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REGILENE D. S. OLIVEIRA
YULIN ZHAO

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ON THE STRUCTURAL STABILITY OF PLANAR QUASI-HOMOGENEOUS POLYNOMIAL VECTOR FIELDS

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ABSTRACT. Let H_{pqm} be the space of planar (p,q)- quasi-homogeneous polynomial vector fields of weight degree m endowed with the coefficient topology. In this paper we characterize the set Ω_{pqm} of the vector fields of H_{pqm} that are structurally stable with respect to perturbations in H_{pqm} and determine the exact number of the topological equivalence classes in Ω_{pqm}. We apply it to give an extension of Hartman-Grobmann Theorem for a kind of planar vector fields.

1. INTRODUCTION AND THE MAIN RESULTS

A function $\mathcal{F}(x,y)$ is called a (p,q)- *quasi-homogeneous function of weight degree* m if $\mathcal{F}(\lambda^p x, \lambda^q y) = \lambda^m \mathcal{F}(x,y)$ for all $\lambda \in \mathbb{R}$. If $P(x,y)$ and $Q(x,y)$ are (p,q)- quasi-homogeneous polynomials of weight degrees $p-1+m$ and $q-1+m$, respectively, we say $X=(P,Q)$ a *planar* (p,q)*-quasi-homogeneous polynomial vector field of weight degree* m. The system of differential equations associated to X is

$$\frac{dx}{dt} = P(x,y) = \sum_{pi+qj=p-1+m} a_{ij}x^i y^j, \quad \frac{dy}{dt} = Q(x,y) = \sum_{pi+qj=q-1+m} b_{ij}x^i y^j. \tag{1}$$

In this paper we always suppose that p, q, m are positive integers and that $P(x,y)$ and $Q(x,y)$ are coprime in the ring $\mathbb{R}[x,y]$, denoted by $(P(x,y),Q(x,y)) = 1$ for short.

In order to study the behavior of the trajectories of a planar differential system near infinity we use the Poincaré-Lyapunov compactification, see for instance [4]. In the Poincaré-Lyapunov compactification we prefer to work on a hemisphere, calling it as Poincaré-Lyapunov disk. The singular points at infinity are spread out along the equator of the sphere.

The extended vector field on Poincaré-Lyapunov sphere is called *the Poincaré-Lyapunov compactification of the vector field* X, denoted by $E(X)$.

Two vector field $X, Y \in H_{pqm}$ are said *topologically equivalent* if there exists a homeomorphism h such that the orbits of the flow induced by $E(X)$ are carried onto orbits of the flow induced by $E(Y)$, preserving sense but not necessarily

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parametrization and as usual regarding X is *structurally stable* with respect to perturbations in H_{pqm} if it is topologically equivalent to every nearby vector field.

Note that a $(1,1)$-quasi-homogeneous polynomial system of weight degree m is a homogeneous polynomial differential systems. Llibre, Perez, and Rodriguez [11] studied the structurally stability of planar homogeneous polynomial vector field in [10]. Some results on planar homogeneous polynomial vector fields can be found in [2, 3] and reference therein. Structurally stability for some vector fields has been investigated by Andronov and Pontrjagin[1], Peixoto[12], Kotus *et al.* [6], Jarque and Libre [7], Llibre, Perez, and Rodriguez [11], *et al.*

Let H_{pqm} be the set of (p,q)-quasi-homogeneous polynomial vector fields of weight degree m. Denote by a and b the number of the non-negative integer solutions of the equations $pi+qj=p-1+m$ and $pi+qj=q-1+m$ respectively. For given p, q, and m, every $X \in H_{pqm}$ is specified in some unique way by the $a+b$ coefficients of $P(x,y)$ and $Q(x,y)$, and hence it may be identified with a unique point in $\mathbb{R}^{a+b}$. Similarly as in the paper [10], we take in H_{pqm} the topology induced by the Euclidean norm of $\mathbb{R}^{a+b}$. Then we say that $X \in H_{pqm}$ is *structurally stable* with respect to perturbations in H_{pqm} if there exists a neighborhood U of X in H_{pqm} such that for all $Y \in U$ the vector fields X and Y are topologically equivalent.

Denoted by Ω_{pqm} the set the vector fields in H_{pqm} which are structurally stable with respect to perturbations in H_{pqm}. In this paper we characterize the vector fields in Ω_{pqm} and and determine the exact number of the topological equivalence classes in Ω_{pqm}. By using this characterization we give an extension of Hartman-Grobman Theorem for a kind of planar vector fields.

To simplify the proof of this paper, firstly we give the following lemma before stating the main results.

Lemma 1. *Suppose $(p,q)=k\geq 2$ in (1), then there exists p', q' and m' such that system (1) is a (p',q')–quasi-homogeneous vector field of weight degree m', provided that p' and q' are coprime.*

We shall prove Lemma 1 in Section 2. Let

$$\eta(x,y)=pxQ(x,y)-qyP(x,y). \tag{2}$$

It is proved in Lemma 9 (see Section 2 below) that, if $\eta(x,y)\equiv 0$, then $(P,Q)=(px,qy)$. This degenerate case will not be considered in the present paper. We give the following conventions.

Convention 2. *For proof's convenience, throughout this paper suppose that*

(a) *p is odd and $(p,q)=1$ unless the opposite is claimed, and*
(b) *$\eta(x,y)\not\equiv 0$.*

In Section 2 we shall study the phase portrait of the vector fields in H_{pqm} and give the following result.

Theorem 3. *Let $X \in H_{pqm}$. If $\eta(1,y)$ has no zero, $\eta(0,1)\neq 0$, then $\eta(x,y)\neq 0$ for $(x,y)\neq(0,0)$, and the origin of system (1) is*

(a) *a global center if and only if* $I_X = 0$, *where*

$$I_X = \int_{-\infty}^{+\infty} \frac{P(1,u)}{\eta(1,u)} du. \tag{3}$$

(b) *a global stable (respectively, unstable) focus if and only if* $I_X < 0$ *(respectively,* $I_X > 0$*).*

In the paper [9], the authors gave the formula of I_X by the integral of (p,q)-trigonometric functions. Here we give the expression I_X by the integral of the rational function $P(x,y)/\eta(x,y)$.

We will show in Section 2 that if either $\eta(1,y) = 0$ has a real zero, or $\eta(0,1) = 0$, then system (1) has an invariant curve. Therefore, Theorem 3 implies that the origin is a center if and only if $\eta(x,y) \neq 0,\ I_X = 0$, if and only if $\eta(1,y) \neq 0,\ \eta(0,1) \neq 0,\ I_X = 0$.

The structurally stable set Ω_{pqm} is characterized by the following theorem,which will be proved in Section 3.

Theorem 4. *Denoted by* Ω_{pqm} *the set the vector fields in* H_{pqm} *which are structurally stable with respect to perturbations in* H_{pqm}. *The vector field* $X \in \Omega_{pqm}$ *if and only if one of the following conditions holds:*

(a) *If* $\eta(1,y)$ *has no zero and* $\eta(0,1) \neq 0$ *then* $I_X \neq 0$;
(b) *If the condition in (a) does not hold, then all the zeros of* $\eta(1,y)$ *are simple if there exist, and* $\partial\eta(0,1)/\partial x \neq 0$ *if* $\eta(0,1) = 0$.

To compute the number of topological equivalence classes in Ω_{pqm}, first of all we give the normal form of the structural stable vector field $X = (P,Q)$ by the following proposition.

Proposition 5. *Let* $X \in \Omega_{pqm}$. *Suppose that* $\eta(x,y)$ *is defined in (2), then* $\eta(x,y)$ *is a* (p,q)*- quasi-homogeneous polynomial of weight degree* $p+q+m-1$ *having the form*

$$\eta(x,y) = \sum_{pi+qj=p+q+m-1} c_{ij}x^i y^j. \tag{4}$$

Moreover, there exist an integer r *such that*

(a) *if* $\eta(0,1) \neq 0,\ \eta(1,0) \neq 0$ *then* $p+q+m-1 = (r+1)pq$ *with* $r \geq 0$ *and*

$$\begin{aligned} \eta(x,y) &= \textstyle\sum_{l=0}^{r+1} c_{lq,(r+1-l)p}(x^q)^l(y^p)^{r+1-l}, \\ P(x,y) &= \textstyle\sum_{l=0}^{r} a_{lq,(r+1-l)p-1}x^{lq}y^{(r+1-l)p-1}, \\ Q(x,y) &= \textstyle\sum_{l=0}^{r} b_{(r+1-l)q-1,lp}x^{(r+1-l)q-1}y^{lp}, \end{aligned} \tag{5}$$

with $a_{0,(r+1)p-1} \neq 0,\ b_{(r+1)q-1,0} \neq 0,\ c_{0,(r+1)p} \neq 0,\ c_{(r+1)q,0} \neq 0;$

(b) *if* $\eta(0,1)\neq 0,\ \eta(1,0)=0,$ *then* $p+m-1=rpq$ *with* $r\geq 0$ *and*

$$\begin{aligned}\eta(x,y) &= \left(\sum_{l=0}^{r} c_{lq,(r-l)p+1}(x^q)^l(y^p)^{r-l}\right)y,\\ P(x,y) &= \sum_{l=0}^{r} a_{lq,(r-l)p}(x^q)^l(y^p)^{r-l},\\ Q(x,y) &= \sum_{l=0}^{r} b_{(r-l)q-1,lp+1}x^{(r-l)q-1}y^{lp+1},\end{aligned}\tag{6}$$

with $a_{0,rp}\neq 0,\ b_{rq-1,1}\neq 0,\ c_{0,rp+1}\neq 0,\ c_{rq,1}\neq 0;$

(c) *if* $\eta(0,1)=0,\ \eta(1,0)\neq 0,$ *then* $q+m-1=rpq$ *with* $r\geq 0$ *and*

$$\begin{aligned}\eta(x,y) &= x\left(\sum_{l=0}^{r} c_{1+lq,(r-l)p}(x^q)^l(y^p)^{r-l}\right),\\ P(x,y) &= x\left(\sum_{l=0}^{r-1} a_{1+lq,(r-l)p-1}x^{lq}y^{(r-l)p-1}\right),\\ Q(x,y) &= \sum_{l=0}^{r} b_{(r-l)q,lp}(x^q)^{r-l}(y^p)^l,\end{aligned}\tag{7}$$

with $a_{1,rp-1}\neq 0,\ b_{rq,0}\neq 0,\ c_{1,rp}\neq 0,\ c_{1+rq,0}\neq 0;$

(d) *if* $\eta(0,1)=0,\ \eta(1,0)=0,$ *then* $m-1=(r-1)pq$ *with* $r\geq 1$ *and*

$$\begin{aligned}\eta(x,y) &= xy\left(\sum_{l=0}^{r-1} c_{1+lq,(r-1-l)p+1}(x^q)^l(y^p)^{r-1-l}\right),\\ P(x,y) &= x\left(\sum_{l=0}^{r-1} a_{1+lq,(r-1-l)p}(x^q)^l(y^p)^{r-1-l}\right),\\ Q(x,y) &= y\left(\sum_{l=0}^{r-1} b_{(r-1-l)q,1+lq}(x^q)^{r-1-l}(y^p)^l\right),\end{aligned}\tag{8}$$

with $a_{1,(r-1)p}\neq 0,\ b_{(r-1)q,1}\neq 0\ c_{1,(r-1)p}\neq 0,\ c_{1+(r-1)q,1}\neq 0.$

Proposition 5 shows that, if $X\in\Omega_{pqm}$, then there exists a non-negative integer r such that one of the following equation holds:

$$\begin{array}{ll} p+q+m-1=(r+1)pq, & p+m-1=rpq,\\ q+m-1=rpq, & m-1=(r-1)pq.\end{array}\tag{9}$$

Denoted by $\mathcal{E}_i,\ i=1,2,3,4,$ the i-th equation of (9). Let

$$\Theta_i=\{(p,q,m)|\text{there exist } r,\ \text{such that } \mathcal{E}_i \text{ holds}\}.\tag{10}$$

We classified (p,q,m) by $\Theta_i,\ i=1,2,3,4.$

Here we give some remarks about Θ_i. Let r_i be solution of $\mathcal{E}_i$. By simple calculation one gets that, if $(p,q,m)\in\Theta_i\cap\Theta_j,\ i\neq j,$ then $r_i=r_j$ and p,q,m have some restrictions. It is easy to prove that, if (p,q,m) satisfies three equations of (9), then $p=q=1$, which means that system (1) is a homogeneous polynomial vector fields. Since $\Theta_i\cap\Theta_j\subset\Theta_i$, we only need to consider Θ_i when compute the number of topological equivalence classes in Ω_{pqm}.

