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NOTAS

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RESUMO

O conceito de determinação Finita Relativa foi introduzido por Porto e Loibel [P-L] em 1978 e se ocupa com subespaços no R^n. Neste trabalho, generalizamos este conceito a alguns conjuntos algébricos e damos uma relação com determinação finita pela direita. Terminamos com algumas observações sobre ideais de Lojasiewicz.

Finite Determination on Algebraic Sets

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Summary

The concept of finite relative determination was introduce by Porto and Loibel [P-L] in 1978 and works with subspaces of $\mathbb{R}^n$. In these notes, we generalize this concept to some algebraic sets and give a relation with finite determination on the right. We finish with some observations on Lojasiewicz ideals.

1 Introduction

In this work we generalize the concept of finite relative determination introduced by Porto-Loibel in P-L to some algebraic sets. We introduce the concept of radical of an ideal and give necessary and sufficient conditions for a germ being finitely determinated relative to the set of common zeroes of an ideal which concides with its radical, in particular for the algebraic set $x^2 - y^3 = 0$. These conditions are detected, as in the Mather and Porto-Loibel cases, by the jacobian ideal of the germ and the ideal of zeroes. We relate this concept with determination on the right and finish with Lojasiewicz ideals and give sufficient conditions for the radical of an ideal being finitely generated.
We suppose in this work that S is a germ of a subset that contains the origin and we are interested with the ideal of germs of $E(n)$ that vanish at S. We denote by G_s the germs of diffeomorphisms of $\mathbb{R}^n$ that are the identity on S.

Definition 1

1) Let f be a germ with $f(\bar{0}) = 0$, we shall say that f is k-determined with respect to G_s if whenever g is a germ such that $j^k f(\bar{0}) = j^k g(\bar{0})$ and $g - f$ vanishes at S, there exist h in G_s with $g = f \circ h$.

2) Let I a ideal of $R[x]$, the ring of polynomials in variables $x_1, \ldots x_n$ and S=$z(I)$ the common zeroes of I . We denote by $\hat{I}$ the ideal of germs of $E(n)$ that vanish at S, and we say that I is radical if $\hat{I} = IE(n)$.

3) In general if $I \subseteq E(n)$, and $z(I)$ the set of common zeroes of I contains the origin. it has sense $\widehat{I} = I(I$ radical).

Example 1

a) If $I = \langle x_1, \ldots, x_n \rangle$, $S = \{\bar{0}\}$ and $\widehat{I} = m(n) = IE(n)$

b) If $I = \langle x_i, \ldots, x_s \rangle, s < n, S = \{\bar{0}\} \times \mathbb{R}^{n-s}$ and $\widehat{I} = IE(n)$

c) If $\langle x_1x_2, x_1x_3, x_2x_3 \rangle, S$ are the coordinate axis and $\widehat{I} = IE(n)$.

For c) we use Hadamard 's lemma, let $f \in \widehat{I}$

$f(x_1, x_2, x_3) = x_1x_2g_{12} + x_1x_3g_{13} + x_2x_3g_{23} + x_1^2g_{11} + x_2^2g_{22} + x_3^2g_{33}$

Now $f(x_1, 0, 0) \equiv 0 \iff g_{11}(x_1, 0, 0) \equiv 0$. By Hadamard's lema there exist germs h_1 and h_2 such that $g_{11}(x_1, x_2, x_3) = x_2h_1(x_1, x_2, x_3) + x_3h_2(x_1, x_2, x_3)$ and $x_1^2g_{11}$ is an element of $IE(n)$. Similarly for g_{22} and g_{33}. □

Theorem 2 (P) *Let $f \in m(n)^\infty$, then there exist germs g and h in $m(n)^\infty$ with $g(x) > 0 \; \forall \; x \neq 0$ and $f = gh$.*

Corollary 3 *Let I ideal of $\mathbb{R}[x]$, the $\widehat{I}m(n)^\infty = \widehat{I} \cap m(n)^\infty$.*

