions in equations (4.2), (4.3), and (4.4). First, by Hodge theorem, there exist an orthonormal base of $\Lambda^{(q)}T^*W$ as stated. Thus, we decompose any form ω in this base. Second, we compute $\Delta\omega$, using this decomposition and the formula in equation (4.4). This gives some differential equations in the functions appearing as coefficients of the forms. All these differential equations reduce to equations of Bessel type. Third, we write all the solutions using Bessel functions. A complete proof for the case of the harmonic forms can be found in [27] Section 5. $\square$

Note that the forms of types 1 and 3 are coexact, those of types 2 and 4 exacts. The operator d sends forms of types 1 and 3 in forms of types 2 and 4, while $d^\dagger$ sends forms of types 2 and 4 in forms of types 1 and 3, respectively. The Hodge operator sends forms of type 1 in forms of type 4, 2 in 3, and E in 0.

Corollary 4.1. *The functions $+$ in Lemma 4.1 are square integrable and satisfy the boundary conditions at $x = 0$ defining the domain of $\Delta_{\text{rel/abs}}$. The functions $-$ either are not square integrable or do not satisfy these conditions.*

Remark 4.1. *All the $-$ solutions are either not square or their exterior derivative are not square integrable. Requiring the last condition in the definition of the domain of $\Delta_{\text{rel/abs}}$, it follows that there are not boundary conditions at zero. This was observed by Cheeger for harmonic forms when the dimension is odd in [9] Section 3.*

Lemma 4.3. *The positive part of the spectrum of the Laplace operator on forms on C_lW , with absolute boundary conditions on ∂C_lW is:*

$$\begin{aligned} \text{Sp}_+\Delta_{\text{abs}}^{(q)} = & \left\{ m_{\text{cex},q,n} : \hat{j}_{\mu_{q,n},\alpha_q,k}^2/l^2 \right\}_{n,k=1}^\infty \cup \left\{ m_{\text{cex},q-1,n} : \hat{j}_{\mu_{q-1,n},\alpha_{q-1},k}^2/l^2 \right\}_{n,k=1}^\infty \\ & \cup \left\{ m_{\text{cex},q-1,n} : j_{\mu_{q-1,n},k}^2/l^2 \right\}_{n,k=1}^\infty \cup \left\{ m_{q-2,n} : j_{\mu_{q-2,n},k}^2/l^2 \right\}_{n,k=1}^\infty \\ & \cup \left\{ m_{\text{har},q,0} : \hat{j}_{|\alpha_q|,\alpha_q,k}^2/l^2 \right\}_{k=1}^\infty \cup \left\{ m_{\text{har},q-1,0} : \hat{j}_{|\alpha_{q-1}|,\alpha_{q-1},k}^2/l^2 \right\}_{k=1}^\infty. \end{aligned}$$

With relative boundary conditions:

$$\begin{aligned} \text{Sp}_+\Delta_{\text{rel}}^{(q)} = & \left\{ m_{\text{cex},q,n} : j_{\mu_{q,n},k}^{-2s}/l^{-2s} \right\}_{n,k=1}^\infty \cup \left\{ m_{\text{cex},q-1,n} : j_{\mu_{q-1,n},k}^{-2s}/l^{-2s} \right\}_{n,k=1}^\infty \\ & \cup \left\{ m_{\text{cex},q-1,n} : \hat{j}_{\mu_{q-1,n},-\alpha_{q-1},k}^{-2s}/l^{-2s} \right\}_{n,k=1}^\infty \cup \left\{ m_{\text{cex},q-2,n} : \hat{j}_{\mu_{q-2,n},-\alpha_{q-2},k}^{-2s}/l^{-2s} \right\}_{n,k=1}^\infty \\ & \cup \left\{ m_{\text{har},q} : j_{|\alpha_q|,k}^{-2s}/l^{-2s} \right\}_{k=1}^\infty \cup \left\{ m_{\text{har},q-1} : j_{|\alpha_{q-1}|,k}^{-2s}/l^{-2s} \right\}_{k=1}^\infty, \end{aligned}$$

where the $j_{\mu,k}$ are the zeros of the Bessel function $J_\mu(x)$, the $\hat{j}_{\mu,c,k}$ are the zeros of the function $\hat{J}_\mu(x) = cJ_\mu(x) + xJ'_\mu(x)$, $c \in \mathbb{R}$, α_q and $\mu_{q,n}$ are defined in Lemma 4.2.

Proof. By the Lemma 4.1, Lemma 4.2 and its corollary, we know that the $+$ solutions of Lemma 4.2 determine a complete system of square integrable solutions of the eigenvalues equation $\Delta^{(q)}u = \lambda u$, with $\lambda \neq 0$, satisfying the boundary condition at $x = 0$. Since $\Delta_{\text{abs/rel}}^{(q)}$ has pure point spectrum, in order to obtain a discrete resolution (more precisely the positive part of it) of $\Delta_{\text{abs/rel}}^{(q)}$, we have to determine among these solutions those that belong to the domain of $\Delta_{\text{abs/rel}}^{(q)}$, namely those that

satisfy the boundary condition at $x = l$. We give details for absolute BC, the analysis for relative BC is analogous. So consider absolute BC, as given in equation (2.9). For a form of type 1 to satisfy this condition means

$$\partial_x \psi_{1,n,\lambda}^{(q)}|_{x=l} = 0,$$

i.e.

$$\alpha_q J_{\mu_{q,n}}(\lambda l) + \lambda l J'_{\mu_{q,n}}(\lambda l) = 0,$$

and this gives $\lambda = \hat{j}_{\mu_{q,n},\alpha_q,k}/l$. For forms of type 2, we get

$$\partial_x \psi_2^{(q)}|_{x=l} = 0,$$

that gives

$$\alpha(q-1) J_{\mu_{q-1,n}}(\lambda l) + \lambda l J'_{\mu_{q-1,n}}(\lambda l) = 0,$$

so $\lambda = \hat{j}_{\mu_{q-1,n},\alpha(q-1),k}/l$. For forms of type 3, we obtain the system

$$\begin{cases} x^{\alpha_{q-1}-1} J_{\mu_{q-1,n}}(\lambda x)|_{x=l} = 0, \\ \partial_x (x^{2\alpha_{q-1}+1} \partial_x (x^{-\alpha_{q-1}} J_{\mu_{q,n}}(\lambda x))) - \bar{\lambda} x^{\alpha_{q-1}-1} J_{\mu_{q-1,n}}(\lambda x)|_{x=l} = 0. \end{cases}$$

Using classical properties of Bessel functions and their derivative, we obtain $\lambda = j_{\mu_{q-1,n},k}/l$. For forms of type 4, we get

$$x^{\alpha_{q-2}+1} J_{\mu_{q-2,n},k}(\lambda l)|_{r=l} = 0,$$

that gives $\lambda = j_{\mu_{q-1,n},k}/l$. Similar analysis gives for forms of types E and O : $\lambda = \hat{j}_{|\alpha_q|,k,\alpha_q}/l$ and $\lambda = \hat{j}_{|\alpha_{q-1}|,k,\alpha_{q-1}}/l$, respectively. $\square$

We conclude with the harmonic forms of Δ . The proofs are similar to the previous ones, so will be omitted.

Lemma 4.4. [9][27] *With the notation of Lemma 4.2, and*

$$a_{\pm,q,n} = \alpha_q \pm \mu_{q,n},$$

then all the solutions of the harmonic equation $\Delta u = 0$, are convergent sums of forms of the following four types:

$$\begin{aligned} \psi_{\pm,1,n}^{(q)} &= x^{a_{\pm,q,n}} \varphi_{\text{ccl},n}^{(q)}, \\ \psi_{\pm,2,n}^{(q)} &= x^{a_{\pm,q-1,n}} \tilde{d} \varphi_{\text{ccl},n}^{(q-1)} + a_{\pm,q-1,n} x^{a_{\pm,q-1,n}-1} dx \wedge \varphi_{\text{ccl},n}^{(q-1)}, \\ \psi_{\pm,3,n}^{(q)} &= x^{a_{\pm,q-1,n}+2} \tilde{d} \varphi_{\text{ccl},n}^{(q-1)} + a_{\mp,q-1,n} x^{a_{\pm,q-1,n}+1} dx \wedge \varphi_{\text{ccl},n}^{(q-1)}, \\ \psi_{\pm,4,n}^{(q)} &= x^{a_{\pm,q-2,n}+1} dx \wedge \tilde{d} \varphi_{\text{ccl},n}^{(q-2)}. \end{aligned}$$

Lemma 4.5. *Assume $\dim W = 2p - 1$ is odd. Then*

$$\begin{aligned} \mathcal{H}_{\text{abs}}^q(C_l W) &= \begin{cases} \mathcal{H}^q(W), & 0 \leq q \leq p-1, \\ \{0\}, & p \leq q \leq 2p-1. \end{cases} \\ \mathcal{H}_{\text{rel}}^q(C_l W) &= \begin{cases} \{0\}, & 0 \leq q \leq p, \\ \{x^{2\alpha_q-1} dx \wedge \varphi^{(q-1)}, \varphi^{(q-1)} \in \mathcal{H}^{q-1}(W)\}, & p+1 \leq q \leq 2p. \end{cases} \end{aligned}$$

Proof. First, by Remark 4.1, we need only to consider the $+$ solutions in Lemma 4.4. The proof then follows by argument similar to the one used in the proof of Lemma 4.3. Let see one case in details. Consider $\psi_{+,1,n}^{(q)} = x^{a_{+,q,n}} \varphi_{\text{ccl},n}^{(q)}$, where $a_{+,q,n} = \alpha_q + \mu_{q,n}$. In order that $\psi_{+,1,n}^{(q)}$ satisfies the absolute boundary condition (2.9), we need that

$$(d\psi_{+,1,n}^{(q)})_{\text{norm}} \Big|_{\partial C_l W} = a_{+,q,n} l^{a_{+,q,n}-1} dx \wedge d\varphi_{\text{ccl},n}^{(q)} = 0$$

and this is true if and only if $a_{+,q,n} = 0$. The condition $a_{+,q,n} = 0$ is equivalent to the conditions $\lambda_{q,n} = 0$, and $\alpha_q = -|\alpha_q|$. Therefore, $\varphi_{\text{ccl},n}^{(q)}$ is harmonic, $0 \leq q \leq p-1$, and $\psi_{+,1,n}^{(q)} = \varphi_{\text{ccl},n}^{(q)}$. $\square$

5. TORSION ZETA FUNCTION AND POINCARÉ DUALITY FOR A CONE

Using the description of the spectrum of the Laplace operator on forms $\Delta_{\text{abs/rel}}^{(q)}$ given in Lemma 4.3, we define the zeta function on q -forms as in Section 2.5, by

$$\zeta(s, \Delta_{\text{abs/rel}}^{(q)}) = \sum_{\lambda \in \text{Sp}_+ \Delta_{\text{abs/rel}}^{(q)}} \lambda^{-s},$$

for $\text{Re}(s) > \frac{m+1}{2}$. Even if we can not apply directly Proposition 3.2, the explicit knowledge of the behaviour of the large eigenvalues allows to completely determine the analytic continuation of the zeta function, by using the tools of Section 3.3. In particular, it is possible to prove that there can be at most a simple pole at $s = 0$. We will not do this here (but the interested reader can compare with [39]), because for our purpose it is more convenient to investigate the analytic properties of other zeta functions, resulting by a suitable different decomposition of the analytic torsion, as described here below. For we define the *torsion zeta function* by

$$t_{\text{abs/rel}}(s) = \frac{1}{2} \sum_{q=1}^{m+1} (-1)^q q \zeta(s, \Delta_{\text{abs/rel}}^{(q)}).$$

It is clear that the analytic torsion of $C_l W$ is (in the following we will use the simplified notation $T(C_l W)$ for $T((C_l W, g); \rho)$)

$$\log T_{\text{abs/rel}}(C_l W) = t'_{\text{abs/rel}}(0).$$

Our first result is a Poincaré duality (compare with Proposition 2.1, [22] and the result of [11]).

Theorem 5.1. Poincaré duality for the analytic torsion of a cone. *Let (W, g) be an orientable compact connected Riemannian manifold of dimension m , without boundary, then*

$$\log T_{\text{abs}}(C_l W) = (-1)^m \log T_{\text{rel}}(C_l W).$$

Proof. By Hodge duality in equation (2.7), the Hodge operator $\star$ sends forms of type 1, 2, 3, 4, E , and O into forms of type 4, 3, 2, 1, O , and E , respectively. Moreover, $\star$ sends q -forms satisfying absolute boundary conditions, as in equation (2.9), into $m+1-q$ -forms satisfying relative boundary conditions, as in equation (2.10). Therefore, using the explicit description of the eigenvalues given in Lemma 4.3, it follows that $\text{Sp} \Delta_{\text{abs}}^{(q)} = \text{Sp} \Delta_{\text{rel}}^{(m+1-q)}$. Using the formulas in equations (4.2), (4.3), and (4.4), and the eigenforms in Lemma 4.2, a straightforward calculation shows that the forms of

type 1, 3, and E are coexact, and those of type 2, 4, and O are exact, and that the operator d sends forms of type 1, 3, and E in forms of type 2, 4, and O , respectively, with inverse $d^\dagger$. Then, set

$$\begin{aligned} F_{\text{ccl,abs}}^{(q)} &= \left\{ m_{\text{ccl},q,n} : \hat{j}_{\mu_q,n,\alpha_q,k}^2 / l^2 \right\}_{n,k=1}^\infty \cup \left\{ m_{\text{ccl},q-1,n} : \hat{j}_{\mu_{q-1},n,k}^2 / l^2 \right\}_{n,k=1}^\infty \\ &\quad \cup \left\{ m_{\text{ccl},q,0} : \hat{j}_{|\alpha_q|,\alpha_q,k}^2 / l^2 \right\}_{k=1}^\infty, \\ F_{\text{cl,abs}}^{(q)} &= \left\{ m_{\text{cl},q-1,n} : \hat{j}_{\mu_{q-1},n,\alpha_{q-1},k}^2 / l^2 \right\}_{n,k=1}^\infty \cup \left\{ m_{\text{cl},q-2,n} : \hat{j}_{\mu_{q-2},n,k}^2 / l^2 \right\}_{n,k=1}^\infty \\ &\quad \cup \left\{ m_{\text{cl},q-1,0} : \hat{j}_{|\alpha_{q-1}|,\alpha_{q-1},k}^2 / l^2 \right\}_{k=1}^\infty. \end{aligned}$$

$F_{\text{ccl,abs}}^{(q)}$ is the set of the eigenvalues of the coclosed q -forms with absolute boundary conditions, and $F_{\text{cl,abs}}^{(q)}$ is the set of the eigenvalues of the closed q -forms with absolute. Since obviously $\text{Sp}\Delta_{\text{abs}}^{(q)} = F_{\text{ccl,abs}}^{(q)} \cup F_{\text{cl,abs}}^{(q)}$, and $F_{\text{ccl,abs}}^{(q)} = F_{\text{cl,abs}}^{(q+1)}$, we have that

$$\begin{aligned} t_{\text{abs}}(s) &= \frac{1}{2} \sum_{q=0}^{m+1} (-1)^q q \zeta(s, \Delta_{\text{abs}}^{(q)}) = \frac{1}{2} \sum_{q=0}^{m+1} (-1)^q q \zeta(s, \Delta_{\text{rel}}^{(m+1-q)}) \\ &= (-1)^m t_{\text{rel}}(s) + \frac{1}{2} (m+1) \sum_{q=0}^{m+1} (-1)^{m+1-q} \zeta(s, \Delta_{\text{rel}}^{(q)}) \\ &= (-1)^m t_{\text{rel}}(s) + \frac{1}{2} (m+1) \sum_{q=0}^{m+1} (-1)^q \left(\zeta(s, F_{\text{ccl,abs}}^{(q+1)}) + \zeta(s, F_{\text{cl,abs}}^{(q)}) \right) \\ &= (-1)^m t_{\text{rel}}(s). \end{aligned}$$

Since by definition $\log T_{\text{abs}}(W) = t'_{\text{abs}}(0)$, the thesis follows. $\square$

6. THE TORSION ZETA FUNCTION OF THE CONE OVER AN ODD DIMENSIONAL MANIFOLD

In this section we develop the main steps in order to obtain the proof of our theorems. This accounts essentially in the application of the tools described in Section 3.3 to some suitable sequences appearing in the definition of the torsion. So our first step is precisely to obtain this suitable description. This we do in this section. In the next two subsections, we will make the calculations necessary for the proof of our main theorems.

We proceed assuming $\dim W = 2p - 1$ odd, and assuming absolute boundary condition; for notational convenience, we will omit the *abs* subscript.

Lemma 6.1.

$$\begin{aligned} t(s) = & \frac{l^{2s}}{2} \sum_{q=0}^{p-2} (-1)^q \left(\sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \left(2j_{\mu_{q,n},k}^{-2s} - \hat{j}_{\mu_{q,n},\alpha_q,k}^{-2s} - \hat{j}_{\mu_{q,n},-\alpha_q,k}^{-2s} \right) \right) \\ & + (-1)^{p-1} \frac{l^{2s}}{2} \left(\sum_{n,k=1}^{\infty} m_{\text{cex},p-1,n} \left(j_{\mu_{p-1,n},k}^{-2s} - (j'_{\mu_{p-1,n},k})^{-2s} \right) \right) \\ & - \frac{l^{2s}}{2} \sum_{q=0}^{p-1} (-1)^q \text{rk} \mathcal{H}_q(\partial C_l W; \mathbb{Q}) \sum_{k=1}^{\infty} \left(j_{-\alpha_{q-1},k}^{-2s} - j_{-\alpha_q,k}^{-2s} \right). \end{aligned}$$

Proof. Using the eigenvalues in Lemma 4.3

$$\begin{aligned} l^{2s} \zeta(s, \Delta^{(q)}) = & \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \hat{j}_{\mu_{q,n},\alpha_q,k}^{-2s} + \sum_{n,k=1}^{\infty} m_{\text{cex},q-1,n} \hat{j}_{\mu_{q-1,n},\alpha_{q-1},k}^{-2s} + \sum_{n,k=1}^{\infty} m_{\text{cex},q-1,n} j_{\mu_{q-1,n},k}^{-2s} \\ & + \sum_{n,k=1}^{\infty} m_{\text{cex},q-2,n} j_{\mu_{q-2,n},k}^{-2s} + \sum_{k=1}^{\infty} m_{\text{har},q,0} \hat{j}_{|\alpha_q|,\alpha_q,k}^{-2s} + \sum_{k=1}^{\infty} m_{\text{har},q-1,0} \hat{j}_{|\alpha_{q-1}|,\alpha_{q-1},k}^{-2s}. \end{aligned}$$

Since for each fixed q , with $0 \leq q \leq 2p-2$,

$$\begin{aligned} & (-1)^q q \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \hat{j}_{\mu_{q,n},\alpha_q,k}^{-2s} + (-1)^{q+1} (q+1) \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \hat{j}_{\mu_{q,n},\alpha_q,k}^{-2s} \\ & + (-1)^{q+1} (q+1) \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} j_{\mu_{q,n},k}^{-2s} + (-1)^{q+2} (q+2) \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} j_{\mu_{q,n},k}^{-2s} \\ & + q(-1)^q \sum_{k=1}^{\infty} m_{\text{har},q,0} \hat{j}_{|\alpha_q|,\alpha_q,k}^{-2s} + (q+1)(-1)^{q+1} \sum_{k=1}^{\infty} m_{\text{har},q,0} \hat{j}_{|\alpha_q|,\alpha_q,k}^{-2s} \\ = & (-1)^q \left(\sum_{n,k=1}^{\infty} m_{\text{cex},q,n} j_{\mu_{q,n},k}^{-2s} - \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \hat{j}_{\mu_{q,n},\alpha_q,k}^{-2s} \right) + (-1)^{q+1} \sum_{k=1}^{\infty} m_{\text{har},q,0} \hat{j}_{|\alpha_q|,\alpha_q,k}^{-2s}. \end{aligned}$$

it follows that

$$t(s) = \frac{l^{2s}}{2} \sum_{q=0}^{2p-2} (-1)^q \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \left(j_{\mu_{q,n},k}^{-2s} - \hat{j}_{\mu_{q,n},\alpha_q,k}^{-2s} \right) + \frac{l^{2s}}{2} \sum_{q=0}^{2p-1} (-1)^{q+1} \sum_{k=1}^{\infty} m_{\text{har},q,0} \hat{j}_{|\alpha_q|,\alpha_q,k}^{-2s}.$$

Next, by Hodge duality on coexact q -forms on the section (see equation (2.7)) $\lambda_{q,n} = \lambda_{2p-2-q,n}$, and recalling the definition of the constants α_q and $\mu_{q,n}$ in Lemma 4.2, we have that $\alpha_q = \frac{1}{2}(1 + 2q - 2p + 1) = q - p + 1 = -\alpha_{2p-2-q}$, and $\mu_{q,n} = \mu_{2p-2-q,n}$. Thus, fixing q with $0 \leq q \leq p-2$,

$$\begin{aligned} & (-1)^q \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \left(j_{\mu_{q,n},k}^{-2s} - \hat{j}_{\mu_{q,n},\alpha_q,k}^{-2s} \right) + (-1)^{(2p-2-q)} \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \left(j_{\mu_{q,n},k}^{-2s} - \hat{j}_{\mu_{q,n},-\alpha_q,k}^{-2s} \right) \\ = & (-1)^q \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \left(2j_{\mu_{q,n},k}^{-2s} - \hat{j}_{\mu_{q,n},\alpha_q,k}^{-2s} - \hat{j}_{\mu_{q,n},-\alpha_q,k}^{-2s} \right), \end{aligned}$$

while when $q = p - 1$, $\lambda_{p-1,n} = \lambda_{2p-1-(p-1),n}$, and $\alpha_q = 0$. Therefore,

$$\begin{aligned} t(s) &= \frac{l^{2s}}{2} \sum_{q=0}^{p-2} (-1)^q \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \left(2j_{\mu_q,n,k}^{-2s} - \hat{j}_{\mu_q,n,\alpha_q,k}^{-2s} - \hat{j}_{\mu_q,n,-\alpha_q,k}^{-2s} \right) \\ &\quad + (-1)^{p-1} \frac{l^{2s}}{2} \sum_{n,k=1}^{\infty} m_{\text{cex},p-1,n} \left(j_{\mu_{p-1},n,k}^{-2s} - (j'_{\mu_{p-1},n,k})^{-2s} \right) \\ &\quad + \frac{l^{2s}}{2} \sum_{q=0}^{2p-1} (-1)^{q+1} \sum_{k=1}^{\infty} m_{\text{har},q,0} \hat{j}_{|\alpha_q|,\alpha_q,k}^{-2s}, \end{aligned}$$

where $j'_{\nu,k} = \hat{j}_{\nu,0,k}$ are the zeros of J'_{ν} . Eventually, consider the sum

$$\sum_{q=0}^{2p-1} (-1)^{q+1} \sum_{k=1}^{\infty} m_{\text{har},q,0} \hat{j}_{|\alpha_q|,\alpha_q,k}^{-2s}.$$

We will use some classical properties of Bessel function, see for example [41]. Recall $m = \dim W = 2p - 1$, and therefore $\alpha_q = q - p + 1$ is an integer. Moreover, α_q is negative for $0 \leq q < p - 1$. Fixed such a q , we study the function $\hat{J}_{-\alpha_q,\alpha_q}(z) = \alpha_q J_{-\alpha_q}(z) + z J'_{-\alpha_q}(z)$. Since

$$z J'_{\mu}(z) = -z J_{\mu+1}(z) + \mu J_{\mu}(z)$$

it follows that $\hat{J}_{-\alpha_q,\alpha_q}(z) = -z J_{-\alpha_q+1}(z) = -z J_{-\alpha_{q-1}}(z)$, and hence $\hat{j}_{|\alpha_q|,\alpha_q,k} = j_{-\alpha_{q-1},k}$. Next, fix q with $p - 1 < q \leq 2p - 1$, such that α_q is a positive integer. Then, since

$$z J'_{\mu}(z) = z J_{\mu-1}(z) - \mu J_{\mu}(z),$$

the function $\hat{J}_{\alpha_q,\alpha_q}(z) = \alpha_q J_{\alpha_q}(z) + z J'_{\alpha_q}(z)$ coincides with $z J_{\alpha_q-1}(z)$, and hence $\hat{j}_{|\alpha_q|,\alpha_q,k} = j_{\alpha_{q-1},k}$. Note that when $q = p - 1$, $\alpha_{p-1} = 0$ and hence $j_{\alpha_{p-1},\alpha_{p-1},k} = j'_{0,k} = j_{1,k}$. Summing up,

$$\begin{aligned} \sum_{q=0}^{2p-1} (-1)^{q+1} \sum_{k=1}^{\infty} m_{\text{har},q,0} \hat{j}_{|\alpha_q|,\alpha_q,k}^{-2s} &= \sum_{q=0}^{p-2} (-1)^{q+1} \sum_{k=1}^{\infty} \frac{m_{\text{har},q,0}}{j_{-\alpha_{q-1},k}^{-2s}} + (-1)^p \sum_{k=1}^{\infty} \frac{m_{\text{har},p-1,0}}{j_{1,k}^{-2s}} \\ &\quad + \sum_{q=p}^{2p-1} (-1)^{q+1} \sum_{k=1}^{\infty} \frac{m_{\text{har},q,0}}{j_{\alpha_{q-1},k}^{-2s}}, \end{aligned}$$

and since by Hodge duality $m_{q,0} = m_{2p-1-q,0}$,

$$\begin{aligned} &= \sum_{q=0}^{p-2} (-1)^{q+1} \sum_{k=1}^{\infty} m_{\text{har},q,0} j_{-\alpha_{q-1},k}^{-2s} + (-1)^p \sum_{k=1}^{\infty} m_{\text{har},p-1,0} j_{1,k}^{-2s} \\ &\quad + \sum_{q=0}^{p-1} (-1)^{2p-q} \sum_{k=1}^{\infty} m_{\text{har},2p-1-q,0} j_{\alpha_{2p-2-q},k}^{-2s} \\ &\quad + \sum_{q=0}^{p-2} (-1)^{q+1} \sum_{k=1}^{\infty} m_{\text{har},q,0} j_{-\alpha_{q-1},k}^{-2s} + (-1)^p \sum_{k=1}^{\infty} m_{\text{har},p-1,0} j_{1,k}^{-2s} + \sum_{q=0}^{p-1} (-1)^q \sum_{k=1}^{\infty} m_{\text{har},q,0} j_{-\alpha_q,k}^{2s} \\ &= \sum_{q=0}^{p-1} (-1)^{q+1} m_{\text{har},q,0} \sum_{k=1}^{\infty} \left(j_{-\alpha_{q-1},k}^{-2s} - j_{-\alpha_q,k}^{-2s} \right). \end{aligned}$$

Since $m_{\text{har},q,0} = \text{rk} \mathcal{H}_q(\partial C_l W; \mathbb{Q})$, this completes the proof. $\square$

It is convenient to introduce the following functions. We set

$$\begin{aligned}
 Z_q(s) &= \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} j_{\mu_{q,n},k}^{-2s}, \\
 \dot{Z}_q(s) &= \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} (j'_{\mu_{q,n},k})^{-2s}, \\
 Z_{q,\pm}(s) &= \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \hat{j}_{\mu_{q,n},\pm\alpha_q,k}^{-2s}, \\
 z_q(s) &= \sum_{k=1}^{\infty} \left(j_{-\alpha_{q-1},k}^{-2s} - j_{-\alpha_q,k}^{-2s} \right),
 \end{aligned}
 \tag{6.1}$$

for $0 \leq q \leq p-1$, and

$$\begin{aligned}
 t_{p-1}(s) &= Z_{p-1}(s) - \dot{Z}_{p-1}(s), \\
 t_q(s) &= 2Z_q(s) - Z_{q,+}(s) - Z_{q,-}(s), \quad 0 \leq q \leq p-2.
 \end{aligned}
 \tag{6.2}$$

Then,

$$\begin{aligned}
 t(s) &= \frac{l^{2s}}{2} \sum_{q=0}^{p-2} (-1)^q (2Z_q(s) - Z_{q,+}(s) - Z_{q,-}(s)) + (-1)^{p-1} \frac{l^{2s}}{2} (Z_{p-1}(s) - \dot{Z}_{p-1}(s)) \\
 &\quad - \frac{l^{2s}}{2} \sum_{q=0}^{p-1} (-1)^q \text{rk} \mathcal{H}_q(\partial C_l W; \mathbb{Q}) z_q(s) \\
 &= \frac{l^{2s}}{2} \sum_{q=0}^{p-1} (-1)^q t_q(s) - \frac{l^{2s}}{2} \sum_{q=0}^{p-1} (-1)^q \text{rk} \mathcal{H}_q(\partial C_l W; \mathbb{Q}) z_q(s),
 \end{aligned}$$

and

$$\begin{aligned}
 \log T(C_l W) &= t'(0) = \frac{\log l^2}{2} \left(\sum_{q=0}^{p-1} (-1)^{q+1} r_q z_q(0) + \sum_{q=0}^{p-1} (-1)^q t_q(0) \right) \\
 &\quad + \frac{1}{2} \left(\sum_{q=0}^{p-1} (-1)^{q+1} r_q z'_q(0) + \sum_{q=0}^{p-1} (-1)^q t'_q(0) \right),
 \end{aligned}
 \tag{6.3}$$

where $r_q = \text{rk} \mathcal{H}_q(\partial C_l W; \mathbb{Q})$. In order to obtain the value of $\log T(C_l W)$ we use Theorem 3.2 and its corollary applied to the functions $z_q(s)$, $Z_q(s)$, $\dot{Z}_q(s)$, $Z_{q,\pm}(s)$. More precisely, the functions z_q were studied in Section 3.2, and we will study the functions t_q in Sections 6.1 and 6.2, and eventually we sum up on the forms degree q in Section 7.