Let C_{pqm} be the number of topological equivalence classes in Ω_{pqm}. The value C_{pqm} is given by the following theorem.

Theorem 6. *Let* r *be an integer, defined in Proposition 5.*

(a) *Suppose that both p and q is odd. If r is odd, then*

$$C_{pqm} = \begin{cases} 1 + \frac{1}{2}\sum_{l=1}^{(r+1)/2}\sum_{n|l}(\mathcal{P}_{2n} + I_{2n}), & \text{if } (p,q,m) \in \Theta_1 \backslash (\Theta_2 \cup \Theta_3 \cup \Theta_4), \\ -1 + \frac{1}{2}\sum_{l=1}^{(r+1)/2}\sum_{n|l}(\mathcal{P}_{2n} + I_{2n}), & \text{if } (p,q,m) \in \Theta_2 \cup \Theta_3 \cup \Theta_4; \end{cases}$$

If r is even, then

$$C_{pqm} = -1 + \frac{1}{2}\sum_{l=1}^{r/2}\sum_{n|2l+1}(\mathcal{P}_{2n} + I_{2n}),$$

where

$$\mathcal{P}_{2n} = \frac{1}{n}\left(2^{2n} - \sum_{l|n, l\neq n} l\mathcal{P}_{2l}\right), \quad I_{2n} = 2^{n+1} - \sum_{l|n,l\neq n} I_{2l},$$

if r is odd, and

$$\mathcal{P}_{2n} = \frac{1}{n}\left(2^{n} - \sum_{l|n, l\neq n} l\mathcal{P}_{2l}\right), \quad I_{2n} = 2^{(n+1)/2} - \sum_{l|n,l\neq n} I_{2l}, \tag{11}$$

if r is even, respectively.

(b) *Suppose that p is odd and q is even.*

(i) *If $(p,q,m) \in \Theta_1 \backslash (\Theta_2 \cup \Theta_3 \cup \Theta_4)$, then*

$$C_{pqm} = \begin{cases} \frac{1}{2}\left(r + 3 + \sum_{l=1}^{(r+1)/2}\sum_{n|l}\mathcal{P}_{2n}\right), & \text{if } r \text{ is odd}, \\ \frac{1}{2}\left(r + 1 + \sum_{l=1}^{r/2}\sum_{n|2l+1}\mathcal{P}_{2n}\right), & \text{if } r \text{ is even}; \end{cases}$$

if $(p,q,m) \in \Theta_2$, then

$$C_{pqm} = \begin{cases} \frac{1}{2}\left(r - 1 + \sum_{l=1}^{(r+1)/2}\sum_{n|l}\mathcal{P}_{2n}\right), & \text{if } r \text{ is odd}, \\ \frac{1}{2}\left(r + 1 + \sum_{l=1}^{r/2}\sum_{n|2l+1}\mathcal{P}_{2n}\right), & \text{if } r \text{ is even}; \end{cases}$$

where $\mathcal{P}_{2n}$ is defined as in (11).

(ii) *If $(p,q,m) \in \Theta_3 \cup \Theta_4$, then*

$$C_{pqm} = \begin{cases} r + \frac{1}{2}\left(\sum_{l=1}^{(r+1)/2}\sum_{n|l}\mathcal{P}_{2n}\right), & \text{if } r \text{ is odd}, \\ \frac{1}{2}\left(r - 2 + \sum_{l=1}^{r/2}\sum_{n|2l+1}\mathcal{P}_{2n}\right), & \text{if } r \text{ is even}, \end{cases}$$

where

$$\mathcal{P}_{2n} = \begin{cases} \frac{1}{n}\left(2^{n+1} - \sum_{l|n,l\neq n} l\mathcal{P}_{2l}\right), & \text{if } n \neq k, \\ \frac{1}{n}\left(2^{n} - \sum_{l|n,l\neq n} l\mathcal{P}_{2l}\right), & \text{if } n = k, \end{cases}$$

with $k \neq 0$.

We say that two analytic vector fields X and Y are *locally topologically equivalent at origin (resp. infinity)* if there exists two neighborhoods U and V of the origin (resp. the infinity) and a homeomorphism $h: U \to V$ that carries orbits of the flow induced by X onto orbits of the flow induced by Y, preserving sense but not necessarily parametrization (see for instance [10]).

Let $X = \sum_{i\geq m} X_i$, where X_i, $i = m, m+1, \cdots$, is an (p,q)-quasi-homogeneous polynomial vector field of weighted degree i. The next theorem shows that X and X_m are locally topologically equivalent at the origin, provided $X_m \in \Omega_{pqm}$. This is an extension of Hartman Grobman Theorem.

Theorem 7. *Let Ω_{pqm} be the vector fields in H_{pqm} that are structurally stable with respect to perturbations in H_{pqm} and $X_m = (P_m(x,y), Q_m(x,y)) \in \Omega_{pqm}$. Then X_m and $X = (\sum_{i\geq m} P_i(x,y), \sum_{i\geq m} Q_i(x,y))$ are locally topologically equivalent at the origin, where $P_i(x,y)$ and $Q_i(x,y)$ are (p,q)-quasi-homogeneous polynomials of weight degree $p-1+i$ and $q-1+i$, respectively.*

Now we can get the analogous of Theorem 7 at infinity.

Theorem 8. *If $X_m = (P_m(x,y), Q_m(x,y)) \in \Omega_{pqm}$, then the phase portraits of X_m and $\hat{X} = (\sum_{i=0}^{m} P_i(x,y), \sum_{i=0}^{m} Q_i(x,y))$ are locally topologically equivalent at infinity, where $P_i(x,y)$ and $Q_i(x,y)$ are are (p,q)-quasi-homogeneous polynomials of weight degree $p-1+i$ and $q-1+i$, respectively.*

The rest of this paper is organized as follows. In Section 2 we study the phase portraits of (p,q)-quasi-homogeneous polynomials of degree m on Poincaré-Lyapunov disk. In Section 3 we prove Theorem 4 that characterizes the vector fields $X \in \Omega_m$. In Section 4 we compute the number C_{pqm} of topological equivalence classes of the structurally stable vector fields in H_{pqm} with respect to perturbation in H_m. Finally we prove Theorem 7 and Theorem 8 in Section 5.

2. Phase portraits of Quasi-homogeneous Vector Fields

In this section we shall study the phase portraits of quasi-homogeneous vector fields. Firstly we prove some preliminary results.

Proof of Lemma 1. Let

$$P(x,y) = \sum_{pi+qj=p-1+m} a_{ij}x^i y^j, \quad Q(x,y) = \sum_{pi'+qj'=q-1+m} b_{i'j'}x^{i'} y^{j'}$$

in (1) and $p = kp_1$, $q = kq_1$ with $(p_1, q_1) = 1$. It follows from $P(\lambda^p x, \lambda^q y) = \lambda^{p-1+m}P(x,y)$ and $Q(\lambda^p x, \lambda^q y) = \lambda^{q-1+m}Q(x,y)$ that $kp_1 - 1 + m = kp_1 i + kq_1 j$ and $kq_1 - 1 + m = kp_1 i' + kq_1 j'$, which gives $m - 1 = k(p_1(i-1) + q_1 j) = k(p_1 i' + q_1(j'-1))$. Let $m_1 = p_1(i-1) + q_1 j + 1 = p_1 i' + q_1(j'-1) + 1$. Then $P(x,y)$ and $Q(x,y)$ are also (p_1,q_1)-quasi-homogeneous polynomials of weight degree $p_1 - 1 + m_1$ and $q_1 - 1 + m_1$ respectively. The assertion follows. $\square$

Lemma 9. *Suppose that $\eta(x,y)$ is defined as (2).*

(a) *$\eta(x,y)$ is a (p,q)-quasi-homogeneous polynomial of weight degree $p+q+m-1$.*
(b) *If $\eta(x,y)\equiv 0$, then $(P,Q)=(px,qy)$.*

Proof. It is easy to verify that $\eta(x,y)$ is a (p,q)-quasi-homogeneous polynomial of weight degree $p+q+m-1$. If $\eta(x,y)\equiv 0$, then $pxQ(x,y)=qyP(x,y)$. Since $P(x,y)$ and $Q(x,y)$ are coprime, one gets $P(x,y)|x$, $Q(x,y)|y$. This finishes proof.

□

Proposition 10. *Let $X\in H_{pqm}$.*
(a) *If $\eta(0,1)=0$, then $x=0$ is invariant under the flow induced by $E(X)$;*
(b) *If $\eta(1,\lambda)=0$, $\lambda\in\mathbb{R}$, then $y^p-\lambda^p x^q=0$ is invariant under the flow induced by $E(X)$;*
(c) *$E(X)$ has no periodic orbit if one of the conditions in (a) and (b) holds.*

Proof. If $\eta(0,1)=0$, then $P(0,1)=0$, which implies $y^{p-1+m}P(0,1)=P(0,y^q)=0$. Hence $x=0$ is an invariant line of system (1). This proves (a)

If $\lambda=0$, then $\eta(1,0)=0$ implies $Q(1,0)=0$. Since $x^{q-1+m}Q(1,0)=Q(x^p,0)\equiv 0$, $y=0$ is an invariant line of system (1).

Suppose $\lambda\neq 0$. Recall that we always assume that p is odd. If $x\neq 0$, then it follows from Lemma 9(a) that

$$\eta(x,y)=(x^{1/p})^{p+q-1+m}\eta\left(1,\frac{y}{x^{q/p}}\right), \tag{12}$$

where $x^{q/p}:=(x^{1/p})^q$. Let $V(x,y)=y^p-\lambda^p x^q$. Then

$$\begin{aligned}\left.\frac{dV}{dt}\right|_{y=\lambda x^{q/p}} &= p\lambda^{p-1}x^{q(p-1)/p}Q(x,\lambda x^{q/p})-q\lambda^p x^{q-1}P(x,\lambda x^{q/p})\\ &= \lambda^{p-1}(x^{1/p})^{pq+m-1}(pQ(1,\lambda)-q\lambda P(1,\lambda))\\ &= \lambda^{p-1}(x^{1/p})^{pq+m-1}\eta(1,\lambda)=0,\end{aligned}$$

which proves (b).

The paper [9] has proved that system (1) has a unique singular point $(0,0)$. If there exists an invariant curve passing through the unique singular point of system (1), then no limit cycle can surround the origin. The assertion (c) is proved. □

To study the singular point $(0,0)$ of system (1), we introduce the (p,q)-*trigonometric functions $z(\phi)=\mathrm{Cs}\phi$ and $\omega(\phi)=\mathrm{Sn}\phi$* [8, 9] as the solution of the following initial problem

$$\dot z=-\omega^{2p-1},\ \dot\omega=z^{2q-1},\ z(0)=p^{-\frac{1}{2q}},\ \omega(0)=0.$$

It is know that $\mathrm{Cs}\phi$ and $\mathrm{Sn}\phi$ are $\mathcal{T}$-periodic functions with

$$\mathcal{T}=2p^{-\frac{1}{2q}}q^{-\frac{1}{2p}}\frac{\Gamma(\frac{1}{2p})\Gamma(\frac{1}{2q})}{\Gamma(\frac{1}{2p}+\frac{1}{2q})}.$$

and satisfies

$$p\mathrm{C}s^{2q}\phi + q\mathrm{S}n^{2p}\phi = 1, \quad \frac{d\mathrm{C}s\phi}{d\phi} = -\mathrm{S}n^{2p-1}\phi, \quad \frac{d\mathrm{S}n\phi}{d\phi} = \mathrm{C}s^{2q-1}\phi.$$

For $(p,q) = (1,1)$, we have that $\mathrm{C}s\phi = \cos\phi$, $\mathrm{S}n\phi = \sin\phi$, i.e. the $(1,1)$-trigonometric functions are the classical ones.