Proof. One of the inclusions is obvious. For the other, let $f \in m(n)^\infty$, by the previous theorem $f = gh$ with $g, h \in m(n)^\infty$. Since $f(x)$ vanishes at S, and $g(x) > 0 \;\; \forall \, x \neq 0$. then $h \in \widehat{I}$ and f is an element of $\widehat{I}m(n)^\infty$. □

Lemma 4 (Artin-Rees) *Let A be a commutative ring with unitary and noetherian. M an A-module finitely generated. If I is ideal of A and M' submodule of M, then there exists k natural number such that $I^nM \cap M' = I^{n-k}(I^kM \cap M') \, \forall n \geq k$.* □

In case $A = \mathbb{R}[[x]]$, the ring of formal power series $M = A$ and I the maximal ideal M of A, we have that for M' ideal of A (denoted by I), there exists k natural number such that $M^n \cap I = M^{n-k}(M^k \cap I\,) \, \forall n \geq k$.

Remark. If $l \geq k$, we have that $M^n \cap I = M^{n-l}(M^l \cap I\,) \, \forall \, n \geq l$.

Notation. The least k with the above property is denoted by $A(I)$.

Example 2

a) If $I = \langle x_1, ..., x_s \rangle$, we have that $A(I) = 1$.

b) If $I = \langle x^2 - y^3 \rangle$, we have that $A(I) = 2$.

Let $\pi : E(n) \to \mathbb{R}[[x]]$ be the canonical map that assigns to each germ its Taylor series. I ideal of $E(n)$, from the above lemma there exist k such that

$\pi(I) \cap M^n = M^{m-k}(\pi(I) \cap M^k)\ \forall\, m \geq k$

then

$I \cap m(n)^m + m(n)^\infty = m(n)^{m-k}(I \cap m(n)^k) + m(n)^\infty\ \forall\, m \geq k$

and intersecting with I, we get

$$(1.1) \qquad I \cap m(n)^m = m(n)^{m-k}(I \cap m(n)^k) + I \cap m(n)^\infty\ \forall\, m \geq k$$

In case $I = \hat{I}$, we have that

$I \cap m(n)^\infty = Im(n)^\infty = \ (Im(n)^\infty)\, m(n)^\infty \subseteq (I \cap m(n)^k)\, m(n)^{m-k}$

and (1.1) will remain

$I \cap m(n)^m = m(n)^{m-k}(I \cap m(n)^k)\ \forall\, m \geq k$

In the cases b) and c) of examples 1, we get that

b) $I \cap m(n)^m = m(n)^{m-1} I$

c) $I \cap m(n)^m = m(n)^{m-2} I$

Moreover, for case b) of examples 2, we can argue in the following way.
Let $\phi(x,y) = (x, x^2 - y^3)$, we get that $E(2)/\phi^*(m(2)) \cong \mathbb{R}\langle 1, \bar{y}, \bar{y}^2\rangle$ and applying Malgrange's preparation theorem, we get that

$f(x,y) = h_0(x, x^2 - y^3) + y h_1(x, x^2 - y^3) + y^2 h_2(x, x^2 - y^3)$.

Let $f \in \hat{I}$ i.e. $f(x,y) \equiv 0$ for $x^2 - y^3 \equiv 0$, this is equivalent to

$0 = f(x^3, x^2) = h_0(x^3, 0) + x^2 h_1(x^3, 0)\ +\ x^4 h_2(x^3, 0)$

then $\pi(h_0(x,0)) = \pi(h_1(x,0)) = \pi(h_2(x,0)) = 0$ and

$$\begin{aligned} f(x,y) \ &= \ (h_0(x, x^2 - y^3) - h_0(x,0)) + y\,(h_1(x, x^2 - y^3) - h_1(x,0)) \\ &\quad + y^2(h_2(x, x^2 - y^3) - h_2(x,0)) + n(x) \ \text{ with } \ n(x) \in m(1)^\infty. \end{aligned}$$

By Hadamard's lemma, we obtain the equality $f(x,y) = (x^2 - y^3) g(x,y) + \eta(x)$.
Since $f(x,y) \equiv 0$ in $x^2 - y^3 = 0$, we get $\eta(x) \equiv 0$ and $f \in I$, hence $\hat{I} \subset I$.
The other inclusion is obvious. □