6.1. The function $t_{p-1}(s)$. In this section we study the function $t_{p-1}(s)$. For we apply Theorems 3.1 and 3.2 to the double sequences $S_{p-1} = \{m_{p-1,n} : j_{\mu_{p-1,n},k}^2\}_{n=1}^{\infty}$ and $\dot{S}_{p-1} = \{m_{p-1,n} : (j'_{\mu_{p-1,n},k})^2\}_{n=1}^{\infty}$, since $Z_{p-1}(s) = \zeta(s, S_{p-1})$, $\dot{Z}_{p-1}(s) = \zeta(s, \dot{S}_{p-1})$. First, we verify Definition 3.1. For we introduce the simple sequence $U_{p-1} = \{m_{p-1,n} : \mu_{p-1,n}\}_{n=1}^{\infty}$.

Lemma 6.2. *The sequence U_{p-1} is a totally regular sequence of spectral type with infinite order, $e(U_{p-1}) = g(U_{p-1}) = 2p - 1$, and $\zeta(s, U_{p-1}) = \zeta_{cex}\left(\frac{s}{2}, \tilde{\Delta}^{(p-1)}\right)$, with possible simple poles at $s = 2p - 1 - h$, $h = 0, 2, 4, \dots$.*

Lemma 6.3. *The logarithmic Gamma functions associated to the sequences $S_{p-1,n}/\mu_{p-1,n}^2$ and $\dot{S}_{p-1,n}/\mu_{p-1,n}^2$ have the following representations, with $\lambda \in D_{\theta,c}$, $0 \leq \theta \leq \pi$, $c = \frac{\min(j_{\mu_{p-1,n}}^2, (j'_{\mu_{p-1,n}})^2)}{2\mu_{p-1,n}^2}$,*

$$\begin{aligned} \log \Gamma(-\lambda, S_{p-1,n}/\mu_{p-1,n}^2) &= -\log \prod_{k=1}^{\infty} \left(1 + \frac{(-\lambda)\mu_{p-1,n}^2}{j_{\mu_{p-1,n},k}^2}\right) \\ &= -\log I_{\mu_{p-1,n}}(\mu_{p-1,n}\sqrt{-\lambda}) + (\mu_{p-1,n}) \log \sqrt{-\lambda} \\ &\quad + \mu_{p-1,n} \log(\mu_{p-1,n}) - \mu_{p-1,n} \log 2 - \log \Gamma(\mu_{p-1,n} + 1), \\ \log \Gamma(-\lambda, \dot{S}_{p-1,n}/\mu_{p-1,n}^2) &= -\log \prod_{k=1}^{\infty} \left(1 + \frac{(-\lambda)(\mu_{p-1,n})^2}{(j'_{\mu_{p-1,n},k})^2}\right) \\ &= -\log I'_{\mu_{p-1,n}}(\mu_{p-1,n}\sqrt{-\lambda}) + (\mu_{p-1,n} - 1) \log \sqrt{-\lambda} \\ &\quad + \mu_{p-1,n} \log(\mu_{p-1,n}) - \mu_{p-1,n} \log 2 - \log \Gamma(\mu_{p-1,n} + 1). \end{aligned}$$

Using the expansions of the Bessel functions, it follows from Lemma 6.3 that there is a complete asymptotic expansion for the functions $\log \Gamma(-\lambda, S_{p-1,n}/\mu_{p-1,n}^2)$ and $\log \Gamma(-\lambda, \dot{S}_{p-1,n}/\mu_{p-1,n}^2)$, and then the sequences $S_{p-1,n}/\mu_{p-1,n}^2$ and $\dot{S}_{p-1,n}/\mu_{p-1,n}^2$ are sequences of spectral type. A simple calculation shows that they are totally regular sequences of infinite order.

Proposition 6.1. *The double sequences S_{p-1} and $\dot{S}_{p-1}$ have relative exponents $(p, \frac{2p-1}{2}, \frac{1}{2})$, relative genus $(p, p-1, 0)$, and are spectrally decomposable over U_{p-1} with power $\kappa = 2$, length $\ell = 2p$ and domain $D_{\theta,c}$. The coefficients σ_h appearing in equation (3.3) are $\sigma_h = h - 1$, with $h = 0, 1, \dots, \ell = 2p$.*

Proof. The values of the exponents and genus of S follow by classical estimates of the zeros of the Bessel functions [41], and zeta function theory. In particular, to determinate $s_0 = p$, we use the Young inequality and the Plana theorem as in [33]. Note that $\alpha > \frac{1}{2}$, since $s_2 = \frac{1}{2}$. The exponents and genus of $\dot{S}$ are the same, since the zeros of the derivative of the Bessel function are correlated by those of the Bessel function, namely $j_{\mu,k} < j'_{\mu,k} < j_{\mu,k+1}$. As observed, the existence of a complete asymptotic expansion of the Gamma function $\log \Gamma(-\lambda, S_{p-1,n})$ and $\log \Gamma(-\lambda, \dot{S}_{p-1,n})$ follows by Lemma 6.3. This implies that $S_{p-1,n}$ and $\dot{S}_{p-1,n}$ are sequences of spectral type. A direct inspection of the expansions shows that $S_{p-1,n}$ and $\dot{S}_{p-1,n}$ are totally regular sequences of infinite order. The existence of the uniform expansion follows using the uniform expansions for the Bessel functions and their derivative given for example in [28] (7.18) and Ex. 7.2, and classical expansion of the Euler Gamma function [15] 8.344. We refer to [18] Section 5 or to [19] Section 4 for details. This proves that $S_{p-1,n}$ and $\dot{S}_{p-1,n}$ are spectrally decomposable over U_{p-1} , with power $\kappa = 2$. The length ℓ of the decomposition is precisely $2p$. For $e(U_{p-1}) = 2p - 1$, and therefore the larger integer such that $\sigma_h = h - 1 \leq 2p - 1$ is $2p$. $\square$

Remark 6.1. *By Theorem 3.1, only the terms with $\sigma_h = 1$, $\sigma_h = 3$, $\dots$, $\sigma_h = 2p - 1$ namely $h = 2, 4, \dots, 2p$, appear in the formula of Theorem 3.2, since the unique non negative poles of $\zeta(s, U_{p-1})$ are at $s = 1$, $s = 3$, $\dots$, $s = 2p - 1$, by Lemma 6.2.*

Since we aim to apply the version of Theorem 3.2 given in Corollary 3.1, for linear combination of two spectrally decomposable sequences, we need more information on the uniform asymptotic expansion of the difference $S_{p-1} - \dot{S}_{p-1}$. This will also give the functions $\phi_{\sigma_h}(\lambda)$, necessary in the following.

Lemma 6.4. *The difference of the logarithmic Gamma functions associated to the sequences $S_{p-1,n}/\mu_{p-1}^2$ and $\dot{S}_{p-1,n}/\mu_{p-1,n}^2$ have the following uniform asymptotic expansions for large n , $\lambda \in D_{\theta,c}$,*

$$\begin{aligned} \log \Gamma(-\lambda, S_{p-1,n}/(\mu_{p-1,n}^p)^2) - \log \Gamma(-\lambda, \dot{S}_{p-1,n}/(\mu_{p-1,n}^p)^2) &= \\ &= -\log I(\mu_{p-1,n}\sqrt{-\lambda}) + \log I'(\mu_{p-1,n}\sqrt{-\lambda}) + \log \sqrt{-\lambda} \\ &= \frac{1}{2} \log(1-\lambda) + \sum_{j=1}^{2p-1} \phi_{j,p-1}(\lambda) \frac{1}{(\mu_{p-1,n}^p)^j} + O\left(\frac{1}{\mu_{p-1,n}^{2p}}\right). \end{aligned}$$

Proof. By Lemma 6.3

$$(6.4) \quad \begin{aligned} \log \Gamma(-\lambda, S_{p-1,n}/(\mu_{p-1,n})^2) - \log \Gamma(-\lambda, \dot{S}_{p-1,n}/(\mu_{p-1,n})^2) &= \\ &= -\log I(\mu_{p-1,n}\sqrt{-\lambda}) + \log I'(\mu_{p-1,n}\sqrt{-\lambda}) + \log \sqrt{-\lambda}. \end{aligned}$$

Recall the uniform expansions for the Bessel functions given for example in [28] (7.18) pg. 376, and Ex. 7.2,

$$I_\nu(\nu z) = \frac{e^{\nu\sqrt{1+z^2}} e^{\nu \log \frac{z}{1+\sqrt{1+z^2}}}}{\sqrt{2\pi\nu}(1+z^2)^{\frac{1}{4}}} \left(1 + \sum_{j=1}^{2p-1} \frac{U_j(z)}{\nu^j} + O\left(\frac{1}{\nu^{2p}}\right) \right),$$

where

$$\begin{aligned} U_0(w) &= 1, \\ U_j(w) &= \frac{1}{2} w^2 (1-w^2) \frac{d}{dw} U_{j-1}(w) + \frac{1}{8} \int_0^w (1-5t^2) U_{j-1}(t) dt, \end{aligned}$$

with $w = \frac{1}{\sqrt{1+z^2}}$, and

$$I'_\nu(\nu z) = \frac{(1+z^2)^{\frac{1}{4}} e^{\nu\sqrt{1+z^2}} e^{\nu \log \frac{z}{1+\sqrt{1+z^2}}}}{\sqrt{2\pi\nu} z} \left(1 + \sum_{j=1}^{2p-1} \frac{V_j(z)}{\nu^j} + O\left(\frac{1}{\nu^{2p}}\right) \right),$$

$$\begin{aligned} V_0(w) &= 1, \\ V_j(w) &= U_j(w) - \frac{w}{2} (1-w^2) U_{j-1}(w) - w^2 (1-w^2) \frac{d}{dw} U_{j-1}(w). \end{aligned}$$

Then,

$$\begin{aligned}\log I_\nu(\nu z) &= \nu\sqrt{1+z^2} + \nu\log z - \nu\log(1+\sqrt{1+z^2}) \\ &\quad - \frac{1}{2}\log 2\pi\nu - \frac{1}{4}\log(1+z^2) + \log\left(1 + \sum_{j=1}^{2p-1} \frac{U_j(z)}{\nu^j} + O\left(\frac{1}{\nu^{2p}}\right)\right) \\ \log I'_\nu(\nu z) &= \nu\sqrt{1+z^2} + \nu\log z - \nu\log(1+\sqrt{1+z^2}) - \log z \\ &\quad - \frac{1}{2}\log 2\pi\nu + \frac{1}{4}\log(1+z^2) + \log\left(1 + \sum_{j=1}^{2p-1} \frac{V_j(z)}{\nu^j} + O\left(\frac{1}{\nu^{2p}}\right)\right),\end{aligned}$$

and substitution in equation (6.4) gives

$$\begin{aligned}\log \Gamma(-\lambda, S_{p-1,n}/(\mu_{p-1,n})^2) - \log \Gamma(-\lambda, \dot{S}_{p-1,n}/(\mu_{p-1,n})^2) \\ = \frac{1}{2}\log(1-\lambda) - \log\left(1 + \sum_{j=1}^{2p-1} \frac{U_j(\sqrt{-\lambda})}{\mu_{p-1,n}^j} + O\left(\frac{1}{\mu_{p-1,n}^{2p}}\right)\right) \\ + \log\left(1 + \sum_{j=1}^{2p-1} \frac{V_j(\sqrt{-\lambda})}{\mu_{p-1,n}^j} + O\left(\frac{1}{\mu_{p-1,n}^{2p}}\right)\right).\end{aligned}$$

Expanding the logarithm as

$$\log\left(1 + \sum_{j=1}^{\infty} \frac{a_j}{z^j}\right) = \sum_{j=1}^{\infty} \frac{l_j}{z^j},$$

where $a_0 = 1$, $a_1 = l_1$ and

$$l_j = a_j - \sum_{k=1}^{j-1} \frac{j-k}{j} a_k l_{j-k},$$

we have that

$$\begin{aligned}\log \Gamma(-\lambda, S_{p-1,n}/(\mu_{p-1,n})^2) - \log \Gamma(-\lambda, \dot{S}_{p-1,n}/(\mu_{p-1,n})^2) &= \frac{1}{2}\log(1-\lambda) \\ &+ \sum_{j=1}^{2p-1} \frac{1}{\mu_{p-1,n}^j} \left(V_j(\sqrt{-\lambda}) - U_j(\sqrt{-\lambda}) + \sum_{k=1}^{j-1} \frac{j-k}{j} (V_k(\sqrt{-\lambda}) l_{j-k}(\lambda) - U_k(\sqrt{-\lambda}) l_{j-k}(\lambda)) \right) \\ &+ O\left(\frac{1}{\mu_{p-1,n}^{2p}}\right),\end{aligned}$$

where we denote by $\dot{l}_j(\lambda)$ the term in the expansion relative to the sequence $\dot{S}$ (thus the one containing the $V_j(z)$) and by $l_j(\lambda)$ the term relative to S (thus the one containing the $U_j(z)$). Setting

$$\begin{aligned}\phi_{p-1,j}(\lambda) &= \dot{l}_j(\lambda) - l_j(\lambda) \\ (6.5) \quad &= V_j(\sqrt{-\lambda}) - U_j(\sqrt{-\lambda}) + \sum_{k=1}^{j-1} \frac{j-k}{j} (V_k(\sqrt{-\lambda}) \dot{l}_{j-k}(\lambda) - U_k(\sqrt{-\lambda}) l_{j-k}(\lambda)),\end{aligned}$$

we have the formula stated in the thesis.

□

Remark 6.2. Note that there are no logarithmic terms $\log \mu_{p-1,n}$ in the asymptotic expansion of the difference of the logarithmic Gamma function given in Lemma 6.4. This permits to apply Corollary 3.1.

Next, give some results on the functions $\phi_{j,p-1}(\lambda)$, and on the functions $\Phi_{j,p-1}(s)$ defined in equation (3.4).

Lemma 6.5. For all j , the functions $\phi_{j,p-1}(\lambda)$ are odd polynomial in $w = \frac{1}{\sqrt{1-\lambda}}$

$$\phi_{j,p-1}(\lambda) = \sum_{k=j}^{3j+1} a_{j,p-1,k} w^{2k+1}.$$

Proof. This follows by the definition in equation (6.5) in the proof of Lemma 6.4. □

Lemma 6.6. For all j , $\phi_{j,p-1}(0) = 0$.

Proof. We use induction on j . For $j = 1$

$$\begin{aligned} \phi_{1,p-1}(\lambda) &= \dot{l}_1(\lambda) - l_1(\lambda) = V_1(\sqrt{-\lambda}) - U_1(\sqrt{-\lambda}) \\ &= -\frac{1}{2} \frac{1}{(1-\lambda)^{\frac{1}{2}}} + \frac{1}{2} \frac{1}{(1-\lambda)^{\frac{3}{2}}}, \end{aligned}$$

and hence $\phi_{1,p-1}(0) = 0$. Assume $\phi_{k,p-1}(0) = 0$ for $k = 1, \dots, j-1$. For simplicity, write ϕ as a function of $w = (1-\lambda)^{-\frac{1}{2}}$. Then, by definition

$$\begin{aligned} \phi_{j,p-1}(w) &= \dot{l}_j(w) - l_j(w) \\ &= V_j(w) - U_j(w) + \sum_{s=1}^{j-1} \frac{j-s}{j} \left(U_s(w) l_{j-s}(w) - V_s(w) \dot{l}_{j-s}(w) \right) \\ &= V_j(w) - U_j(w) + \sum_{s=1}^{j-1} \frac{j-s}{j} \left(U_s(w) (l_{j-s}(w) - \dot{l}_{j-s}(w)) \right) \\ &\quad + \sum_{s=1}^{j-1} (1-w^2) \left(\frac{w}{2} U_{s-1}(w) + w^2 \frac{d}{dw} U_{s-1}(w) \right) \dot{l}_{j-s}(w). \end{aligned}$$

Since $U_j(1) = V_j(1)$ for all j , and $\phi_{j-1,p-1}(w) = \dot{l}_{j-1}(w) - l_{j-1}(w)$, using the induction's hypothesis, the thesis follows. □

Corollary 6.1. For all j , the Laurent expansion of the functions $\Phi_{2j+1,p-1}(s)$ at $s = 0$ has coefficients: for $1 \leq j \leq p-1$

$$\operatorname{Res}_{s=0} \Phi_{2j+1,p-1}(s) = 2 \sum_{k=j+1}^{3j+1} a_{j,p-1,k} \sum_{t=2}^k \frac{1}{2t-1}, \quad \operatorname{Res}_{s=0} \Phi_{2j+1,p-1}(s) = 0,$$

and for $j = 0$

$$\operatorname{Res}_{s=0} \Phi_{1,p-1}(s) = 2a_{0,p-1,1} = 1, \quad \operatorname{Res}_{s=0} \Phi_{1,p-1}(s) = 0.$$

Proof. The proof follows from the definition in equation (3.4), classical results on the Euler Gamma function (see equation (10.5) in the appendix), and calculation based on the formula (10.6) in the appendix. See [17] for further details. The formula for $j = 0$ follows by explicit knowledge of the coefficients $a_{0,1}$. $\square$

Next, we determine the terms $A_{0,0}(0)$ and $A'_{0,1}(0)$, defined in equation (3.6). Using an upper dot to denote the ones relative to the sequence $\dot{S}_{p-1}$, we have the following result.

Lemma 6.7.

$$\begin{aligned} \mathcal{A}_{0,0,p-1}(s) &= A_{0,0,p-1}(s) - \dot{A}_{0,0,p-1}(s) = 0, \\ \mathcal{A}_{0,1,p-1}(s) &= A_{0,1,p-1}(s) - \dot{A}_{0,1,p-1}(s) = \frac{1}{2}\zeta(2s, U_{p-1}). \end{aligned}$$

Proof. The defining equation (3.6) reads

$$\begin{aligned} A_{0,0,p-1}(s) &= \sum_{n=1}^{\infty} m_{\text{cex},p-1,n} \left(a_{0,0,n,p-1} - \sum_{j=1}^p b_{2j-1,0,0,p-1} \mu_{p-1,n}^{-2j+1} \right) \mu_{p-1,n}^{-2s}, \\ A_{0,1,p-1}(s) &= \sum_{n=1}^{\infty} m_{\text{cex},p-1,n} \left(a_{0,1,n,p-1} - \sum_{j=1}^p b_{2j-1,0,1,p-1} \mu_{p-1,n}^{-2j+1} \right) \mu_{p-1,n}^{-2s}. \end{aligned}$$

in the present case. We need the expansion of the functions $\log \Gamma(-\lambda, S_{p-1,n}/\mu_{p-1,n}^2)$, $l_{2j-1}(\lambda)$, $\log \Gamma(-\lambda, \dot{S}_{p-1,n}/\mu_{p-1,n}^2)$, and $\dot{l}_{2j-1}(\lambda)$, for $j = 1, 2, \dots, p$, and large λ . Using classical expansions for the Bessel functions and their derivative and the formulas in equation (3.5), we obtain (see [17] for further details)

$$\begin{aligned} a_{0,0,n,p-1} &= \frac{1}{2} \log 2\pi + \left(\mu_{p-1,n} + \frac{1}{2} \right) \log \mu_{p-1,n} - \mu_{p-1,n} \log 2 - \log \Gamma(\mu_{p-1,n} + 1), \\ a_{0,1,n,p-1} &= \frac{1}{2} \left(\mu_{p-1,n} + \frac{1}{2} \right), \\ b_{2j-1,0,0,p-1} &= 0, \quad b_{2j-1,0,1,p-1} = 0, \quad j = 1, 2, \dots, p, \end{aligned}$$

note that $b_{2j-1,0,0,p-1} = 0$ since $l_{2j-1}(\lambda)$ don't have constant term.

$$\begin{aligned} \dot{a}_{0,0,n,p-1} &= \frac{1}{2} \log 2\pi + \left(\mu_{p-1,n} + \frac{1}{2} \right) \log \mu_{p-1,n} - \mu_{p-1,n} \log 2 - \log \Gamma(\mu_{p-1,n} + 1), \\ \dot{a}_{0,1,n,p-1} &= \frac{1}{2} \left(\mu_{p-1,n} - \frac{1}{2} \right), \\ \dot{b}_{2j-1,0,0,p-1} &= 0, \quad \dot{b}_{2j-1,0,1,p-1} = 0, \quad j = 1, 2, \dots, p, \end{aligned}$$

and $\dot{b}_{2j-1,0,0,p-1} = 0$ since $\dot{l}_{2j-1}(\lambda)$ don't have constant term. The thesis follows. $\square$

We now have all the necessary information to apply Theorem 3.2 and its corollary. We obtain the following result, where we distinguish the *regular part* and the *singular part*, as in Remark 3.1.

Proposition 6.2.

$$\begin{aligned} t_{p-1}(0) &= t_{p-1,\text{reg}}(0) + t_{p-1,\text{sing}}(0), \\ t'_{p-1}(0) &= t'_{p-1,\text{reg}}(0) + t'_{p-1,\text{sing}}(0), \end{aligned}$$

where

$$\begin{aligned} t_{p-1,\text{reg}}(0) &= -\frac{1}{2}\zeta(0, U_{p-1}) = -\frac{1}{2}\zeta_{\text{cex}}\left(0, \tilde{\Delta}^{(p-1)}\right), \\ t_{p-1,\text{sing}}(0) &= 0, \\ t'_{p-1,\text{reg}}(0) &= -\zeta'(0, U_{p-1}) = -\frac{1}{2}\zeta'_{\text{cex}}\left(0, \tilde{\Delta}^{(p-1)}\right), \\ t'_{p-1,\text{sing}}(0) &= \frac{1}{2}\sum_{j=0}^{p-1} \text{Res}_0 \Phi_{2j+1,q}(s) \text{Res}_{s=2j+1} \zeta(s, U_{p-1}) \\ &= \frac{1}{2}\sum_{j=0}^{p-1} \text{Res}_0 \Phi_{2j+1,q}(s) \text{Res}_{s=2j+1} \zeta_{\text{cex}}\left(\frac{s}{2}, \tilde{\Delta}^{(p-1)}\right). \end{aligned}$$

Proof. By definition in equations (6.1) and (6.2),

$$\begin{aligned} t_{p-1}(0) &= Z_{p-1}(0) - \dot{Z}_{p-1}(0), \\ t'_{p-1}(0) &= Z'_{p-1}(0) - \dot{Z}'_{p-1}(0), \end{aligned}$$

where $Z_{p-1}(s) = \zeta(s, S_{p-1})$, and $\dot{Z}_{p-1}(s) = \zeta(s, \dot{S}_{p-1})$. By Proposition 6.1 and Lemma 6.4, we can apply Theorem 3.2 and its Corollary to the difference of these double zeta functions. The regular part of $Z_{p-1}(0) - \dot{Z}_{p-1}(0)$ is then given in Lemma 6.7, while the singular part vanishes, since, by Corollary 6.1, the residues of the functions $\Phi_{k,p-1}(s)$ at $s = 0$ vanish. The regular part of $Z'_{p-1}(0) - \dot{Z}'_{p-1}(0)$ again follows by Lemma 6.7. For the singular part, since by Proposition 6.1, $\kappa = 2$, $\ell = 2p$, and $\sigma_h = h - 1$, with $0 \leq h \leq 2p$, by Remark 6.1 we need only the odd values of $h - 1 = 2j + 1$, $0 \leq j \leq p - 1$, and this gives the formula stated for $t'_{p-1,\text{sing}}(0)$. $\square$

6.2. The functions $t_q(s)$, $0 \leq q \leq p - 2$. In this section we study the functions $t_q(s)$. For we apply Theorems 3.1 and 3.2 to the double sequences $S_q = \{m_{q,n} : j_{\mu_{q,n},k}^2\}_{n=1}^\infty$ and $S_{q,\pm} = \{m_{q,n} : j_{\mu_{q,n}\pm\alpha_q,k}^2\}_{n=1}^\infty$, since we have that $Z_q(s) = \zeta(s, S_q)$, $Z_{q,\pm}(s) = \zeta(s, S_{q,\pm})$, where $q = 0, 1, \dots, p - 2$, $\alpha_q = p - q - 1$. Note that the sequence S_q coincides with the sequence S_{p-1} analysed in Section 6.1, with $q = p - 1$. So we just need to study the other two sequences. First, we verify Definition 3.1. For we introduce the simple sequence $U_q = \{m_{q,n} : \mu_{q,n}\}_{n=1}^\infty$.

Lemma 6.8. *For all $0 \leq q \leq p - 1$, the sequence U_q is a totally regular sequence of spectral type with infinite order, $e(U_q) = g(U_q) = 2p - 1$, and*

$$\zeta(s, U_q) = \zeta_{\text{cex}}\left(\frac{s}{2}, \tilde{\Delta}^{(q)} + \alpha_q^2\right).$$

The possible poles of $\zeta(s, U_q)$ are at $s = 2p - 1 - h$, $h = 0, 2, 4, \dots$, and the residues are completely determined by the residues of the function $\zeta_{\text{ceex}}(s, \tilde{\Delta}^{(q)})$, namely:

$$\text{Res}_{s=2k+1} \zeta(s, U_q) = \sum_{j=0}^{p-1-k} \binom{-\frac{2k+1}{2}}{j} \text{Res}_{s=2(k+j)+1} \zeta_{\text{ceex}}\left(\frac{s}{2}, \tilde{\Delta}^{(q)}\right) \alpha_q^{2j}.$$

Proof. By definition $U_q = \{m_{\text{ceex},q,n} : \mu_{q,n}\}_{n=1}^{\infty}$, where by Lemmas 4.2 and 4.3

$$\mu_{q,n} = \sqrt{\lambda_{q,n} + \alpha_q^2},$$

and the $\lambda_{q,n}$ are the eigenvalues of the operator $\tilde{\Delta}^{(q)}$ on the compact manifold W . Counting such eigenvalues according to multiplicity of the associated coexact eigenform, since the dimension of the eigenspace of $\lambda_{q,n}$ are finite, by Proposition 3.2, $\lambda_{q,n} \sim n^{\frac{2}{m}}$ for large n . This gives order and genus. Next, by definition

$$\begin{aligned} \zeta(s, U_q) &= \sum_{n=1}^{\infty} m_{\text{ceex},q,n} (\lambda_{q,n} + \alpha_q^2)^{-\frac{s}{2}} = \sum_{j=0}^{\infty} \binom{-\frac{s}{2}}{j} \sum_{n=1}^{\infty} m_{\text{ceex},q,n} \lambda_{q,n}^{-\frac{s}{2}-j} \alpha_q^{2j} \\ &= \sum_{j=0}^{\infty} \binom{-\frac{s}{2}}{j} \zeta_{\text{ceex}}\left(\frac{s}{2} + j, \tilde{\Delta}^{(q)}\right) \alpha_q^{2j} = \zeta_{\text{ceex}}\left(\frac{s}{2}, \tilde{\Delta}^{(q)}\right) - \frac{s}{2} \zeta_{\text{ceex}}\left(\frac{s}{2} + 1, \tilde{\Delta}^{(q)}\right) \alpha_q^2 + \dots \end{aligned}$$

The last statement follows by Proposition 3.3. □

The analysis of the double sequences $S_{q,\pm}$ is little bit harder than that of the sequences of the previous Section 6.1, since now the elements of the sequences are not multiple of zeros of Bessel functions (and their derivative). However, they are the zeros of some linear combinations of Bessel functions and their derivative, and this makes possible the following analysis.

For $c \in \mathbb{C}$, let define the functions

$$\hat{J}_{\nu,c}(z) = cJ_{\nu}(z) + zJ'_{\nu}(z).$$

Recalling the series definition of the Bessel function

$$J_{\nu}(z) = \frac{z^{\nu}}{2^{\nu}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k} k! \Gamma(\nu + k + 1)},$$

we obtain that near $z = 0$

$$\hat{J}_{\nu,c}(z) = \left(1 + \frac{c}{\nu}\right) \frac{z^{\nu}}{2^{\nu} \Gamma(\nu)}.$$

This means that the function $z^{-\nu} \hat{J}_{\nu,c}(z)$ is an even function of z . Let $\hat{j}_{\nu,c,k}$ be the positive zeros of $\hat{J}_{\nu,c}(z)$ arranged in increasing order. By the Hadamard factorization theorem, we have the product expansion

$$z^{-\nu} \hat{J}_{\nu(z),c} = z^{-\nu} \hat{J}_{\nu,c}(z) \prod_{k=-\infty}^{+\infty} \left(1 - \frac{z}{\hat{j}_{\nu,c,k}}\right),$$

and therefore

$$\hat{J}_{\nu,c}(z) = \left(1 + \frac{c}{\nu}\right) \frac{z^{\nu}}{2^{\nu} \Gamma(\nu)} \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{\hat{j}_{\nu,c,k}^2}\right).$$

Next, recalling that (when $-\pi < \arg(z) < \frac{\pi}{2}$)

$$\begin{aligned} J_\nu(iz) &= e^{\frac{\pi}{2}i\nu} I_\nu(z), \\ J'_\nu(iz) &= e^{\frac{\pi}{2}i\nu} e^{-\frac{\pi}{2}i} I'_\nu(z), \end{aligned}$$

we obtain

$$\hat{J}_{\nu,c}(iz) = e^{\frac{\pi}{2}i\nu} (cI_\nu(z) + zI'_\nu(z)).$$

Thus, we define (for $-\pi < \arg(z) < \frac{\pi}{2}$)

$$(6.6) \quad \hat{I}_{\nu,c}(z) = e^{-\frac{\pi}{2}i\nu} \hat{J}_{\nu,c}(iz),$$

and hence

$$(6.7) \quad \hat{I}_{\nu,\pm\alpha_q}(z) = \pm\alpha_q I_\nu(z) + zI'_\nu(z) = \left(1 \pm \frac{\alpha_q}{\nu}\right) \frac{z^\nu}{2^\nu \Gamma(\nu)} \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{\hat{j}_{\nu,\pm\alpha_q,k}^2}\right).$$

Recalling the definition in equation (3.1) we have proved the following fact.