In the (p,q)-*polar coordinates* (r,ϕ)

$$x = r^p \mathrm{C}s\phi, \quad y = r^q \mathrm{S}n\phi, \tag{13}$$

the planar (p,q)-quasi-homogeneous system (1) of weight degree m becomes [9]

$$\dot r = r^m F(\phi), \quad \dot\phi = r^{m-1} G(\phi),$$

where

$$F(\phi) = \xi\left(\mathrm{C}s\phi, \mathrm{S}n\phi\right), \quad G(\phi) = \eta\left(\mathrm{C}s\phi, \mathrm{S}n\phi\right),$$

with

$$\xi(x,y) = x^{2q-1}P(x,y) + y^{2p-1}Q(x,y). \tag{14}$$

Taking the changes

$$\frac{ds}{dt} = r^{m-1}, \quad \theta = \frac{2\pi\phi}{\mathcal{T}}, \tag{15}$$

the above system goes over to

$$r' = rf(\theta), \quad \theta' = g(\theta). \tag{16}$$

where prime denotes derivative with respect to s,

$$f(\theta) = F\left(\frac{\mathcal{T}\theta}{2\pi}\right), \quad g(\theta) = \frac{2\pi}{\mathcal{T}} G\left(\frac{\mathcal{T}\theta}{2\pi}\right). \tag{17}$$

It is easy to check that $f(\theta)$ and $g(\theta)$ are 2π-periodic functions.

Proof of Theorem 3. Firstly we are going to prove $\eta(x,y) \neq 0$, which implies that $g(\theta) \neq 0$.

Recall that we always suppose that p is odd. It follows from (12) that $\eta(x,y) \neq 0$ for $x \neq 0$. Suppose that there exists $y^* \neq 0$ such that $0 = \eta(0,y^*) = -qy^*P(0,y^*)$, then

$$0 = P(0,y^*) = (|y^*|^{1/q})^{p-1+m} P(0, \mathrm{sgn}(y^*)).$$

Hence $P(0,\mathrm{sgn}(y^*)) = 0$. If $y^* > 0$, then $P(0,1) = 0$. If $y^* < 0$ and $y < 0$, then

$$P(0,y) = (|y|^{1/q})^{p-1+m} P(0,-1) = 0.$$

Since $P(0,y)$ is polynomial of y, $P(0,y)$ is also identically equals to zero for $y > 0$, which also gives $P(0,1) = 0$. We note that $\eta(0,1) = qP(0,1) = 0$. This contradicts the assumption. Therefore, $\eta(0,y) \neq 0$ for $y \neq 0$. Summing the above discussion one gets $\eta(x,y) \neq 0$ for $(x,y) \neq (0,0)$.

Secondly we prove the assertions (a) and (b). If $\eta(x,y) \neq 0$, then $g(\theta) \neq 0$ (see (17)). It follows from (16) that

$$\frac{dr}{d\theta} = \frac{rf(\theta)}{g(\theta)},$$

which implies that the return map is given by

$$r(2\pi) = \begin{cases} r(0)e^{\tilde{I}_X}, & \text{if } f(\theta) \not\equiv 0, \\ r(0), & \text{if } f(\theta) \equiv 0. \end{cases}$$

where

$$\tilde{I}_X = \int_0^{2\pi} \frac{f(\theta)}{g(\theta)} d\theta,$$

and $(r(0), 0)$ is the point of the positive x-axis. Hence we deduce from the return map that, if $\tilde{I}_X = 0$, then the origin is a center, and if $\tilde{I}_X < 0$ (resp.$\tilde{I}_X > 0$) , then it is a stable (resp. unstable) focus.

To end the proof we must show that the sign of I_X is equal to the sign $\tilde{I}_X$.

Let $x = \mathrm{Cs}\phi$, $y = \mathrm{Sn}\phi$. Then we have

$$\tilde{I}_X = \frac{2\pi}{T}\int_0^T \frac{F(\phi)}{G(\phi)} d\phi = \frac{2\pi}{T}\oint_{px^{2q}+qy^{2p}=1} \frac{\xi(x,y)}{x^{2q-1}\eta(x,y)} dy = \frac{2\pi}{T}(\tilde{I}_+ - \tilde{I}_-),$$

where

$$\tilde{I}_\pm = \int_{-q^{-1/(2p)}}^{q^{-1/(2p)}} \frac{\xi(x,y)}{x^{2q-1}\eta(x,y)}\bigg|_{x=x_\pm=\pm p^{-1/(2q)}(1-qy^{2p})^{1/(2q)}} dy.$$

If p is odd and $x \neq 0$, then

$$\xi(x,y) = (x^{1/p})^{2pq-1+m}\xi\left(1, \frac{y}{x^{q/p}}\right), \tag{18}$$

where $\xi(x, y)$ is defined in (14). It follows from (12) and (18) that

$$\tilde{I}_+ = \int_{-q^{-1/(2p)}}^{q^{-1/(2p)}} \frac{\xi(1, \frac{y}{x^{q/p}})}{x^{q/p}\eta(1, y/x^{q/p})}\bigg|_{x=x_+} dy.$$

Let $x = x_\pm$ and $u = y/x^{q/p}$. Then $u^{2p} = y^{2p}/x^{2q}$, which gives $y^{2p} = u^{2p}/(p+qu^{2p})$. Hence

$$\begin{aligned} \tilde{I}_+ &= \int_{-\infty}^{\infty} \frac{\xi(1,u)}{x^{q/p}\eta(1,u)}\bigg|_{x=x_+} \frac{1}{2py^{2p-1}} \frac{d}{du}\left(\frac{u^{2p}}{p+qu^{2p}}\right) du \\ &= \int_{-\infty}^{\infty} \frac{p\xi(1,u)}{(p+qu^{2p})\eta(1,u)} du. \end{aligned}$$

If p is odd, then (12) and (18) also hold for $x = x_-$. Using the same arguments as above, we get $\tilde{I}_- = -\tilde{I}_+$ if p is odd, and hence $\tilde{I}_X = 2\tilde{I}_+$.

Since

$$\int_{-\infty}^{\infty} \frac{p\xi(1,u)}{(p+qu^{2p})\eta(1,u)} du - \int_{-\infty}^{\infty} \frac{P(1,u)}{\eta(1,u)} du = \int_{-\infty}^{\infty} \frac{u^{2p-1}}{p+qu^{2p}} du = 0,$$

one obtains the assertions (a) and (b), provided p is odd. The proof is finished. □

By (p,q)-polar coordinates changes one gets the vector field (16) which is defined on $\mathbb{S}^1 \times \mathbb{R}$. The origin of system (1) is blown up to to a circle of system (16). Although the cylinder $\mathbb{S}^1 \times \mathbb{R}$ is good surface for getting the phase portrait near the origin, it is often less appropriate for making calculations, since we have to deal with expressions of (p,q)-trigonometric functions. Hence we prefer to make the calculations in different charts.

We are going to use the method of quasi-homogeneous blow-up to study the singular point $(0,0)$ of system (1). We first blow up the vector field in the positive x-direction by

$$x=\bar{x}^p,\ \ y=\bar{x}^q\bar{y},\ \ d\tau=\frac{\bar{x}^{m-1}dt}{p},\ \ \bar{x}>0,$$

yielding

$$\dot{\bar{x}}=\bar{x}P(1,\bar{y}),\ \ \dot{\bar{y}}=\eta(1,\bar{y}). \tag{19}$$

The points $(0,\lambda)$ satisfying $\eta(1,\lambda)=0$ are the isolated singular points of (19) on the line $\{\bar{x}=0\}$. The singular point has a linear part

$$\begin{pmatrix} P(1,\lambda) & 0 \\ 0 & \frac{\partial\eta}{\partial\bar{y}}(1,\lambda) \end{pmatrix}.$$

If $P(1,\lambda)=0$, then $\eta(1,\lambda)=pQ(1,\lambda)=0$, which implies that $P(x,\lambda x^{q/p})=Q(x,\lambda x^{q/p})=0$ for $x>0$. Hence the algebraic curves $P(x,y)=0$ and $Q(x,y)=0$ have infinite intersection points. Since $P(x,y)$ and $Q(x,y)$ are coprime polynomials, it follows from Bézout theorem [5] that $P(x,y)=0$ and $Q(x,y)=0$ have finite intersection points on $\mathbb{R}^2$, which yields contradictions. Therefore, $P(1,\lambda)\neq 0$ if $\eta(1,\lambda)=0$, $\eta(1,\lambda)\not\equiv 0$. Hence all singular points are elementary.

We will come back to discuss the singular points of system (19) in this section when we study the singular points at infinity.

Next we blow up the vector field in the negative x-direction by

$$x=-\bar{x}^p,\ \ y=\bar{x}^q\bar{y},\ \ d\tau=\frac{\bar{x}^{m-1}dt}{p},\ \ \bar{x}>0,$$

which gives

$$\dot{\bar{x}}=-\bar{x}P(-1,\bar{y}),\ \ \dot{\bar{y}}=-\eta(-1,\bar{y}). \tag{20}$$

The points $(0,\lambda)$ which satisfy $\eta(-1,\lambda)=0$ are the singular points of system (20). By the same arguments as in system (19), we know that all singular points of system (20) are elementary.

Finally we have to blow up the positive and negative y-direction by

$$x=\bar{x}\bar{y}^p,\ \ y=\pm\bar{y}^q,\ \ d\tau=\frac{\bar{y}^{m-1}dt}{q},\ \ \bar{y}>0,$$

which gives

$$\bar{X}_\pm:\quad \dot{\bar{x}}=\mp\eta(\bar{x},\pm 1),\ \ \dot{\bar{y}}=\pm\bar{y}Q(\bar{x},\pm 1). \tag{21}$$

We only need to determine whether the origin is a singular point of the vector fields $\bar{X}_\pm$. Since $\eta(0,1)=0$ implies that $x=0$ is an invariant line of $E(X)$ (see Proposition 10), one gets $P(0,y)\equiv 0$, which gives $P(0,-1)=0$, and hence $\eta(0,-1)=0$. It can also be proved that $\eta(0,-1)=0$ implies $\eta(0,1)$. This yields that the origin is a singular point of system $\bar{X}_+$ if and only if it is a singular point of system $\bar{X}_-$. We also have $Q(0,\pm 1) \neq 0$ if $\eta(0,\pm 1) = 0$, and hence $(0,0)$ is elementary singular point of system $\bar{X}_\pm$ if $\eta(0,\pm 1) = 0$.

We have blown up the origin to the elementary singular points, which implies that the singular point $(0,0)$ of (1) has been desingularized. After blowing down we get the phase portrait of system (1) near the origin.

To study the singularities at infinity, we use *Poincaré-Lyapunov Compactification*, see for instance [4]. First we transfer system (1) in the positive x-direction using the transformation

$$x = \frac{1}{z^p}, \quad y = \frac{u}{z^q}, d\tau = \frac{z^{1-m}dt}{p}, \quad z > 0.$$

This yields the vector field

$$\frac{dz}{d\tau} = -zP(1,u), \quad \frac{du}{d\tau} = \eta(1,u). \tag{22}$$

The singular points $(0,\lambda)$ that satisfy $\eta(1,\lambda) = 0$ are the infinity singularity of system (1). Each one of them has, as a linear part

$$\begin{pmatrix} -P(1,\lambda) & 0 \\ 0 & \frac{\partial \eta}{\partial u}(1,\lambda) \end{pmatrix}. \tag{23}$$

Since we have proved that $P(1,\lambda) \neq 0$, all singular points of system (22) are elementary.

Second we transfer system (1) in the negative x-direction using the transformation

$$x = -\frac{1}{z^p}, \quad y = \frac{u}{z^q}, d\tau = \frac{z^{1-m}dt}{p}, \quad z > 0,$$

one gets

$$\frac{dz}{d\tau} = zP(-1,u), \quad \frac{du}{d\tau} = -\eta(-1,u). \tag{24}$$

The points $(0,\lambda)$ which satisfy $\eta(-1,\lambda) = 0$ are the singular points of system (24). Each one of them has, as a linear part

$$\begin{pmatrix} P(-1,\lambda) & 0 \\ 0 & -\frac{\partial \eta}{\partial u}(-1,\lambda) \end{pmatrix}. \tag{25}$$

These points are studied in the same way as the ones of system (22).