Proposition 5 *Let I be an ideal of $\mathbb{R}[x]$, then $IE(n) = \hat{I}$ if and only if $\pi(IE(n)) = \pi(\hat{I})$ and $\hat{I}$ is finitely generated.*

Proof. The necessity is obvious.
For the sufficiency, we have that

$IE(n) + m(n)^\infty = \hat{I} + m(n)^\infty$

and intersecting with $\hat{I}$, the equality that we obtain is

$IE(n) + \hat{I}\, m(n)^\infty = \hat{I}$

Nakayama's lemma concludes the proof. □

Remark. The equality $\pi(\hat{I}) = \widehat{\pi(I)}$ its not true in general. Let $f(x)$ be defined by $f(x) = e^{\frac{-1}{x^2}}$ if $x \neq 0$ and $f(0) = 0$. We have thath if $I = \langle f \rangle$, $z(I) = \{0\}$ and $\hat{I} = m(1)$, but $\widehat{\pi(I)} = \widehat{\langle 0 \rangle} = \{0\}$.
Up to now we have been using a lemma of algebra that we shall now prove.

Lemma 6 *Let A be a commutative ring with unitary, I, J and K be ideals of A, and $I \subset K$, then we have the following equality:*
$(I + J) \cap K = I + (J \cap K)$.

Proof. Let $x \in (I + J) \cap K$, we write $x = y + z$, $y \in I$ and $z \in J$, but $x \in K$ then: $z = x - y \in K$ and $z \in J \cap K$. The other inclusion is obvius. $\square$

Remark. Let I be an a ideal of $\mathbb{R}[x]$ and $z(I)$ be a coherent subset, then $\hat{I}$ is finitely generated.

Lemma 7 *Let $\{f_i\}_{i=1}^{n}$ be polynomials in $E(n)$, I the ideal generated by them, $S = z(I)$ and I radical. Consider $\tilde{I} = \langle \tilde{f}_1, ...\tilde{f}_n, t\rangle$ in $E(n+1)$ with $\tilde{f}_i(x_1, ..., x_n, t) = f_i(x_1, ..., x_n)$. Then $\tilde{I}$ is radical in $E(n+1)$ and $A(I) = A(\tilde{I})$.*

Proof. Clearly $z(\tilde{I}) = S \times \{0\}$ and we have a map
$\phi : \left\{g \in E(n+1)\ |g|_{s\times\{0\}} \equiv 0\right\} \longrightarrow \{f \in E(n) | f|_s \equiv 0\ \} \times \langle t\rangle\, E\ (n+1)$
$\phi(g) = (g(x_1, ...x_{n,}0),\ g(x_1, ..., x_{n+1}) - g(x_1, ...x_{n,}0))$.
and a map
$\psi : \{f \in E(n) | f|_s \equiv 0\ \} \times \langle t\rangle\, E\ (n+1) \longrightarrow \left\{g \in E(n+1)\ |g|_{s\times\{0\}} \equiv 0\right\}$
$\psi(f(x_1, ..., x_n), th(x_1, ..., x_{n+1})) = f(x_1, ..., x_n) + th(x_1, ..., x_{n+1})$.
Its clear that $\phi \circ \psi(f, th) = (f, th)$ and $\psi \circ \phi(g) = g$, then ϕ is a bijection. Moreover, since $\tilde{I} = \langle \tilde{f}_1, ...\tilde{f}_n, t\rangle$ we get that:
$\phi(\tilde{I}) = \left\langle \tilde{f}_i\, |_{\mathbb{R}^n\times\{0\}} \right\rangle \times \langle t\rangle\, E(n+1) = \hat{I} \times \langle t\rangle\, E(n+1) = I \times \langle t\rangle\, E(n+1)$.
Also $\phi(\hat{\tilde{I}}) = I \times \langle t\rangle\, E(n+1)$, then $\tilde{I} = \hat{\tilde{I}}$ and $\tilde{I}$ is radical. $\square$

Notation. If f is a germ of $E(n)$, then $\langle df\rangle$ will denote the ideal generated by its partial derivatives.