Lemma 6.9. *The logarithmic Gamma functions associated to the sequences $S_{q,\pm,n}$ have the following representations, when $\lambda \in D_{\theta,c'}$, with $c' = \frac{1}{2} \min(j_{\mu_{q,0}}^2, j_{\mu_{q,0},\pm\alpha_q}^2)$,*

$$\begin{aligned} \log \Gamma(-\lambda, S_{q,\pm,n}) &= -\log \prod_{k=1}^{\infty} \left(1 + \frac{(-\lambda)}{\hat{j}_{\mu_{q,n},\pm\alpha_q,k}^2}\right) \\ &= -\log \hat{I}_{\mu_{q,n},\pm\alpha_q}(\sqrt{-\lambda}) + \mu_{q,n} \log \sqrt{-\lambda} - \mu_{q,n} \log 2 \\ &\quad - \log \Gamma(\mu_{q,n}) + \log \left(1 \pm \frac{\alpha_q}{\mu_{q,n}}\right). \end{aligned}$$

Proposition 6.3. *The double sequences $S_{q,\pm}$ have relative exponents $(p, \frac{2p-1}{2}, \frac{1}{2})$, relative genus $(p, p-1, 0)$, and are spectrally decomposable over U_q with power $\kappa = 2$, length $\ell = 2p$ and domain $D_{\theta,c'}$. The coefficients σ_h appearing in equation (3.3) are $\sigma_h = h-1$, with $h = 1, 2, \dots, \ell = 2p$.*

Proof. The proof is the same of the one of Proposition 6.1. □

Remark 6.3. *By Theorem 3.1, only the term with $\sigma_h = 1, \sigma_h = 3, \dots, \sigma_h = 2p-1$ namely $h = 2, 4, \dots, 2p$, appear in the formula of Theorem 3.2, since the unique poles of $\zeta(s, U_q)$ are at $s = 1, s = 3, \dots, s = 2p-1$.*

Since we aim to apply the version of Theorem 3.2 given in Corollary 3.1, for linear combination of two spectrally decomposable sequences, we inspect directly the uniform asymptotic expansion of $2S_q - S_{q,-} - S_{q,+}$. This give the functions ϕ_{σ_h} .

Lemma 6.10. *We have the the following asymptotic expansions for large n , uniform in λ , for $\lambda \in D_{\theta, c'}$,*

$$\begin{aligned} & 2 \log \Gamma(-\lambda, S_{q,n}/\mu_{q,n}^2) - \log \Gamma(-\lambda, S_{q,+,n}/\mu_{q,n}^2) - \log \Gamma(-\lambda, S_{q,-,n}/\mu_{q,n}^2) \\ &= -2 \log I_{\mu_{q,n}}(\mu_{q,n} \sqrt{-\lambda}) + \log \hat{I}_{\mu_{q,n}, \alpha_q}(\mu_{q,n} \sqrt{-\lambda}) + \log \hat{I}_{\mu_{q,n}, -\alpha_q}(\mu_{q,n} \sqrt{-\lambda}) \\ &\quad - 2 \log \mu_{q,n} - \log \left(1 - \frac{\alpha_q^2}{\mu_{q,n}^2} \right) \\ &= \log(1 - \lambda) + \sum_{j=1}^{2p-1} \phi_{j,q}(\lambda) \frac{1}{\mu_{q,n}^j} + O\left(\frac{1}{(\mu_{q,n})^{2p}}\right). \end{aligned}$$

Proof. Using the representations given in Lemmas 6.4 and 6.9, we obtain

$$\begin{aligned} & 2 \log \Gamma(-\lambda, S_{q,n}/\mu_{q,n}^2) - \log \Gamma(-\lambda, S_{q,+,n}/\mu_{q,n}^2) - \log \Gamma(-\lambda, S_{q,-,n}/\mu_{q,n}^2) \\ &= -2 \log I_{\mu_{q,n}}(\mu_{q,n} \sqrt{-\lambda}) + \log \hat{I}_{\mu_{q,n}, \alpha_q}(\mu_{q,n} \sqrt{-\lambda}) + \log \hat{I}_{\mu_{q,n}, -\alpha_q}(\mu_{q,n} \sqrt{-\lambda}) \\ &\quad - 2 \log \mu_{q,n} - \log \left(1 - \frac{\alpha_q^2}{\mu_{q,n}^2} \right). \end{aligned}$$

Using the expansion given in Lemma 6.4 for $I_\nu(\nu z)$ and $I'_\nu(\nu z)$, we obtain the following expansion for $\hat{I}_{\nu, \pm \alpha_q}(\nu z)$,

$$\begin{aligned} \hat{I}_{\nu, \pm \alpha_q}(\nu z) &= \pm \alpha_q I_\nu(\nu z) + \nu z I'_\nu(\nu z) \\ &= \sqrt{\nu}(1+z^2)^{\frac{1}{4}} \frac{e^{\nu \sqrt{1+z^2}} e^{\nu \log \frac{z}{1+\sqrt{1+z^2}}}}{\sqrt{2\pi}} \left(1 + \sum_{j=1}^{2p-1} W_{\pm \alpha_q, j}(z) \frac{1}{\nu^j} + O\left(\frac{1}{\nu^{2p}}\right) \right), \end{aligned}$$

where $W_{\pm \alpha_q, j}(z) = V_j(z) \pm \frac{\alpha_q}{\sqrt{1+z^2}} U_j(z)$. Thus,

$$\begin{aligned} \log \hat{I}_{\nu, \pm \alpha_q}(\nu z) &= \nu \sqrt{1+z^2} + \nu \log z - \nu \log(1 + \sqrt{1+z^2}) + \log \nu + \frac{1}{4} \log(1+z^2) \\ &\quad - \frac{1}{2} \log 2\pi \nu + \log \left(1 + \sum_{j=1}^{2p-1} W_{\pm \alpha_q, j}(z) \frac{1}{\nu^j} + O\left(\frac{1}{\nu^{2p}}\right) \right). \end{aligned}$$

This gives,

$$\begin{aligned} & 2 \log \Gamma(-\lambda, S_{q,n}/\mu_{q,n}^2) - \log \Gamma(-\lambda, S_{q,+,n}/\mu_{q,n}^2) - \log \Gamma(-\lambda, S_{q,-,n}/\mu_{q,n}^2) \\ &= \log(1 - \lambda) - 2 \log \left(1 + \sum_{j=1}^{2p-1} \frac{U_j(\sqrt{-\lambda})}{\mu_{q,n}^j} + O\left(\frac{1}{\mu_{q,n}^{2p}}\right) \right) \\ &\quad + \log \left(1 + \sum_{j=1}^{2p-1} \frac{W_{+\alpha_q, j}(\sqrt{-\lambda})}{\mu_{q,n}^j} + O\left(\frac{1}{\mu_{q,n}^{2p}}\right) \right) + \log \left(1 + \sum_{j=1}^{2p-1} \frac{W_{-\alpha_q, j}(\sqrt{-\lambda})}{\mu_{q,n}^j} + O\left(\frac{1}{\mu_{q,n}^{2p}}\right) \right). \end{aligned}$$

Using the same expansion for the logarithm as in the proof of Lemma 6.4

$$\begin{aligned} & 2 \log \Gamma(-\lambda, S_{q,n}/\mu_{q,n}^2) - \log \Gamma(-\lambda, S_{q,+,n}/\mu_{q,n}^2) - \log \Gamma(-\lambda, S_{q,-,n}/\mu_{q,n}^2) \\ &= \log(1-\lambda) + \sum_{j=1}^p (-2l_{2j-1}(\lambda) + l_{2j-1}^+(\lambda) + l_{2j-1}^-(\lambda)) \frac{1}{\mu_{q,n}^{2j-1}} \\ &+ \sum_{j=1}^{p-1} \left(-2l_{2j}(\lambda) + l_{2j}^+(\lambda) + l_{2j}^-(\lambda) + \frac{\alpha_q^{2j}}{j} \right) \frac{1}{\mu_{q,n}^{2j}} + O\left(\frac{1}{\mu_{q,n}^{2p}}\right), \end{aligned}$$

where we denote by $l_j(\lambda)$ the term in the expansion relative to the sequence S (thus the one containing the $U_j(z)$) and by $l_j^\pm(\lambda)$ the terms relative to $S_\pm$ (thus the ones containing the $W_{\pm\alpha_q,j}(z)$). Setting

$$\begin{aligned} \phi_{2j-1,q}(\lambda) &= -2l_{2j-1}(\lambda) + l_{2j-1}^+(\lambda) + l_{2j-1}^-(\lambda) \\ \phi_{2j,q}(\lambda) &= -2l_{2j}(\lambda) + l_{2j}^+(\lambda) + l_{2j}^-(\lambda) + \frac{\alpha_q^{2j}}{j}, \end{aligned} \tag{6.8}$$

the result follows. $\square$

Remark 6.4. Note that there are no logarithmic terms $\log \mu_{q,n}$ in the asymptotic expansion of the difference of the logarithmic Gamma function given in Lemma 6.10. This permits to apply Corollary 3.1.

Next, we give some results on the functions $\phi_{j,q}(\lambda)$, and on the functions $\Phi_{j,q}(s)$ defined in equation (3.4).

Lemma 6.11. For all j and all $0 \leq q \leq p-2$, the functions $\phi_{j,q}(\lambda)$ are odd polynomial in $w = \frac{1}{\sqrt{1-\lambda}}$

$$\begin{aligned} \phi_{2j-1,q}(\lambda) &= \sum_{k=0}^{2j-1} a_{2j-1,q,k} w^{2k+2j-1}, \\ \phi_{2j,q}(\lambda) &= \sum_{k=0}^{2j} a_{2j,q,k} w^{2k+2j} + \frac{\alpha_q^{2j}}{j}. \end{aligned}$$

The coefficients $a_{j,q,k}$ are completely determined by the coefficients of the expansion given in Lemma 6.10.

Proof. This follows by direct inspection of the last equality in the statement of Lemma 6.10. $\square$

Lemma 6.12. For all j and all $0 \leq q \leq p-2$, $\phi_{j,q}(0) = 0$.

Proof. The proof is by induction on j . We will consider all the functions as functions of $w = \frac{1}{\sqrt{1-\lambda}}$. We use the following hypothesis for the induction, for $1 \leq k \leq j-1$:

$$\phi_{2k-1,q}(1) = 0, \tag{6.9}$$

$$\phi_{2k,q}(1) = 0, \tag{6.10}$$

$$l_{2k-1}^-(1) - l_{2k-1}^+(1) = \frac{-2\alpha_q^{2k-1}}{2k-1}, \tag{6.11}$$

$$l_{2k}^-(1) - l_{2k}^+(1) = 0, \tag{6.12}$$

where the functions $\phi_{j,q}(\lambda)$ are defined in equation (6.8), and the function $l(\lambda)$ in the course of the proof of Lemma 6.10.

First, we verify the hypothesis for $j = 1$. Equations (6.11) and (6.12) follow by the definition when $k = 1$. For equations (6.9) and (6.10), we have by definition when $k = 1$ that

$$\begin{aligned}\phi_{1,q}(\lambda) &= -2l_1(\lambda) + l_1^+(\lambda) + l_1^-(\lambda) \\ &= -2U_1(\sqrt{-\lambda}) + V_1(\sqrt{-\lambda}) + \alpha_q U_0(\sqrt{-\lambda}) + V_1(\sqrt{-\lambda}) - \alpha_q U_0(\sqrt{-\lambda}) \\ &= -\frac{1}{(1-\lambda)^{\frac{1}{2}}} + \frac{1}{(1-\lambda)^{\frac{3}{2}}},\end{aligned}$$

and

$$\begin{aligned}\phi_{2,q}(\lambda) &= -2l_2(\lambda) + l_2^+(\lambda) + l_2^-(\lambda) + \alpha_q^2 \\ &= -2U_2(\sqrt{-\lambda}) + 2V_2(\sqrt{-\lambda}) + U_1(\sqrt{-\lambda})^2 - V_1(\sqrt{-\lambda})^2 \\ &= -\frac{3}{2} \frac{1}{(1-\lambda)} + 2 \frac{1}{(1-\lambda)^2} - \frac{3}{2} \frac{1}{(1-\lambda)^3} + 1,\end{aligned}$$

and hence equations (6.9) and (6.10) are also verified when $k = 1$.

Second we prove that the equations (6.9), (6.10), (6.11), and (6.12) hold for $k = j$. Recalling that $U_k(1) = V_k(1)$ for all k , we have from the definition that

$$\begin{aligned}l_{2j-1}^-(1) - l_{2j-1}^+(1) &= U_{2j-1}(1) - \alpha_q U_{2j-2}(1) - U_{2j-1}(1) - \alpha_q U_{2j-2}(1) \\ &\quad - \sum_{k=1}^{2j-2} \frac{2j-1-k}{2j-1} \left(U_k(1)(l_{2j-1-k}^-(1) - l_{2j-1-k}^+(1)) \right) \\ &\quad + \sum_{k=1}^{2j-2} \frac{2j-1-k}{2j-1} \left(\alpha_q U_{k-1}(1)(l_{2j-1-k}^-(1) + l_{2j-1-k}^+(1)) \right),\end{aligned}$$

and hence, using the hypothesis we obtain

$$\begin{aligned}l_{2j-1}^-(1) - l_{2j-1}^+(1) &= -2\alpha_q U_{2j-2}(1) + \sum_{k=1}^{j-1} \frac{2(j-k)}{2j-1} \alpha_q U_{2k-2}(1) \left(2l_{2(j-k)} - \frac{\alpha_q^{2(j-k)}}{j-k} \right) \\ &\quad - \sum_{k=1}^{j-1} U_{2k}(1) \frac{-2\alpha_q^{2(j-k)-1}}{2j-1} + \sum_{k=1}^{j-1} \frac{2(j-k)-1}{2j-1} 2\alpha_q U_{2k-1}(1) l_{2(j-k)-1}(1) \\ &= -\frac{2\alpha_q^{2j-1}}{2j-1} - 2\alpha_q U_{2j-2}(1) + \frac{2\alpha_q}{2j-1} U_{2j-2}(1) \\ &\quad + \frac{2\alpha_q}{2j-1} \left(2(j-1)l_{2j-2} + \sum_{k=1}^{2j-3} (2j-2-k)\alpha_q U_k(1)l_{2j-2-k}(1) \right) \\ &= -\frac{2\alpha_q^{2j-1}}{2j-1} - 2\alpha_q U_{2j-2}(1) + \frac{2\alpha_q}{2j-1} U_{2j-2}(1) + \frac{2\alpha_q(2j-2)U_{2j-2}}{2j-1} \\ &= -\frac{2\alpha_q^{2j-1}}{2j-1},\end{aligned}$$

thus proving (6.11) for $k = j$. For (6.9), note that

$$\begin{aligned}\phi_{2j-1,q}(1) &= -2l_{2j-1}(1) + l_{2j-1}^+(1) + l_{2j-1}^-(1) \\ &= \sum_{k=1}^{2j-2} \frac{2j-1-k}{2j-1} \left(U_k(1)(2l_{2j-1-k}(1) - l_{2j-1-k}^+(1) - l_{2j-1-k}^-(1)) \right) \\ &\quad - \sum_{k=1}^{2j-2} \frac{2j-1-k}{2j-1} \alpha_q U_{k-1}(1) \left(l_{2j-1-k}^-(1) - l_{2j-1-k}^+(1) \right),\end{aligned}$$

and using the induction hypothesis, and (6.11) with $k = j$ just proved, this means that

$$\begin{aligned}\phi_{2j-1,q}(1) &= \sum_{k=1}^j \frac{2j-1-(2k-1)}{2j-1} U_{2k-1}(1) \frac{\alpha(i)^{2(j-k)}}{j-k} \\ &\quad - 2 \sum_{k=1}^j \frac{2j-1-2k}{2j-1} \alpha(i) U_{2k-1}(1) \left(\frac{\alpha(i)^{2(j-k)-1}}{2(j-k)-1} \right) = 0.\end{aligned}$$

and (6.9) with $k = j$ follows. For (6.12), using the hypothesis, we have

$$\begin{aligned}&l_{2j}^-(1) - l_{2j}^+(1) \\ &= U_{2j}(1) - \alpha_q U_{2j-1}(1) - U_{2j}(1) - \alpha_q U_{2j-1}(1) \\ &\quad - \sum_{k=1}^{2j-1} \frac{2j-k}{2j} \left(U_k(1)(l_{2j-k}^-(1) - l_{2j-k}^+(1)) \right) \\ &\quad + \sum_{k=1}^{2j-1} \frac{2j-k}{2j} \left(\alpha_q U_{k-1}(1)(l_{2j-k}^-(1) + l_{2j-k}^+(1)) \right) \\ &= -2\alpha_q U_{2j-1}(1) + \sum_{k=1}^{j-1} \frac{2j-2k}{2j} \alpha_q U_{2k-1}(1) \left(2l_{2j-2k}(1) - \frac{\alpha_q^{2j-2k}}{j-k} \right) \\ &\quad + \sum_{k=1}^j \frac{2(j-k)+1}{2j} \left(U_{2k-1}(1) \frac{2\alpha_q^{2(j-k)+1}}{2(j-k)+1} + \alpha_q U_{2k-2}(1) 2l_{2(j-k)+1}(1) \right) \\ &= -2\alpha_q U_{2j-1}(1) + \frac{2\alpha_q U_{2j-1}(1)}{2j} + \frac{2\alpha_q}{2j} (2j-1) l_{2j-1}(1) + 2\alpha_q \sum_{k=2}^{2j-1} \frac{2j-k}{2j} U_{k-1}(1) l_{2j-k}(1) \\ &= -2\alpha_q U_{2j-1}(1) + \frac{\alpha_q}{j} \left(U_{2j-1}(1) + (2j-1) l_{2j-1}(1) + \sum_{k=1}^{2j-2} (2j-1-k) U_k(1) l_{2j-1-k}(1) \right) \\ &= -2\alpha_q U_{2j-1}(1) + \frac{(\alpha_q(2j-1) + \alpha_q) U_{2j-1}(1)}{j} = 0.\end{aligned}$$

Eventually, for (6.10)

$$\begin{aligned}
\phi_{2j,q}(1) &= -2l_{2j}(1) + l_{2j}^+(1) + l_{2j}^-(1) + \frac{\alpha_q^{2j}}{j} \\
&= \frac{\alpha_q^{2j}}{j} + \sum_{k=1}^{2j-1} \frac{2j-k}{2j} \left(U_k(1)(2l_{2j-k}(1) - l_{2j-k}^+(1) - l_{2j-k}^-(1)) \right) \\
&\quad - \sum_{k=1}^{2j-1} \frac{2j-k}{2j} \alpha_q U_{k-1}(1) \left(l_{2j-k}^-(1) - l_{2j-k}^+(1) \right) \\
&= \sum_{k=1}^{j-1} \frac{2j-2k}{2j} U_{2k}(1) \frac{\alpha_q^{2(j-k)}}{j-k} - 2 \sum_{k=2}^j \alpha_q U_{2k-2}(1) \frac{\alpha_q^{2(j-k)+1}}{2j} = 0,
\end{aligned}$$

□

We also give a recurrence relation satisfied by the functions $\phi_{j,q}(\lambda)$, that will be fundamental in the proof of Theorem 1.3.

Lemma 6.13. *For all j and all $0 \leq q \leq p-2$, the functions $\phi_{j,q}(w)$ satisfy the following recurrence relations (where $w = \frac{1}{\sqrt{1-\lambda}}$)*

$$\begin{aligned}
\phi_{2j-1,q}(\lambda) &= w^{2j-2} \alpha_q^{2j-2} \phi_{q,1}(w) + \sum_{t=1}^{j-2} K_{2j-1,t}(w) \alpha_q^{2t} + 2\phi_{2j-1,p-1}(w) \\
\phi_{2j,q}(\lambda) &= -\frac{(w^{2j}-1)\alpha_q^{2j}}{j} + \sum_{t=1}^{j-1} K_{2j,t}(w) \alpha_q^{2t} + 2\phi_{2j,p-1}(w),
\end{aligned}$$

where the $K_{j,t}(w)$ are polynomials in w .

Proof. The proof is by induction on j . For $j = 1$,

$$\begin{aligned}
\phi_{1,q}(w) &= 2\phi_{1,p-1}(w) = -w + w^3 \\
\phi_{2,q}(w) &= -(w^2-1)\alpha_q^2 + 2\phi_{2,p-1}(w) = -(w^2-1)\alpha_q^2 + \left(-\frac{w^2}{2} - 2w^4 - \frac{3w^6}{2}\right).
\end{aligned}$$

Assuming the formulas hold for $1 \leq k \leq j-2$. Then, by definition of the functions $\phi_{j,q}(\lambda)$ and $l(\lambda)$ in the proof of Lemma 6.10, we have that

$$\begin{aligned}
l_{2s-1}^+(w) + l_{2s-1}^-(w) &= 2l_{2s-1}(w) + w^{2s-2} \alpha_q^{2s-2} (\phi_{q,1}(w)) + \sum_{t=1}^{s-2} K_{2s-1,t}(w) \alpha_q^{2t}, \\
l_{2s}^+(w) + l_{2s}^-(w) &= 2l_{2s}(w) - \frac{w^{2s} \alpha_q^{2s}}{s} + \sum_{t=1}^{s-1} K_{2s,t}(w) \alpha_q^{2t}, \\
l_{2s-1}^+(w) - l_{2s-1}^-(w) &= \frac{2}{2s-1} \alpha_q^{2s-1} w^{2s-1} + \alpha_q \sum_{t=0}^{s-2} D_{2s-1,t}(w) \alpha_q^{2t}, \\
l_{2s}^+(w) - l_{2s}^-(w) &= -\alpha_q^{2s-1} w^{2s-1} \phi_{q,1}(w) + \alpha_q \sum_{t=0}^{s-2} D_{2s,t}(w) \alpha_q^{2t},
\end{aligned}$$

for all $s = 1, 2, \dots, j-1$, and where the $D_{s,t}$ are polynomials in w . We proceed as in the proof of Lemma 6.12 (see [17] for further details). For the odd index we have:

$$\begin{aligned}
l_{2j-1}^+(w) - l_{2j-1}^-(w) &= 2\alpha_q U_{2j-2}(w) - \sum_{k=1}^{2j-2} \frac{2j-1-k}{2j-1} V_k(w) (l_{2j-1-k}^+(w) - l_{2j-1-k}^-(w)) \\
&\quad + \sum_{k=1}^{2j-2} \frac{2j-1-k}{2j-1} w \alpha_q U_{k-1}(w) (l_{2j-1-k}^+(w) + l_{2j-1-k}^-(w)), \\
&= 2\alpha_q U_{2j-2}(w) - \sum_{k=1}^{j-1} \frac{2j-1-2k}{2j-1} V_{2k}(w) (l_{2j-1-2k}^+(w) - l_{2j-1-2k}^-(w)) \\
&\quad - \sum_{k=1}^{j-1} \frac{2j-1-2k}{2j-1} w \alpha_q U_{2k-1}(w) (l_{2j-1-2k}^+(w) + l_{2j-1-2k}^-(w)) \\
&\quad - \sum_{k=1}^{j-1} \frac{2j-2k}{2j-1} V_{2k-1}(w) (l_{2j-2k}^+(w) - l_{2j-2k}^-(w)) \\
&\quad - \sum_{k=1}^{j-1} \frac{2j-2k}{2j-1} w \alpha_q U_{2k-2}(w) (l_{2j-2k}^+(w) + l_{2j-2k}^-(w)) \\
&= \frac{2}{2j-1} \alpha_q^{2j-1} w^{2j-1} + \alpha_q \sum_{t=0}^{j-2} D_{2j-1,t}(w) \alpha_q^{2t},
\end{aligned}$$

and this gives

$$\begin{aligned}
\phi_{2j-1,q}(w) &= -2U_{2j-1}(w) + 2V_{2j-1}(w) + \sum_{k=1}^{2j-2} \frac{2j-1-k}{2j-1} (2U_k(w) l_{2j-1-k}(w)) \\
&\quad - \sum_{k=1}^{j-1} \frac{2j-1-2k}{2j-1} (V_{2k}(w) (l_{2j-1-2k}^+(w) + l_{2j-1-2k}^-(w))) \\
&\quad - \sum_{k=1}^{j-1} \frac{2j-1-2k}{2j-1} (w \alpha_q U_{2k-1}(w) (l_{2j-1-2k}^+(w) - l_{2j-1-2k}^-(w))) \\
&\quad - \sum_{k=1}^{j-1} \frac{2j-2k}{2j-1} (V_{2k-1}(w) (l_{2j-2k}^+(w) + l_{2j-2k}^-(w))) \\
&\quad - \sum_{k=1}^{j-1} \frac{2j-2k}{2j-1} (w \alpha_q U_{2k-2}(w) (l_{2j-2k}^+(w) - l_{2j-2k}^-(w))) \\
&= w^{2j-2} \alpha_q^{2j-2} (\phi_{1,q}(w)) + \sum_{t=1}^{j-2} K_{2j-1,t}(w) \alpha_q^{2t} + 2\phi_{2j-1,p-1}(w).
\end{aligned}$$

For the even index, using the result proved for the odd index, we get

$$\begin{aligned}
l_{2j}^+(w) - l_{2j}^-(w) &= 2\alpha_q U_{2j-1}(w) - \sum_{k=1}^{j-1} \frac{2j-2k}{2j} V_{2k}(w)(l_{2j-2k}^+(w) - l_{2j-2k}^-(w)) \\
&\quad - \sum_{k=1}^{j-1} \frac{2j-2k}{2j} w\alpha_q U_{2k-1}(w)(l_{2j-2k}^+(w) + l_{2j-2k}^-(w)) \\
&\quad - \sum_{k=1}^{j-1} \frac{2j-2k+1}{2j} V_{2k-1}(w)(l_{2j-2k+1}^+(w) - l_{2j-2k+1}^-(w)) \\
&\quad - \sum_{k=1}^{j-1} \frac{2j-2k+1}{2j} w\alpha_q U_{2k-2}(w)(l_{2j-2k+1}^+(w) + l_{2j-2k+1}^-(w)) \\
&= -\alpha_q^{2j-1} w^{2j-1} \phi_{1,q}(w) + \alpha_q \sum_{t=0}^{j-2} D_{2j,t}(w) \alpha_q^{2t},
\end{aligned}$$

and proceeding as before, this gives the last formula in the thesis. $\square$

Corollary 6.2. *For all j and all $0 \leq q \leq p-2$, the Laurent expansion of the functions $\Phi_{2j+1,q}(s)$ at $s=0$ has coefficients: for $1 \leq j \leq p-1$*

$$\text{Res}_{s=0} \Phi_{2j+1,q}(s) = \frac{2}{2j+1} \alpha_q^{2j} + \sum_{t=1}^{j-1} k_{2j+1,t} \alpha_q^{2t} + 2 \text{Res}_{z=0} \Phi_{2j+1,p-1}(s), \quad \text{Res}_{s=0} \Phi_{2j+1,q}(s) = 0,$$

where the $k_{j,t}$ are real numbers, and for $j=0$

$$\text{Res}_{s=0} \Phi_{1,q}(s) = 2 \text{Res}_{s=0} \Phi_{1,p-1}(s) = 2, \quad \text{Res}_{s=0} \Phi_{1,q}(s) = 0.$$

Proof. By Lemma 6.11,

$$\phi_{2j+1,q}(\lambda) = \sum_{k=0}^{2j+1} a_{2j+1,q,k} w^{2k+2j+1},$$

where $w = \frac{1}{\sqrt{1-\lambda}}$, and $\phi_{2j+1,q}(0) = 0$, therefore $\sum_{k=0}^{2j-1} a_{2j+1,q,k} = 0$. Using the formula in equation (10.6) and the residues for the Gamma function in equation (10.5) in the appendix, we obtain

$$\text{Res}_{s=0} \Phi_{2j+1,q}(s) = \sum_{k=0}^{2j+1} a_{2j+1,q,k} = 0.$$

Using the same formulas in the appendix, but the result of Lemma 6.13, we prove the formula for the finite part. The formula for $j=0$ follows by explicit knowledge of the coefficients $a_{0,1}$. $\square$

Next, we determine the terms $A_{0,0}(0)$ and $A'_{0,1}(0)$, defined in equation (3.6).