Finally we consider the two transformations

$$x = \frac{v}{z^p}, \quad y = \pm\frac{1}{z^q}, d\tau = \frac{z^{1-m}dt}{q}, \quad z > 0,$$

which yields two vector fields of the forms

$$\bar{X}^\infty_\pm : \quad \frac{dz}{d\tau} = \mp zQ(v,\pm 1), \quad \frac{dv}{d\tau} = \mp\eta(v,\pm 1). \tag{26}$$

For those two vector fields we only have to determine whether the point $(0,0)$ is a singular point. The origin is a singular point of X^∞_+ (resp. X^∞_-) if and only if $\eta(0,1) = 0$ (resp. $\eta(0,-1) = 0$). By the same way as in the study of system $\bar{X}_\pm$, the origin is a singular point of system $\bar{X}_+^\infty$ if and only if it is a singular point of system $\bar{X}_-^\infty$.

Lemma 11. *Suppose that p is odd.*

(a) $(0,\lambda)$ *is a singular points of system* (22) *if and only if* $(0,(-1)^q\lambda)$ *is also a singular point of system* (24). *Suppose that λ is a zero of $\eta(1,\lambda)$ with multiplicity k, then*
   (i) *If k is even, then $(0,\lambda)$ and $(0,(-1)^q\lambda)$ are saddle-node of system* (22) *and* (24) *respectively;*
   (ii) *if k is odd and $P(1,\lambda)(\partial^k\eta/\partial u^k)(1,\lambda)>0$, then $(0,\lambda)$ and $(0,(-1)^q\lambda)$ are saddles of system* (22) *and* (24) *respectively;*
   (iii) *if k is odd and $P(1,\lambda)(\partial^k\eta/\partial u^k)(1,\lambda)<0$, $P(1,\lambda)>0$ (resp. $P(1,\lambda)<0$), then $(0,\lambda)$ is a stable (resp. unstable) node of* (22), *and $(0,(-1)^q\lambda)$ is a unstable (resp. stable) node if m is even, a stable (resp. unstable) node of system* (24) *if m is odd.*

(b) *System $\bar{X}_+^\infty$ has a singular point at $(0,0)$, if and only if system $\bar{X}_-^\infty$ has also a singular point at the origin, if and only if $\eta(0,1)=0$. Moreover, suppose $\eta(0,1)=0$, then there exists positive integers k, l such that $m=(k-1)p+(l-1)q+1$, and*
   (i) *if k is even, then system $\bar{X}_+^\infty$ (resp. $\bar{X}_-^\infty$) has a saddle-node at the origin.*
   (ii) *if k is odd and $Q(0,1)(\partial^k\eta/\partial x^k)(0,1)<0$, then system $\bar{X}_+^\infty$ (resp. $\bar{X}_-^\infty$) has a saddle at the origin.*
   (iii) *if k is odd and $Q(0,1)(\partial^k\eta/\partial x^k)(0,1)>0$, $Q(0,1)>0$ (resp. $Q(0,1)<0$), then system $\bar{X}_+^\infty$ has a stable (resp. unstable) node at the origin, and $\bar{X}_-^\infty$ has a stable (resp. unstable) node at origin if l is odd, a unstable (resp. stable) node at origin if l is even, respectively.*

Proof. (a) Since p is odd, it follows from (12) that

$$\eta(-1,(-1)^q\lambda)=(-1)^{p+q-1+m}\eta(1,\lambda), \tag{27}$$

which implies that $\eta(1,\lambda)=0$ if and only if $\eta(-1,(-1)^q\lambda)=0$, and hence $(0,\lambda)$ is a singular point of system (22), if and only if $(0,(-1)^q\lambda)$ is a singular point of system (24). The equality (27) also yields

$$\frac{\partial^k\eta(-1,(-1)^q\lambda)}{\partial u^k}=(-1)^{(k-1)q+m}\frac{\partial^k\eta(1,\lambda)}{\partial u^k}. \tag{28}$$

Since $P(-1,(-1)^q\lambda)=(-1)^mP(1,\lambda)$, the assertions (i),(ii) and (iii) with $k\geq 2$ in (a) follow from Theorem 2.19 of [4],(23), (25) and (28). If $k=1$,then all singular points are hyperbolic and the assertion in (a) follows from Theorem 2.15 of the same book [4].

(b) Using the same arguments as in the case for system $\bar{X}_\pm$, we get that system $\bar{X}_+^\infty$ has a singular point at $(0,0)$, if and only if system $\bar{X}_-^\infty$ a singular point at the origin, if and only if $\eta(0,1)=0$.

Suppose that $\eta(x,y)$ has the form (4). If $\eta(0,1)=0$, then there exist k, l such that $pk+ql=p+q+m-1$, $c_{kl}\neq 0$ and $c_{ij}=0$ for $i\leq k-1$, where $k\geq 1,\ l\geq 1$. This gives $(\partial^k\eta/\partial x^k)(0,\pm 1)=k!c_{kl}(\pm 1)^l$.

If $\eta(0,1)=0$, then $P(0,1)=0$, which implies that $P(x,y)$ has a divisor x. On the other hand, $Q(0,1)=0$ also means that $Q(x,y)$ has a divisor x. Since $P(x,y)$ and $Q(x,y)$ are coprime, we have that $Q(0,1)\neq 0$ if $\eta(0,1)=0$. The other assertions in (b) follows by the same arguments as in the proof of (a). □

To compare the finite singular point $(0,0)$ and the singular points at infinity, we give the following lemma.

Lemma 12. *Suppose that λ is a zero of $\eta(1,u)$ with multiplicity k, $k\geq 1$.*

(a) *If k is even, then system* (19) *(resp.* (22)*) has a saddle-node at $(0,\lambda)$ respectively.*
(b) *If k is odd and $P(1,\lambda)(\partial^k\eta/\partial u^k)(1,\lambda)<0$ (resp. >0) , then $(0,\lambda)$ is a saddle of system* (19) *(resp.* (22)*);*
(c) *If k is odd and $P(1,\lambda)\partial^k\eta/\partial u^k)(1,\lambda)>0$ (resp.<0) then $(0,\lambda)$ is node of system* (19) *(resp.* (22)*). Moreover, it is a stable node of system* (19) *(resp.* (22)*) if $P(1,\lambda)<0$ (resp. >0) and a unstable node if $P(1,\lambda)>0$ (resp. <0) respectively.*

The singular points of system (20) *and* (24)*, $\bar{X}_\pm$ and $\bar{X}_\pm^\infty$ can be studied in the same way.*

Proof. Suppose that λ is a simple zero of $\eta(1,\lambda)$, then $P(1,\lambda)\neq 0$, $\partial\eta/\partial u)(1,\lambda)\neq 0$, which implies the linear part of the system (19) (resp. (22)) at the singular point $(0,\lambda)$ has two real eigenvalue. This yields the assertion (b) and (c) for $k=1$.

The other assertions of this lemma follows from Theorem 2.19 of [4]. □

Before the end of this part, we give some comments. If p is odd, then it follows from (27) that $(0,\lambda)$ is a singular point of system (19) (resp. (22)), if and only if, $(0,(-1)^q\lambda)$ is also a singular point of system (20) (resp. (24)). In fact, the information found in the positive x-direction (resp. z-direction) also covers the one in the negative x-direction (resp. z-direction). However, to the proof's convenience we study the behavior of both the positive and the negative x-direction (resp. z-direction) yet.

Proposition 13. (i) *All singular points at infinity and those of system* (19)*,* (20) *and $\bar{X}_\pm$ are elementary and they are nodes, saddles, or saddle-node.*

(ii) *The singular points at infinity are determined by the equation $\eta(1,\lambda)=0$ and whether $\eta(0,1)=0$ hold.*
(iii) *Suppose that L_{λ_i}, $L_{\lambda_{i+1}}$ are two consecutive invariant algebraic curve of system* (1)*, defined in Proposition 10, then they are characteristic orbits at $(0,0)$ and determine one sector at the origin:*
   (a) *two consecutive parabolic sectors at infinity give a hyperbolic sector at $(0,0)$,*

(b) *two consecutive hyperbolic sectors at infinity give a elliptic sector at* $(0,0)$, *and*
(c) *one parabolic and one hyperbolic sectors at infinity give a parabolic sector at* $(0,0)$.

(iv) *Assume that* $(0,0)$ *is a* α*-limit set or* ω*-limit set of the orbit* γ.
(a) *if* $(p,q) \neq (1,1)$, *then* γ *contacts either* x*-axis or* y*-axis at the origin;*
(b) *If* $p = q = 1$, *then* γ *contacts one of the invariant straight line, define in Proposition 10, at the origin.*

Proof. We have shown that all singular points are elementary in the above discussions. The other assertions follows from Proposition 10 and Lemma 11 and Lemma 12. □

We describe the phase portrait of system (1) on Poincaré-Lyapunov disk by Proposition 10 and Proposition 13. In fact, the phase portrait of system (1) is determined by the zeros of equations $\eta(1,\lambda) = 0$ and whether $\eta(0,1) = 0$ hold.

3. Structural Stability

In this section we prove Theorem 4 which characterize the structurally stable set Ω_{pqm}. Recall that we used the Poincaré-Lyapunov compactification to study the trajectories of system (1) and work on the Poincaré-Lyapunov sphere. The extended vector field on Poincaré-Lyapunov sphere is denoted by $E(X)$.

Proof of Theorem 4. Assume that X is structurally stable with respect to perturbations in H_{pqm}. Let us prove that (a) and (b) holds. If $\eta(1,y)$ has no zero and $\eta(0,1) \neq 0$, then it follows from Theorem 3 that the flow of the system (1) is a global center if $I_X = 0$, and a global stable or unstable focus if $I_X \neq 0$, respectively. When $I_X = 0$ there exists a vector field Y in any neighborhood of X in H_{pqm} such that the phase portrait of Y is a global focus. Therefore X is not a structural stable vector field with respect to perturbations in H_{pqm} and (a) holds.

Moreover, the origin is the unique singular point of system (1)[9], and the phase portrait of X is completely determined by the zeros of $\eta(1,y)$ and $\eta(0,1)$ (see Proposition 13, Lemma 11 and Lemma 12). If $\eta(1,y)$ has a zero or $\eta(0,1) = 0$, then it follows from Proposition 10(c) there is no periodic orbits in the set of all solutions of the system (1).

If $\eta(1,y)$ has a multiple zero, by Lemma 11 we conclude that there exist vector field Y in a neighborhood of X in H_{pqm} such that X and Y have different number of sectors. So X is not structural stable with respect to H_{pqm}. If $\eta(0,1) = 0$ and $\partial\eta(0,1)/\partial x = 0$ by Lemma 11 and 12 we can repeat the same argument, so (b) holds.

Now, we assume that (a) or (b) holds and we shall prove that X is a structural stable vector field with respect to perturbations in H_{pqm}. First we suppose that (a) holds. Then there exists a neighborhood V of y and a a neighborhood W of X in H_{pqm} such that if $y* \in V$ then $\eta(1,y*) \neq 0$ and for all $Y \in W$

$sign(I_X) = sing(I_Y)$. From Theorem 3 X and Y are topologically equivalent and X is structural stable with respect to perturbations in H_{pqm}.

Now we are assuming that there exists s points $(1, y_i)$ such that $\eta(1, y_i) = 0$ or $\eta(0,1) = 0$ and that these zeros are simples. It follows from Lemma 11 and Lemma 12 that there exist a neighborhood U of X in H_{pqm} such that if $Y \in U$ then there exists exactly s points such that $\eta(1, \lambda_i) = 0$ or $\eta(0,1) = 0$. We can choose this neighborhood such that $sign(P_X \partial\eta_X/\partial u)(1, \lambda_i) = sign(P_Y \partial\eta_Y/\partial u)(1, \lambda_i))$, and $sign(Q_X \partial\eta_X/\partial x)(0,1) = sign(P_Y(1,\lambda)\partial\eta_Y/\partial x)(0,1)$ if $\eta(0,1) = 0$. By Proposition 13 each vector field $Y \in U$ has the same local sectors as the vector field Y. So X is structurally stable with respect to perturbations in H_{pqm}. □

4. The number of topological equivalence classes in Ω_{pqm}

In this section we compute the number of topological equivalence classes in Ω_{pqm} which is the set of all structurally stable vector fields with respect to perturbation in H_{pqm}.