Theorem 8 *Suppose that I is radical, $I \cap m(n)^k$ is finitely generated and $I \cap m(n)^k \subseteq Im(n)\langle df\rangle$, with $k \geq A(I)$. Then f is k-determined relative to G_s, where $S = z(I)$.*

Proof. Let $t_0 \in \mathbb{R}$, g be a fixed germ such that $g - f \in \hat{I}$ and $j^k f(\bar{0}) = j^k g(\bar{0})$. Consider the map $F : (\mathbb{R}^n \times \mathbb{R}, (\bar{0}, t_0)) \longrightarrow \mathbb{R}$ defined by $F(x,t) = (1-t)f(x) + tg(x)$. We shall show that $F_t(x) \equiv F(x,t)$ is G_s equivalent to F_{t_0} for t close to t_0.
For this is sufficient to give $h : (\mathbb{R}^n \times \mathbb{R},\ (\bar{0}, t_0)) \longrightarrow \mathbb{R}^n$, with the following properties

I) $\sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x,t) h_i(x,t) + \frac{\partial F}{\partial t}(x,t) \equiv 0$

II) $h_i(x,t) \equiv 0$ for $x \in S$ and t close to t_0.

We have that $\frac{\partial F}{\partial t} = g - f \in m(n)^{k+1} \cap \hat{I} = m(n)^{k+1} \cap I$. also $\frac{\partial f}{\partial x_i} = \frac{\partial F}{\partial x_i} + t(\frac{\partial f}{\partial x_i} - \frac{\partial g}{\partial x_i})$

Then, since $(\frac{\partial f}{\partial x_i} - \frac{\partial g}{\partial x_i}) \in m(n)^k \; \forall \, i$ we get

$$\begin{aligned}(m(n)^k \cap I)E(n+1) &\subseteq m(n)I \langle df \rangle E(n+1) \\ &\subseteq m(n)I \left\langle \frac{\partial F}{\partial x_i} \right\rangle E(n+1) + m(n)I \, (m(n)^k E(n+1)) \\ &\subseteq m(n)I \left\langle \frac{\partial F}{\partial x_i} \right\rangle E(n+1) + m(n)(I \cap m(n)^k)E(n+1)\end{aligned}$$

Since $I \cap m(n)^k$ is finitely generated, Nakayama's lemma implies $(m(n)^k \cap I)E(n+1) \subseteq m(n)I \left\langle \frac{\partial F}{\partial x_i} \right\rangle E(n+1)$ and $\frac{\partial F}{\partial t} = \sum_{i=1}^n \frac{\partial F}{\partial x_i}(x,t)h_i(x,t)$ with $h_i \in m(n)I$.
This equality gives conditions I) and II). □

Remark

a) If $I = m(n)$,we have $m(n)^k \subseteq m(n)^2 \langle df \rangle$.

b) If $I = \langle x_1, ..., x_s \rangle$,we have $m(n)^{k-1} I \subseteq Im(n) \langle df \rangle$.

Now we shall proof that finite determination implies an inclusion of the above kind.

Notation. Let $z \in j_0^q(n,1)$, the space of q-jets that vanish at the origin and f representative of z. We shall denote by $j_0^q(f,S,n)$ the set of q-jets $j^q g(0)$ in $j_0^q(n,1)$ such that $g \equiv f$ in S, and by
$\bar{\pi}_q : f + \hat{I} \longrightarrow j_0^q(f,S,n)$, and $\tilde{\pi} : \hat{I} \longrightarrow j_0^q(0,S,n) \equiv j_s^q(n)$
the restrictions of the canonical map $\pi_q : E(n) \longrightarrow j^q(n,1)$.