Lemma 6.14. *For all $0 \leq q \leq p-2$,*

$$\begin{aligned} \mathcal{A}_{0,0,q}(s) &= 2A_{0,0,q}(s) - A_{0,0,q,+}(s) - A_{0,0,q,-}(s) = - \sum_{n=1}^{\infty} \log \left(1 - \frac{\alpha_q^2}{\mu_{q,n}^2} \right) \frac{m_{q,n}}{\mu_{q,n}^{2s}}, \\ \mathcal{A}_{0,1,q}(s) &= 2A_{0,1,q}(s) - A_{0,1,q,+}(s) - A_{0,1,q,-}(s) = \zeta(2s, U_q). \end{aligned}$$

Proof. For S_q equation (3.6) reads

$$\begin{aligned} A_{0,0,q}(s) &= \sum_{n=1}^{\infty} m_{\text{cex},q,n} \left(a_{0,0,n,q} - \sum_{j=1}^p b_{2j-1,0,0,q} \mu_{q,n}^{-2j+1} \right) \mu_{q,n}^{-2s}, \\ A_{0,1,q}(s) &= \sum_{n=1}^{\infty} m_{\text{cex},q,n} \left(a_{0,1,n,q} - \sum_{j=1}^p b_{2j-1,0,1,q} \mu_{q,n}^{-2j+1} \right) \mu_{q,n}^{-2s}. \end{aligned}$$

for $S_{q,\pm}$:

$$\begin{aligned} A_{0,0,q,\pm}(s) &= \sum_{n=1}^{\infty} m_{\text{cex},q,n} \left(a_{0,0,n,q,\pm} - \sum_{j=1}^p b_{2j-1,0,0,q,\pm} \mu_{q,n}^{-2j+1} \right) \mu_{q,n}^{-2s}, \\ A_{0,1,q,\pm}(s) &= \sum_{n=1}^{\infty} m_{\text{cex},q,n} \left(a_{0,1,n,q,\pm} - \sum_{j=1}^p b_{2j-1,0,1,q,\pm} \mu_{q,n}^{-2j+1} \right) \mu_{q,n}^{-2s}. \end{aligned}$$

We need the expansions for large λ of $l_{2j-1}(\lambda)$, $l_{2j-1}^{\pm}(\lambda)$, for $j = 1, 2, \dots, p$, $\log \Gamma(-\lambda, S_{q,n}/\mu_{q,n}^2)$ and $\log \Gamma(-\lambda, S_{q,\pm,n}/\mu_{q,n}^2)$. Using classical expansion for Bessel functions and their derivative (see [19] or [17] for details), we obtain

$$\begin{aligned} \log \Gamma(-\lambda, S_{q,n}/\mu_{q,n}^2) &= \frac{1}{2} \log 2\pi + \left(\mu_{q,n} + \frac{1}{2} \right) \log \mu_{q,n} - \mu_{q,n} \log 2 \\ &\quad - \log \Gamma(\mu_{q,n} + 1) + \frac{1}{2} \left(\mu_{q,n} + \frac{1}{2} \right) \log(-\lambda) + O(e^{-\mu_{q,n}\sqrt{-\lambda}}). \end{aligned}$$

For $S_{q,\pm}$, by the same expansions in the definition of the function $\hat{I}$, equation (6.7), we obtain

$$\hat{I}_{\nu,\pm\alpha_q}(z) \sim \frac{\sqrt{z}e^z}{\sqrt{2\pi}} \left(1 + \sum_{k=1}^{\infty} b_k z^{-k} \right) + O(e^{-z}),$$

and hence

$$\begin{aligned} \log \Gamma(-\lambda, S_{q,\pm,n}/\mu_{q,n}^2) &= \mu_{q,n} \sqrt{-\lambda} + \frac{1}{2} \log 2\pi + \left(\mu_{q,n} - \frac{1}{2} \right) \log \mu_{q,n} - \mu_{q,n} \log 2 \\ &\quad - \log \Gamma(\mu_{q,n}) + \frac{1}{2} \left(\mu_{q,n} - \frac{1}{2} \right) \log(-\lambda) + \log \left(1 \pm \frac{\alpha_q}{\mu_{q,n}} \right) + O(e^{-\mu_{q,n}\sqrt{-\lambda}}). \end{aligned}$$

This gives

$$\begin{aligned} a_{0,0,n,q} &= \frac{1}{2} \log 2\pi + \left(\mu_{q,n} + \frac{1}{2} \right) \log \mu_{q,n} - \mu_{q,n} \log 2 - \log \Gamma(\mu_{q,n} + 1), \\ a_{0,1,n,q} &= \frac{1}{2} \left(\mu_{q,n} + \frac{1}{2} \right), \\ a_{0,0,n,q,\pm} &= \frac{1}{2} \log 2\pi + \left(\mu_{q,n} - \frac{1}{2} \right) \log \mu_{q,n} - \log 2^{\mu_{q,n}} \Gamma(\mu_{q,n}) + \log \left(1 \pm \frac{\alpha_q}{\mu_{q,n}} \right), \\ a_{0,1,n,q,\pm} &= \frac{1}{2} \left(\mu_{q,n} - \frac{1}{2} \right), \end{aligned}$$

while the $b_{2j-1,0,0,q}$, $b_{2j-1,0,0,q,\pm}$ all vanish since the functions $l_{2j-1}(\lambda)$, $l_{2j-1}^{\pm}(\lambda)$ do not have constant terms. Therefore,

$$\begin{aligned} 2a_{0,0,n,q} - a_{0,0,n,q,+} - a_{0,0,n,q,-} &= -\log \left(1 - \frac{\alpha_q^2}{\mu_{q,n}^2} \right), \\ 2a_{0,1,n,q} - a_{0,1,n,q,+} - a_{0,1,n,q,-} &= 1, \end{aligned}$$

and the thesis follows. $\square$

Applying Theorem 3.2 and its corollary, we obtain the values of $t_q(0)$ and $t'_q(0)$.

Proposition 6.4. *For $0 \leq q \leq p-2$,*

$$\begin{aligned} t_q(0) &= t_{q,\text{reg}}(0) + t_{q,\text{sing}}(0) \\ t'_q(0) &= t'_{q,\text{reg}}(0) + t'_{q,\text{sing}}(0), \end{aligned}$$

where

$$\begin{aligned} t_{q,\text{reg}}(0) &= -\zeta(0, U_q) = -\zeta_{\text{cex}} \left(0, \tilde{\Delta}^{(q)} + \alpha_q^2 \right), \\ t_{q,\text{sing}}(0) &= 0, \\ t'_{q,\text{reg}}(0) &= -\mathcal{A}_{q,0,0}(0) - \mathcal{A}'_{q,0,1}(0), \\ t'_{q,\text{sing}}(0) &= \frac{1}{2} \sum_{j=0}^{p-1} \text{Res}_0 \Phi_{2j+1,q}(s) \text{Res}_1 \zeta(s, U_q) \\ &= \frac{1}{2} \sum_{j=0}^{p-1} \text{Res}_0 \Phi_{2j+1,q}(s) \text{Res}_{s=2j+1} \zeta_{\text{cex}} \left(\frac{s}{2}, \tilde{\Delta}^{(q)} + \alpha_q^2 \right). \end{aligned}$$

Proof. By definition in equations (6.1) and (6.2),

$$\begin{aligned} t_q(0) &= 2Z_q(0) - Z_{q,+}(0) - Z_{q,-}(0), \\ t'_q(0) &= 2Z'_q(0) - Z'_{q,+}(0) - Z'_{q,-}(0). \end{aligned}$$

where $Z_q(s) = \zeta(s, S_q)$, and $Z_{q,\pm}(s) = \zeta(s, S_{q,\pm})$. By Proposition 6.3 and Lemma 6.10, we can apply Theorem 3.2 and its Corollary to the linear combination above of these double zeta functions. The regular part of $2Z_q(0) - Z_{q,+}(0) - Z_{q,-}(0)$ is then given in Lemma 6.14, while the singular part vanishes, since, by Corollary 6.2, the residues of the functions $\Phi_{k,q}(s)$ at $s=0$ vanish. The regular part of $2Z'_q(0) - Z'_{q,+}(0) - Z'_{q,-}(0)$ again follows by Lemma 6.7. For the singular part, since by

Proposition 6.3, $\kappa = 2$, $\ell = 2p$, and $\sigma_h = h - 1$, with $0 \leq h \leq 2p$, by Remark 6.3 we need only the odd values of $h - 1 = 2j + 1$, $0 \leq j \leq p - 1$, and this gives the formula stated for $t'_{p-1,\text{sing}}(0)$. $\square$

7. THE ANALYTIC TORSION, AND THE PROOF OF THEOREM 1.1

In this section we collect all the results obtained in the previous one in order to produce our formulas for the analytic torsion, thus proving Theorem 1.1, that follows from Propositions 7.1 and 7.2 below. By equation (6.3), the torsion is

$$\begin{aligned} \log T(C_l W) = t'(0) &= \frac{\log l^2}{2} \left(\sum_{q=0}^{p-1} (-1)^{q+1} r_q z_q(0) + \sum_{q=0}^{p-1} (-1)^q t_q(0) \right) \\ &+ \frac{1}{2} \left(\sum_{q=0}^{p-1} (-1)^{q+1} r_q z'_q(0) + \sum_{q=0}^{p-1} (-1)^q t'_q(0) \right). \end{aligned}$$

However, it is convenient to split the torsion in *regular* and *singular* part, accordingly to remark 3.1 and the results in Propositions 6.2 and 6.4. First, observe that the functions $z_q(s)$ where studied in Section 3.2, where it is showed that there is no singular contribution to $z_q(0)$ and $z'_q(0)$. So $z_q(0) = z_{q,\text{reg}}(0)$, and $z'_q(0) = z'_{q,\text{reg}}(0)$. Therefore, we set

$$\log T(C_l W) = \log T_{\text{reg}}(C_l W) + \log T_{\text{sing}}(C_l W),$$

with

$$\begin{aligned} (7.1) \quad \log T_{\text{reg}}(C_l W) = t'_{\text{reg}}(0) &= \frac{\log l^2}{2} \left(\sum_{q=0}^{p-1} (-1)^{q+1} r_q z_q(0) + \sum_{q=0}^{p-1} (-1)^q t_{q,\text{reg}}(0) \right) \\ &+ \frac{1}{2} \left(\sum_{q=0}^{p-1} (-1)^{q+1} r_q z'_q(0) + \sum_{q=0}^{p-1} (-1)^q t'_{q,\text{reg}}(0) \right), \end{aligned}$$

$$(7.2) \quad \log T_{\text{sing}}(C_l W) = t'_{\text{sing}}(0) = \frac{\log l^2}{2} \sum_{q=0}^{p-1} (-1)^q t_{q,\text{sing}}(0) + \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q t'_{q,\text{sing}}(0).$$

Lemma 7.1. *For all $0 \leq q \leq p - 1$,*

$$\begin{aligned} z_q(0) &= -\frac{1}{2}, \\ z'_q(0) &= \log 2 + \log(p - q). \end{aligned}$$

Proof. This follows by equation (3.2). $\square$

Lemma 7.2.

$$\begin{aligned} t_{q,\text{reg}}(0) &= -\zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)}), & 0 \leq q \leq p - 2, \\ t'_{q,\text{reg}}(0) &= -\zeta'_{\text{cex}}(0, \tilde{\Delta}^{(q)}), & 0 \leq q \leq p - 2, \\ t_{p-1,\text{reg}}(0) &= -\frac{1}{2} \zeta_{\text{cex}}(0, \tilde{\Delta}^{(p-1)}), \\ t'_{p-1,\text{reg}}(0) &= -\frac{1}{2} \zeta'_{\text{cex}}(0, \tilde{\Delta}^{(p-1)}). \end{aligned}$$

Proof. The first and the third formulas follows by Propositions 6.2 and 6.4, and the fact that for the zeta function associated to any sequence S , and any number b , $\zeta(0, S + b) = \zeta(0, S)$. For the derivatives, when $0 \leq q \leq p - 2$, by Proposition 6.4,

$$t'_{q,\text{reg}}(0) = -\mathcal{A}_{0,0,q}(0) - \mathcal{A}'_{0,1,q}(0).$$

By Lemma 6.14

$$\begin{aligned} \mathcal{A}_{0,0,q}(s) &= -\sum_{n=1}^{\infty} \log \left(1 - \frac{\alpha_q^2}{\mu_{q,n}^2} \right) \frac{m_{\text{cex},q,n}}{\mu_{q,n}^{2s}}, \\ \mathcal{A}_{0,1,q}(s) &= \zeta(2s, U_q) = \sum_{n=1}^{\infty} \frac{m_{\text{cex},q,n}}{\mu_{q,n}^{2s}}. \end{aligned}$$

Recalling that $\mu_{q,n} = \sqrt{\lambda_{q,n} + \alpha_q}$, and expanding the binomial, we obtain

$$\begin{aligned} -\mathcal{A}_{0,0,q}(s) - \mathcal{A}'_{0,1,q}(s) &= \sum_{n=1}^{\infty} \log \left(1 - \frac{\alpha_q^2}{\mu_{q,n}^2} \right) \frac{m_{\text{cex},q,n}}{\mu_{q,n}^{2s}} - \sum_{n=1}^{\infty} \frac{m_{\text{cex},q,n}}{\mu_{q,n}^{2s}} \log \mu_{q,n}^2 \\ &= \sum_{n=1}^{\infty} \log \lambda_{q,n} \frac{m_{\text{cex},q,n}}{\mu_{q,n}^{2s}} \\ &= \sum_{n=1}^{\infty} \log \lambda_{q,n} \sum_{j=0}^{\infty} \binom{-s}{j} \frac{m_{\text{cex},q,n}}{\lambda_{q,n}^{s+j}} \alpha_q^{2j} \\ &= -\sum_{j=0}^{\infty} \binom{-s}{j} \zeta'_{\text{ccl}}(s+j, \tilde{\Delta}^{(q)}) \alpha_q^{2j}, \end{aligned}$$

that gives the second formula. Eventually, the result for $t'_{p-1,\text{reg}}(0)$ follows by Proposition 6.2 and the fact that $\alpha_{p-1} = 0$ since the dimension is $m = 2p - 1$. $\square$

Proposition 7.1.

$$\begin{aligned} \log T_{\text{reg}}(C_l W) &= \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q r_q \log \frac{l}{2} - \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q r_q \log(p-q) + \frac{1}{2} \log T(W, g) \\ &\quad - \left(\sum_{q=0}^{p-2} (-1)^q \zeta_{\text{ccl}}(0, \tilde{\Delta}^{(q)}) + \frac{1}{2} (-1)^{p-1} \zeta_{\text{ccl}}(0, \tilde{\Delta}^{(p-1)}) \right) \log l \\ &= \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q r_q \log \frac{l}{2} - \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q r_q \log(p-q) + \frac{1}{2} \log T(W, l^2 g), \end{aligned}$$

where $r_q = \text{rk} \mathcal{H}_q(\partial C_l W; \mathbb{Q})$

Proof. Substitution in the formula in equation (7.1) of the values given in Lemmas 7.1 and 7.2 gives

$$\begin{aligned} \log T_{\text{reg}}(C_l W) &= \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q r_q \log \frac{l}{2} - \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q r_q \log(p-q) \\ &\quad - \left(\sum_{q=0}^{p-2} (-1)^q \zeta_{\text{ccl}}(0, \tilde{\Delta}^{(q)}) + \frac{1}{2} (-1)^{p-1} \zeta_{\text{ccl}}(0, \tilde{\Delta}^{(p-1)}) \right) \log l \\ &\quad + \frac{1}{4} \left(2 \sum_{q=0}^{p-2} (-1)^{q+1} \zeta'_{\text{ccl}}(0, \tilde{\Delta}^{(q)}) + (-1)^p \zeta'_{\text{ccl}}(0, \tilde{\Delta}^{(p-1)}) \right). \end{aligned}$$

By the second formula in equation (2.11)

$$\frac{1}{4} \left(2 \sum_{q=0}^{p-2} (-1)^{q+1} \zeta'_{\text{ccl}}(0, \tilde{\Delta}^{(q)}) + (-1)^p \zeta'_{\text{ccl}}(0, \tilde{\Delta}^{(p-1)}) \right) = \frac{1}{2} \log T(W, g),$$

and this gives the first formula stated. For the second formula, note that the boundary of the cone $\partial C_l W$ is the manifold W with metric $l^2 g$. The restriction of the Laplace operator on the boundary is then $\Delta_{\partial C_l W} = \frac{\tilde{\Delta}}{l^2}$. Since for the zeta function associated to any sequence S , and any number a ,

$$\zeta'(0, aS) = -\zeta(0, S) \log a + \zeta'(0, S),$$

a simple calculation shows that

$$\begin{aligned} & - \left(\sum_{q=0}^{p-2} (-1)^q \zeta_{\text{ccl}}(0, \tilde{\Delta}^{(q)}) + \frac{1}{2} (-1)^{p-1} \zeta_{\text{ccl}}(0, \tilde{\Delta}^{(p-1)}) \right) \log l^2 \\ & + \frac{1}{2} \left(2 \sum_{q=0}^{p-2} (-1)^{q+1} \zeta'_{\text{ccl}}(0, \tilde{\Delta}^{(q)}) + (-1)^p \zeta'_{\text{ccl}}(0, \tilde{\Delta}^{(p-1)}) \right) \\ & = t(0, W) \log l^2 + t'(0, W) = \log T(\partial C_l W). \end{aligned}$$

□

Proposition 7.2.

$$\begin{aligned} \log T_{\text{sing}}(C_l W) &= \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q \sum_{j=0}^{p-1} \text{Res}_0 \Phi_{2j+1}(s) \text{Res}_{s=j+\frac{1}{2}} \zeta_{\text{cex}} \left(s, \tilde{\Delta}^{(q)} + \alpha_q^2 \right) \\ &= \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q \sum_{j=0}^{p-1} \text{Res}_0 \Phi_{2j+1}(s) \sum_{l=0}^q (-1)^l \text{Res}_{s=j+\frac{1}{2}} \zeta \left(s, \tilde{\Delta}^{(l)} + \alpha_q^2 \right) \\ &= \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q \sum_{j=0}^{p-1} \sum_{k=0}^j \text{Res}_0 \Phi_{2k+1,q}(s) \binom{-\frac{1}{2}-k}{j-k} \text{Res}_{s=j+\frac{1}{2}} \zeta_{\text{cex}} \left(s, \tilde{\Delta}^{(q)} \right) \alpha_q^{2(j-k)} \\ &= \frac{1}{2} \sum_{q=0}^{p-1} \sum_{j=0}^{p-1} \sum_{k=0}^j \text{Res}_0 \Phi_{2k+1,q}(s) \binom{-\frac{1}{2}-k}{j-k} \sum_{l=0}^q (-1)^l \text{Res}_{s=j+\frac{1}{2}} \zeta \left(s, \tilde{\Delta}^{(l)} \right) \alpha_q^{2(j-k)}. \end{aligned}$$

Proof. The first formula follows by substitution in equation (7.2) of the values given in Propositions 6.2 and 6.4, and observing that, for the zeta function associated to any sequence S

$$a \operatorname{Res}_{s=s_0} \zeta(as, S) = \operatorname{Res}_{s=as_0} \zeta(s, S).$$

The second by duality, see Section 2.5,

$$\zeta_{ccl}(s, \tilde{\Delta}^{(q)}) = \zeta(s, \tilde{\Delta}^{(q)}) - \zeta_{cl}(s, \tilde{\Delta}^{(q)}) = \zeta(s, \tilde{\Delta}^{(q)}) - \zeta_{ccl}(s, \tilde{\Delta}^{(q-1)}) = \sum_{k=0}^q (-1)^{q+k} \zeta(s, \tilde{\Delta}^{(k)}).$$

The third formula follows by Lemmas 6.2 and 6.8, and some combinatorics, and the last by the previous ones. $\square$

8. THE PROOF OF THEOREM 1.2

On order to prove Theorem 1.2 we calculate the regular and the singular parts of the torsion in the case $W = S_{\sin \alpha}^m$, according to Propositions 7.1 and 7.2. Recall we are considering the absolute BC case. The result for the regular part follows easily, the one for the singular part requires more works, that will be developed in the following subsections. Here we recall the underlying geometric setting. Let S_b^m be the sphere of radius $b > 0$ in $\mathbb{R}^{m+1}$, $S_b^m = \{x \in \mathbb{R}^{m+1} \mid |x| = b\}$ (we simply write S^m for S_1^m). Let $C_l S_{\sin \alpha}^m$ denotes the cone of angle α over $S_{\sin \alpha}^m$ in $\mathbb{R}^{m+2}$. We embed $C_l S_{\sin \alpha}^m$ in $\mathbb{R}^{m+2}$ as the subset of the segments joining the origin to the sphere $S_{l \cos \alpha}^m \times \{(0, \dots, 0, l \cos \alpha)\}$. We parametrize the cone by

$$C_l S_{\sin \alpha}^m = \begin{cases} x_1 &= r \sin \alpha \sin \theta_m \sin \theta_{m-1} \cdots \sin \theta_3 \sin \theta_2 \cos \theta_1 \\ x_2 &= r \sin \alpha \sin \theta_m \sin \theta_{m-1} \cdots \sin \theta_3 \sin \theta_2 \sin \theta_1 \\ x_3 &= r \sin \alpha \sin \theta_m \sin \theta_{m-1} \cdots \sin \theta_3 \cos \theta_2 \\ &\vdots \\ x_{m+1} &= r \sin \alpha \cos \theta_m \\ x_{m+2} &= r \cos \alpha \end{cases}$$

with $r \in [0, l]$, $\theta_1 \in [0, 2\pi]$, $\theta_2, \dots, \theta_m \in [0, \pi]$, and where α is a fixed positive real number and $0 < a = \frac{1}{\nu} = \sin \alpha \leq 1$. The induced metric is ($r > 0$)

$$\begin{aligned} g_E &= dr \otimes dr + r^2 g_{S_{\sin \alpha}^m} \\ &= dr \otimes dr + r^2 a^2 \left(\sum_{i=1}^{m-1} \left(\prod_{j=i+1}^m \sin^2 \theta_j \right) d\theta_i \otimes d\theta_i + d\theta_m \otimes d\theta_m \right), \end{aligned}$$

and $\sqrt{|\det g_E|} = (r \sin \alpha)^m (\sin \theta_m)^{m-1} (\sin \theta_{m-1})^{m-2} \cdots (\sin \theta_3)^2 (\sin \theta_2)$.

8.1. The regular part of the torsion.

Proposition 8.1.

$$\log T_{\text{reg}}(C_l S_{\sin \alpha}^{2p-1}) = \frac{1}{2} \log \text{Vol}(C_l S_{\sin \alpha}^{2p-1}).$$

Proof. By Proposition 7.1, when $W = S_{\sin \alpha}^{2p-1}$ with the standard Euclidean metric g_E ,

$$\log T_{\text{reg}}(C_l S_{\sin \alpha}^{2p-1}) = \frac{1}{2} \log \frac{l}{2} - \frac{1}{2} \log p + \frac{1}{2} \log T(S_{\sin \alpha}^{2p-1}, l^2 g_E).$$

By [23], $T(S_{\sin \alpha}^{2p-1}, l^2 g_E) = \text{Vol}(S_{\sin \alpha}^{2p-1}, g_E)$, and this proves the proposition since, if W has metric g and dimension m , then

$$\text{Vol}(C_l W) = \int_{C_l W} \sqrt{\det(x^2 g)} dx \wedge d\text{vol}_g = \int_0^l x^m \int_W d\text{vol}_g = \frac{l^{m+1}}{m+1} \text{Vol}(W, g),$$

and

$$\text{Vol}(S_b^m, g_E) = \frac{2\pi^{\frac{m+1}{2}} b^m}{\Gamma(\frac{m+1}{2})}.$$

□

8.2. The conjecture for the singular part. Assuming that the formula for the anomaly boundary term $A_{\text{BM,abs}}(\partial W)$ of Brüning and Ma [3] is valid in the case of $C_l S_{\sin \alpha}^{2p-1}$, we computed in [18] (note the slight different notation), by applying the definition given equation (2.15) of in Section 2.6, that

$$A_{\text{BM,abs}}(\partial C_l S_{\sin \alpha}^{2p-1}) = \sum_{j=0}^{p-1} \frac{2^{p-j}}{j!(2(p-j)-1)!!} \sum_{h=0}^j \binom{j}{h} \frac{(-1)^h \nu^{-2(p-j+h)+1}}{(2(p-j+h)-1)} \frac{(2p-1)!}{4^p(p-1)!},$$

where $\frac{1}{\nu} = \sin \alpha$. Our purpose now is to prove that (this was proved in [18] for $m < 4$)

$$(8.1) \quad \log T_{\text{sing}}(C_l S_{\sin \alpha}^{2p-1}) = A_{\text{BM,abs}}(\partial C_l S_{\sin \alpha}^{2p-1})$$

where $\log T_{\text{sing}}(C_l S_{\sin \alpha}^{2p-1})$ is given in Proposition 7.2. For it is convenient to rewrite the second term as follows:

$$\begin{aligned} A_{\text{BM,abs}}(\partial C_l S_{\sin \alpha}^{2p-1}) &= \sum_{j=0}^{p-1} \frac{2^{p-j}}{j!(2(p-j)-1)!!} \sum_{h=0}^j \binom{j}{h} \frac{(-1)^h \nu^{-2(p-j+h)+1}}{(2(p-j+h)-1)} \frac{(2p-1)!}{4^p(p-1)!} \\ &= \sum_{j=0}^{p-1} \frac{2^{j+1}}{(p-1-j)!(2j+1)!!} \sum_{h=0}^{p-1-j} \binom{p-1-j}{h} \frac{(-1)^h \nu^{-2(j+1+h)+1}}{2(j+1+h)-1} \frac{(2p-1)!}{4^p(p-1)!} \\ &= \frac{(2p-1)!}{4^p(p-1)!} \sum_{k=0}^{p-1} \frac{1}{(2k+1)\nu^{2k+1}} \sum_{j=0}^k \frac{(-1)^{k-j} 2^{j+1}}{(p-1-j)!(2j+1)!!} \binom{p-1-j}{k-j} \\ &= \frac{(2p-1)!}{4^p(p-1)!} \sum_{k=0}^{p-1} \frac{1}{(p-1-k)!(2k+1)} \sum_{j=0}^k \frac{(-1)^{k-j} 2^{j+1}}{(k-j)!(2j+1)!!} \frac{1}{\nu^{2k+1}}, \end{aligned}$$

8.3. The eigenvalues of the Laplacian over $C_l S_{\sin \alpha}^m$. Let Δ be the self adjoint extension of the formal Laplace operator on $C_l S_{\sin \alpha}^m$ as defined in section 4.3. Then, the positive part of the spectrum of Δ (with absolute BC) is given in Lemma 4.3, once we know the eigenvalues of the restriction of the Laplacian on the section and their coexact multiplicity, according to Lemma

4.2. These information are available by work of Ikeda and Taniguchi [20]. The eigenvalues of the Laplacian on q -forms on $S_{\sin \alpha}^{2p-1}$ are

$$\begin{cases} \lambda_{0,n} = \nu^2 n(n+2p-2), \\ \lambda_{q,n} = \nu^2 (n+q)(n+2p-q-2), \quad 1 \leq q < p-2, \\ \lambda_{p-2,n} = \nu^2 ((n-1+p)^2 - 1), \\ \lambda_{p-1,n} = \nu^2 (n-1+p)^2, \end{cases}$$

with coexact multiplicity

$$\begin{aligned} m_{\text{cex},0,n} &= \frac{2}{(2p-2)!} \prod_{j=2}^p (n-1+j)(2p+n-1-j), \\ m_{\text{cex},q,n} &= \frac{2}{q!(2p-q-2)!} \prod_{\substack{j=1, \\ j \neq q+1}}^p (n-1+j)(2p+n-1-j), \quad 1 \leq q < p-2, \\ m_{\text{cex},p-2,n} &= \frac{2}{(p-2)!p!} \prod_{\substack{j=1 \\ j \neq p-1}}^p (n-1+j)(2p+n-1-j), \\ m_{\text{cex},p-1,n} &= \frac{2}{[(p-1)!]^2} \prod_{j=1}^{p-1} (n-1+j)(2p+n-1-j), \end{aligned}$$

thus the indices $\mu_{q,n}$ are

$$\begin{cases} \mu_{0,n} = \sqrt{\nu^2(n(n+2p-2)) + (p-1)^2}, \\ \mu_{q,n} = \sqrt{\nu^2(n+q)(n+2p-q-2) + \alpha_q^2}, \quad 1 \leq q < p-2, \\ \mu_{p-2,n} = \sqrt{\nu^2((n-1+p)^2 - 1) + 1}, \\ \mu_{p-1,n} = \nu(n-1+p). \end{cases}$$

8.4. **Some combinatorics.** We introduce some notation. Let

$$U_{q,S^{2p-1}} = \{m_{\text{cex},q,n} : \lambda_{q,n,S^{2p-1}}\},$$

denotes the sequence of the eigenvalues of the coexact q -forms of the Laplace operator over the sphere of dimension $2p-1$ and radius 1. Let $a_1, \dots, a_m$ be a finite sequence of real numbers. Then,

$$\prod_{j=1}^m (x + a_j) = \sum_{j=0}^m e_{m-j}(a_1, \dots, a_m) x^j$$

where the $e_1, \dots, e_m$ are elementary symmetric polynomials in $a_1, \dots, a_m$. Let define the numbers:

$$d_j^q := (j-q-1)(2p-q-j-1),$$

for $q = 0, \dots, p-1, j \neq q+1$, and

$$d^q := (d_1^q, d_2^q, \dots, \hat{d}_{q+1}^q, \dots, d_p^q),$$

where, as usual, the hat means the underling term is delated.

Lemma 8.1. *The sequence U_{p-1} is a totally regular sequence of spectral type with infinite order, exponent and genus: $e(U_{p-1}) = g(U_{p-1}) = 2p - 1$, and*

$$\zeta(s, U_{p-1}) = \frac{2\nu^{-s}}{(p-1)!^2} \sum_{j=0}^{p-1} e_{p-1-j}(d^{p-1}) \zeta_R(s-2j).$$

Proof. The first part of the statement follows from Lemma 6.2. In order to prove the formula, note that $\zeta(s, U_{p-1}) = \nu^{-s} \zeta\left(\frac{s}{2}, U_{p-1, S^{2p-1}}\right)$, where

$$\zeta\left(\frac{s}{2}, U_{p-1, S^{2p-1}}\right) = \sum_{n=1}^{\infty} \frac{m_{\text{cex}, p-1, n}}{\lambda_{p-1, n, S^{2p-1}}^{\frac{s}{2}}} = \sum_{n=1}^{\infty} \frac{m_{\text{cex}, p-1, n}}{(n+p-1)^s}.$$

Shifting n to $n-p+1$, and observing that the numbers $1, \dots, p-1$ are roots of the polynomial $\sum_{j=0}^{p-1} e_{p-1-j}(d^{p-1})n^{2j}$, we obtain

$$\begin{aligned} \zeta(s, U_{p-1}) &= \nu^{-s} \sum_{n=p}^{\infty} \frac{m_{\text{cex}, p-1, n-p+1}}{n^s} = \frac{2\nu^{-s}}{(p-1)!^2} \sum_{n=p}^{\infty} \frac{\prod_{j=1}^{p-1} n^2 - (p-j)^2}{n^s} \\ &= \frac{2\nu^{-s}}{(p-1)!^2} \sum_{j=0}^{p-1} e_{p-1-j}(d^{p-1}) \zeta_R(s-2j). \end{aligned}$$

□

Note that, using the formula of the lemma, $\zeta(s, U_{p-1})$ has an expansion near $s = 2k+1$, with $k = 0, 1, \dots, p-1$, of the following type:

$$\zeta(s, U_{p-1}) = \frac{2}{\nu^{2k+1}(p-1)!^2} e_{p-1-k}(d^{p-1}) \frac{1}{s-2k-1} + L_{p-1, 2k+1}(s),$$

where the $L_{p-1, 2k+1}(s)$ are regular function for $k = 0, 1, \dots, p-1$.