Proof of Proposition 5. It is easy to verify that $\eta(x, y)$ is a (p, q)-quasi-homogeneous polynomial of weight degree $p+q+m-1$ having the form (4). Then (i, j) in (4) is a solution of the equation

$$pi + qj = p + q + m - 1. \tag{29}$$

(i, j) is called a integer (resp. non-negative integer) solution of (29) if i and j are integers (resp. non-negative integers). If (i_0, j_0) is a integer solution of (29), then all the other integer solution are

$$\{(i_0 + lq,\ j_0 - lp) : l \in \mathbb{Z}\}.$$

If $\eta(0,1) \neq 0$, $\eta(1,0) \neq 0$, then the equation (29) has two integer solutions $(0, j')$ and $(i', 0)$ satisfying $qj' = p+q+m-1$ and $pi' = p+q+m-1$ respectively. Since $(p, q) = 1$, we have $pq|(p+q+m-1)$. Hence there exists r such that $p+q+m-1 = (r+1)pq$. Let $(i_0, j_0) = (0, j') = (0, (p+q+m-1)/q) = (0, (r+1)p)$. Then all non-negative integer solution of (29) are $\{(lq, (r+1-l)p) :\ l = 0, 1, 2, \cdots, r+1\}$, which yields $\eta(x, y)$ has the form, defined in (5). On the other hand, if $\eta(0,1) \neq 0$, $\eta(1,0) \neq 0$, then $P(0,1) \neq 0$, $Q(1,0) \neq 0$. This gives that $q|(p+m-1)$ and $p|(q+m-1)$. Noting $(p+m-1)/q = (p+q+m-1)/q - 1 = (r+1)p - 1$ and $(q+m-1)/p = (p+q+m-1)/p - 1 = (r+1)q - 1$, we get that $P(x, y)$ and $Q(x, y)$ has the form, defined in (5), by the same arguments as above.

Suppose that $\eta(0,1) \neq 0$, $\eta(1,0) = 0$. Since $X \in \Omega_{pqm}$, it follows from Theorem 4 that $y = 0$ is a simple zero of $\eta(1, y)$. By the assumption, the equation (29) has two integer solutions $(0, j')$, $(i', 1)$ respectively and $c_{0,j'} \neq 0$, $c_{i',1} \neq 0$, which implies that $q|p+m-1$ and $p|m-1$. Let $m-1 = k'p$. Then $p+m-1 = (k'+1)p$. Since $(p, q) = 1$, we have $q|(k'+1)$. Hence there exists r such that $p+m-1 = rpq$. All non-negative integer solution of (29) are $\{lq, (r-l)p+1 :\ l = 0, 1, 2, \cdots, r\}$, which yields that $\eta(x, y)$ has the form as in (6). On the other hand, if $\eta(0,1) \neq 0$, $\eta(1,0) = 0$, then $P(0,1) \neq 0$, $Q(1,0) = 0$. We get $P(x, y)$ and $Q(x, y)$, defined in (6), by the same arguments as above.

If $\eta(0,1)=0$ and $\eta(1,0)\neq 0$, then (29) has two integer solutions $(1,j')$, $(i',0)$ and $x=0$ is a simple zero of $\eta(x,1)$. Using the same arguments as above, we get (c).

If $\eta(0,1)=0$ and $\eta(1,0)=0$, then (29) has two integer solutions $(1,j')$, $(i',1)$ and $x=0$ and $y=0$ are simple zeros of $\eta(x,1)$ and $\eta(1,y)$ respectively. Using the same arguments as above, one gets (d). $\square$

Proposition 14. *Let p, q, m, r be the integers defined in Proposition 5.*

(a) *Suppose that p and q are odd, then m is odd (resp. even) if and only if r is odd (resp. even),*

(b) *Suppose that p is odd and q is even.*

  (i) *If either $p+q+m-1=(r+1)pq$ or $p+m-1=rpq$, then m is even,*

  (ii) *if either $q+m-1=rpq$ or $m-1=(r-1)pq$, then m is odd.*

Proof. Suppose $p+q+m-1=(r+1)pq$. If p and q are odd, then $p+q-1$ is odd. Since $m=(r+1)pq-(p+q-1)$, we have that m is odd (resp. even) if and only if r is odd (resp. even). If p is odd and q is even, then $p-1$ is even, which implies that $m=(r+1)pq-q-(p-1)$ is even.

The other assertions can be proved by the same arguments. $\square$

Proposition 15. *If $X\in\Omega_{pqm}$, then there exists a number k such that X has $2k$ singular points at infinity with $k\leq r+1$ and $k\equiv r+1$ (mod 2), where r is defined in Proposition 5.*

Proof. Consider the quasi-homogeneous polynomial $\eta(x,y)$, defined in Proposition 5. Since p is odd, $\eta(1,y)$ has at most $r+1$ real zeros if $\eta(0,1)\neq 0$ and r real zeros if $\eta(0,1)=0$. On the other hand, $x=0$ is a simple zero of $\eta(x,1)$ if $\eta(0,1)=0$. We note $(0,\lambda)$ is a singular point of system (22) if $\eta(1,\lambda)=0$ and $(0,0)$ is a singular point of system $\bar{X}^{\infty}_{+}$ if $\eta(0,1)=0$. The assertion follows from Lemma 11. $\square$

Let

$$\Omega^{2k}_{pqm}=\{X\in\Omega_{pqm}:E(X)\text{ has }2k\text{ singular point at infinity}\}$$

The following corollary follows from Proposition 15:

Corollary 16. *Suppose that r is defined in Proposition 5, then $\Omega_{pqm}=\bigcup_{k\in J_{m,r}}\Omega^{2k}_{pqm}$, where*

$$J_{m,r}=\left\{k=2l:\ 0\leq l\leq\frac{r+1}{2}\right\}.$$

if r is odd, and

$$J_{m,r}=\left\{k=2l+1:\ 0\leq l\leq\frac{r}{2}\right\}.$$

if r is even, respectively.

If X and Y are two topological equivalent vector fields in Ω_{pqm}, then they have the same number singular points at infinity. Let C^k_m be the number of topological equivalence classes in Ω^{2k}_{pqm}. To simplify notation here we do not indicate the dependence on p and q and write C^k_m for the number of topological equivalence classes in Ω^{2k}_{pqm}.

It follows from Corollary 16 that $C_m = \sum_{k\in J_{m,r}} C^k_m$. If $k=0$, then by Theorem 3 $C^0_m = 2$ (a global stable focus and a global unstable focus).

To convenience, we call $\lambda = +\infty$ (resp. $\lambda = -\infty$) a simple zero of $\eta(1,u)$ (resp. $\eta(-1,u)$) if $\eta(0,1)=0$ (resp. $\eta(0,-1)=0$) and $\partial\eta(0,1)/\partial x \neq 0$. Define $\mathrm{sgn}(\partial\eta(\pm1,\pm\infty)/\partial u) = -\mathrm{sgn}(\partial\eta(0,\pm1)/\partial x)$ if $\eta(0,\pm1)=0$. In what follows we suppose that $\lambda_1, \lambda_2, \cdots, \lambda_k$ are zeros of $\eta(1,u)$ and $\lambda_{k+1}, \lambda_{k+2}, \cdots, \lambda_{2k}$ are zeros of $\eta(-1,u)$ with $-\infty < \lambda_1 < \lambda_2 < \cdots < \lambda_k$ and $+\infty > \lambda_{k+1} > \lambda_{k+2} > \cdots > \lambda_{2k}$ respectively, provided $X=(P,Q)\in\Omega^{2k}_{pqm}$.

Proposition 17. *Let $X=(P,Q)\in\Omega^{2k}_{pqm}$, Then*

(a) $(\partial\eta(1,\lambda_i)/\partial u)(\partial\eta(1,\lambda_{i+1})/\partial u) < 0$, $i=1,2,\cdots,k-1$, $k\geq 2$;
(b) $(\partial\eta(-1,\lambda_i)/\partial u)(\partial\eta(-1,\lambda_{i+1})/\partial u) < 0$ *for* $i=k+1,k+2,\cdots,2k-1$, $k\geq 2$;
(c) $P(1,\lambda_i)\neq 0$, $i=1,2,\cdots,k$, *if* $\lambda_k \neq +\infty$, *and* $P(-1,\lambda_i)\neq 0$, $i=k+1,2,\cdots,2k$, *if* $\lambda_{2k}\neq -\infty$;
(d) $Q(0,\pm1)\neq 0$ *if* $\eta(0,\pm1)=0$; *and*
(e) $(\partial\eta(1,\lambda_k)/\partial u)(\partial\eta(-1,\lambda_{k+1})/\partial u) > 0$, $(\partial\eta(1,\lambda_1)/\partial u)(\partial\eta(-1,\lambda_{2k})/\partial u) > 0$.

Proof. Since $X\in\Omega_{pqm}$, all zeros of $\eta(\pm1,u)$ are simple. The assertions (a) and (b) follow if $\lambda_k\neq+\infty$ and $\lambda_{2k}\neq-\infty$.

Suppose $\lambda_k=+\infty$, which yields $\eta(0,1)=0$ and $\eta(1,u)\neq 0$ for $u>\lambda_{k-1}$. Proposition 5 shows that $\eta(x,y)$ has the form defined in either (7) or (8). If $\partial\eta(1,\lambda_{k-1})/\partial u>0$, then $\eta(1,u)>0$, $u\in(\lambda_{k-1},+\infty)$. This implies either $c_{1,rp}>0$ in (7) or $c_{1,(r-1)p}>0$ in (8). Since either $\partial\eta(0,1)/\partial x = c_{1,rp}$ or $\partial\eta(0,1)/\partial x = c_{1,(r-1)p}$ holds and $\mathrm{sgn}(\partial\eta(\pm1,\pm\infty)/\partial u) = -\mathrm{sgn}(\partial\eta(0,\pm1)/\partial x)$, one gets (a) for $i=k$ if $\partial\eta(1,\lambda_{k-1})/\partial u>0$. By the same arguments we get $\partial\eta(1,\lambda_k)/\partial u>0$ if $\lambda_k=+\infty$, $\partial\eta(1,\lambda_{k-1})/\partial u<0$. This proves (a).

If $\lambda_{2k}=-\infty$, then $\eta(-1,u)\neq0$ for $u\in(-\infty,\lambda_{2k-1})$. Repeating the same arguments as the proof of (a), one gets (b).

The assertion (c) and (d) have been proved in Section 2.

Finally we prove (e). Suppose $\eta(0,1)\neq 0$, then $\eta(x,y)$ has the form as either (5) or (6). From the definition of λ_k and λ_{k+1} we have $\eta(1,u)\neq0$ in the interval $(\lambda_k,+\infty)$ and $\eta(-1,u)\neq 0$ in the interval $(\lambda_{k+1},+\infty)$ respectively. If $\eta(x,y)$ has the form (5), then

$$\eta(\pm1,u) = c_{0,(r+1)p}u^{(r+1)p} + (\pm1)c_{1,rp}u^{rp} + \cdots + c_{rq,p}(\pm1)^{rq}u^p + c_{(r+1)q,0}(\pm1)^{(r+1)q},$$

which gives $\mathrm{sgn}\eta(1,u)=\mathrm{sgn}\eta(-1,u)$ as $u\to+\infty$. Since λ_k and λ_{k+1} are simple zeros of $\eta(1,u)$ and $\eta(-1,u)$ respectively, one gets $(\partial\eta(1,\lambda_k)/\partial u)(\partial\eta(-1,\lambda_{k+1})/\partial u)>0$. On the other hand, $\mathrm{sgn}\eta(1,u)=\mathrm{sgn}\eta(-1,u)$ as $u\to-\infty$, which implies $(\partial\eta(1,\lambda_1)/\partial u)(\partial\eta(-1,\lambda_{2k})/\partial u)>0$.

If $\eta(x,y)$ is defined as (6), the assertion in (e) follows by the same arguments.