Proposition 9 *Let I be an ideal of $\mathbb{R}[x]$. If $\bar{0} \in S = z(I)$, then $G_s^q = \{j^q h(\bar{0}) \mid h \in G_s\}$ is a Lie group.*

Proof. We shall show the following equality $G_s^q = \left\{ j^q(Id + (h_1, ...; h_n)) \mid h_i \in \hat{I} \right\} \cap G_{\{0\}}^q$, then G_s^q is a closed subgroup of the Lie group $G_{\{0\}}^q$ and in fact we have $T_{Id}G_s^q = \pi_q(\hat{I}\,) \times ... \times \pi_q(\hat{I}\,)$.
If $\sigma = j^q\phi(0) \in G_s^q$, we have that $\phi = (\phi_1, ..., \phi_n)$ is a diffeomorphism with $\phi(x) = x \, \forall x \in S$. Then $\phi = Id + (\phi - Id)$ and if $h = \phi - Id$ then $h_1(x) = 0, ..., h_n(x) = 0 \; \forall x \in S$, i.e. $h_i \in \hat{I}$ for $1 \leq i \leq n$. □

Lemma 10 *If zG_s^q denotes the orbit of z in $j_s^q(f,S,n)$, then $\tilde{\pi}_q^{-1}(T_z zG_s^q) = \hat{I} \langle df \rangle + \hat{I} \cap m(n)^{q+1}$.*

Proof. Let $\beta \in T_{Id} \, G_s^q$ tangent vector, then $\beta = j^q\beta'(0)$ with $\beta' \in \hat{I} \times ... \times \hat{I}$. Define $\delta(t) = Id + t\beta'$ and consider the curve $\pi_q \circ \delta : (-\epsilon, \epsilon) \longrightarrow G_s^q$, then $\beta = \frac{d}{dt}(\pi_q \circ \delta)(0)$. We also have that $\frac{d}{dt}(z \cdot (\pi_q \circ \delta)(0) = \frac{d}{dt}(\pi_q(f \circ \delta)(0) = \pi_q(\sum_{i=1}^n \frac{\partial f}{\partial x_i}\beta_i')$. We have showed that $T_z zG_s^q = \tilde{\pi}_q(\langle df \rangle \hat{I})$, then $\tilde{\pi}_q^{-1}(T_z zG_s^q) = \hat{I} \langle df \rangle + \ker \tilde{\pi}_q = \hat{I} \langle df \rangle + \hat{I} \cap m(n)^{q+1}$ □

Lemma 11 *Let q be a non negative integer, $z = j^q f(0)$ and $l < q$. If z is l-determined relative to G_s^q, then $\hat{I} \cap m(n)^{l+1} \subseteq \hat{I}\langle df\rangle + \hat{I} \cap m(n)^{q+1}$.*

Proof. Let $A = \left\{z' \in j_s^q \mid \pi_q^l(z') = \pi_q^l(z)\right\}$ ($\pi_q^l : j^q(n,1) \longrightarrow j^l(n,1)$ is the canonical projection). Since A is an affine space, we have that $T_z A = \tilde{\pi}_q(\hat{I} \cap m(n)^{l+1})$. Also by hypothesis we have that $A \subseteq zG_s^l$, then $T_z A \subseteq T_z zG_s^l$. This is equivalent to $\tilde{\pi}_q(\hat{I} \cap m(n)^{l+1}) \subseteq \tilde{\pi}_q(\langle df\rangle \hat{I})$ and we conclude that $\hat{I} \cap m(n)^{l+1} \subseteq \langle df\rangle \hat{I} + \hat{I} \cap m(n)^{q+1}$. □

Theorem 12 *Let f be a k-determinated germ relative to G_s, $S = z(I)$ and I radical. Then $I \cap m(n)^{k+1} \subset I\langle df\rangle + m(n)^{k+2} \cap I$.*

Proof. Since f is k-determined relative to G_s,we have that $\bar{\pi}_{k+1} f$ is also k-determinated. From the previous lemma, for $q = k+1$ and $l = k$, we get that $\hat{I} \cap m(n)^{k+1} \subseteq \langle df\rangle \hat{I} + \hat{I} \cap m(n)^{k+2}$. Since $I = \hat{I}$, we get the inclusion we proposed. □

Remark. If f is a k-determinated germ relative to G_s, $S = z(I)$, with I radical, $k \geq A(I)$ and $m(n)^k \cap I$ is finitely generated, then $I \cap m(n)^{k+1} \subseteq \hat{I}\langle df\rangle$. We just have to observe that $m(n)^{k+2} \cap I = m(n)(m(n)^{k+1} \cap I)$ and used Nakayama's lemma.
Joining theorems 8 and 12, we get the following