Corollary 8.1. *The function $\zeta(s, U_{p-1})$ has simple poles at $s = 2k+1$, for $k = 0, 1, \dots, p-1$, with residues*

$$\text{Res}_{s=2k+1} \zeta(s, U_{p-1}) = \frac{2}{\nu^{2k+1}(p-1)!^2} e_{p-1-k}(d^{p-1}).$$

Lemma 8.2. *The sequence U_q is a totally regular sequence of spectral type with infinite order, exponent and genus: $e(U_q) = g(U_q) = 2p-1$, and (where $i = \sqrt{-1}$)*

$$\zeta(s, U_q) = \frac{2\nu^{-s}}{q!(2p-q-2)!} \sum_{t=0}^{\infty} \binom{-\frac{s}{2}}{t} \sum_{j=0}^{p-1} e_{p-1-j}(d^q) z \left(\frac{s+2t-2j}{2}, i\alpha_q \right) \frac{\alpha_q^{2t}}{\nu^{2t}}.$$

The function $\zeta(s, U_q)$ has simple poles at $s = 2(p-k) - 1$, with $k = 0, 1, 2, \dots$

Proof. The first statement follows by Lemma 6.8. For the second one, consider the sequence $H_{q,h} = \{m_{\text{cex}, q, n} : \sqrt{\lambda_{q, n, S^{2p-1}} + h}\}_{n=1}^{\infty}$. Then $\zeta(s, U_q) = \nu^{-s} \zeta(s, H_{q, \frac{\alpha_q^2}{\nu^2}})$, and

$$\zeta(s, H_{q,h}) = \sum_{n=1}^{\infty} \frac{m_{\text{cex}, q, n}}{(\lambda_{q, n, S^{2p-1}} + h)^{\frac{s}{2}}} = \sum_{n=1}^{\infty} \sum_{t=0}^{\infty} \binom{-\frac{s}{2}}{t} \frac{m_{\text{cex}, q, n}}{\lambda_{q, n, S^{2p-1}}^{\frac{s}{2}+t}} h^t = \sum_{t=0}^{\infty} \binom{-\frac{s}{2}}{t} \zeta(s+2t, H_{q,0}) h^t.$$

Next observe that the zeta function associated to the sequence $H_{q,0}$ is

$$\begin{aligned}\zeta(2s, H_{q,0}) &= \zeta(s, U_{q,S^{2p-1}}) = \sum_{n=1}^{\infty} \frac{m_{\text{cex},q,n}}{\lambda_{q,n,S^{2p-1}}^s} = \sum_{n=p}^{\infty} \frac{m_{q,n-p+1}}{\lambda_{q,n-p+1,S^{2p-1}}^s} \\ &= \frac{2}{q!(2p-q-2)!} \sum_{n=p}^{\infty} \frac{\prod_{\substack{j=1, \\ j \neq q+1}}^p (n^2 - (p-j)^2)}{(n^2 - \alpha_q^2)}.\end{aligned}$$

Recall that $\alpha_q^2 = d_p^q$, and note that

$$\sum_{j=0}^{p-1} e_{p-j-1}(d^q)(n^2 - \alpha_q^2)^j = \sum_{j=0}^{p-1} e_{p-j-1}(d^q)(n^2 - d_p^q)^j = \prod_{\substack{j=1, \\ j \neq q+1}}^p (n^2 - d_p^q + d_j^q) = \prod_{\substack{j=1, \\ j \neq q+1}}^p (n^2 - (p-j)^2),$$

and that the numbers $n = 1, 2, \dots, -\alpha_q$ are roots of this polynomial. Therefore, we can write

$$\begin{aligned}\zeta(2s, H_{q,0}) &= \frac{2}{q!(2p-q-2)!} \sum_{j=0}^{p-1} e_{p-1-j}(d^q) \left(z(s-j, i\alpha_q) - \sum_{n=1}^{p-q-2} (n^2 - \alpha_q^2)^{-s+j} \right) \\ &= \frac{2}{q!(2p-q-2)!} \sum_{j=0}^{p-1} e_{p-1-j}(d^q) z(s-j, i\alpha_q),\end{aligned}$$

and

$$z(s-j, i\alpha_q) = \sum_{n=1}^{\infty} \frac{1}{(n^2 - \alpha_q^2)^{s-j}}.$$

Expanding the binomial, as in [36], Section 2, $z(s, a) = \sum_{k=0}^{\infty} \binom{-s}{k} a^{2k} \zeta_R(2s+2k)$, and hence $z(s, a)$ has simple poles at $s = \frac{1}{2} - k$, $k = 0, 1, 2, \dots$. Since

$$\zeta(2s, H_{q,0}) = \frac{2}{q!(2p-q-2)!} \sum_{j=0}^{p-1} e_{p-1-j}(d^q) z(s-j, i\alpha_q),$$

$\zeta(2s, H_{q,0})$ has simple poles at $s = \frac{1}{2} + p - 1 - k$, $k = 0, 1, 2, \dots$, $\zeta(s, H_{q,0})$ has simple poles at $s = 2(p-k) - 1$, $k = 0, 1, 2, \dots$, and this completes the proof. $\square$

Corollary 8.2. *The function $\zeta(s, U_q)$ has simple poles at $s = 2k + 1$, for $k = 0, 1, \dots, p-1$, with residues*

$$\text{Res}_{s=2k+1} \zeta(s, U_q) = \frac{2\nu^{-2k-1}}{q!(2p-q-2)!} \sum_{t=0}^{p-1-k} \frac{1}{\nu^{2t}} \binom{-\frac{2k+1}{2}}{t} \sum_{j=k+t}^{p-1} e_{p-1-j}(d^q) \binom{-\frac{1}{2}}{j-k-t} \alpha_q^{2(j-k)}$$

Proof. Since the value of the residue of the Riemann zeta function at $s = 1$ is 1,

$$\text{Res}_{s=\frac{1}{2}-k} z(s-j, a) = \text{Res}_{s=\frac{1}{2}-j-k} z(s, a) = \binom{-\frac{1}{2}+j+k}{j+k} \frac{a^{2j+2k}}{2},$$

for $k = 0, 1, 2, \dots$. Considering $\zeta(2s, H_{q,0})$, we have, for $k = 0, 1, \dots, p-1$,

$$\text{Res}_{s=\frac{1}{2}+k} \zeta(2s, H_{q,0}) = \frac{2}{q!(2p-q-2)!} \sum_{j=k}^{p-1} e_{p-1-j}(d^q) (-1)^{j-k} \binom{-\frac{1}{2}+j-k}{j-k} \frac{\alpha_q^{2j-2k}}{2},$$

and the thesis follows. $\square$

The result contained in the next lemma follows by geometric reasons. However, we present here a purely combinatoric proof.

Lemma 8.3. *For all $0 \leq q \leq p-1$, $\zeta(0, U_{q, S^{2p-1}}) = (-1)^{q+1}$.*

Proof. Consider the function

$$\zeta_{t,c}(s) = \sum_{n=1}^{\infty} \frac{1}{(n(n+2t))^{s-c}} = \sum_{n=t+1}^{\infty} \frac{1}{(n^2 - t^2)^{s-c}}.$$

Since

$$z(s-c, it) = \sum_{n=1}^{\infty} \frac{1}{(n^2 - t^2)^{s-c}} = \sum_{j=0}^{\infty} \binom{-s+c}{j} (-1)^j t^{2j} \zeta_R(2s+2j-2c),$$

we have when $s=0$, that $z(-c, it) = (-1)^c t^{2c} \zeta_R(0) = (-1)^{c+1} \frac{t^{2c}}{2}$, and hence

$$\zeta_{t,c}(s) = z(s-c, it) - \sum_{n=1}^t \frac{1}{(n^2 - t^2)^{s-c}},$$

and for $c=0$ and $s=0$ $\zeta_{t,0}(0) = -\frac{1}{2} - t$. Next, consider $c > 0$, then:

$$\zeta_{t,c}(0) = (-1)^{c+1} \frac{t^{2c}}{2} - \sum_{n=1}^{t-1} (n^2 - t^2)^c.$$

For $q=0, \dots, p-1$, we have

$$\begin{aligned} \zeta(s, U_{q, S^{2p-1}}) &= \sum_{n=1}^{\infty} \frac{m_{\text{cex}, q, n}}{\lambda_{q, n, S^{2p-1}}} = \sum_{n=1}^{\infty} \frac{m_{\text{cex}, q, n}}{((n+q)(n+2p-q-2))^s} \\ &= \sum_{n=q+1}^{\infty} \frac{m_{\text{cex}, q, n-q}}{(n(n-2\alpha_q))^s}. \end{aligned}$$

Recalling the relation given in Section 8.4

$$\begin{aligned} m_{\text{cex}, q, n-q} &= \frac{2}{q!(2p-q-2)!} \prod_{\substack{j=1, \\ j \neq q+1}}^p (n-q-1+j)(n+2p-q-1-j) \\ &= \frac{2}{q!(2p-q-2)!} \prod_{\substack{j=1, \\ j \neq q+1}}^p n(n-2\alpha_q) + d_j^q \\ &= \frac{2}{q!(2p-q-2)!} \sum_{j=0}^{p-1} e_{p-1-j}(d^q) (n(n-2\alpha_q))^j. \end{aligned}$$

Thus

$$\begin{aligned}\zeta(s, U_q, S^{2p-1}) &= \frac{2}{q!(2p-q-2)!} \sum_{j=0}^{p-1} e_{p-j-1}(d^q) \left(\zeta_{-\alpha_q, j}(s) - \sum_{n=1}^q \frac{1}{(n(n-2\alpha_q))^{s-j}} \right) \\ &= \frac{2}{q!(2p-i-2)!} \sum_{j=0}^{p-1} e_{p-j-1}(d^q) \zeta_{-\alpha_q, j}(s).\end{aligned}$$

where

$$\sum_{j=0}^{p-1} e_{p-j-1}(d^q) \frac{1}{(n(n+2p-2q-2))^{s-j}} = 0,$$

for $1 \leq n \leq q$, by result of [42]. For $s = 0$, we obtain

$$\begin{aligned}\zeta(0, U_q, S^{2p-1}) &= \frac{2}{q!(2p-q-2)!} \sum_{j=0}^{p-1} e_{p-j-1}(d^q) \zeta_{-\alpha_q, j}(0) \\ &= \frac{2}{q!(2p-q-2)!} \left(e_{p-1}(d^q) \left(-\frac{1}{2} - ((p-q-2)+1) \right) \right. \\ &\quad \left. + \sum_{j=1}^{p-1} e_{p-j-1}(d^q) \left((-1)^{j+1} \frac{\alpha_q^{2j}}{2} - \sum_{n=1}^{p-q-2} (n^2 - \alpha_q^2)^j \right) \right) \\ &= \frac{2}{q!(2p-q-2)!} \left(-e_{p-1}(d^q) \right. \\ &\quad \left. + \sum_{j=0}^{p-1} e_{p-j-1}(d^q) \left((-1)^{j+1} \frac{\alpha_q^{2j}}{2} - \sum_{n=1}^{p-q-2} (n^2 - \alpha_q^2)^j \right) \right) \\ &= \frac{2}{q!(2p-q-2)!} \left((-1)^{q+1} \frac{q!(2p-q-2)!}{2} \right. \\ &\quad \left. + \sum_{j=0}^{p-1} e_{p-j-1}(d^q) \left((-1)^{j+1} \frac{\alpha_q^{2j}}{2} - \sum_{n=1}^{p-q-2} (n^2 - \alpha_q^2)^j \right) \right).\end{aligned}$$

To conclude the proof, note that the second term vanishes. For first, as showed in the proof of Lemma 8.2, the numbers $n = 1, 2, \dots, -\alpha_q$ are roots of the polynomial $\sum_{j=0}^{p-1} e_{p-j-1}(d^q)(n^2 - \alpha_q^2)^j$, and second:

$$\sum_{j=0}^{p-1} e_{p-j-1}(d^q)(-1)^j \alpha_q^{2j} = \sum_{j=0}^{p-1} e_{p-j-1}(d^q)(-d_p^q)^j = \prod_{\substack{j=1, \\ j \neq q+1}}^p (-d_p^q + d_j^q) = - \prod_{\substack{j=1, \\ j \neq i+1}}^p (p-j)^2 = 0.$$

□

8.5. **The proof of the conjecture.** We need some notation. Set

$$D(q, k, t) = \frac{2}{q!(2p - q - 2)!} \binom{-\frac{2k+1}{2}}{t} \sum_{l=k+t}^{p-1} e_{p-1-l}(d^q) (-1)^{l-k} \binom{-\frac{1}{2} - k - t + l}{l - k - t} \alpha_q^{2(l-k)},$$

$$F(q, k) = \operatorname{Res}_{s=0} \Phi_{2k+1, q}(s), \quad 1 \leq k \leq p-1,$$

for $0 \leq q \leq p-1$, $F(q, 0) = 2$ for $0 \leq q \leq p-2$, and $F(p-1, 0) = 1$. Then, by Corollary 8.2, the residues of $\zeta(s, U_q)$, for $0 \leq q \leq p-2$, are

$$\operatorname{Res}_{s=2k+1} \zeta(s, U_q) = \frac{1}{\nu^{2k+1}} \sum_{t=0}^{p-1-k} \frac{1}{\nu^{2t}} D(q, k, t),$$

for $k = 0, \dots, p-1$, and when $q = p-1$:

$$\operatorname{Res}_{s=2k+1} \zeta(s, U_{p-1}) = \frac{1}{\nu^{2k+1}} D(p-1, k, 0),$$

with $k = 0, \dots, p-1$. Now, for $0 \leq q \leq p-1$, it is easy to see that

$$\operatorname{Res}_{s=0} \Phi_{2k+1, q}(s) \operatorname{Res}_{s=2k+1} \zeta(s, U_q) = \frac{F(q, k)}{\nu^{2k+1}} \sum_{t=0}^{p-1-k} \frac{1}{\nu^{2t}} D(q, k, t)$$

and hence

$$t'_{q, \text{sing}}(0) = \frac{1}{2} \sum_{k=0}^{p-1} \operatorname{Res}_{s=0} \Phi_{2k+1, q}(s) \operatorname{Res}_{s=2k+1} \zeta(s, U_q) = \frac{1}{2} \sum_{k=0}^{p-1} \frac{F(q, k)}{\nu^{2k+1}} \sum_{t=0}^{p-1-k} \frac{1}{\nu^{2t}} D(q, k, t).$$

On the other side, set:

$$A_{\text{BM, abs}}(C_l S_{\sin \alpha}^{2p-1}) = \sum_{k=0}^{p-1} \frac{1}{\nu^{2k+1}} \tilde{Q}_p(k), \quad \tilde{Q}_p(k) = \sum_{j=0}^k N_j(p, k),$$

where

$$N_j(p, k) = \frac{(2p-1)!}{4^p(p-1)!} \frac{1}{(p-1-k)!(2k+1)} \frac{(-1)^{k-j} 2^{j+1}}{(k-j)!(2j+1)!}.$$

Lemma 8.4. $\frac{1}{2} \sum_{q=0}^{p-1} (-1)^q t'_{q, \text{sing}}(0)$ is an odd polynomial in $\frac{1}{\nu}$.

Proof. This follows by rearrangement of the finite sum:

$$\begin{aligned} \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q t'_{q, \text{sing}}(0) &= \frac{1}{4} \sum_{q=0}^{p-1} (-1)^q \sum_{k=0}^{p-1} F(q, k) \sum_{t=0}^{p-1-k} \frac{1}{\nu^{2(t+k)+1}} D(q, k, t) \\ &= \frac{1}{4} \sum_{k=0}^{p-1} \frac{1}{\nu^{2k+1}} \sum_{q=0}^{p-1} (-1)^q \sum_{j=0}^k F(q, j) D(q, j, k-j) \\ &= \frac{1}{4} \sum_{k=0}^{p-1} \frac{1}{\nu^{2k+1}} \sum_{j=0}^k \sum_{q=0}^{p-1} (-1)^q F(q, j) D(q, j, k-j). \end{aligned}$$

□

Then, set:

$$\frac{1}{2} \sum_{q=0}^{p-1} (-1)^q t'_{q, \text{sing}}(0) = \sum_{k=0}^{p-1} \frac{1}{\nu^{2k+1}} Q_p(k), \quad Q_p(k) = \sum_{j=0}^k M_j(p, k),$$

where

$$\begin{aligned} M_j(p, k) &= \sum_{q=0}^{p-1} (-1)^q F(q, j) D(q, j, k-j) \\ &= \sum_{q=0}^{p-1} (-1)^q \frac{2F(q, j)}{4(2p-2)!} \binom{2p-2}{q} \binom{-\frac{1}{2}-j}{k-j} \alpha_q^{-2j} \sum_{l=k}^{p-1} e_{p-1-l}(d^q) \alpha_q^{2l} \binom{-\frac{1}{2}}{l-k}. \end{aligned}$$

This shows that all we need to prove to prove the conjecture is the identity: $M_j(p, k) = N_j(p, k)$. This is in the next two lemmas. Before, we need some further notation and combinatorics. First, recall that if

$$f_h(x) = e_h(x^2 - (p-1)^2, x^2 - (p-2)^2, \dots, x^2 - 1^2, x^2),$$

then $f_h(\alpha_q) = e_h(d^q)$, and $f_h(x)$, for $h \geq 1$, is a polynomial of the following type:

$$(8.2) \quad f_h(x) = \sum_{0 \leq j_1 \leq j_2 \leq \dots \leq j_h \leq p-1} (x^2 - j_1^2)(x^2 - j_2^2) \dots (x^2 - j_h^2) = \binom{p}{h} x^{2h} + \sum_{s=0}^{h-1} c_s^h x^{2s}.$$

Second, we have the following four identities. The first three can be found in [15], 0.151.4, 0.154.5 and 0.154.6, but see [21] for a proof. The fourth is in [16], equation (5.3).

$$(8.3) \quad \sum_{k=0}^n \frac{(-1)^k}{(2n)!} \binom{2n}{k} = \frac{(-1)^n}{(2n)!} \binom{2n-1}{n} = \frac{(-1)^n}{2(2n)!} \binom{2n}{n},$$

$$(8.4) \quad \sum_{k=0}^n (-1)^k \binom{n}{k} (\alpha + k)^n = (-1)^n n!$$

$$(8.5) \quad \sum_{k=0}^N (-1)^k \binom{N}{k} (\alpha + k)^{n-1} = 0,$$

$$(8.6) \quad \sum_{l=0}^n \binom{n+1}{l+1} \binom{-\frac{1}{2}}{l-k} = \binom{n+\frac{1}{2}}{n-k} = \frac{(2n+1)!!}{2^{n-k}(n-k)!(2k+1)!!}.$$

with $1 \leq n \leq N$ and $\alpha \in \mathbb{R}$.

Lemma 8.5. *For $0 \leq k \leq p-1$, we have that $M_0(p, k) = N_0(p, k)$.*

Proof. Since $j = 0$,

$$\begin{aligned} M_0(p, k) &= \sum_{q=0}^{p-1} (-1)^q \frac{2F(q, 0)}{4(2p-2)!} \binom{2p-2}{q} \binom{-\frac{1}{2}}{k} \sum_{l=k}^{p-1} e_{p-1-l}(d^q) \alpha_q^{2l} \binom{-\frac{1}{2}}{l-k}, \\ N_0(p, k) &= \frac{(2p-1)!}{2^{2p-1}(p-1)!} \frac{1}{(p-1-k)!(2k+1)} \frac{(-1)^k}{k!}. \end{aligned}$$

Consider first $k \neq 0$. Then,

$$\begin{aligned}
M_0(p, k) &= \binom{-\frac{1}{2}}{k} \sum_{q=0}^{p-1} (-1)^q \frac{1}{(2p-2)!} \binom{2p-2}{q} \sum_{l=k}^{p-1} f_{p-1-l}(\alpha_q) \alpha_q^{2l} \binom{-\frac{1}{2}}{l-k} \\
&= \binom{-\frac{1}{2}}{k} \sum_{q=0}^{p-1} (-1)^q \frac{1}{(2p-2)!} \binom{2p-2}{q} \sum_{l=k}^{p-1} \binom{p}{p-1-l} \alpha_q^{2p-2-2l} \alpha_q^{2l} \binom{-\frac{1}{2}}{l-k} \\
&\quad + \binom{-\frac{1}{2}}{k} \sum_{q=0}^{p-1} (-1)^q \frac{1}{(2p-2)!} \binom{2p-2}{q} \sum_{l=k}^{p-2} \sum_{s=0}^{p-2-l} c_s^{p-1-l} \alpha_q^{2s} \alpha_q^{2l} \binom{-\frac{1}{2}}{l-k} \\
&= \binom{-\frac{1}{2}}{k} \sum_{q=0}^{p-1} (-1)^q \frac{1}{(2p-2)!} \binom{2p-2}{q} \alpha_q^{2p-2} \sum_{l=k}^{p-1} \binom{p}{p-1-l} \binom{-\frac{1}{2}}{l-k} \\
&\quad + \sum_{l=k}^{p-2} \sum_{s=0}^{p-2-l} c_s^{p-1-l} \binom{-\frac{1}{2}}{k} \binom{-\frac{1}{2}}{l-k} \sum_{q=0}^{p-1} (-1)^q \frac{1}{(2p-2)!} \binom{2p-2}{q} \alpha_q^{2s+2l}.
\end{aligned}$$

Using the identity in equation (8.5), the second term in the last line vanishes since $2s+2l < 2p-2$. Thus,

$$\begin{aligned}
M_0(p, k) &= \binom{-\frac{1}{2}}{k} \sum_{q=0}^{p-1} (-1)^q \frac{1}{(2p-2)!} \binom{2p-2}{q} \alpha_q^{2p-2} \sum_{l=k}^{p-1} \binom{p}{p-1-l} \binom{-\frac{1}{2}}{l-k} \\
&= \frac{1}{2} \binom{-\frac{1}{2}}{k} \sum_{l=k}^{p-1} \binom{p}{p-1-l} \binom{-\frac{1}{2}}{l-k} \\
&= \frac{1}{2} \binom{-\frac{1}{2}}{k} \binom{p}{k+1} \frac{(k+1)!}{p!} \frac{(2p-1)!!}{(2k+1)!!} \frac{2^{k+1}}{2^p} \\
&= \frac{(-1)^k}{k!} \frac{1}{(p-k-1)!} \frac{(2p-1)!}{(2k+1)} \frac{1}{2^{2p-1}(p-1)!} = N_0(p, k).
\end{aligned}$$

Next, consider $k = 0$. Then,

$$\begin{aligned}
M_0(p, 0) &= \sum_{q=0}^{p-2} (-1)^q \frac{1}{(2p-2)!} \binom{2p-2}{q} \sum_{l=0}^{p-1} f_{p-1-l}(\alpha_q) \alpha_q^{2l} \binom{-\frac{1}{2}}{l} + \frac{1}{2} \\
&= \sum_{q=0}^{p-1} (-1)^q \frac{1}{(2p-2)!} \binom{2p-2}{q} \sum_{l=0}^{p-1} f_{p-1-l}(\alpha_q) \alpha_q^{2l} \binom{-\frac{1}{2}}{l} - 1 + \frac{1}{2} \\
&= \sum_{q=0}^{p-1} (-1)^q \frac{1}{(2p-2)!} \binom{2p-2}{q} \sum_{l=0}^{p-1} \binom{p}{p-1-l} \alpha_q^{2p-2} \binom{-\frac{1}{2}}{l} \\
&\quad + \sum_{q=0}^{p-1} (-1)^q \frac{1}{(2p-2)!} \binom{2p-2}{q} c_0^{p-1} - \frac{1}{2} \\
&= \sum_{q=0}^{p-1} (-1)^q \frac{1}{(2p-2)!} \binom{2p-2}{q} \sum_{l=0}^{p-1} \binom{p}{p-1-l} \alpha_q^{2p-2} \binom{-\frac{1}{2}}{l} \\
&\quad + \frac{(-1)^{p-1}}{2(2p-2)!} \binom{2p-2}{p-1} (-1)^{p-1} (p-1)! (p-1)! - \frac{1}{2} \\
&= \sum_{q=0}^{p-1} (-1)^q \frac{1}{(2p-2)!} \binom{2p-2}{q} \sum_{l=0}^{p-1} \binom{p}{p-1-l} \alpha_q^{2p-2} \binom{-\frac{1}{2}}{l} + \frac{1}{2} - \frac{1}{2} \\
&= \sum_{q=0}^{p-1} (-1)^q \frac{1}{(2p-2)!} \binom{2p-2}{q} \alpha_q^{2p-2} \sum_{l=0}^{p-1} \binom{p}{p-1-l} \binom{-\frac{1}{2}}{l} \\
&= \frac{1}{2} \sum_{l=0}^{p-1} \binom{p}{p-1-l} \binom{-\frac{1}{2}}{l} = \frac{1}{2^p} \frac{(2p-1)!!}{(p-1)!} = \frac{2p-1}{2^{2p-1}} \binom{2p-2}{p-1} = N_0(p, 0)
\end{aligned}$$

□

Lemma 8.6. For $1 \leq j \leq p-1$, we have that $M_j(p, k) = N_j(p, k)$.

Proof. Note that $j \leq k$, and hence $1 \leq j \leq k \leq p-1$. Recall that

$$F(q, j) = \frac{2}{2j+1} \alpha_q^{2j} + \sum_{t=1}^{j-1} k_{2j+1,t} \alpha_q^{2t} + 2 \operatorname{Res}_{z=0} \Phi_{2j+1,p-1}(s),$$

by Corollary 6.2. Set $k_{2j+1,0} = 2 \operatorname{Res}_{z=0} \Phi_{2j+1,p-1}(s)$. We split the proof in three cases. First, for $j = k < p-1$, we have

$$\begin{aligned}
M_j(p, j) &= \sum_{q=0}^{p-1} (-1)^q \frac{F(q, j)}{2(2p-2)!} \binom{2p-2}{q} \sum_{l=j}^{p-1} e_{p-1-l}(d^q) \alpha_q^{2l-2j} \binom{-\frac{1}{2}}{l-j} \\
&= \sum_{q=0}^{p-1} (-1)^q \frac{\alpha_q^{2j}}{(2j+1)(2p-2)!} \binom{2p-2}{q} \sum_{l=j}^{p-1} f_{p-1-l}(\alpha_q) \alpha_q^{2l-2j} \binom{-\frac{1}{2}}{l-j} \\
&\quad + \sum_{t=0}^{j-1} k_{t,j} \sum_{q=0}^{p-1} (-1)^q \frac{\alpha_q^{2t}}{(2p-2)!} \binom{2p-2}{q} \sum_{l=j}^{p-1} f_{p-1-l}(\alpha_q) \alpha_q^{2l-2j} \binom{-\frac{1}{2}}{l-j}
\end{aligned}$$

Using the formula in equation (8.2) for the functions $f_{p-1-l}(\alpha_q)$, we get

$$\begin{aligned}
M_j(p, j) &= \sum_{q=0}^{p-1} (-1)^q \frac{1}{(2j+1)(2p-2)!} \binom{2p-2}{q} \sum_{l=j}^{p-1} \binom{p}{p-1-l} \alpha_q^{2p-2-2l} \alpha_q^{2l} \binom{-\frac{1}{2}}{l-j} \\
&\quad + \sum_{q=0}^{p-1} (-1)^q \frac{1}{(2j+1)(2p-2)!} \binom{2p-2}{q} \sum_{l=j}^{p-2} \sum_{s=0}^{p-2-l} c_s \alpha_q^{2s+2l} \binom{-\frac{1}{2}}{l-j} \\
&\quad + \sum_{t=0}^{j-1} k_{2j+1,t} \sum_{q=0}^{p-1} (-1)^q \frac{1}{(2p-2)!} \binom{2p-2}{q} \sum_{l=j}^{p-1} \binom{p}{p-1-l} \alpha_q^{2p-2+2t-2j} \binom{-\frac{1}{2}}{l-j} \\
&\quad + \sum_{t=0}^{j-1} k_{2j+1,t} \sum_{q=0}^{p-1} (-1)^q \frac{1}{(2p-2)!} \binom{2p-2}{q} \sum_{l=j}^{p-2} \sum_{s=0}^{p-2-l} c_s \alpha_q^{2s+2l+2t-2j} \binom{-\frac{1}{2}}{l-j} \\
&= \frac{1}{(2j+1)} \sum_{q=0}^{p-1} (-1)^q \frac{1}{(2p-2)!} \binom{2p-2}{q} \alpha_q^{2p-2} \sum_{l=j}^{p-1} \binom{p}{p-1-l} \binom{-\frac{1}{2}}{l-j} \\
&= \frac{1}{2(2j+1)} \sum_{l=j}^{p-1} \binom{p}{p-1-l} \binom{-\frac{1}{2}}{l-j} = \frac{1}{2(2j+1)} \frac{1}{(p-1-j)!} \frac{(2p-1)!!}{(2j+1)!!} \frac{2^{j+1}}{2^p} \\
&= \frac{1}{(2j+1)(p-1-j)!} \frac{(2p-1)!}{(2j+1)!!} \frac{2^j}{2^{2p-1}} \frac{1}{(p-1)!} = N_j(p, j),
\end{aligned}$$

where the first three terms in the first equation vanish because $s + l < p - 1$ and $t - j \leq -1$. The second case is $j = k = p - 1$. Then,

$$\begin{aligned}
M_{p-1}(p, p-1) &= \sum_{q=0}^{p-1} (-1)^q \frac{F(q, p-1)}{2(2p-2)!} \binom{2p-2}{q} \\
&= \sum_{q=0}^{p-1} (-1)^q \frac{\alpha_q^{2p-2}}{(2p-1)(2p-2)!} \binom{2p-2}{q} \\
&\quad + \sum_{t=0}^{p-2} k_{2j+1,t} \sum_{q=0}^{p-2} (-1)^q \frac{\alpha_q^{2t}}{2(2p-2)!} \binom{2p-2}{q} \\
&\quad + \sum_{t=0}^{p-2} \frac{k_{2j+1,t}}{2} (-1)^{p-1} \frac{\alpha_{p-1}^{2t}}{2(2p-2)!} \binom{2p-2}{p-1} \\
&= \frac{1}{2(2p-1)} + k_{2j+1,0} \sum_{q=0}^{p-1} (-1)^q \frac{1}{2(2p-2)!} \binom{2p-2}{q} \\
&\quad - k_{2j+1,0} (-1)^{p-1} \frac{1}{2(2p-2)!} \binom{2p-2}{p-1} \\
&\quad + \frac{k_{2j+1,0}}{2} (-1)^{p-1} \frac{1}{2(2p-2)!} \binom{2p-2}{p-1} \\
&= \frac{1}{2(2p-1)} + \frac{k_{2j+1,0}}{2} \frac{(-1)^{p-1}}{(p-1)!(p-1)!} - \frac{k_{2j+1,0}}{2} \frac{(-1)^{p-1}}{(p-1)!(p-1)!} \\
&= \frac{1}{2(2p-1)} = N_{p-1}(p, p-1).
\end{aligned}$$

The last case is $1 \leq j < k$. Then,

$$\begin{aligned}
M_j(p, k) &= \sum_{i=0}^{p-1} (-1)^q \frac{2F(q, j)}{4(2p-2)!} \binom{2p-2}{q} \binom{-\frac{1}{2}-j}{k-j} \alpha_q^{-2j} \sum_{l=k}^{p-1} e_{p-1-l}(d^q) \alpha_q^{2l} \binom{-\frac{1}{2}}{l-k}, \\
&= \binom{-\frac{1}{2}-j}{k-j} \sum_{q=0}^{p-1} (-1)^q \frac{1}{(2j+1)(2p-2)!} \binom{2p-2}{q} \sum_{l=k}^{p-1} \binom{p}{p-1-l} \alpha_q^{2p-2-2l} \alpha_q^{2l} \binom{-\frac{1}{2}}{l-k} \\
&\quad + \binom{-\frac{1}{2}-j}{k-j} \sum_{q=0}^{p-1} (-1)^q \frac{1}{(2j+1)(2p-2)!} \binom{2p-2}{q} \sum_{l=k}^{p-2} \sum_{s=0}^{p-2-l} c_s \alpha_q^{2s+2l} \binom{-\frac{1}{2}}{l-k} \\
&\quad + \binom{-\frac{1}{2}-j}{k-j} \sum_{t=0}^{j-1} k_{2j+1,t} \sum_{q=0}^{p-1} (-1)^q \frac{1}{(2p-2)!} \binom{2p-2}{q} \sum_{l=k}^{p-1} \binom{p}{p-1-l} \alpha_q^{2p-2+2t-2j} \binom{-\frac{1}{2}}{l-k} \\
&\quad + \binom{-\frac{1}{2}-j}{k-j} \sum_{t=0}^{j-1} k_{2j+1,t} \sum_{q=0}^{p-1} (-1)^q \frac{1}{(2p-2)!} \binom{2p-2}{q} \sum_{l=k}^{p-2} \sum_{s=0}^{p-2-l} c_s \alpha_q^{2s+2l+2t-2j} \binom{-\frac{1}{2}}{l-k}
\end{aligned}$$

$$\begin{aligned}
&= \binom{-\frac{1}{2}-j}{k-j} \sum_{q=0}^{p-1} (-1)^q \frac{1}{(2j+1)(2p-2)!} \binom{2p-2}{q} \alpha_q^{2p-2} \sum_{l=k}^{p-1} \binom{p}{p-1-l} \binom{-\frac{1}{2}}{l-k} \\
&= \binom{-\frac{1}{2}-j}{k-j} \frac{1}{2(2j+1)} \sum_{l=k}^{p-1} \binom{p}{p-1-l} \binom{-\frac{1}{2}}{l-k} \\
&= \binom{-\frac{1}{2}-j}{k-j} \frac{1}{2(2j+1)} \frac{(2p-1)!!}{(p-1-k)!(2k+1)!!2^{p-k-1}} \\
&= \frac{(-1)^{k-j}}{(k-j)!} \frac{2^j}{2^k} \frac{(2k-1)!!}{(2j-1)!!} \frac{1}{2(2j+1)} \frac{(2p-1)!!}{(p-1-k)!(2k+1)!!2^{p-k-1}} \\
&= \frac{(-1)^{k-j}}{(k-j)!} \frac{2^j}{2^{2p-1}(p-1)!} \frac{1}{(2j+1)!!} \frac{(2p-1)!}{(p-1-k)!(2k+1)} = N_j(p, k).
\end{aligned}$$

□

9. THE PROOF OF THEOREM 1.3: LOW DIMENSIONAL CASES

We prove the two cases $m = 2p - 1 = 3$, and $m = 2p - 1 = 5$ independently, in the following two subsections. In the last subsection we give some remarks on the general case. Even if the proofs of the two cases $m = 3$ and $m = 5$ follow the same line, we prefer to give details separately, for two reasons: first, in order to improve readability, and second to avoid a problem that will be made clear in Section 9.3 below. In each case the proof is in two parts: in the first we compute the anomaly boundary term, as defined in Section 4.2, in the second we compute the singular term in the analytic torsion, using Proposition 7.2.