Suppose $\eta(0,1)=0$, then $\eta(x,y)$ has the form as either (7) or (8) and $\lambda_k = +\infty$, $\lambda_{2k}=-\infty$. If $\eta(x,y)$ is defined in (7), then

$$\eta(x,\pm1)=x(c_{1,rp}(\pm1)^{rp}+c_{1+q,(r-1)p}(\pm1)^{(r-1)p}x^q+\cdots+c_{1+rq,0}x^{rq}),$$

which yields $\mathrm{sgn}(\partial\eta(\pm1,\pm\infty)/\partial u) = -\mathrm{sgn}(\partial\eta(0,\pm1)/\partial x) = -\mathrm{sgn}(c_{1,rp}(\pm1)^{rp})$. Since

$$\eta(\pm1,u)=(\pm1)(c_{1,rp}u^{rp}+c_{1+q,(r-1)p}(\pm1)^q u^{(r-1)p}+\cdots+c_{1+rq,0}(\pm1)^{rq}),$$

$\mathrm{sgn}(\eta(-1,u))=-\mathrm{sgn}(c_{1,rp})$ for $u\in(\lambda_{k+1},+\infty)$ and $\mathrm{sgn}(\eta(1,u))=\mathrm{sgn}(c_{1,rp}(-1)^{rp})$ for $u\in(-\infty,\lambda_1)$, which shows that $\mathrm{sgn}(\partial\eta(-1,\lambda_{k+1})/\partial u) = -\mathrm{sgn}(c_{1,rp})$ and $\mathrm{sgn}(\partial\eta(1,\lambda_1)/\partial u) = -\mathrm{sgn}(c_{1,rp}(-1)^{rp})$. The assertion in (e) follows if $\eta(x,y)$ is defined in (7).

If $\eta(x,y)$ has the form as (8), then one gets (e) by the same arguments. □

Let S be the set of all sequence $(\sigma,\nu)=\{(\sigma_i,\nu_i)\}_{i\in\mathbb{Z}}$ such that $\sigma_i,\nu_i\in\{-1,1\}$ and $\sigma_i\sigma_{i+1}<0$ for all $i\in\mathbb{Z}$. For each $k\in\mathbb{N}$ we denote

$$S^{2k}=(\sigma,\nu)\in S:(\sigma_i,\nu_i)=(\sigma_{i+2k},\nu_{i+2k}),\ \text{for all } i\in\mathbb{Z}\}.$$

A sequence (σ,ν) is *periodic of period* l (or l*-periodic*) if l is the smallest natural number such that $(\sigma_i,\nu_i)=(\sigma_{i+l},\nu_{i+l})$ for all $i\in\mathbb{Z}$. Obviously each sequence (σ,ν) in S^{2k} is periodic and is completely determined if the elements (σ_i,ν_i) are given for $i=1,2,\cdots,l$. Since $\sigma_i\sigma_{i+1}=-1$, the period l is an even divisor of $2k$.

The above notation has been used in the study of the homogeneous vector fields in [10].

¿From Proposition 17 we can associate a sequence of S^{2k} to each vector field $X\in\Omega^{2k}_{pqm}$ by taking

$$(30)\quad (\sigma_i,\nu_i)=\begin{cases}\left(\mathrm{sgn}\left(\frac{\partial\eta(1,\lambda_i)}{\partial u}\right),\mathrm{sgn}(-P(1,\lambda_i))\right), & i=1,2,\cdots,k-1;\\ \left(\mathrm{sgn}\left(\frac{\partial\eta(1,\lambda_i)}{\partial u}\right),\mathrm{sgn}(-P(1,\lambda_i))\right), & \text{if } i=k,\ \lambda_k\neq+\infty;\\ \left(\mathrm{sgn}\left(\frac{-\partial\eta(0,1)}{\partial v}\right),\mathrm{sgn}(-Q(0,1))\right), & \text{if } i=k,\ \lambda_k=+\infty;\\ \left(\mathrm{sgn}\left(\frac{-\partial\eta(-1,\lambda_i)}{\partial u}\right),\mathrm{sgn}(P(-1,\lambda_i))\right), & i=k+1,\cdots,2k-1;\\ \left(\mathrm{sgn}\left(\frac{-\partial\eta(-1,\lambda_i)}{\partial u}\right),\mathrm{sgn}(P(-1,\lambda_i))\right), & \text{if } i=2k,\ \lambda_{2k}\neq-\infty;\\ \left(\mathrm{sgn}\left(\frac{\partial\eta(0,-1)}{\partial v}\right),\mathrm{sgn}(Q(0,-1))\right), & \text{if } i=2k,\ \lambda_{2k}=-\infty,\end{cases}$$

where $\eta(v,u)$ is defined by (2).

Following the idea in [10], we say that (σ,ν) is m*-admissible* if there exists $X\in\Omega^{2k}_{pqm}$ satisfying (30). Denote by S^{2k}_m the set of all sequence in S^{2k} that are m-admissible.

Proposition 18. *Let* $X=(P,Q)\in\Omega^{2k}_{pqm}$ *with* $0\neq k\in J_{m,r}$ *and let* (σ,ν) *be the sequence associated to* X *according to* (30).

(a) *Suppose that both p and q are odd, then* $(\sigma_{i+k}, \nu_{i+k}) = (-1)^{m-1}(\sigma_i, \nu_i)$ *for* $i = 1, 2, \cdots, k$*;*
(b) *Suppose that p is odd and q is even.*
   (i) *If* $\eta(0,1) \neq 0$*, then m is even and* $(\sigma_{2k-i+1}, \nu_{2k-i+1}) = -(\sigma_i, \nu_i)$ *for* $i = 1, 2, \cdots, k$.
   (ii) *If* $\eta(0,1) = 0$*, then m is odd,* $(\sigma_{2k-i}, \nu_{2k-i}) = (\sigma_i, \nu_i)$ *for* $i = 1, 2, \cdots, k-1$*, and* $(\sigma_{2k}, \nu_{2k}) = (-1)^{r-1}(\sigma_k, \nu_k)$*, where r is defined in either (c) or (d) of Proposition 5.*

Proof. It follows from Lemma 11 (or (27)) that if $\eta(1, \lambda_i) = 0$, $i = 1, 2, \cdots, k$ and $\lambda_k \neq +\infty$, then $\eta(-1, (-1)^q\lambda_i) = 0$.

(a) Suppose that both p and q are odd. Since $\lambda_1 < \lambda_2 < \cdots < \lambda_k$, we have $(-1)^q\lambda_1 > (-1)^q\lambda_2 > \cdots > (-1)^q\lambda_k$, which implies $\lambda_{i+k} = (-1)^q\lambda_i$, $i = 1, 2, \cdots, k$. Since $P(-1, \lambda_{i+k}) = P((-1)^p, (-1)^q\lambda_i) = (-1)^m P(1, \lambda_i)$, we get $(\sigma_{i+k}, \nu_{i+k}) = (-1)^{m-1}(\sigma_i, \nu_i)$ from (28) for $i = 1, 2, \cdots, k-1$, and for $i = k$ if $\lambda_k \neq +\infty$.

If $\lambda_k = +\infty$, then it follows from Lemma 11 $\eta(0,1) = \eta(0,-1) = 0$, which implies that $\eta(x,y)$ has the form as either (7) or (8). If $\eta(x,y)$ is defined in (7), then $\partial\eta(0,\pm1)/\partial x = c_{1,rp}(\pm1)^{rp} = c_{1,rp}(\pm1)^r$ and $q+m-1 = rpq$. By Proposition 14 $\partial\eta(0,\pm1)/\partial x = c_{1,rp}(\pm1)^m$. Since $Q(0,-1) = (-1)^{q-1+m}Q(0,1) = (-1)^m Q(0,1)$, one gets $(\sigma_{2k}, \nu_{2k}) = (-1)^{m-1}(\sigma_k, \nu_k)$, provided that $\eta(x,y)$ has the form in (7).

If $\eta(x,y)$ is defined in (8), then we get $(\sigma_{2k}, \nu_{2k}) = (-1)^{m-1}(\sigma_k, \nu_k)$ by the same arguments.

(b) Suppose that p is odd and q is even. If $\eta(0,1) \neq 0$, then $\lambda_{2k-i+1} = \lambda_i$. By the same arguments we get $(\sigma_{2k-i+1}, \nu_{2k-i+1}) = (-1)^{m-1}(\sigma_i, \nu_i)$ for $i = 1, 2, \cdots, k$. On the other hand, if $\eta(0,1) \neq 0$, then Proposition 5 and Proposition 14(i) imply that that m is even. This proves (i).

If $\eta(0,1) = 0$ and q is even, then it follows from Proposition 5 and Proposition 14(ii) that m is odd and $\lambda_k = -\lambda_{2k} = +\infty$. By the same arguments as (a), we have $\lambda_{2k-i} = \lambda_i$ and hence $(\sigma_{2k-i}, \nu_{2k-i}) = (\sigma_i, \nu_i)$ for $i = 1, 2, \cdots, k-1$.

If $\eta(x,y)$ is defined in (7), then it follows from (7) that $\partial\eta(0,\pm1)/\partial v = c_{1,rp}(-1)^r$, $Q(0,\pm1) = b_{0,rp}(\pm1)^r$, which gives $(\sigma_{2k}, \nu_{2k}) = (-1)^{r-1}(\sigma_k, \nu_k)$. If $\eta(x,y)$ has the form (8), then we get (ii) by the same arguments. □

The next two proposition characterizes the sequences in S^{2k} that are m-admissible. It follows from Proposition 5 that, if $X \in \Omega_{pqm}$, then it is only necessary to consider the numbers p, q, m and r that satisfy one of the equations in (9).

Proposition 19. *Suppose that r is defined as in Proposition 5. Denote by s the number of changes of sign in the sequence* $\{\nu_1, \nu_2, \cdots, \nu_k\}$. *Let* $0 \neq k \in J_{m,r}$ *and* $(\sigma, \nu) = \{(\sigma_i, \nu_i)\}_{i \in \mathbb{Z}}$ *verifying*
(i) $(\sigma_{i+k}, \nu_{i+k}) = (-1)^{m-1}(\sigma_i, \nu_i)$ *if both p and q are odd,*
(ii) $(\sigma_{2k-i+1}, \nu_{2k-i+1}) = -(\sigma_i, \nu_i)$ *if p is odd and q, m are even,*
(iii) $(\sigma_{2k-i}, \nu_{2k-i}) = (\sigma_i, \nu_i)$ *for* $i = 1, 2, \cdots, k-1$ *and* $(\sigma_{2k}, \nu_{2k}) = (-1)^{r-1}(\sigma_k, \upsilon_k)$ *if p, m are odd and q is even.*

Then

(a) *If either* $k < r+1$ *or* $k = r+1,\ s < r$, *then* (σ, ν) *is* m*-admissible.*
(b) *If* $k = r+1,\ s = r$, *then* (σ, ν) *is* m*-admissible if and only if there exists* $j \in \mathbb{Z}$ *such that* $\sigma_j = \nu_j$.

Proof. (a) To prove the assertion (a) it is sufficient to find $X = (P(x, y), Q(x, y))$ such that $X \in \Omega_{pqm}^{2k}$ and (30) holds for $i = 1, 2, \cdots, k$.

We note that $k < r+1$ implies $s < r$.

Case 1. $(p, q, m) \in \Theta_1$.

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ be the finite real number with $\lambda_j \neq 0,\ j = 1, 2, \cdots, k$ and

$$\eta(x, y) = a(x^{2q} + y^{2p})^{(r+1-k)/2} \prod_{j=1}^{k} (y^p - \lambda_j^p x^q)$$

with $a \in \mathbb{R}$, $a \neq 0$. Since $k \in J_{m,r}$, it follows from Proposition 15 that $(r+1-k)/2 \in \mathbb{N}$, and hence $\eta(x, y)$ is a (p, q) quasi-homogeneous polynomial of weight degree $p+q+m-1 = (r+1)pq$, which implies that $\eta(1, u)$ has simple real zeros $\lambda_1, \lambda_2, \cdots, \lambda_k$. Furthermore,

$$\frac{\partial \eta(1, \lambda_i)}{\partial u} = ap\lambda_i^{p-1}(1 + \lambda_i^{2p})^{(r+1-k)/2} \prod_{j=1, j\neq i}^{k} (\lambda_i^p - \lambda_j^p) \neq 0, \quad i = 0, 1, \cdots, k.$$

If we choose $a = \pm 1$ in such way that $\operatorname{sgn}\partial\eta(1, \lambda_1)/\partial u = \sigma_1$, then $\eta(x, y)$ satisfies $\operatorname{sgn}\partial\eta(1, \lambda_i)/\partial u = \sigma_i,\ i = 1, 2, \cdots, k$.