Theorem 13 *Let f be a germ, I a radical ideal in $\mathbb{R}[x]$, $S = z(I)$. Suppose that the ideal $I \cap m(n)^k$ is finitely generated for $k \geq A(I)$. Then f is finitely determined relative to G_s if and only if there exist a natural number l greater or equal than k such that $m(n)^l \cap I \subseteq I\langle df\rangle$.*

Remark. Let I ideal of $E(n)$ and $\pi(I \cap m(n)^k)$ in $\mathbb{R}[[x]]$ generated by $\{h_1, ..., h_s\}$. Let f_i germs such that $\pi(f_i) = h_i$. If g_i in $I \cap m(n)^k$ is such that $\pi(g_i) = h_i$, we will have that

$$\langle g_1, ..., g_s\rangle + I \cap m(n)^\infty = I \cap m(n)^k. \tag{1.2}$$

Theorem 14 *Consider the equality (1.2) with I radical, then the following assertions are equivalent.*

(1) $I \cap m(n)^k$ is finitely generated

(2) $I \cap m(n)^k = \langle g_1, ..., g_s\rangle$

(3) $\langle g_1, ..., g_s\rangle \supseteq I \cap m(n)^\infty$.

Remark. In general, if I is an ideal of the algebra $E(n)$, we have $Im(n)^\infty = (I \cap m(n)^\infty)m(n)^\infty$ and if I is radical $I \cap m(n)^\infty = (I \cap m(n)^\infty)m(n)^\infty$.

Proof .
(1) $\Rightarrow$ (2)

$$\begin{aligned} I \cap m(n)^k &= \langle g_1, ..., g_s\rangle + Im(n)^\infty \supseteq \langle g_1, ..., g_s\rangle + (I \cap m(n)^k)m(n)^\infty \supseteq \langle g_1, ..., g_s\rangle \\ &+ (I \cap m(n)^\infty)m(n)^\infty = \langle g_1, ..., g_s\rangle + Im(n)^\infty . \end{aligned}$$

Then $I \cap m(n)^k = \langle g_1, ..., g_s\rangle + (I \cap m(n)^k)m(n)^\infty$. Since $I \cap m(n)^k$ is finitely generated, Nakayama's lemma concludes this implication.

(2) $\Rightarrow$ (3) it is obvious.

(3) $\Rightarrow$ (1) from equality (1.2) we have that $I \cap m(n)^k = \langle g_1, ..., g_s\rangle$ □

Lemma 15 *Let $\{p_j\}$ monomials of degree α of the form $x_1^{i_1}...x_n^{i_n}$ with $0 \leq i_j \leq 1$ for all j. If I is the ideal of $E(n)$ generated by these monomials, we have that*

(1) $I = \hat{I}$

(2) $I \cap m(n)^m = m(n)^{m-\alpha}\ I \ \forall m \geq \alpha$.

Theorem 16 *Let f be a finitely determinated germ on the right, then f is finitely determinated relative to G_s, $S = z(I)$, I as above.*

Proof. Since $m(n)^l \subseteq \langle df\rangle$, we have that $m(n)^l I \subseteq \langle df\rangle\, I$ and $m(n)^{l+\alpha} \cap I \subseteq \langle df\rangle\, I$. Moreover, $l+\alpha$ can be used for Artin-Rees, since α too. □

Theorem 17 *Let I ideal as in lemma 14. Suppose that f germ is finitely determined relative to Gs, $S = z(I)$ and that the closure of $Wi = \cap_{j=1, j\neq i} z(p_j) - z(p_i)$ contains $z(I)$. Then the germ f is finitely determined on the right.*

Proof. We have m natural such that $m(n)^m \cap I \subseteq \langle df\rangle\, I$. Let $x \in m(n)^{2(m-\alpha)}, x = yy'$ with y and y' elements of $m(n)^{m-\alpha}$.
Let i fixed, then we can write

$yp_i = p_i \sum_{j=1}^n \frac{\partial f}{\partial x_j} h_{ij} + \sum_{k\neq i} \sum_{j=1}^n p_k \frac{\partial f}{\partial x_j} h_{kj}$.