9.1. Case $m = 3$.

9.1.1. *Part 1.* Since $m = 3$, the unique terms that give non trivial contribution in the Berezin integral appearing in equation (2.14) are those homogeneous of degree 3. By definition

$$e^{-\frac{1}{2}\hat{\Omega}-u^2\mathcal{S}_1^2} = 1 - \frac{1}{2}\hat{\Omega} - u^2\mathcal{S}_1^2 + \dots,$$

(recall that $\Theta = \hat{\Omega}$, see Section 4.2) and therefore the terms of degree 3 in

$$e^{-\frac{1}{2}\hat{\Omega}-u^2\mathcal{S}_1^2} \sum_{k=1}^{\infty} \frac{1}{\Gamma(\frac{k}{2}+1)} u^{k-1} \mathcal{S}_1^k,$$

are

$$-\frac{2}{3\sqrt{\pi}}u^2\mathcal{S}_1^3 - \frac{1}{\sqrt{\pi}}\hat{\Omega}\mathcal{S}_1.$$

Applying the definition in equation (2.14), this gives

$$\begin{aligned}
(9.1) \quad B(\nabla_1) &= \frac{1}{2} \int_0^1 \int^B e^{-\frac{1}{2}\hat{\Omega}-u^2\mathcal{S}_j^2} \sum_{k=1}^{\infty} \frac{1}{\Gamma(\frac{k}{2}+1)} u^{k-1} \mathcal{S}_j^k du \\
&= \frac{1}{2} \int_0^1 \int^B \left(-\frac{2}{3\sqrt{\pi}}u^2\mathcal{S}_1^3 - \frac{1}{\sqrt{\pi}}\hat{\Omega}\mathcal{S}_1 \right) du \\
&= -\frac{1}{2\sqrt{\pi}} \int^B \hat{\Omega}\mathcal{S}_1 - \frac{1}{9\sqrt{\pi}} \int^B \mathcal{S}_1^3.
\end{aligned}$$

By equation (4.7)

$$(9.2) \quad S_1^3 = -\frac{1}{8} \left(\sum_{k=1}^m b_k^* \wedge \hat{e}_k^* \right)^3 = \frac{3}{4} dvol_g \wedge \hat{e}_1^* \wedge \hat{e}_2^* \wedge \hat{e}_3^*.$$

This follows by direct calculation. For example

$$(9.3) \quad \begin{aligned} S_1^2 &= \frac{1}{4} \left(\sum_{k=1}^3 b_k^* \wedge \hat{e}_k^* \right)^2 \\ &= \frac{1}{4} (b_1^* \wedge \hat{e}_1^* + b_2^* \wedge \hat{e}_2^* + b_3^* \wedge \hat{e}_3^*)^2 \\ &= \frac{1}{4} (b_1^* \wedge \hat{e}_1^* \wedge b_2^* \wedge \hat{e}_2^* + \dots + b_2^* \wedge \hat{e}_2^* \wedge b_1^* \wedge \hat{e}_1^* + \dots) \\ &= \frac{1}{4} (-b_1^* \wedge b_2^* \wedge \hat{e}_1^* \wedge \hat{e}_2^* + \dots - b_2^* \wedge b_1^* \wedge \hat{e}_2^* \wedge \hat{e}_1^* + \dots) \\ &= \frac{1}{4} (-2b_1^* \wedge b_2^* \wedge \hat{e}_1^* \wedge \hat{e}_2^* + \dots) \\ &= -\frac{1}{2} \sum_{j < k=1}^3 b_j^* \wedge b_k^* \wedge \hat{e}_j^* \wedge \hat{e}_k^*, \end{aligned}$$

while

$$(b_1^* \wedge b_2^* \wedge \hat{e}_1^* \wedge \hat{e}_2^*) \wedge (b_3^* \wedge \hat{e}_3^*) = b_1^* \wedge b_2^* \wedge b_3^* \wedge \hat{e}_1^* \wedge \hat{e}_2^* \wedge \hat{e}_3^*.$$

Thus,

$$\int^B S_1^3 = \frac{3}{4\pi^{\frac{3}{2}}} dvol_g.$$

By equations (4.7) and (2.13),

$$\hat{\Omega} S_1 = -\frac{1}{4} \left(\sum_{k,l=1}^3 \tilde{\Omega}_{kl} \wedge \hat{e}_k^* \wedge \hat{e}_l^* \right) \wedge \left(\sum_{k=1}^3 b_k^* \wedge \hat{e}_k^* \right).$$

Direct calculations give

$$\begin{aligned} \hat{\Omega} S_1 &= -\frac{1}{2} (\Omega_{23} \wedge b_1^* - \Omega_{13} \wedge b_2^* + \Omega_{12} \wedge b_3^*) \wedge \hat{e}_1^* \wedge \hat{e}_2^* \wedge \hat{e}_3^* \\ &= -\frac{1}{2} (R_{2332} + R_{1331} + R_{1221}) \hat{e}_1^* \wedge \hat{e}_2^* \wedge \hat{e}_3^* \\ &= -\frac{1}{4} \tilde{\tau} \hat{e}_1^* \wedge \hat{e}_2^* \wedge \hat{e}_3^*, \end{aligned}$$

and hence

$$\int^B \hat{\Omega} S_1 = \frac{1}{4\pi^{\frac{3}{2}}} \sum_{k,l=1}^3 \tilde{R}_{klkk} dvol_g.$$

Substitution in equation (9.1) gives

$$\begin{aligned} B(\nabla_1) &= \frac{1}{4\sqrt{\pi}} \int^B \hat{\Omega} S_1 - \frac{1}{9\sqrt{\pi}} \int^B S_1^3 \\ &= \frac{1}{8\pi^2} \tilde{\tau} d\text{vol}_g - \frac{1}{12\pi^2} d\text{vol}_g. \end{aligned}$$

By the formula in equation (2.15), the anomaly boundary term is

$$A_{\text{BM,abs}}(\partial C_l W) = \frac{1}{16\pi^2} \int_{\partial C_l W} \tilde{\tau} d\text{vol}_g - \frac{1}{24\pi^2} \int_{\partial C_l W} d\text{vol}_g.$$

9.1.2. *Part 2.* By Proposition 7.2, with $p = 2$,

$$\log T_{\text{sing}}(C_l W) = \frac{1}{2} \sum_{q=0}^1 (-1)^q \sum_{j=0}^1 \text{Res}_0 \Phi_{2j+1,q}(s) \text{Res}_{s=j+\frac{1}{2}} \zeta_{\text{cex}}(s, \tilde{\Delta}^{(q)} + \alpha_q^2).$$

Since $p = 2$, $\alpha_0 = -1$ and $\alpha_1 = 0$. Since there are no exact 0-forms

$$\zeta_{\text{cex}}(s, \tilde{\Delta}^{(0)} + \alpha_0^2) = \zeta(s, \tilde{\Delta}^{(0)} + \alpha_0^2).$$

By Lemma 6.8,

$$\begin{aligned} \text{Res}_{s=\frac{3}{2}} \zeta(s, \tilde{\Delta}^{(0)} + \alpha_0^2) &= \text{Res}_{s=\frac{3}{2}} \zeta(s, \tilde{\Delta}^{(0)}), \\ \text{Res}_{s=\frac{1}{2}} \zeta(s, \tilde{\Delta}^{(0)} + \alpha_0^2) &= \text{Res}_{s=\frac{1}{2}} \zeta(s, \tilde{\Delta}^{(0)}) - \frac{1}{2} \text{Res}_{s=\frac{3}{2}} \zeta(s, \tilde{\Delta}^{(0)}). \end{aligned}$$

By duality (see Section 2.5)

$$\zeta_{\text{cex}}(s, \tilde{\Delta}^{(1)}) = \zeta(s, \tilde{\Delta}^{(1)}) - \zeta_{\text{ex}}(s, \tilde{\Delta}^{(1)}) = \zeta(s, \tilde{\Delta}^{(1)}) - \zeta_{\text{cex}}(s, \tilde{\Delta}^{(0)}),$$

and also

$$\begin{aligned} \text{Res}_{s=\frac{1}{2}} \zeta(s, \tilde{\Delta}^{(1)}) &= -3 \text{Res}_{s=\frac{1}{2}} \zeta(s, \tilde{\Delta}^{(0)}), \\ \text{Res}_{s=\frac{3}{2}} \zeta(s, \tilde{\Delta}^{(1)}) &= 3 \text{Res}_{s=\frac{3}{2}} \zeta(s, \tilde{\Delta}^{(0)}). \end{aligned}$$

Putting all together, we obtain

$$\begin{aligned} \log T_{\text{sing}}(C_l W) &= \frac{1}{2} \left(\text{Res}_{s=0} \Phi_{1,0}(s) + \text{Res}_{s=0} \Phi_{1,1}(s) + 3 \text{Res}_{s=0} \Phi_{1,1}(s) \right) \text{Res}_{s=\frac{1}{2}} \zeta(s, \tilde{\Delta}^{(0)}) \\ &\quad + \frac{1}{2} \left(\text{Res}_{s=0} \Phi_{3,1}(s) + \text{Res}_{s=0} \Phi_{3,0}(s) - \frac{1}{2} \text{Res}_{s=0} \Phi_{1,0}(s) - 3 \text{Res}_{s=0} \Phi_{3,1}(s) \right) \text{Res}_{s=\frac{3}{2}} \zeta(s, \tilde{\Delta}^{(0)}). \end{aligned}$$

By Corollaries 6.1 (when $q = 1$), and 6.2 (when $q = 0$)

$$\begin{aligned} \text{Res}_{s=0} \Phi_{1,1}(s) &= 1, & \text{Res}_{s=0} \Phi_{3,1}(s) &= \frac{2}{315}, \\ \text{Res}_{s=0} \Phi_{1,0}(s) &= 2, & \text{Res}_{s=0} \Phi_{3,0}(s) &= \frac{214}{315}. \end{aligned}$$

This gives

$$\log T_{\text{sing}}(C_l W) = 3 \operatorname{Res}_{s=\frac{1}{2}} \zeta(s, \tilde{\Delta}^{(0)}) - \frac{1}{6} \operatorname{Res}_{s=\frac{3}{2}} \zeta(s, \tilde{\Delta}^{(0)}),$$

and by Propositions 3.1 and 3.3

$$\log T_{\text{sing}}(C_l W) = \frac{1}{16\pi^2} \int_{\partial C_l W} \tilde{\tau} d\text{vol}_g - \frac{1}{24\pi^2} \int_{\partial C_l W} d\text{vol}_g.$$

9.2. Case $m = 5$.

9.2.1. *Part 1.* Since $m = 5$, the unique terms that gives non trivial contribution in the Berezin integral appearing in equation (2.14) are those homogeneous of degree 5. After some calculation, the terms of degree 3 in

$$e^{-\frac{1}{2}\hat{\Omega} - u^2 S_1^2} \sum_{k=1}^{\infty} \frac{1}{\Gamma(\frac{k}{2} + 1)} u^{k-1} S_1^k,$$

are

$$-\frac{1}{5\sqrt{\pi}} u^4 S_1^5 + \frac{1}{3\sqrt{\pi}} u^2 \hat{\Omega} S_1^3 + \frac{1}{4\sqrt{\pi}} \hat{\Omega}^2 S_1.$$

Integration in u , as in equation (2.14) gives

$$(9.4) \quad B(\nabla_1) = \frac{1}{2\sqrt{\pi}} \int^B \left(\frac{1}{25} S_1^5 + \frac{1}{9} u^2 \hat{\Omega} S_1^3 + \frac{1}{4} \hat{\Omega}^2 S_1 \right).$$

We calculate the three terms appearing in the integrand.

Proceeding as for equations (9.3), starting from equation (4.7), we obtain

$$S_1^2 = -\frac{1}{2} \sum_{j < k=1}^5 b_j^* \wedge b_k^* \wedge \hat{e}_j^* \wedge \hat{e}_k^*,$$

and

$$\begin{aligned} S_1^5 &= (S_1^2)^2 S_1 = -\frac{1}{2^5} \left(-\frac{1}{2} \sum_{j < k=1}^5 b_j^* \wedge b_k^* \wedge \hat{e}_j^* \wedge \hat{e}_k^* \right)^2 \sum_{k=1}^5 b_k^* \wedge \hat{e}_k^* \\ &= -\frac{5}{4} d\text{vol}_g \wedge \hat{e}_1^* \wedge \hat{e}_2^* \wedge \hat{e}_3^* \wedge \hat{e}_4^* \wedge \hat{e}_5^*, \end{aligned}$$

and recalling the definition in (2.13) of $\hat{\Omega}$,

$$\begin{aligned} \hat{\Omega} S_1^3 &= \frac{3}{8} \left(\sum_{k,l=1}^3 \tilde{\Omega}_{kl} \wedge \hat{e}_k^* \wedge \hat{e}_l^* \right) \wedge d\text{vol}_g \wedge \hat{e}_1^* \wedge \hat{e}_2^* \wedge \hat{e}_3^* \wedge \hat{e}_4^* \wedge \hat{e}_5^* \\ &= \frac{3}{8} \tilde{\tau} \wedge d\text{vol}_g \wedge \hat{e}_1^* \wedge \hat{e}_2^* \wedge \hat{e}_3^* \wedge \hat{e}_4^* \wedge \hat{e}_5^*. \end{aligned}$$

The last term is

$$\hat{\Omega}^2 S_1 = -\frac{1}{2^5} \left(\sum_{k,l=1}^3 \tilde{\Omega}_{kl} \wedge \hat{e}_k^* \wedge \hat{e}_l^* \right)^2 \left(\sum_{k=1}^5 b_k^* \wedge \hat{e}_k^* \right).$$

This term requires some noisy explicit calculations, that we omit here. The result is

$$\begin{aligned}\hat{\Omega}^2 \mathcal{S}_1 &= -\frac{1}{2^5} \left(4|\tilde{R}|^2 - 16|\tilde{Ric}|^2 + 4\tilde{\tau}^2 \right) dvol_g \wedge \hat{e}_1^* \wedge \hat{e}_2^* \wedge \hat{e}_3^* \wedge \hat{e}_4^* \wedge \hat{e}_5^* \\ &= \left(-\frac{1}{8}|\tilde{R}|^2 + \frac{1}{2}|\tilde{Ric}|^2 - \frac{1}{8}\tilde{\tau}^2 \right) dvol_g \wedge \hat{e}_1^* \wedge \hat{e}_2^* \wedge \hat{e}_3^* \wedge \hat{e}_4^* \wedge \hat{e}_5^*.\end{aligned}$$

Substitution in equation (9.4) gives

$$B(\nabla_1) = \frac{3dvol_g}{40\pi^3} - \frac{\tilde{\tau}dvol_g}{48\pi^3} + \frac{|\tilde{R}|^2 dvol_g}{64\pi^3} - \frac{|\tilde{Ric}|^2 dvol_g}{16\pi^3} + \frac{\tilde{\tau}^2 dvol_g}{64\pi^3}.$$

By the formula in equation (2.15), the anomaly boundary term is

$$\begin{aligned}A_{\text{BM,abs}}(\partial C_l W) &= \frac{3}{80\pi^3} \int_{\partial C_l W} dvol_g - \frac{1}{96\pi^3} \int_{\partial C_l W} \tilde{\tau} dvol_g + \frac{1}{128\pi^3} \int_{\partial C_l W} |\tilde{R}|^2 dvol_g \\ &\quad - \frac{1}{32\pi^3} \int_{\partial C_l W} |\tilde{Ric}|^2 dvol_g + \frac{1}{128\pi^3} \int_{\partial C_l W} \tilde{\tau}^2 dvol_g.\end{aligned}$$

9.2.2. *Part 2.* By Proposition 7.2, with $p = 3$,

$$\log T_{\text{sing}}(C_l W) = \frac{1}{2} \sum_{q=0}^2 (-1)^q \sum_{j=0}^2 \text{Res}_0 \Phi_{2j+1,q}(s) \text{Res}_1 \zeta_{\text{cex}} \left(s, \tilde{\Delta}^{(q)} + \alpha_q^2 \right).$$

Since $p = 3$, $\alpha_0 = -2$, $\alpha_1 = -1$, and $\alpha_2 = 0$. By duality (see Section 2.5)

$$\begin{aligned}\zeta_{\text{cex}}(s, \tilde{\Delta}^{(0)}) &= \zeta(s, \tilde{\Delta}^{(0)}), \\ \zeta_{\text{cex}}(s, \tilde{\Delta}^{(1)}) &= \zeta(s, \tilde{\Delta}^{(1)}) - \zeta_{\text{ex}}(s, \tilde{\Delta}^{(1)}) = \zeta(s, \tilde{\Delta}^{(1)}) - \zeta_{\text{cex}}(s, \tilde{\Delta}^{(0)}), \\ \zeta_{\text{cex}}(s, \tilde{\Delta}^{(2)}) &= \zeta(s, \tilde{\Delta}^{(2)}) - \zeta_{\text{ex}}(s, \tilde{\Delta}^{(2)}) = \zeta(s, \tilde{\Delta}^{(2)}) - \zeta_{\text{cex}}(s, \tilde{\Delta}^{(1)}) \\ &= \zeta(s, \tilde{\Delta}^{(2)}) - \zeta(s, \tilde{\Delta}^{(1)}) + \zeta(s, \tilde{\Delta}^{(0)}).\end{aligned}$$

By Lemma 6.8, with $q = 0$ and 1 ,

$$\begin{aligned}\text{Res}_1 \zeta \left(s, \tilde{\Delta}^{(q)} + \alpha_q^2 \right) &= \text{Res}_1 \zeta \left(s, \tilde{\Delta}^{(q)} \right), \\ \text{Res}_1 \zeta \left(s, \tilde{\Delta}^{(q)} + \alpha_q^2 \right) &= \text{Res}_1 \zeta \left(s, \tilde{\Delta}^{(q)} \right) - \frac{3}{2} \text{Res}_1 \zeta \left(s, \tilde{\Delta}^{(q)} \right), \\ \text{Res}_1 \zeta \left(s, \tilde{\Delta}^{(q)} + \alpha_q^2 \right) &= \text{Res}_1 \zeta \left(s, \tilde{\Delta}^{(q)} \right) - \frac{1}{2} \text{Res}_1 \zeta \left(s, \tilde{\Delta}^{(q)} \right) + \frac{3}{8} \text{Res}_1 \zeta \left(s, \tilde{\Delta}^{(q)} \right).\end{aligned}$$

By Proposition 3.1

$$\begin{aligned}\text{Res}_1 \zeta(s, \tilde{\Delta}^{(0)}) &= \frac{e_{0,4}}{\Gamma(1/2)} = \frac{1}{2^5 3^2 5 \pi^3} \left(5 \int_{\partial C_l W} \tilde{\tau}^2 dvol_g - 2 \int_{\partial C_l W} |\tilde{Ric}|^2 dvol_g + 2 \int_{\partial C_l W} |\tilde{R}|^2 dvol_g \right), \\ \text{Res}_1 \zeta(s, \tilde{\Delta}^{(0)}) &= \frac{e_{0,2}}{\Gamma(3/2)} = \frac{1}{96\pi^3} \int_{\partial C_l W} \tilde{\tau} dvol_g, \\ \text{Res}_1 \zeta(s, \tilde{\Delta}^{(0)}) &= \frac{e_{0,0}}{\Gamma(5/2)} = \frac{1}{24\pi^3} \int_{\partial C_l W} dvol_g;\end{aligned}$$

$$\begin{aligned}
\operatorname{Res}_1 \zeta(s, \tilde{\Delta}^{(1)}) &= \frac{e_{1,4}}{\Gamma(1/2)} \\
&= \frac{1}{2^5 3^2 5 \pi^3} \left(-35 \int_{\partial C_l W} \tilde{\tau}^2 d\operatorname{vol}_g + 170 \int_{\partial C_l W} |\tilde{Ric}|^2 d\operatorname{vol}_g - 20 \int_{\partial C_l W} |\tilde{R}|^2 d\operatorname{vol}_g \right), \\
\operatorname{Res}_1 \zeta(s, \tilde{\Delta}^{(1)}) &= \frac{e_{1,2}}{\Gamma(3/2)} = -\frac{1}{96\pi^3} \int_{\partial C_l W} \tilde{\tau} d\operatorname{vol}_g = -\operatorname{Res}_1 \zeta(s, \tilde{\Delta}^{(0)}), \\
\operatorname{Res}_1 \zeta(s, \tilde{\Delta}^{(1)}) &= \frac{e_{1,0}}{\Gamma(5/2)} = \frac{5}{24\pi^3} \int_{\partial C_l W} d\operatorname{vol}_g = 5 \operatorname{Res}_1 \zeta(s, \tilde{\Delta}^{(0)});
\end{aligned}$$

$$\begin{aligned}
\operatorname{Res}_1 \zeta(s, \tilde{\Delta}^{(2)}) &= \frac{e_{2,4}}{\Gamma(1/2)} \\
&= \frac{1}{2^5 3^2 5 \pi^3} \left(50 \int_{\partial C_l W} \tilde{\tau}^2 d\operatorname{vol}_g - 200 \int_{\partial C_l W} |\tilde{Ric}|^2 d\operatorname{vol}_g + 110 \int_{\partial C_l W} |\tilde{R}|^2 d\operatorname{vol}_g \right), \\
\operatorname{Res}_1 \zeta(s, \tilde{\Delta}^{(2)}) &= \frac{e_{2,2}}{\Gamma(3/2)} = -\frac{8}{96\pi^3} \int_{\partial C_l W} \tilde{\tau} d\operatorname{vol}_g = -8 \operatorname{Res}_1 \zeta(s, \tilde{\Delta}^{(0)}), \\
\operatorname{Res}_1 \zeta(s, \tilde{\Delta}^{(2)}) &= \frac{e_{2,0}}{\Gamma(5/2)} = \frac{10}{24\pi^3} \int_{\partial C_l W} d\operatorname{vol}_g = 10 \operatorname{Res}_1 \zeta(s, \tilde{\Delta}^{(0)}).
\end{aligned}$$

Summing up, after some calculations, we obtain

$$\begin{aligned}
&\log T_{\operatorname{sing}}(C_l W) \\
&= \frac{1}{2} \left(\operatorname{Res}_0 \Phi_{1,0}(s) + \operatorname{Res}_0 \Phi_{1,1}(s) + \operatorname{Res}_0 \Phi_{1,2}(s) \right) \operatorname{Res}_1 \zeta(s, \tilde{\Delta}^{(0)}) \\
&\quad - \frac{1}{2} \left(\operatorname{Res}_0 \Phi_{1,1}(s) + \operatorname{Res}_0 \Phi_{1,2}(s) \right) \operatorname{Res}_1 \zeta(s, \tilde{\Delta}^{(1)}) \\
&\quad + \frac{1}{2} \operatorname{Res}_0 \Phi_{1,2}(s) \operatorname{Res}_1 \zeta(s, \tilde{\Delta}^{(2)}) \\
&\quad + \operatorname{Res}_0 \left(\frac{1}{2} \Phi_{3,0}(s) + \Phi_{3,1}(s) - \Phi_{3,2}(s) - \Phi_{1,0}(s) - \frac{1}{2} \Phi_{1,1}(s) \right) \operatorname{Res}_1 \zeta(s, \tilde{\Delta}^{(0)}) \\
&\quad + \operatorname{Res}_0 \left(\frac{1}{2} \Phi_{5,0}(s) - 2\Phi_{5,1}(s) + 3\Phi_{5,2}(s) - 3\Phi_{3,0}(s) + \frac{3}{4} \Phi_{1,0}(s) + \frac{3}{4} \Phi_{1,1}(s) \right) \operatorname{Res}_1 \zeta(s, \tilde{\Delta}^{(0)}).
\end{aligned}$$

By Corollaries 6.1 (when $q = 2$), and 6.2 (when $q = 0, 1$)

$$\begin{array}{lll}
\operatorname{Res}_0 \Phi_{1,2}(s) = 1, & \operatorname{Res}_0 \Phi_{3,2}(s) = \frac{2}{315}, & \operatorname{Res}_0 \Phi_{5,2}(s) = -\frac{346}{22522} \\
\operatorname{Res}_0 \Phi_{1,1}(s) = 2, & \operatorname{Res}_0 \Phi_{3,1}(s) = \frac{214}{315}, & \operatorname{Res}_0 \Phi_{5,1}(s) = \frac{31706}{75075} \\
\operatorname{Res}_0 \Phi_{1,0}(s) = 2, & \operatorname{Res}_0 \Phi_{3,0}(s) = \frac{844}{315}, & \operatorname{Res}_0 \Phi_{5,0}(s) = \frac{487876}{75075}.
\end{array}$$

This gives

$$\begin{aligned} \log T_{\text{sing}}(C_l W) &= \frac{5}{2} \operatorname{Res}_1 \zeta(s, \tilde{\Delta}^{(0)}) - \frac{3}{2} \operatorname{Res}_1 \zeta(s, \tilde{\Delta}^{(1)}) + \frac{1}{2} \operatorname{Res}_1 \zeta(s, \tilde{\Delta}^{(2)}) \\ &\quad - \operatorname{Res}_1 \zeta(s, \tilde{\Delta}^{(0)}) + \frac{9}{10} \operatorname{Res}_1 \zeta(s, \tilde{\Delta}^{(0)}), \end{aligned}$$

and by Propositions 3.1 and 3.3

$$\begin{aligned} \log T_{\text{sing}}(C_l W) &= \frac{3}{80\pi^3} \int_{\partial C_l W} d\operatorname{vol}_g - \frac{1}{96\pi^3} \int_{\partial C_l W} \tilde{\tau} d\operatorname{vol}_g + \frac{1}{128\pi^3} \int_{\partial C_l W} |\tilde{R}|^2 d\operatorname{vol}_g \\ &\quad - \frac{1}{32\pi^3} \int_{\partial C_l W} |\tilde{Ric}|^2 d\operatorname{vol}_g + \frac{1}{128\pi^3} \int_{\partial C_l W} \tilde{\tau}^2 d\operatorname{vol}_g. \end{aligned}$$

9.3. A remark on the general case. Assume $m = 2p - 1$ is odd, $p \geq 1$. Then, $\log T_{\text{sing}}(C_l W)$ depends only the functions $\Phi_{k,q}(s)$ and $\zeta(s, \tilde{\Delta}^{(q)})$, by Proposition 7.2. It follows from the definition in Section 3.3 that the functions $\Phi_{k,q}(s)$ are universal functions that depend only on the decomposition of the spectrum of the Laplace operator on forms on the cone on the spectrum of the Laplace operator on forms on the section. This decomposition is independent from the section, so the functions $\Phi_{k,q}(s)$ do not depend on the particular section (they obviously depend on the dimension). This follows also from Corollaries 6.1 and 6.2. Therefore, we can use the functions $\Phi_{k,q}(s)$ calculated when the section is a sphere of odd dimension. It follows that $\log T_{\text{sing}}(C_l W)$ is a polynomial in the residues of the functions $\zeta(s, \tilde{\Delta}^{(q)})$, with coefficients that are the same as in the case when W is a sphere. Now, the residues of $\zeta(s, \tilde{\Delta}^{(q)})$ are polynomials in the coefficients $e_{q,j}$ of the asymptotic expansion of the heat kernel of the Laplacian on forms on (W, g) . In turn, the $e_{q,j}$ are the integrals of some polynomial in the metric tensor g , its inverse and its derivatives. By work of P. Gilkey [14], the coefficients of these polynomial are universal, namely are the same for any manifold W [14] Theorem 1.8.3. More precisely, by the above considerations and invariance theory as developed in [14] Theorem 4.1.9, it follows that

$$\log T_{\text{sing}}(C_l W) = \int_{\partial C_l W} P(x),$$

where P belongs to the ring of all invariant polynomials in the derivative of the metric defined for manifolds of dimension m , $\mathcal{B}_m(g)$ (see [14] Section 2.1.4, and Lemma 2.4.2). Bases for this ring are given in terms of covariant derivatives of the curvature tensor using H. Weyl invariants of the orthogonal groups [14] Section 2.4.3. It is possible to prove that these polynomial are universal up to a constant factor depending on the dimension $P = c_m Q$, where Q does not depends on the dimension [14] Lemma 4.1.4 and Theorem 4.1.9.