To determine $P(x, y)$ we choose $\mu_i \in (\lambda_j, \lambda_{j+1})$ if $\nu_j \neq \nu_{j+1}$. So we obtain s real numbers $\mu_1 < \mu_2 < \cdots < \mu_s$ where s is the number of changes of sign in the sequence $\{\nu_1, \nu_2, \cdots, \nu_k\}$. Let

$$h(x, y) = \prod_{i=1}^{s} (y^p - \mu_i^p x^q).$$

If $\operatorname{sgn}(ah(1, \lambda_1)) = -\nu_1$, then define

$$\alpha(x, y) = \begin{cases} (x^{2q} + y^{2p})^{(r-s-1)/2}(y^p + (1 - \lambda_1^p)x^q), & \text{if } r-s \text{ is odd,} \\ (x^{2q} + y^{2p})^{(r-s)/2}, & \text{if } r-s \text{ is even,} \end{cases}$$

If $\operatorname{sgn}(ah(1, \lambda_1)) = \nu_1$, then define

$$\alpha(x, y) = \begin{cases} (x^{2q} + y^{2p})^{(r-s-1)/2}(y^p - (1 + \lambda_k^p)x^q), & \text{if } r-s \text{ is odd,} \\ (x^{2q} + y^{2p})^{(r-s-2)/2}(y^p + (1 - \lambda_1^p)x^q)(y^p - (1 + \lambda_k^p)x^q), & \text{if } r-s \text{ is even.} \end{cases}$$

Define $P(x, y)$ and $Q(x, y)$ as follows

$$P(x, y) = -\frac{a}{q}\alpha(x, y)h(x, y)y^{p-1}, \ Q(x, y) = \frac{1}{px}(\eta(x, y) + qyP(x, y)).$$

Since $\eta(0, y) + qyP(0, y) \equiv 0$, $Q(x, y)$ is also a polynomial. It is easy to prove that $X = (P, Q)$ is a (p, q)-quasi-homogeneous vector filed of weight degree m. From Theorem 4 and Proposition 18 it follows that $X \in \Omega_{pqm}^{2k}$ and (σ, ν) associated to X verifies either (i) or (ii). Therefore (σ, ν) is m-admissible.

Case 2. $(p, q, m) \in \Theta_2$.

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ be the finite real number with $0 \in \{\lambda_1, \lambda_2, \cdots, \lambda_k\}$. Without loss of generality suppose $\lambda_l = 0$. Let

$$\begin{aligned} \eta(x,y) &= a(x^{2q}+y^{2p})^{(r+1-k)/2} y \prod_{i=1, i\neq l}^{k} (y^p - \lambda_i^p x^q), \\ P(x,y) &= -\frac{a}{q}\alpha(x,y)h(x,y), \end{aligned}$$

with $a \in \mathbb{R}$, $a \neq 0$, where $\alpha(x,y)$, $h(x,y)$ are defined in the same way as in Case 1. Define $Q(x,y)$ by the equation (2). Using the same arguments as in Case 1, $X = (P,Q) \in \Omega_{pqm}^{2k}$ and (σ, ν) associated to X verifies either (i) or (ii). Therefore (σ,ν) is m-admissible.

Case 3. $(p,q,m) \in \Theta_3$.

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_{k-1}$ be the finite real number with $\lambda_i \neq 0$, $i = 1, 2, \cdots, k-1$, $\lambda_k = +\infty$, and

$$\eta(x,y) = ax(x^{2q}+y^{2p})^{(r+1-k)/2} \prod_{i=1}^{k-1} (y^p - \lambda_i^p x^q).$$

If $\operatorname{sgn}(ah(1,\lambda_1)) = -\nu_1$, then define

$$\alpha(x,y) = \begin{cases} (x^{2q}+y^{2p})^{(r-s-1)/2}, & \text{if } r-s \text{ is odd}, \\ (x^{2q}+y^{2p})^{(r-s-2)/2}(y^p + (1-\lambda_1^p)x^q), & \text{if } r-s \text{ is even}. \end{cases}$$

If $\operatorname{sgn}(ah(1,\lambda_1)) = \nu_1$, then define

$$\alpha(x,y) = \begin{cases} -x^q(x^{2q}+y^{2p})^{(r-s-3)/2}(y^p + (1-\lambda_1^p)x^q), & \text{if } r-s \text{ is odd}, \\ -x^q(x^{2q}+y^{2p})^{(r-s-2)/2}, & \text{if } r-s \text{ is even}. \end{cases}$$

Let

$$P(x,y) = -\frac{a}{q}xy^{p-1}\alpha(x,y)h(x,y), \quad Q(x,y) = \frac{1}{px}(\eta(x,y) + qyP(x,y)),$$

where $h(x,y)$ is defined as in Case 1. Using the same arguments as in Case 1, $X = (P,Q) \in \Omega_{pqm}^{2k}$ and (σ,ν) associated to X verifies either (i) or (iii). Therefore (σ,ν) is m-admissible.

Case 4. $(p,q,m) \in \Theta_4$.

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_{k-1}$ be the finite real number with $0 \in \{\lambda_1, \lambda_2, \cdots, \lambda_{k-1}\}$, and $\lambda_k = +\infty$. Without loss of generality suppose $\lambda_l = 0$. Let

$$\eta(x,y) = axy(x^{2q}+y^{2p})^{(r+1-k)/2} \prod_{i=1, i\neq l}^{k-1} (y^p - \lambda_i^p x^q), \quad P(x,y) = -\frac{a}{q}x\alpha(x,y)h(x,y),$$

where $h(x,y)$ and $\alpha(x,y)$ are defined as in Case 1 and Case 3 respectively. Define $Q(x,y)$ by the equation (2). We get a a (p,q)-quasi-homogeneous vector filed $X = (P,Q)$ of weight degree m which is structurally stable. The sequence (σ,ν) associated to X verifies either (i) or (iii). Therefore (σ,ν) is m-admissible.

(b) Let $k = r+1$, $s = r$. Suppose that (σ,ν) is m-admissible, then there exists $X = (P,Q) \in \Omega_{pqm}^{2k}$ such that (σ,ν) associated to X verifies one of (i), (ii) and (iii) and $\eta(x,y)$, $P(x,y)$, $Q(x,y)$ have one of the forms listed in Proposition 5.

If $\eta(0,1) \neq 0$, $\eta(1,0) \neq 0$, then $\eta(x,y)$ has the form (5), which implies that $\eta(1,u) = a\prod_{i=1}^{r+1}(u^p - \lambda_i^p)$, where $\lambda_1 < \lambda_2 < \cdots < \lambda_{r+1}$ with $\lambda_i \neq 0$, $i = 1, 2, \cdots, r+1$ and $a \in \mathbb{R}$, $a \neq 0$. This gives

$$\eta(x,y) = (x^{1/p})^{p+q+m-1}\eta\left(1, \frac{y}{x^{q/p}}\right) = a\prod_{i=1}^{r+1}(y^p - \lambda_i^p x^q).$$

Since $s = r$, $P(1,y)$ has r real zeros which belongs to the interval $[\lambda_1, \lambda_{k+1}]$.. It follows from (5) that $P(x,y)$ is necessarily of the form

$$P(x,y) = b\left(\prod_{i=1}^{r}(y^p - \mu_i^p x^q)\right)y^{p-1}, \quad \lambda_1 < \mu_1 < \lambda_2 < \cdots < \lambda_r < \mu_r < \lambda_{r+1},$$

with $b \in \mathbb{R}$, $b \neq 0$. $Q(x,y)$ is defined by the equation (2). Hence we have $\eta(0,1) = a = -qb = -qP(0,1)$. It follows easily that $\sigma_1 = \mathrm{sgn}(\partial\eta(1,\lambda_1)/\partial u) = \mathrm{sgn}((-1)^r a) = \mathrm{sgn}((-1)^{r+1}b) = -\mathrm{sgn}(P(1,\lambda_1)) = \nu_1$.

If $\eta(0,1) = 0$, $\eta(1,0) \neq 0$, then $\eta(1,u)$ has $r+1$ zeros $\lambda_1 < \lambda_2 < \cdots < \lambda_r < \lambda_{r+1} = +\infty$ with $\lambda_i \neq 0$, $i = 1, 2, \cdots, r$. By the same arguments as above, we have

$$\eta(x,y) = ax\prod_{i=1}^{r}(y^p - \lambda_i^p x^q), \quad P(x,y) = bx\left(\prod_{i=1}^{r-1}(y^p - \mu_i^p x^q)\right)y^{p-1}$$

with $ab \neq 0$, $a, b \in \mathbb{R}$, and $\lambda_1 < \mu_1 < \lambda_2 < \cdots < \mu_{r-1} < \lambda_r < +\infty$. This gives

$$Q(x,y) = \frac{1}{p}\left(a\prod_{i=1}^{r}(y^p - \lambda_i^p x^q) + qb\left(\prod_{i=1}^{r-1}(y^p - \mu_i^p x^q)\right)y^p\right).$$

Hence $\nu_r = \mathrm{sgn}(-P(1,\lambda_r)) = \mathrm{sgn}(-b)$, $\nu_{r+1} = \mathrm{sgn}(-Q(0,1)) = \mathrm{sgn}(-a-qb)$. Since we suppose $s = r$, we have $\nu_r\nu_{r+1} = -1$, which implies $ab < 0$. This yields $\sigma_1 = \mathrm{sgn}(\partial\eta(1,\lambda_1)/\partial v) = \mathrm{sgn}(a(-1)^{r-1})$, $\nu_1 = \mathrm{sgn}(-P(1,\lambda_1)) = \mathrm{sgn}(-b(-1)^{r-1})$. Finally, one gets $\sigma_1 = \nu_1$.

If either $\eta(0,1) \neq 0$, $\eta(1,0) = 0$ or $\eta(0,1) = \eta(1,0) = 0$, one gets $\sigma_1 = \nu_1$ by the same arguments.

Conversely, if $\sigma_1 = \nu_1$ and $k = r+1$, $s = r$, then there exist the vector field $X = (P, Q)$ such that (σ, ν) verifies one of (i), (ii) and (iii), where $P(x,y)$, $Q(x,y)$ are defined as above. This finishes proof. □

Proposition 20. *Let $0 \neq k \in J_{m,r}$ and X, $X' \in \Omega_{pqm}^{2k}$. Denote by (σ,ν), $(\sigma',\nu') \in S_m^{2k}$ the sequences associated to X and X' respectively. Then X and X' are topologically equivalent if and only if there exists $\tau \in \mathbb{N}$ $(0 \leq \tau \leq 2k-1)$ such that one of the following conditions holds for all $i \in \mathbb{Z}$:*

$$(\sigma_i, \nu_i) = (\sigma'_{i+\tau}, \nu'_{i+\tau}),$$

$$(\sigma_i, \nu_i) = (\sigma'_{2k-i+1+\tau}, \nu'_{2k-i+1+\tau}).$$

Proof. The assertion is proved by following the arguments in the proof of Proposition 11 of [10]. □

Let $w = (\sigma, \nu) = \{w_i = (\sigma_i, \nu_i),\ i \in \mathbb{Z}\} \in S_m^{2k}$. It follows from Proposition 19 that the sequence $\bar{w} = \{\bar{w}_i = (\sigma_{i+1}, \nu_{i+1}),\ i \in \mathbb{Z}\}$ also belongs S_m^{2k}. Define the application

$$\mathfrak{R}: \ S_m^{2k} \longrightarrow S_m^{2k}$$

such that if $w = (\sigma, \nu) \in S_m^{2k}$, then $\mathcal{R}(w)$ is the sequence of S_m^{2k} verifying

$$(\mathfrak{R}(w))_i = w_{i+1} \quad \text{for all } i \in \mathbb{Z}. \tag{31}$$

Whenever $f: \ A \longrightarrow A$ is an arbitrary application then $a \in A$ is called a *l-periodic point* of f if $f^l(a) = a$ and $f^i(a) \neq a$ for $i = 1, 2, \cdots, l-1$. The integer l is called the *period* of a. The set $C = \{a, f(a), \cdots, f^{l-1}(a)\} \subset A$ is a *cycle of order l (l-cycle)* of f if $a \in A$ is a l-periodic point of f. These notation can be found in many papers, see for instance [10].