If $\phi = y - \sum_{j=1}^n \frac{\partial f}{\partial x_j} h_{ij}$, we have that ϕ is zero in the closure of $Wi = \cap_{j=1, j\neq i}^k z(p_j) - z(p_i)$ and by hypothesis ϕ is zero in $z(I)$, then ϕ belongs to $\hat{I} = I$.
Write $p_i\, y = p_i\gamma + p_i\phi$ where $\gamma = \sum_{j=1}^n \frac{\partial f}{\partial x_j} h_{ij}$, and clearly we have that $y = \gamma + \phi$, $\gamma \in \langle df\rangle$. Since $\phi \in I \Rightarrow \phi y' \in I(m(n))^{m-\alpha}$ and by the hypothesis of this theorem, $\phi y' \in I\,\langle df\rangle$. Then we get

$x = yy' = \gamma y + \phi y' \in \langle df\rangle \,.\, m(n)^{m-\alpha} + \langle df\rangle\, I$, and then $m(n)^{2(m-\alpha)} \subseteq m(n)\,\langle df\rangle$.

We conclude that f is $2(m-\alpha)+1$ determined on the right. □

For the case treated in [P], $\alpha = 1$ and we will have that $m(n)^{m-1} I = m(n)^m \cap I \subset \langle df\rangle\, I$ implies that f is $2m-1$ determined on the right. □

Definition 18 *Let f be a germ finitely determined. Suppose there exists a natural k such that $\langle df\rangle$ contains $m(n)^k$, but $\langle df\rangle\, m(n)$ does not contain $m(n)^k$. Then $l(f)$ is the smallest natural number with the above properties.*

Remark. If $f(x,y) = x^3 + xy^3$, we have that $\langle df\rangle \supseteq m(2)^5$, $\langle df\rangle\, m(2) \supseteq m(2)^5$ but $\langle df\rangle \not\supset m(2)^4$ (This will be an example where the cancellation of ideals is not true in our algebra $E(n)$.

Proposition 19 *Let f be a analytic germ finitely determined, I be the ideal generated by its partial derivaties and $A(I) = s$, then $s = l(f)$.*

Proof. If $l = l(f)$ its easy to see that $\langle df\rangle \cap m(n)^{l+r} = m(n)^r(\langle df\rangle \cap m(n)^l)$, and we get $l \geq s$.
In case that $l > s$ we have
$m(n)^l \subseteq \langle df\rangle \cap m(n)^{s+1} = m(n)(\langle df\rangle \cap m(n)^s) \subseteq m(n)\langle df\rangle$, which is a contradiction. □

In case that the hypothesis of definition 18 are not satisfied, we look for a natural number k such that $m(n)\langle df\rangle \supset m(n)^k$ but $m(n)^2\langle df\rangle \not\supset m(n)^k$, the proof will run the same but now the ideal is $m(n)\langle df\rangle$. Continuing the process up to infinity, we will have that for a fixed k_0 $m(n)^s\langle df\rangle \supseteq m(n)^{k_0}$ for every natural number s, but this impossible, then there exist l and k such that $m(n)^l\langle df\rangle \supset m(n)^k$ and $m(n)^{l+1}\langle df\rangle \not\supset m(n)^k$.

2 Ideals of Lojasiewicz

Definition 20 *Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ a C^∞ function, X closed subset of $\mathbb{R}^n$. We say that f satisfies the Lojasiewicz inequality in X if for every compact subset K of Ω, there exist constants $C > 0$ and $\alpha \geq 0$ depending on K, such that:*
$|f(x)| \geq Cd(x, X)^\alpha \;\forall\, x \in K$.
An ideal I in the space of de functions $C^\infty(\Omega, \mathbb{R})$ will be a Lojasiewicz ideal, if for X, the set of common zeroes of I, there exists f, in the ideal I, satisfying the Lojasiewics inequality for X.