On the other side, by inspection of [3], the term $A_{\text{BM,abs}}(\partial C_l W)$ is also the integral of some universal polynomial in some tensorial quantities constructed from the metric g . Therefore

$$A_{\text{BM,abs}}(\partial C_l W) = \int_{\partial C_l W} R(x),$$

$R \in \mathcal{B}_m(g)$. Fixing a base for $\mathcal{B}_m(g)$, the proof of Theorem 1.3 in the general case follows if we are able to prove that $P(x) - R(x) = 0$, for (W, g) .

We have some remarks on this point. First, by Theorem 1.2, it follows that $P(x) - R(x) = 0$, for $(W, g) = (S_{\sin \alpha}^{2p-1}, g_E)$. Unfortunately, this does not implies the general case. For using the base for $\mathcal{B}_m(g)$ in Theorem 4.1.9 of [14], we see that there are variable in $P(x) - R(x)$ that involves

derivative of the curvature when $m > 3$. Second, by the same argument, the proof of Theorem 1.3 in the case $m = 5$ is a fundamental indication for the general case: indeed if $m < 5$ the proof of Theorem 1.3 follows by that of Theorem 1.2, by the previous considerations. Third, the proof of the general case along this line depends on the availability of some further information on the coefficients of the heat asymptotic. A recursion relation should probably be sufficient. However, this seems at present an hard problem (see remarks and references in Section 4.1.7 of [14]).

10. THE PROOF OF THEOREM 1.3: THE GENERAL CASE

Since the argument is very closed to the one described in details in the previous sections, we will just sketch it here. We consider the *conical frustum* (or more precisely its external surface) that is the compact connected oriented Riemannian manifold

$$C_{[l_1, l_2]}W = [l_1, l_2] \times W,$$

with $0 < l_1 < l_2$, and with metric

$$g_1 = dx \otimes dx + x^2 g.$$

We study the analytic torsion of $C_{[l_1, l_2]}W$ with relative boundary conditions at $x = l_1$ and absolute boundary condition at $x = l_2$, and we respect to the trivial representation for the fundamental group. We denote by $\partial_{1/2} C_{[l_1, l_2]}W$, or simply $\partial_{1/2}$, the two boundaries, and by $\log T_{\text{rel } \partial_{1/2}, \text{abs } \partial_2}(C_{[l_1, l_2]}W)$ the torsion.

10.1. Spectrum. First, we describe the spectrum of the Laplace operator on forms. The proofs of the next lemmas are analogous to the proofs of Lemmas 4.2 and 4.3 and will be omitted.

Lemma 10.1. *With the notation of Lemma 4.2, assuming that $\mu_{q,n}$ is not an integer, all the solutions of the equation $\Delta u = \lambda^2 u$, with $\lambda \neq 0$, are convergent sums of forms of the following*

twelve types:

$$\begin{aligned}
\psi_{+,1,n,\lambda}^{(q)} &= x^{\alpha_q} J_{\mu_{q,n}}(\lambda x) \varphi_{\text{cex},n}^{(q)}, \\
\psi_{-,1,n,\lambda}^{(q)} &= x^{\alpha_q} Y_{\mu_{q,n}}(\lambda x) \varphi_{\text{cex},n}^{(q)}, \\
\psi_{+,2,n,\lambda}^{(q)} &= x^{\alpha_{q-1}} J_{\mu_{q-1,n}}(\lambda x) \tilde{d}\varphi_{\text{cex},n}^{(q-1)} + \partial_x(x^{\alpha_{q-1}} J_{\mu_{q-1,n}}(\lambda x)) dx \wedge \varphi_{\text{cex},n}^{(q-1)} \\
\psi_{-,2,n,\lambda}^{(q)} &= x^{\alpha_{q-1}} Y_{\mu_{q-1,n}}(\lambda x) \tilde{d}\varphi_{\text{cex},n}^{(q-1)} + \partial_x(x^{\alpha_{q-1}} Y_{\mu_{q-1,n}}(\lambda x)) dx \wedge \varphi_{\text{cex},n}^{(q-1)} \\
\psi_{+,3,n,\lambda}^{(q)} &= x^{2\alpha_{q-1}+1} \partial_x(x^{-\alpha_{q-1}} J_{\mu_{q-1,n}}(\lambda x)) \tilde{d}\varphi_{\text{cex},n}^{(q-1)} \\
&\quad + x^{\alpha_{q-1}-1} J_{\mu_{q-1,n}}(\lambda x) dx \wedge \tilde{d}^\dagger \tilde{d}\varphi_{\text{cex},n}^{(q-1)} \\
\psi_{-,3,n,\lambda}^{(q)} &= x^{2\alpha_{q-1}+1} \partial_x(x^{-\alpha_{q-1}} Y_{\mu_{q-1,n}}(\lambda x)) \tilde{d}\varphi_{\text{cex},n}^{(q-1)} \\
&\quad + x^{\alpha_{q-1}-1} Y_{\mu_{q-1,n}}(\lambda x) dx \wedge \tilde{d}^\dagger \tilde{d}\varphi_{\text{cex},n}^{(q-1)} \\
\psi_{+,4,n,\lambda}^{(q)} &= x^{\alpha_{q-2}+1} J_{\mu_{q-2,n}}(\lambda x) dx \wedge \tilde{d}\varphi_{\text{cex},n}^{(q-2)} \\
\psi_{-,4,n,\lambda}^{(q)} &= x^{\alpha_{q-2}+1} Y_{\mu_{q-2,n}}(\lambda x) dx \wedge \tilde{d}\varphi_{\text{cex},n}^{(q-2)} \\
\psi_{+,E,\lambda}^{(q)} &= x^{\alpha_q} J_{|\alpha_q|}(\lambda x) \varphi_{\text{har},n}^{(q)} \\
\psi_{-,E,\lambda}^{(q)} &= x^{\alpha_q} Y_{|\alpha_q|}(\lambda x) \varphi_{\text{har},n}^{(q)} \\
\psi_{+,O,\lambda}^{(q)} &= \partial_x(x^{\alpha_{q-1}} J_{|\alpha_{q-1}|}(\lambda x)) dx \wedge \varphi_{\text{har},n}^{(q-1)} \\
\psi_{-,O,\lambda}^{(q)} &= \partial_x(x^{\alpha_{q-1}} Y_{|\alpha_{q-1}|}(\lambda x)) dx \wedge \varphi_{\text{har},n}^{(q-1)}.
\end{aligned}$$

When $\mu_{q,n}$ is an integer the $-$ solutions must be modified including some logarithmic term (see for example [41] for a set of linear independent solutions of the Bessel equation).

Note that the forms of types 1, 3 and E are coexact, those of types 2, 4 and O exacts. The operator d sends forms of types 1, 3 and E in forms of types 2, 4 and O , while $d^\dagger$ sends forms of types 2, 4 and O in forms of types 1, 3 and E , respectively. The Hodge operator sends forms of type 1 in forms of type 4, 2 in 3, and E in 0. Define the functions, for $c \neq 0$,

$$\begin{aligned}
F_{\mu,c}(x) &= J_\mu(l_2 x)(cY_\mu(l_1 x) + l_1 x Y'_\mu(l_1 x)) - Y_\mu(l_2 x)(cJ_\mu(l_1 x) + l_1 x J'_\mu(l_1 x)), \\
\hat{F}_{\mu,c}(x) &= J_\mu(l_1 x)(cY_\mu(l_2 x) + l_2 x Y'_\mu(l_2 x)) - Y_\mu(l_1 x)(cJ_\mu(l_2 x) + l_2 x J'_\mu(l_2 x)),
\end{aligned}$$

and when $c = 0$,

$$\begin{aligned}
F_{\mu,0}(x) &= J_\mu(l_2 x) Y'_\mu(l_1 x) - Y_\mu(l_2 x) J'_\mu(l_1 x), \\
\hat{F}_{\mu,0}(x) &= J_\mu(l_1 x) Y'_\mu(l_2 x) - Y_\mu(l_1 x) J'_\mu(l_2 x).
\end{aligned}$$

Lemma 10.2. *The positive part of the spectrum of the Laplace operator on forms on $C_{[l_1, l_2]}W$, with relative boundary conditions on $\partial_1 C_{[l_1, l_2]}W$ and absolute boundary conditions on $\partial_2 C_{[l_1, l_2]}W$ is:*

$$\begin{aligned}
\text{Sp}_+ \Delta_{\text{rel } \partial_1, \text{abs } \partial_2}^{(q)} &= \left\{ m_{\text{cex}, q, n} : \hat{f}_{\mu_{q,n}, \alpha_q, k}^2 \right\}_{n,k=1}^\infty \cup \left\{ m_{\text{cex}, q-1, n} : \hat{f}_{\mu_{q-1,n}, \alpha_{q-1}, k}^2 \right\}_{n,k=1}^\infty \\
&\quad \cup \left\{ m_{\text{cex}, q-1, n} : \hat{f}_{\mu_{q-1,n}, -\alpha_{q-1}, k}^2 \right\}_{n,k=1}^\infty \cup \left\{ m_{\text{cex}, q-2, n} : \hat{f}_{\mu_{q-2,n}, -\alpha_{q-2}, k}^2 \right\}_{n,k=1}^\infty \\
&\quad \cup \left\{ m_{\text{har}, q, 0} : \hat{f}_{|\alpha_q|, \alpha_q, k}^2 \right\}_{k=1}^\infty \cup \left\{ m_{\text{har}, q-1, 0} : \hat{f}_{|\alpha_{q-1}|, \alpha_{q-1}, k}^2 \right\}_{k=1}^\infty.
\end{aligned}$$

With absolute boundary conditions on $\partial_1 C_{[l_1, l_2]} W$ and relative boundary conditions on $\partial_2 C_{[l_1, l_2]} W$ is:

$$\begin{aligned} \mathrm{Sp}_+ \Delta_{\mathrm{abs} \partial_1, \mathrm{rel} \partial_2}^{(q)} = & \left\{ m_{\mathrm{cex}, q, n} : f_{\mu_{q, n}, \alpha_q, k}^{-2s} \right\}_{n, k=1}^{\infty} \cup \left\{ m_{\mathrm{cex}, q-1, n} : f_{\mu_{q-1, n}, \alpha_{q-1}, k}^{-2s} \right\}_{n, k=1}^{\infty} \\ & \cup \left\{ m_{\mathrm{cex}, q-1, n} : \hat{f}_{\mu_{q-1, n}, -\alpha_{q-1}, k}^{-2s} \right\}_{n, k=1}^{\infty} \cup \left\{ m_{\mathrm{cex}, q-2, n} : \hat{f}_{\mu_{q-1, n}, -\alpha_{q-2}, k}^{-2s} \right\}_{n, k=1}^{\infty} \\ & \cup \left\{ m_{\mathrm{har}, q} : f_{|\alpha_q|, \alpha_q, k} \right\}_{k=1}^{\infty} \cup \left\{ m_{\mathrm{har}, q-1} : f_{|\alpha_{q-1}|, \alpha_{q-1}, k} \right\}_{k=1}^{\infty}, \end{aligned}$$

where the $f_{\mu, c, k}$ are the zeros of the function $F_{\mu, c}(x)$, the $\hat{f}_{\mu, c, k}$ are the zeros of the function $\hat{F}_{\mu, c}(x)$, $c \in \mathbb{R}$, α_q and $\mu_{q, n}$ are defined in Lemma 4.2.

10.2. Torsion zeta function. We define the *torsion zeta function* as in Section 2.5 by (for $\mathrm{Re}(s) > \frac{m+1}{2}$)

$$t_{\mathrm{abs}, \mathrm{rel}}(s) = \frac{1}{2} \sum_{q=1}^{m+1} (-1)^q q \zeta(s, \Delta_{\mathrm{abs}, \mathrm{rel}}^{(q)}).$$

By a proof similar to the one of Theorem 5.1, we obtain the expected duality:

$$\log T_{\mathrm{abs} \partial_1, \mathrm{rel} \partial_2}(C_{[l_1, l_2]} W) = (-1)^m \log T_{\mathrm{rel} \partial_1, \mathrm{abs} \partial_2}(C_{[l_1, l_2]} W).$$

We proceed assuming $\dim W = 2p - 1$ odd, and assuming relative boundary condition on $\partial_1 C_{[l_1, l_2]} W$ and absolute boundary condition on ∂_2 ; for notational convenience, we will omit the *abs, rel* subscript. We define the functions

$$\begin{aligned} \hat{F}_c(x) &= J_c(l_2 x) Y_{c-1}(l_1 x) - Y_c(l_2 x) J_{c-1}(l_1 x), \\ F_c(x) &= J_c(l_1 x) Y_{c-1}(l_2 x) - Y_c(l_1 x) J_{c-1}(l_2 x). \end{aligned}$$

Note that, with these definitions $\hat{F}_0(x) = F_1(x)$ and $F_0(x) = \hat{F}_1(x)$ (remember that $Y_{-n}(x) = (-1)^n Y_n(x)$ and $J_{-n}(x) = (-1)^n J_n(x)$). The proof of the following lemma is analogous to the proof of Lemma 6.1. The main step is to prove that $\hat{f}_{|\alpha_q|, \alpha_q, k} = f_{-\alpha_{q-1}, k}$, that $\hat{f}_{|\alpha_q|, \alpha_q, k} = \hat{f}_{\alpha_q, k}$, when $p-1 < q \leq 2p-1$, and that $\hat{f}_{0,0,k} = f_{1,k}$, where the $f_{c,k}$, *hat* $f_{c,k}$ are the zeros of the functions F_c , $\hat{F}_c$, respectively.

Lemma 10.3.

$$\begin{aligned} t(s) = & \frac{1}{2} \sum_{q=0}^{p-2} (-1)^q \sum_{n, k=1}^{\infty} m_{\mathrm{cex}, q, n} \left(f_{\mu_{q, n}, \alpha_q, k}^{-2s} + f_{\mu_{q, n}, -\alpha_q, k}^{-2s} - \hat{f}_{\mu_{q, n}, \alpha_q, k}^{-2s} - \hat{f}_{\mu_{q, n}, -\alpha_q, k}^{-2s} \right) \\ & + (-1)^{p-1} \frac{1}{2} \sum_{n, k=1}^{\infty} m_{\mathrm{cex}, p-1, n} \left(f_{\mu_{p-1, n}, 0, k}^{-2s} - \hat{f}_{\mu_{p-1, n}, 0, k}^{-2s} \right) \\ & - \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q \mathrm{rk} \mathcal{H}_q(W; \mathbb{Q}) \sum_{k=1}^{\infty} \left(f_{-\alpha_{q-1}, k}^{-2s} - \hat{f}_{-\alpha_{q-1}, k}^{-2s} \right). \end{aligned}$$

We set

$$(10.1) \quad \begin{aligned} Z_{q,\pm}(s) &= \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} f_{\mu_q,n,\pm\alpha_q,k}^{-2s}, & \hat{Z}_{q,\pm}(s) &= \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \hat{f}_{\mu_q,n,\pm\alpha_q,k}^{-2s}, \\ Z_{p-1}(s) &= \sum_{n,k=1}^{\infty} m_{\text{cex},p-1,n} f_{\mu_{p-1},n,0,k}^{-2s}, & \hat{Z}_{p-1,\pm}(s) &= \sum_{n,k=1}^{\infty} m_{\text{cex},p-1,n} \hat{f}_{\mu_{p-1},n,0,k}^{-2s}, \\ z_q(s) &= \sum_{k=1}^{\infty} \left(f_{-\alpha_{q-1},k}^{-2s} - \hat{f}_{-\alpha_{q-1},k}^{-2s} \right), \end{aligned}$$

for $0 \leq q \leq p-1$, and

$$(10.2) \quad \begin{aligned} t_{p-1}(s) &= Z_{p-1}(s) - \hat{Z}_{p-1}(s), \\ t_q(s) &= Z_{q,+}(s) + Z_{q,-}(s) - \hat{Z}_{q,+}(s) - \hat{Z}_{q,-}(s), \quad 0 \leq q \leq p-2. \end{aligned}$$

Then,

$$\begin{aligned} t(s) &= \frac{1}{2} \sum_{q=0}^{p-2} (-1)^q \left(Z_{q,+}(s) + Z_{q,-}(s) - \hat{Z}_{q,+}(s) - \hat{Z}_{q,-}(s) \right) + (-1)^{p-1} \frac{1}{2} \left(Z_{p-1}(s) - \hat{Z}_{p-1}(s) \right) \\ &\quad - \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q \text{rk} \mathcal{H}_q(W; \mathbb{Q}) z_q(s) \\ &= \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q t_q(s) - \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q \text{rk} \mathcal{H}_q(\partial C_l W; \mathbb{Q}) z_q(s), \end{aligned}$$

and

$$(10.3) \quad \log T_{\text{rel } \partial_1, \text{abs } \partial_2}(C_{[l_1, l_2]} W) = t'(0) = \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q t'_q(0) - \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q \text{rk} \mathcal{H}_q(\partial C_l W; \mathbb{Q}) z'_q(0).$$

10.3. Expansions of the logarithmic Gamma functions. We study the zeta functions $Z_{q,\pm}$, $\hat{Z}_{q,\pm}$, by the method of Section 3. The double series associated to these zeta functions, as defined in equation (10.1), are denoted by $S_{\pm\alpha_q}$, $\hat{S}_{\pm\alpha_q}$. We show that all these double sequences are spectrally decomposable on the sequence U_q , defined at the beginning of Section 6.2. We verify all requirements precisely as in Sections 6.1 and 6.2. First, we need suitable representation for the associated logarithmic Gamma functions. Proceeding as in Section 6.2, consider for example the function

$$F_{\mu,c}(z) = J_{\mu}(l_2 z)(cY_{\mu}(l_1 z) + l_1 z Y'_{\mu}(l_1 z)) - Y_{\mu}(l_2 z)(cJ_{\mu}(l_1 z) + l_1 z J'_{\mu}(l_1 z)).$$

Recalling the series definition of the Bessel function [15]pg. 910, near $z = 0$,

$$F_{\mu,c}(z) = \frac{1}{\pi} \left(\left(\frac{l_2^{\mu}}{l_1^{\mu}} + \frac{l_1^{\mu}}{l_2^{\mu}} \right) - \frac{c}{\mu} \left(\frac{l_2^{\mu}}{l_1^{\mu}} - \frac{l_1^{\mu}}{l_2^{\mu}} \right) \right)$$

Thus $F_{\mu,c}(z)$ is an even function of z , and we obtain the product representation

$$F_{\mu,c}(z) = \frac{1}{\pi} \left(\left(\frac{l_2^{\mu}}{l_1^{\mu}} + \frac{l_1^{\mu}}{l_2^{\mu}} \right) - \frac{c}{\mu} \left(\frac{l_2^{\mu}}{l_1^{\mu}} - \frac{l_1^{\mu}}{l_2^{\mu}} \right) \right) \prod_{k=1}^{+\infty} \left(1 - \frac{z^2}{f_{\mu,c,k}^2} \right).$$

Recalling that

$$Y_\mu(z) = \frac{\cos \mu\pi}{\sin \mu\pi} J_\mu(z) - \frac{1}{\sin \mu\pi} J_{-\mu}(z), \quad I_{-\mu}(z) = \frac{2}{\pi} \sin \mu\pi K_\mu(z) + I_\mu(z),$$

and that (when $-\pi < \arg(z) \leq \frac{\pi}{2}$) $J_\mu(iz) = e^{\frac{\pi}{2}i\mu} I_\mu(z)$, and $J'_\mu(iz) = e^{\frac{\pi}{2}i\mu} e^{-\frac{\pi}{2}i} I'_\mu(z)$, we obtain

$$\begin{aligned} Y_\mu(iz) &= \left(\frac{\cos \mu\pi}{\sin \mu\pi} e^{\frac{\pi}{2}i\mu} + \frac{e^{-\frac{\pi}{2}i\mu}}{\sin \mu\pi} \right) I_\mu(z) - \frac{2}{\pi} e^{-\frac{\pi}{2}i\mu} K_\mu(z), \\ Y'_\mu(iz) &= e^{-\frac{\pi}{2}i} \left(\frac{\cos \mu\pi}{\sin \mu\pi} e^{\frac{\pi}{2}i\mu} + \frac{e^{-\frac{\pi}{2}i\mu}}{\sin \mu\pi} \right) I'_\mu(z) - \frac{2}{\pi} e^{-\frac{\pi}{2}i} e^{-\frac{\pi}{2}i\mu} K'_\mu(z). \end{aligned}$$

So

$$F_{\mu,c}(iz) = \frac{2}{\pi} \left(-K_\mu(l_2z)(cI_\mu(l_1z) + l_1zI'_\mu(l_1z)) + I_\mu(l_2z)(cK_\mu(l_1z) + l_1zK'_\mu(l_1z)) \right),$$

and if we define (for $-\pi < \arg(z) \leq \frac{\pi}{2}$) $G_{\mu,c}(z) = i^2 F_{\mu,c}(iz)$,

$$G_{\mu,c}(z) = \frac{1}{\pi} \left(\left(\frac{l_2^\mu}{l_1^\mu} + \frac{l_1^\mu}{l_2^\mu} \right) - \frac{c}{\mu} \left(\frac{l_2^\mu}{l_1^\mu} - \frac{l_1^\mu}{l_2^\mu} \right) \right) \prod_{k=1}^{+\infty} \left(1 + \frac{z^2}{f_{\mu,c,k}^2} \right).$$

Proceeding in the similar way

$$\begin{aligned} \hat{F}_{\mu,c}(iz) &= \frac{2}{\pi} \left(K_\mu(l_1z)(cI_\mu(l_2z) + l_2zI'_\mu(l_2z)) - I_\mu(l_1z)(cK_\mu(l_2z) + l_2zK'_\mu(l_2z)) \right), \\ \hat{G}_{\mu,c}(z) &= \hat{F}_{\mu,c}(iz) = \frac{1}{\pi} \left(\left(\frac{l_2^\mu}{l_1^\mu} + \frac{l_1^\mu}{l_2^\mu} \right) + \frac{c}{\mu} \left(\frac{l_2^\mu}{l_1^\mu} - \frac{l_1^\mu}{l_2^\mu} \right) \right) \prod_{k=1}^{+\infty} \left(1 + \frac{z^2}{\hat{f}_{\mu,c,k}^2} \right); \\ F_{\mu,0}(iz) &= \frac{2}{\pi} \left(-K_\mu(l_2z)I'_\mu(l_1z) + I_\mu(l_2z)K'_\mu(l_1z) \right), \\ G_{\mu,0}(z) &= i^2 F_{\mu,0}(iz) = \frac{1}{l_1z\pi} \left(\frac{l_2^\mu}{l_1^\mu} + \frac{l_1^\mu}{l_2^\mu} \right) \prod_{k=1}^{+\infty} \left(1 + \frac{z^2}{f_{\mu,0,k}^2} \right); \\ \hat{F}_{\mu,0}(iz) &= \frac{2}{\pi} \left(K_\mu(l_1z)I'_\mu(l_2z) - I_\mu(l_1z)K'_\mu(l_2z) \right), \\ \hat{G}_{\mu,0}(z) &= \hat{F}_{\mu,0}(iz) = \frac{1}{l_2z\pi} \left(\frac{l_2^\mu}{l_1^\mu} + \frac{l_1^\mu}{l_2^\mu} \right) \prod_{k=1}^{+\infty} \left(1 + \frac{z^2}{\hat{f}_{\mu,0,k}^2} \right). \end{aligned}$$

These give the following representations for the logarithmic Gamma functions with $z = \sqrt{-\lambda}$,

$$\begin{aligned} \log \Gamma(-\lambda, S_{\pm\alpha_q}) &= -\log \prod_{k=1}^{\infty} \left(1 + \frac{(-\lambda)}{f_{\mu_q,n,\pm\alpha_q,k}^2} \right) \\ &= -\log G_{\mu_q,n,\pm\alpha_q}(\sqrt{-\lambda}) + \log \frac{1}{\pi} + \log \left(\left(\frac{l_2^{\mu_q,n}}{l_1^{\mu_q,n}} + \frac{l_1^{\mu_q,n}}{l_2^{\mu_q,n}} \right) \mp \frac{\alpha_q}{\mu_{q,n}} \left(\frac{l_2^{\mu_q,n}}{l_1^{\mu_q,n}} - \frac{l_1^{\mu_q,n}}{l_2^{\mu_q,n}} \right) \right); \\ \log \Gamma(-\lambda, \hat{S}_{\pm\alpha_q}) &= -\log \prod_{k=1}^{\infty} \left(1 + \frac{(-\lambda)}{\hat{f}_{\mu_q,n,\pm\alpha_q,k}^2} \right) \\ &= -\log \hat{G}_{\mu_q,n,\pm\alpha_q}(\sqrt{-\lambda}) + \log \frac{1}{\pi} + \log \left(\left(\frac{l_2^{\mu_q,n}}{l_1^{\mu_q,n}} + \frac{l_1^{\mu_q,n}}{l_2^{\mu_q,n}} \right) \pm \frac{\alpha_q}{\mu_{q,n}} \left(\frac{l_2^{\mu_q,n}}{l_1^{\mu_q,n}} - \frac{l_1^{\mu_q,n}}{l_2^{\mu_q,n}} \right) \right) \end{aligned}$$

$$\begin{aligned}
\log \Gamma(-\lambda, S_0) &= -\log \prod_{k=1}^{\infty} \left(1 + \frac{(-\lambda)}{f_{\mu_{p-1},n,0,k}^2} \right) \\
&= -\log G_{\mu_{n,p-1},0}(\sqrt{-\lambda}) - \frac{1}{2} \log -\lambda - \log l_1 + \log \frac{1}{\pi} + \log \left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} + \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) \\
\log \Gamma(-\lambda, \hat{S}_0) &= -\log \prod_{k=1}^{\infty} \left(1 + \frac{(-\lambda)}{\hat{f}_{\mu_{p-1},n,0,k}^2} \right) \\
&= -\log \hat{G}_{\mu_{n,p-1},0}(\sqrt{-\lambda}) - \frac{1}{2} \log -\lambda - \log l_2 + \log \frac{1}{\pi} + \log \left(\frac{l_2^{\mu_{p-1,n}}}{l_1^{\mu_{p-1,n}}} + \frac{l_1^{\mu_{p-1,n}}}{l_2^{\mu_{p-1,n}}} \right)
\end{aligned}$$

These representations and uniform asymptotic expansions of Bessel functions and their derivative (see the proof of Lemma 6.4 for the functions I_ν and [28] pg. 376 for the functions K_ν) will give the expansion required in equation (3.3) of Definition 3.1. Let see one case in some details. We have

$$\begin{aligned}
\log \Gamma(-\lambda, S_{n,\pm\alpha_q}/\mu_{q,n}^2) &= \\
&= -\log G_{\mu_{n,q},\pm\alpha_q}(\mu_{q,n}\sqrt{-\lambda}) + \log \frac{1}{\pi} + \log \left(\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} + \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) \mp \frac{\alpha_q}{\mu_{q,n}} \left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} - \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) \right).
\end{aligned}$$