Notation. If I is an ideal of $C^\infty(\Omega, \mathbb{R})$, then $J_k(I)$ is the ideal generated by I and by the $k \times k$ minors of the matriz $(\frac{\partial f_i}{\partial x_j})$ with $1 \leq i \leq s$, $1 \leq j \leq n$ and $f_1, ..., f_s$ elements of the ideal I.

Proposition 21 (Tougeron) *Let I the ideal generated by $\varphi_1, ..., \varphi_p$, suppose that $J_p(I)$ is an ideal of Lojasiewicz, then we have that:*

1) I is also an ideal of Lojasiewicz.

2) If f is flat in $z(J_p(I))$, and f vanishes at $z(I)$, then f belongs to I.

Example 3 Let $I = \langle x^2 + y^2\rangle$, we have that $J_1(I) = \langle x, y\rangle$ and $z(J_1(I)) = \{(0,0)\} = z(I)$. Then the ideal I contains the flat functions at the origin, any of them factorizes by $x^2 + y^2$. (Compare with theorem 2).

Corollary 22 *If I is the ideal generated by $\varphi_1, ..., \varphi_p$, and the ideal $J_p(\varphi_1, ..., \varphi_p)$ is of Lojasiewicz with common zeroes the origin, then we have:*

1) $m(n)^\infty \cap \hat{I} = m(n)^\infty \hat{I} = m(n)^\infty I = m(n)^\infty \cap I$

2) $\hat{I}$ is finitely generated.

Proof.

1) Since $m(n)^\infty \cap \hat{I} \subset I$ we have that
$m(n)^\infty \cap \hat{I} = m(n)^\infty \hat{I} = m(n)^\infty\ (m(n)^\infty \hat{I}\ ) = m(n)^\infty\ (m(n)^\infty \cap \hat{I}) \subseteq m(n)^\infty I \subseteq m(n)^\infty \cap I \subseteq m(n)^\infty \cap \hat{I}$, then $m(n)^\infty \cap \hat{I} = m(n)^\infty \hat{I} = m(n)^\infty I = m(n)^\infty \cap I$.

2) If $I = \langle \varphi_1, ..., \varphi_p \rangle$, as in the observation following theorem 13, we will have that:
$\hat{I} = \langle g_1, ..., g_s \rangle + m(n)^\infty \cap \hat{I} = \langle g_1, ..., g_s, \varphi_1, ..., \varphi_p \rangle + m(n)^\infty \cap \hat{I}$
but $m(n)^\infty \cap \hat{I} \subset I$, then $\hat{I} = \langle g_1, ..., g_s, \varphi_1, ..., \varphi_p \rangle$

□

Corollary 23 *Let I be the ideal generated by $\varphi_1, ..., \varphi_p$ analitic germs and suppose that $J_p(I)$ is an ideal of Lojasiewicz with common zeroes the origin. Then if $k = A(\pi(I))$, we have that*

$$I \cap m(n)^m = m(n)^{m-k}(I \cap m(n)^k)\ \forall\, m \geq k\,. \tag{1.3}$$

Proof. From (1.1), we get
$I \cap m(n)^m = m(n)^{m-k}(I \cap m(n)^k) + I \cap m(n)^\infty$, but from corollary 22
$I \cap m(n)^\infty = Im(n)^\infty = (Im(n)^\infty)m(n)^\infty \subseteq (I \cap m(n)^k)m(n)^{m-k}$.
Finally we get $I \cap m(n)^m = m(n)^{m-k}(I \cap m(n)^k)\ \forall\, m \geq k$ □

Remark. We got equallity (1.3) without requiring that the ideal I being radical.

Corollary 24 *Let f be an analytic germ finitely determined, then the conditions of the above corollary are satisfied and if $I = \langle f \rangle$, we have that $\hat{I}$ is finitely generated.*

Proof. Since f is finitely determined on the right, we have that $m(n)^l \subset \langle df \rangle$ and there exists k with $(x_1^2 + ... + x_n^2\ )^k \in \langle df \rangle$. From here, $z(J_p(f)) = \{0\}$ and $J_1(f)$ is an ideal of Lojasiewics. □

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