Using the cited expansions we obtain

$$\begin{aligned}
\log G_{\mu,c}(\mu z) &= \log \frac{1}{\pi} + \mu \left(\sqrt{1 + l_2^2 z^2} - \sqrt{1 + l_1^2 z^2} \right) + \mu \log \frac{l_2(1 + \sqrt{1 + l_1^2 z^2})}{l_1(1 + \sqrt{1 + l_2^2 z^2})} + \frac{1}{4} \log \frac{(1 + l_1^2 z^2)}{(1 + l_2^2 z^2)} \\
&+ \log \left(1 + \sum_{j=1}^{2p-1} \frac{1}{\mu^j} \left(U_j(l_2 z) + (-1)^j W_{c,j}(l_1 z) + \sum_{k=1}^{j-1} (-1)^{j-k} U_k(l_2 z) W_{c,j-k}(l_1 z) \right) + O(\mu^{-2p}) \right),
\end{aligned}$$

where

$$W_{c,j}(z) = V_j(z) - \frac{c}{\sqrt{1+z^2}} U_{j-1}(z).$$

Thus

$$\begin{aligned}
\log G_{\mu_{q,n},\pm\alpha_q}(\mu_{q,n}\sqrt{-\lambda}) &= \mu_{q,n} \left(\sqrt{1 - l_2^2 \lambda} - \sqrt{1 - l_1^2 \lambda} \right) + \mu_{q,n} \log \frac{l_2(1 + \sqrt{1 - l_1^2 \lambda})}{l_1(1 + \sqrt{1 - l_2^2 \lambda})} \\
&+ \log \frac{1}{\pi} + \frac{1}{4} \log \frac{(1 - l_1^2 \lambda)}{(1 - l_2^2 \lambda)} + \sum_{j=1}^{2p-1} \frac{l_{j,\pm\alpha_q}(\lambda)}{\mu_{q,n}^j} + O(\mu_{q,n}^{-2p}),
\end{aligned}$$

with

$$a_{0,\pm\alpha_q}(\lambda) = 1,$$

$$a_{j,\pm\alpha_q}(\lambda) = U_j(l_2\sqrt{-\lambda}) + (-1)^j W_{\pm\alpha_q,j}(l_1\sqrt{-\lambda}) + \sum_{k=1}^{j-1} U_k(l_2\sqrt{-\lambda}) (-1)^{j-k} W_{\pm\alpha_q,j-k}(l_1\sqrt{-\lambda}),$$

$$l_{1,\pm\alpha_q}(\lambda) = a_{1,\pm\alpha_q}(\lambda),$$

$$l_{j,\pm\alpha_q}(\lambda) = a_{j,\pm\alpha_q}(\lambda) - \sum_{k=1}^{j-1} \frac{j-k}{j} a_{k,\pm\alpha_q}(\lambda) l_{j-k,\pm\alpha_q}(\lambda).$$

Substituting in the $\log \Gamma(-\lambda, S_{n,\pm\alpha_q}/\mu_{q,n}^2)$, we have

$$\begin{aligned} \log \Gamma(-\lambda, S_{n,\pm\alpha_q}/\mu_{q,n}^2) &= -\mu_{q,n} \left(\sqrt{1-l_2^2\lambda} - \sqrt{1-l_1^2\lambda} \right) - \mu_{q,n} \log \frac{l_2(1+\sqrt{1-l_1^2\lambda})}{l_1(1+\sqrt{1-l_2^2\lambda})} \\ &\quad - \frac{1}{4} \log \frac{(1-l_1^2\lambda)}{(1-l_2^2\lambda)} - \sum_{j=1}^{2p-1} \frac{l_{j,\pm\alpha_q}(\lambda)}{\mu_{q,n}^j} + \log \left(\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} + \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) \mp \frac{\alpha_q}{\mu_{q,n}} \left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} - \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) \right) + O(\mu_{q,n}^{-2p}). \end{aligned}$$

Proceeding in a similar way we obtain

$$\begin{aligned} \log \Gamma(-\lambda, \hat{S}_{n,\pm\alpha_q}/\mu_{q,n}^2) &= -\mu_{q,n} \left(\sqrt{1-l_2^2\lambda} - \sqrt{1-l_1^2\lambda} \right) - \mu_{q,n} \log \frac{l_2(1+\sqrt{1-l_1^2\lambda})}{l_1(1+\sqrt{1-l_2^2\lambda})} \\ &\quad - \frac{1}{4} \log \frac{(1-l_1^2\lambda)}{(1-l_2^2\lambda)} - \sum_{j=1}^{2p-1} \frac{\hat{l}_{j,\pm\alpha_q}(\lambda)}{\mu_{q,n}^j} + \log \left(\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} + \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) \pm \frac{\alpha_q}{\mu_{q,n}} \left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} - \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) \right) + O(\mu_{q,n}^{-2p}), \end{aligned}$$

with

$$\begin{aligned} \hat{a}_{0,\pm\alpha_q}(\lambda) &= 1, \\ \hat{a}_{j,\pm\alpha_q}(\lambda) &= \hat{W}_{\pm\alpha_q,j}(l_2\sqrt{-\lambda}) + (-1)^j U_j(l_1\sqrt{-\lambda}) + \sum_{k=1}^{j-1} (-1)^k U_k(l_1\sqrt{-\lambda}) \hat{W}_{\pm\alpha_q,j-k}(l_2\sqrt{-\lambda}), \\ \hat{l}_{1,\pm\alpha_q}(\lambda) &= \hat{a}_{1,\pm\alpha_q}(\lambda), \\ \hat{l}_{j,\pm\alpha_q}(\lambda) &= \hat{a}_{j,\pm\alpha_q}(\lambda) - \sum_{k=1}^{j-1} \frac{j-k}{j} \hat{a}_{k,\pm\alpha_q}(\lambda) \hat{l}_{j-k,\pm\alpha_q}(\lambda). \end{aligned}$$

$$\begin{aligned} \log \Gamma(-\lambda, \hat{S}_{n,0}/\mu_{p-1,n}^2) &= -\mu_{p-1,n} \left(\sqrt{1-l_2^2\lambda} - \sqrt{1-l_1^2\lambda} \right) - \mu_{p-1,n} \log \frac{l_2(1+\sqrt{1-l_1^2\lambda})}{l_1(1+\sqrt{1-l_2^2\lambda})} \\ &\quad - \frac{1}{4} \log \frac{(1-l_1^2\lambda)}{(1-l_2^2\lambda)} - \sum_{j=1}^{2p-1} \frac{\hat{l}_{j,0}(\lambda)}{\mu_{p-1,n}^j} + \log \left(\frac{l_2^{\mu_{p-1,n}}}{l_1^{\mu_{p-1,n}}} + \frac{l_1^{\mu_{p-1,n}}}{l_2^{\mu_{p-1,n}}} \right) + O(\mu_{p-1,n}^{-2p}), \end{aligned}$$

with

$$\begin{aligned} \hat{a}_{0,0}(\lambda) &= 1, \\ \hat{a}_{j,0}(\lambda) &= V_j(l_2\sqrt{-\lambda}) + (-1)^j U_j(l_1\sqrt{-\lambda}) + \sum_{k=1}^{j-1} (-1)^k U_k(l_1\sqrt{-\lambda}) V_{j-k}(l_2\sqrt{-\lambda}), \\ \hat{l}_{1,0}(\lambda) &= \hat{a}_{1,0}(\lambda), \\ \hat{l}_{j,0}(\lambda) &= \hat{a}_{j,0}(\lambda) - \sum_{k=1}^{j-1} \frac{j-k}{j} \hat{a}_{k,0}(\lambda) \hat{l}_{j-k,0}(\lambda). \end{aligned}$$

$$\begin{aligned} \log \Gamma(-\lambda, S_{n,0}/\mu_{q,n}^2) &= -\mu_{p-1,n} \left(\sqrt{1-l_2^2\lambda} - \sqrt{1-l_1^2\lambda} \right) - \mu_{p-1,n} \log \frac{l_2(1+\sqrt{1-l_1^2\lambda})}{l_1(1+\sqrt{1-l_2^2\lambda})} \\ &- \frac{1}{4} \log \frac{(1-l_1^2\lambda)}{(1-l_2^2\lambda)} - \sum_{j=1}^{2p-1} \frac{l_{j,0}(\lambda)}{\mu_{p-1,n}^j} + \log \left(\frac{l_2^{\mu_{p-1,n}}}{l_1^{\mu_{p-1,n}}} + \frac{l_1^{\mu_{p-1,n}}}{l_2^{\mu_{p-1,n}}} \right) + O(\mu_{p-1,n}^{-2p}), \end{aligned}$$

with

$$\begin{aligned} a_{0,0}(\lambda) &= 1, \\ a_{j,0}(\lambda) &= U_j(l_2\sqrt{-\lambda}) + (-1)^j V_j(l_1\sqrt{-\lambda}) + \sum_{k=1}^{j-1} U_k(l_2\sqrt{-\lambda}) (-1)^{j-k} V_{j-k}(l_1\sqrt{-\lambda}), \\ l_{1,0}(\lambda) &= a_{1,0}(\lambda), \\ l_{j,0}(\lambda) &= a_{j,0}(\lambda) - \sum_{k=1}^{j-1} \frac{j-k}{j} a_{k,0}(\lambda) l_{j-k}(\lambda). \end{aligned}$$

We conclude this section with the expansions for large λ , accordingly to equation (3.5). Using classical expansions of Bessel functions $I_n u$ and K_ν and their derivative for large argument, we obtain the expansions of the functions G and $\hat{G}$, and then those for the Gamma functions:

$$\begin{aligned} \log \Gamma(-\lambda, S_{n,\pm\alpha_q}/\mu_{q,n}^2) &\sim \\ &\sim -\mu_{q,n}(l_2 - l_1)\sqrt{-\lambda} - \frac{1}{2} \log \frac{l_1}{l_2} + \log \left(\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} + \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) \mp \frac{\alpha_q}{\mu_{q,n}} \left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} - \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) \right) + O\left(\frac{1}{\sqrt{-\lambda}}\right), \\ \log \Gamma(-\lambda, \hat{S}_{n,\pm\alpha_q}/\mu_{q,n}^2) &\sim \\ &\sim -\mu_{q,n}(l_2 - l_1)\sqrt{-\lambda} - \frac{1}{2} \log \frac{l_2}{l_1} + \log \left(\left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} + \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) \pm \frac{\alpha_q}{\mu_{q,n}} \left(\frac{l_2^{\mu_{q,n}}}{l_1^{\mu_{q,n}}} - \frac{l_1^{\mu_{q,n}}}{l_2^{\mu_{q,n}}} \right) \right) + O\left(\frac{1}{\sqrt{-\lambda}}\right), \\ \log \Gamma(-\lambda, S_{n,0}/\mu_{p-1,n}^2) &\sim -\mu_{p-1,n}(l_2 - l_1)\sqrt{-\lambda} + \frac{1}{2} \log \frac{l_2}{l_1} + \log \left(\frac{l_2^{\mu_{p-1,n}}}{l_1^{\mu_{p-1,n}}} + \frac{l_1^{\mu_{p-1,n}}}{l_2^{\mu_{p-1,n}}} \right) + O\left(\frac{1}{\sqrt{-\lambda}}\right), \\ \log \Gamma(-\lambda, \hat{S}_{n,0}/\mu_{p-1,n}^2) &\sim -\mu_{p-1,n}(l_2 - l_1)\sqrt{-\lambda} + \frac{1}{2} \log \frac{l_1}{l_2} + \log \left(\frac{l_2^{\mu_{p-1,n}}}{l_1^{\mu_{p-1,n}}} + \frac{l_1^{\mu_{p-1,n}}}{l_2^{\mu_{p-1,n}}} \right) + O\left(\frac{1}{\sqrt{-\lambda}}\right). \end{aligned}$$

10.4. The function $t_q(s)$. By definition in equation (10.2), we need to consider the difference between $\log \Gamma(-\lambda, S_{n,\pm\alpha_q}/\mu_{q,n})$ and $\log \Gamma(-\lambda, \hat{S}_{n,\pm\alpha_q}/\mu_{q,n})$. The expansions given in the previous subsection give expansion for large μ

$$\begin{aligned} &\log \Gamma(-\lambda, S_{n,\alpha_q}/\mu_{q,n}) + \log \Gamma(-\lambda, S_{n,\alpha_q}/\mu_{q,n}) - \log \Gamma(-\lambda, \hat{S}_{n,\alpha_q}/\mu_{q,n}) - \log \Gamma(-\lambda, \hat{S}_{n,-\alpha_q}/\mu_{q,n}) = \\ &= \log \frac{(1-\lambda l_2^2)}{(1-\lambda l_1^2)} + \sum_{j=1}^{2p-1} \frac{1}{\mu_{q,n}^j} (\hat{l}_{j,\alpha_q}(\lambda) + \hat{l}_{j,-\alpha_q}(\lambda) - l_{j,\alpha_q}(\lambda) - l_{j,-\alpha_q}(\lambda)) + O(\mu_{q,n}^{-2p}), \end{aligned}$$

and for large λ

$$\begin{aligned} &\log \Gamma(-\lambda, S_{n,\alpha_q}/\mu_{q,n}) + \log \Gamma(-\lambda, S_{n,\alpha_q}/\mu_{q,n}) - \log \Gamma(-\lambda, \hat{S}_{n,\alpha_q}/\mu_{q,n}) - \log \Gamma(-\lambda, \hat{S}_{n,-\alpha_q}/\mu_{q,n}) = \\ &= 2 \log \frac{l_2}{l_1} + O\left(\frac{1}{\sqrt{-\lambda}}\right). \end{aligned}$$

Proceeding as in the proof of Lemma 6.14, we obtain

$$\begin{aligned} a_{0,0,q,n} &= 2 \log \frac{l_2}{l_1}, \\ a_{0,1,q,n} &= 0, \\ b_{2j-1,0,0,q} &= 0, & b_{2j-1,0,1,q} &= 0, \end{aligned}$$

and hence

$$\begin{aligned} A_{0,0,q}(s) &= 2 \log \frac{l_2}{l_1} \sum_{n=1}^{\infty} \frac{m_{q,n}}{\mu_{q,n}^{2s}} = 2 \log \frac{l_2}{l_1} \sum_{j=0}^{\infty} \binom{-s}{j} \alpha_q^{2j} \zeta_{\text{ccl}}(s+j, \tilde{\Delta}^{(q)}), \\ A_{0,1,q}(s) &= 0. \end{aligned}$$

This gives

$$A_{0,0,q}(0) = 2 \log \frac{l_2}{l_1} \zeta_{\text{ccl},q}(0, \tilde{\Delta}^{(q)}) = 2(-1)^q \log \frac{l_2}{l_1} \sum_{k=0}^q (-1)^k \text{rk} \mathcal{H}^k(W, \mathbb{Q}),$$

and

$$t'_{q,\text{reg}}(0) = 2(-1)^{q+1} \log \frac{l_2}{l_1} \sum_{k=0}^q (-1)^k \text{rk} \mathcal{H}^k(W, \mathbb{Q})$$

Similarly, we consider the difference of $\log \Gamma(-\lambda, S_{n,0}/\mu_{p-1,n})$ and $\log \Gamma(-\lambda, \hat{S}_{n,0}/\mu_{p-1,n})$ for the function t_{p-1} , and we obtain

$$\begin{aligned} a_{0,0,n,p-1} &= \log \frac{l_2}{l_1}, \\ a_{0,1,n,p-1} &= 0, \\ b_{2j-1,0,0,p-1} &= 0, & b_{2j-1,0,1,p-1} &= 0, \\ A_{0,0,p-1}(s) &= \log \frac{l_2}{l_1} \sum_{n=1}^{\infty} \frac{m_{p-1,n}}{\mu_{p-1,n}^{2s}} = \log \frac{l_2}{l_1} \zeta_{\text{ccl},p-1}(s, \tilde{\Delta}^{(q)}), \\ A_{0,1}(s) &= 0, \end{aligned}$$

$$A_{0,0,p-1}(0) = \log \frac{l_2}{l_1} \zeta_{\text{ccl},q}(0, \tilde{\Delta}^{(p-1)}) = (-1)^{p-1} \log \frac{l_2}{l_1} \sum_{k=0}^{p-1} (-1)^k \text{rk} \mathcal{H}^k(W, \mathbb{Q}),$$

and

$$t'_{p-1,\text{reg}}(0) = (-1)^p \log \frac{l_2}{l_1} \sum_{k=0}^{p-1} (-1)^k \text{rk} \mathcal{H}^k(W, \mathbb{Q})$$

10.5. The regular term of the torsion. We use equation (10.3). First, note that as in Section 7 there is no singular contribution by the functions $z_q(s)$. Using equation 3.2, and recalling that $-\alpha_{q-1} = -(q-1-p+1) = p-q$, we compute as in Lemma 7.1

$$z'_q(0) = \log \frac{l_2}{l_1} - 2(p-q) \log \frac{l_2}{l_1}.$$

Therefore, substitution in equation (10.3) gives

$$\log T_{\text{rel } \partial_1, \text{abs } \partial_2, \text{reg}}(C_{[l_1, l_2]} W) = t'_{\text{reg}}(0) = 0.$$

10.6. The singular term of the torsion. We show that the singular part of the torsion is twice the singular part of the torsion on the cone, namely that

$$(10.4) \quad \log T_{\text{rel } \partial_1, \text{abs } \partial_2, \text{sing}}(C_{[l_1, l_2]}W) = 2 \log T_{\text{abs, sing}}(C_l W).$$

For we need the following lemma.

Lemma 10.4. *We have the following equations:*

$$\begin{aligned} l_{j, \pm \alpha_q}(\lambda) &= l_j(l_2^2 \lambda) + (-1)^j l_j^\mp(l_1^2 \lambda), \\ \hat{l}_{j, \pm \alpha_q}(\lambda) &= l_j^\pm(l_2^2 \lambda) + (-1)^j l_j(l_1^2 \lambda), \\ l_{j, 0}(\lambda) &= l_j(l_2^2 \lambda) + (-1)^j \dot{l}_j(l_1^2 \lambda), \\ \hat{l}_{j, 0}(\lambda) &= \dot{l}_j(l_2^2 \lambda) + (-1)^j l_j(l_1^2 \lambda), \end{aligned}$$

where the functions $l_j, \dot{l}_j$ are defined in the proof of Lemma 6.4, the functions $l_j^\pm$ in the proof of Lemma 6.10, and the other function in Subsection 10.3.

Proof. The proof is by induction. We give details for the first equation. For $j = 1$, we have

$$\begin{aligned} l_{1, \pm \alpha_q}(\lambda) &= U_1(l_2 \sqrt{-\lambda}) - W_{\mp, 1}(l_1 \sqrt{-\lambda}) \\ &= l_1(l_2^2 \lambda) + (-1)^1 l_1(l_1^2 \sqrt{-\lambda}). \end{aligned}$$

Assume the equation is valid for all $n < j$. By definition

$$\begin{aligned} l_{j, \pm \alpha_q}(\lambda) &- \sum_{k=1}^{j-1} U_k(l_2 \sqrt{-\lambda}) (-1)^{j-k} W_{\mp \alpha_q, j-k}(l_1 \sqrt{-\lambda}) \\ &= U_j(l_2 \sqrt{-\lambda}) + (-1)^j W_{\mp \alpha_q, j}(l_1 \sqrt{-\lambda}) - \sum_{k=1}^{j-1} \frac{j-k}{j} a_{k, \mp \alpha_q}(\lambda) l_{j-k, \mp \alpha_q}(\lambda) \end{aligned}$$

and using the inductive hypothesis for $l_{j-k, \mp \alpha_q}(\lambda)$, and collecting similar terms, this gives

$$\begin{aligned} l_{j, \pm \alpha_q}(\lambda) &- \sum_{k=1}^{j-1} U_k(l_2 \sqrt{-\lambda}) (-1)^{j-k} W_{\mp \alpha_q, j-k}(l_1 \sqrt{-\lambda}) \\ &= l_j(l_2^2 \lambda) + (-1)^j l_j^\mp(l_1^2 \lambda) - \sum_{k=1}^{j-1} \frac{j-k}{j} (-1)^k W_{\mp \alpha_q, k}(l_1 \sqrt{-\lambda}) l_{j-k}(l_2^2 \lambda) \\ &\quad - \sum_{k=1}^{j-1} \frac{j-k}{j} (U_k(l_2 \sqrt{-\lambda})) (-1)^{j-k} l_{j-k}^\mp(l_1^2 \lambda) \\ &\quad - \sum_{k=1}^{j-1} \frac{j-k}{j} \sum_{h=1}^{k-1} U_h(l_2 \sqrt{-\lambda}) (-1)^{k-h} W_{\mp \alpha_q, k-h}(l_1 \sqrt{-\lambda}) l_{j-k}(l_2^2 \lambda) \\ &\quad - \sum_{k=1}^{j-1} 1^{j-1} \frac{j-k}{j} \sum_{h=1}^{k-1} U_h(l_2 \sqrt{-\lambda}) (-1)^{k-h} W_{\mp \alpha_q, k-h}(l_1 \sqrt{-\lambda}) (-1)^{j-k} l_{j-k}^\mp(l_1^2 \lambda). \end{aligned}$$

Rearranging the summation's indices, this reads

$$\begin{aligned}
l_{j,\pm\alpha_q}(\lambda) &= \sum_{k=1}^{j-1} U_k(l_2\sqrt{-\lambda})(-1)^{j-k} W_{\mp\alpha_q, j-k}(l_1\sqrt{-\lambda}) \\
&= l_j(l_2^2\lambda) + (-1)^j l_j^\mp(l_1^2\lambda) - \sum_{k=1}^{j-1} (-1)^k W_{\mp\alpha_q, k}(l_1\sqrt{-\lambda}) U_{j-k}(l_2\sqrt{-\lambda}) \\
&\quad + \sum_{k=1}^{j-1} (-1)^{j-k} W_{\mp\alpha_q, j-k}(l_1\sqrt{-\lambda}) \sum_{h=1}^{k-1} \frac{k-h}{j} U_h(l_2\sqrt{-\lambda}) l_{k-h}(l_2^2\lambda) \\
&\quad + \sum_{k=1}^{j-1} (-1)^k U_{j-k}(l_2\sqrt{-\lambda}) \sum_{h=1}^{k-1} \frac{h}{j} W_{\mp\alpha_q, k-h}(l_1\sqrt{-\lambda}) l_h^\mp(l_1^2\lambda) \\
&\quad - \sum_{k=1}^{j-1} \frac{j-k}{j} l_{j-k}(l_2^2\lambda) \sum_{h=1}^{k-1} U_h(l_2\sqrt{-\lambda}) (-1)^{k-h} W_{\mp\alpha_q, k-h}(l_1\sqrt{-\lambda}) \\
&\quad - \sum_{k=1}^{j-1} \frac{j-k}{j} (-1)^{j-k} l_{j-k}^\mp(l_1^2\lambda) \sum_{h=1}^{k-1} U_h(l_2\sqrt{-\lambda}) (-1)^{k-h} W_{\mp\alpha_q, k-h}(l_1\sqrt{-\lambda})
\end{aligned}$$

Reordering the first two double sums as

$$\begin{aligned}
&\sum_{k=1}^{j-1} (-1)^{j-k} W_{\mp\alpha_q, j-k}(l_1\sqrt{-\lambda}) \sum_{h=1}^{k-1} \frac{k-h}{j} U_h(l_2\sqrt{-\lambda}) l_{k-h}(l_2^2\lambda) \\
&= \sum_{k=1}^{j-1} \frac{j-k}{j} l_{j-k}(l_2^2\lambda) \sum_{h=1}^{k-1} U_h(l_2\sqrt{-\lambda}) (-1)^{k-h} W_{\mp\alpha_q, k-h}(l_1\sqrt{-\lambda}), \\
&\sum_{k=1}^{j-1} (-1)^k U_{j-k}(l_2\sqrt{-\lambda}) \sum_{h=1}^{k-1} \frac{k-h}{j} W_{\mp\alpha_q, h}(l_1\sqrt{-\lambda}) l_{k-h}^\mp(l_1^2\lambda) \\
&= \sum_{k=1}^{j-1} \frac{j-k}{j} (-1)^{j-k} l_{j-k}^\mp(l_1^2\lambda) \sum_{h=1}^{k-1} U_h(l_2\sqrt{-\lambda}) (-1)^{k-h} W_{\mp\alpha_q, k-h}(l_1\sqrt{-\lambda}),
\end{aligned}$$

the result follows. $\square$

We are now in the position of proving equation (10.4). Proceeding as in the proof of Propositions 6.2 and 6.4, the singular part of the torsion is given by some residua of the zeta function associated to the sequence U and some residua of the functions Φ . Since the sequence U is the same for the conical frustum and for the cone, and the range of the indices are the same, we only need to compare the functions Φ in the two cases. The functions Ψ are defined in equation (3.4), we introduce the linear operation

$$\Phi_{\sigma_h}(s) = \mathcal{T}(\phi_{\sigma_h}(\lambda))(s) = \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_{\theta,c}} \frac{e^{-\lambda t}}{-\lambda} \phi_{\sigma_h}(\lambda) d\lambda dt.$$

Let use the notation ϕ^{cone} and ϕ^{frust} . We have

$$\begin{aligned}\phi_{2j-1,q}^{\text{cone}} &= -2l_{2j-1}(\lambda) + l_{2j-1}^+(\lambda) + l_{2j-1}^-(\lambda), \\ \phi_{j,q}^{\text{frust}} &= -l_{j,+}(l_2^2\lambda) - l_{j,-}(l_1^2\lambda) + \hat{l}_{j,+}(l_2^2\lambda) + \hat{l}_{j,-}(l_1^2\lambda).\end{aligned}$$

Note that all the functions appearing in the definition of the functions $\phi(\lambda)$ are polynomial in $w = \frac{1}{\sqrt{1-\lambda}}$. Applying the formula in equation (10.6), we have that

$$\mathcal{T}(l_{j,+}(l_2^2\lambda))(s) = l_2^{2s} \mathcal{T}(l_{j,+}(l_2^2\lambda))(s),$$

and similarly for the other. Using Lemma 10.3, and odd indices, we obtain for example

$$\Phi_{2j-1,q}^{\text{frust}}(s) = (l_1^{2s} + l_2^{2s}) \Phi_{2j-1,q}^{\text{cone}}(s)$$

Since by Corollaries 6.1 and 6.2 all the residua Res_1 of the function $\Phi_{2j-1}^{\text{cone}}(s)$ at $s = 0$ vanish, equation 10.4 follows.

10.7. Conclusion. As recalled in Section 2.6, if $\partial W = \partial_1 W \sqcup \partial_2 W$, is the union of two disjoint components, and since the boundary term is local,

$$\log T_{\text{rel } \partial_1, \text{abs } \partial_2}((W, g); \rho) = \log \tau((W, \partial_1 W), g; \rho) + A_{\text{BM,rel}}(\partial_1 W) + A_{\text{BM,abs}}(\partial_2 W).$$

Applying this formula to the conical frustum we have

$$\log T_{\text{rel } \partial_1, \text{abs } \partial_2}(C_{[l_1, l_2]} W) = \log \tau(C_{[l_1, l_2]} W, \partial_1 C_{[l_1, l_2]} W) + A_{\text{BM,rel}}(\partial_1) + A_{\text{BM,abs}}(\partial_2).$$

Let X be a manifold of dimension $2p$ with boundary $\partial X = \partial_2 C_{[l_1, l_2]} W$, and assume there is an isometry of a collar neighborhood of the boundary of X onto a collar neighborhood of $\partial_2 C_{[l_1, l_2]} W$. Let Z be the manifold obtained by glueing smoothly X to $C_{[l_1, l_2]} W$ along the boundary $\partial_2 C_{[l_1, l_2]} W$. Applying duality of analytic torsion [22] Proposition 2.10 to Z , and since the anomaly boundary term is local, it follows that $A_{\text{BM,rel}}(\partial_1 C_{[l_1, l_2]} W) = -A_{\text{BM,abs}}(\partial_1 C_{[l_1, l_2]} W)$. Since it follows by the definition that $A_{\text{BM,abs}}(\partial_1 C_{[l_1, l_2]} W) = -A_{\text{BM,abs}}(\partial_2 C_{[l_1, l_2]} W)$, we obtain

$$\log T_{\text{rel } \partial_1, \text{abs } \partial_2}(C_{[l_1, l_2]} W) = \log \tau(C_{[l_1, l_2]} W, \partial_1 C_{[l_1, l_2]} W) + 2A_{\text{BM,abs}}(\partial_2 C_{[l_1, l_2]} W).$$

Considering the exact sequence of chain complex associated to the pair $(C_{[l_1, l_2]} W, \partial_1 C_{[l_1, l_2]} W)$, it is not difficult to see (see for example [24] Section 3) that the Reidemeister torsion of the pair vanishes, and hence

$$\log T_{\text{rel } \partial_1, \text{abs } \partial_2}(C_{[l_1, l_2]} W) = 2A_{\text{BM,abs}}(\partial_2 C_{[l_1, l_2]} W).$$

Since the anomaly boundary term is local $A_{\text{BM,abs}}(\partial_2 C_{[l_1, l_2]} W) = A_{\text{BM,abs}}(\partial C_l W)$, and hence

$$\log T_{\text{rel } \partial_1, \text{abs } \partial_2}(C_{[l_1, l_2]} W) = 2A_{\text{BM,abs}}(\partial C_l W).$$

This formulas also follows using the formulas for the variation of the torsion with mixed boundary conditions given in the new paper of Brüning and Ma [4]. We thanks the authors for making available to us this part of the results of their still unpublished paper. Since by the calculations of the previous subsections

$$\log T_{\text{rel } \partial_1, \text{abs } \partial_2}(C_{[l_1, l_2]} W) = \log T_{\text{rel } \partial_1, \text{abs } \partial_2, \text{sing}}(C_{[l_1, l_2]} W) = 2 \log T_{\text{abs, sing}}(C_l W) = 2S(\partial C_l W),$$

this completes the proof of Theorem 1.3.

Appendix

The next two formulas follow from the definition of the Euler Gamma function. Here j is any positive integer.

$$(10.5) \quad \begin{aligned} \operatorname{Res}_0 \frac{\Gamma(s + \frac{2j+1}{2})}{\Gamma(\frac{2j+1}{2})s} &= -\gamma - 2\log 2 + 2 \sum_{k=1}^j \frac{1}{2k-1}, \\ \operatorname{Res}_1 \frac{\Gamma(s + \frac{2j+1}{2})}{\Gamma(\frac{2j-1}{2})s} &= 1, \end{aligned}$$

The next formula is proved in [35] Section 4.2. Let $\Lambda_{\theta,c} = \{\lambda \in \mathbb{C} \mid |\arg(\lambda - c)| = \theta\}$, $0 < \theta < \pi$, $0 < c < 1$, a real, then

$$(10.6) \quad \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_{\theta,c}} \frac{e^{-\lambda t}}{-\lambda} \frac{1}{(1-\lambda)^a} d\lambda dt = \frac{\Gamma(s+a)}{\Gamma(a)s}.$$

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(Luiz Hartmann) UFSCar, Universidade Federal de São Carlos, São Carlos, Brazil.

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E-mail address: hartmann@dm.ufscar.br

(Mauro Spreafico) ICMC, Universidade São Paulo, São Carlos, Brazil.

E-mail address: mauros@icmc.usp.br

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