¿From (31) $w \in S_m^{2k}$ is a l-periodic point of $\mathfrak{R}$ if and only if w is a l-periodic sequence. Therefore $C \subset S_m^{2k}$ is a l-cycle of $\mathfrak{R}$ if there exists a l-periodic sequence $w \in S_m^{2k}$ such that $C = \{w, \mathfrak{R}(w), \cdots, \mathfrak{R}^{l-1}(w)\}$. Each sequence $w \in S_m^{2k}$ belongs to some cycle of order l of $\mathfrak{R}$ where l is an even division of $2k$[10].

Define the application

$$\Psi: \ S_m^{2k} \longrightarrow S_m^{2k},$$

where if $w \in S_m^{2k}$ then $\Psi(w)$ is given by

$$(\Psi(w))_i = w_{2k-i+1}.$$

By the definition we know that Ψ is the application that reverses the order of the elements $(\sigma_1, \nu_1), (\sigma_2, \nu_2), \cdots, (\sigma_{2k}, \nu_{2k})$.

The following two propositions have been obtained in [10] for homogeneous polynomial vector fields. They can be proved in the same way for (p, q)-quasi-homogenous polynomial vector fields. Here we omit the details.

Proposition 21. *Let $0 \neq k \in J_{m,r}$. The following statements hold:*

(a) $\Psi \circ \Psi = Id$;
(b) $\Psi \circ \mathfrak{R} = \mathfrak{R}^{-1} \circ \Psi$.

Proposition 22. *Let $0 \neq k \in J_{m,r}$ and X, $X' \in \Omega_{pqm}^{2k}$. Denote by w and w' the sequences of S_m^{2k} associated X and X' respectively. Then X and X' are topologically equivalent if and only if either w and w', or w and $\Psi(w')$ belong to the same cycle of $\mathfrak{R}$.*

Following the idea in [10], we define in S_m^{2k} the equivalent relation: $w \sim \bar{w}$ if and only if one of the sequence w and $\Psi(w)$ belongs to the cycle $\{\bar{w}, \mathfrak{R}(\bar{w}), \cdots, \mathfrak{R}^{l-1}(\bar{w})\}$. According Proposition 22, the number of equivalence class of $\sim$ in S_m^{2k} coincide with the number C_m^k of topological equivalence classes in Ω_{pqm}^{2k}.

Consider the cycle C of $\mathfrak{R}$ such that $w \in C$. From Proposition 21(b), $\Psi(\mathfrak{R}^i(w)) = \mathfrak{R}^{-i}(\Psi(w))$, and hence the cycle of $\Psi(w)$ is $C' = \{\Psi(w) : w \in C\}$. Therefore the equivalence class of w is $C \cup C'$.

A cycle C of $\mathfrak{R}$ is called *symmetric* if $C = C'$. Denote by D_m^k the number of cycles of $\mathfrak{R}$ in S_m^{2k} and E_m^k the number of symmetric cycles respectively. Then

$$C_m^k = E_m^k + \frac{D_m^k - E_m^k}{2} = \frac{E_m^k + D_m^k}{2}. \tag{32}$$

The above expression has been obtained in [10] for homogeneous polynomial systems.

Proposition 23. *Suppose that p and q are odd and $0 \neq k \in J_{m,r}$.*

(a) *If r is odd and $k < r+1$, then*

$$D_m^k = \sum_{2n|k} \mathcal{P}_{2n}, \ \mathcal{P}_{2n} = \frac{1}{n}\left(2^{2n} - \sum_{l|n, l\neq n} l\mathcal{P}_{2l}\right).$$

(b) *If r is even and $k < r+1$, then*

$$D_m^k = \sum_{n|k} \mathcal{P}_{2n}, \ \mathcal{P}_{2n} = \frac{1}{n}\left(2^{n} - \sum_{l|n, l\neq n} l\mathcal{P}_{2l}\right).$$

(c) *If r is odd, then $D_m^{r+1} = (\sum_{2n|r+1} \mathcal{P}_{2n}) - 1$ with $\mathcal{P}_{2n}$ defined as in (a).*

(d) *If r is even, then $D_m^{r+1} = (\sum_{n|r+1} \mathcal{P}_{2n}) - 1$ with $\mathcal{P}_{2n}$ defined as in (b).*

Proof. It follows from Proposition 14 that m is odd (resp. even) if and only if r is odd (resp. even). The proposition follows by the same arguments as in [10]. □

The sequences verifying Proposition 18(b) are not occurred in [10]. We study D_m^k for these sequences in the following proposition.

Proposition 24. *Suppose that p is odd and q is even, $0 \neq k \in J_{m,r}$.*

(a) *If $(p, q, m) \in \Theta_1 \cup \Theta_2$, then*

$$D_m^k = \sum_{n|k} \mathcal{P}_{2n}, \ \mathcal{P}_{2n} = \frac{1}{n}\left(2^{n} - \sum_{l|n, l\neq n} l\mathcal{P}_{2l}\right),$$

for $k < r+1$, and $D_m^{r+1} = (\sum_{n|r+1} \mathcal{P}_{2n}) - 1$ for $k = r+1$, respectively.

(b) *If either $(p, q, m) \in \Theta_3 \cup \Theta_4$, then $D_m^k = \sum_{n|k} \mathcal{P}_{2n}$ for $0 \neq k < r+1$, where $\mathcal{P}_{2n}$ is defined in Theorem 6(b)(ii), and $D_m^k = (\sum_{n|r+1} \mathcal{P}_{2n}) - 1$ for $k = r+1$ respectively.*

Proof. We note that each sequence $w = (\sigma, \nu) \in S_m^{2k}$ is periodic and its period is an even divisor of $2k$. Let w be the $2n$-periodic sequence in S_m^{2k} with $2n|2k$. Then w is completely determined if the elements (σ_i, ν_i) is given for $i = l+1, l+2, \cdots, l+2n$ for $l \in \mathbb{Z}$. It is obvious that w belongs a $2n$-cycle of $\mathfrak{R}$. If $\mathcal{P}_{2n}$ denotes the number of $2n$-cycles of $\mathfrak{R}$ in S_m^{2k}, then $\mathcal{P}_{2n} = \mathcal{N}_{2n}/(2n)$, where $\mathcal{N}_{2n}$ is the number of $2n$-periodic sequences in S_m^{2k}.

(a) Suppose $(p, q, m) \in \Theta_1 \cup \Theta_2$. If $k < r+1$, then it follows from Proposition 18(b)(i) that there exists 2^{n+1} ways of choosing the elements $(\sigma_{k-n+1}, \nu_{k-n+1})$,

$(\sigma_{k-n+2}, \nu_{k-n+2}), \cdots, (\sigma_k, \nu_k), \cdots, (\sigma_{k+n}, \nu_{k+n})$. Therefore

$$\mathcal{N}_{2n} = 2^{n+1} - \sum_{l|n, l\neq n} N_{2l}, \text{ and } \mathcal{P}_{2n} = \frac{1}{n}\left(2^n - \sum_{l|n,l\neq n} lP_{2l}\right).$$

The assertion (a) for $k < r+1$ follows by adding $\mathcal{P}_{2n}$ for all divisors $2n$ of $2k$.

We note that the above computation are valid for the case $k = r+1$, but, by Proposition 19(b), we have to rule out the two sequences satisfying $\sigma_j = v_j$ for all $j \in \mathbb{Z}$. Since these sequence belong to the same 2-cycle of $\mathfrak{R}$, the assertion for $k = r+1$ follows.

(b). Suppose $(p, q, m) \in \Theta_3 \cup \Theta_4$. If $k < r+1$ and $n \neq k$, then it follows from Proposition 18(b)(ii) that there exists 2^{n+2} ways of choosing the elements $(\sigma_{k-n+1}, \nu_{k-n+1}), (\sigma_{k-n+2}, \nu_{k-n+2}), \cdots, (\sigma_k, \nu_k), \cdots, (\sigma_{k+n}, \nu_{k+n})$. Therefore

$$\mathcal{N}_{2n} = 2^{n+2} - \sum_{l|n, l\neq n} N_{2l}, \text{ and } \mathcal{P}_{2n} = \frac{1}{n}\left(2^{n+1} - \sum_{l|n,l\neq n} lP_{2l}\right).$$

If $k < r+1$ and $n = k$, then it follows from Proposition 18(b)(ii) that there exists 2^{n+1} ways of choosing the elements $(\sigma_1, \nu_1), (\sigma_2, \nu_2) \cdots, (\sigma_{2k}, \nu_{2k})$. Therefore

$$\mathcal{N}_{2k} = 2^{k+1} - \sum_{l|k, l\neq n} N_{2l}, \text{ and } \mathcal{P}_{2k} = \frac{1}{k}\left(2^{k} - \sum_{l|k,l\neq k} lP_{2l}\right),$$

which proves (b) for $k < r+1$.

If $k = r+1$, then one gets D_m^k by the same arguments as in (a). □

Next we are going to calculate E_m^k. The following two propositions have been obtained in [10] for homogeneous polynomial vector fields. They can be proved by the same arguments as in [10].

Proposition 25. *Let $k \in J_{m,r}$. Then C is symmetrical if and only if there exists $w \in C$ such that $\Psi(w) = \mathfrak{R}(w)$.*

Proposition 26. *Under the assumptions of Proposition 25, if C is a symmetrical cycle, then there exists two elements of C satisfying $\Psi(w) = \mathfrak{R}(w)$.*

Proposition 27. *Let $0 \neq k \in J_{m,r}$. Suppose that both p and q are odd.*

(a) *If r is odd and $k < r+1$, then*

$$E_m^k = \sum_{2n|k} I_{2n}, \quad \textit{where } I_{2n} = 2^{n+1} - \sum_{l|n,l\neq n} I_{2l}.$$

(b) *If r is even and $k < r+1$, then*

$$E_m^k = \sum_{n|k} I_{2n}, \quad \textit{where } I_{2n} = 2^{(n+1)/2} - \sum_{l|n,l\neq n} I_{2l}.$$

(c) *If r is odd, then $E_m^{r+1} = (\sum_{2n|r+1} I_{2n}) - 1$ with I_{2n} defined as in (a).*

(d) *If r is even, then $E_m^{r+1} = (\sum_{n|r+1} I_{2n}) - 1$ with I_{2n} defined as in (b).*

Proof. It follows from Proposition 14 that r is odd (resp. even) if and only if m is odd (resp. even). The proposition follows by the same arguments as in [10]. □

Proposition 28. *Let $0 \neq k \in J_{m,r}$. Suppose that p is odd and q is even.*

(a) *If $(p, q, m) \in \Theta_1 \cup \Theta_2$, then $E_m^k = 2$ for $k < r+1$, and $E_m^{r+1} = 1$ respectively.*

(b) *If $(p, q, m) \in \Theta_3 \cup \Theta_4$, then*

$$E_m^k = \begin{cases} 4 & \text{if } r \text{ is odd}, \; k < r+1, \\ 2 & \text{if } r \text{ is even}, \; k < r+1. \end{cases} \quad \text{and } E_m^{r+1} = \begin{cases} 3 & \text{if } r \text{ is odd}, \\ 1 & \text{if } r \text{ is even}, \end{cases}$$

respectively.

Proof. We have known that the period of a symmetrical cycle of $\mathfrak{R}$ is an even divisor of $2k$. From Proposition 25 and Proposition 26, the number of symmetrical $2n$-cycles of $\mathfrak{R}$ will be obtained if we divide by 2 the number of $2n$-periodic sequence w in S_m^{2k} such that $\Psi(w) = \mathfrak{R}(w)$. Since the sequences w in a symmetric cycle verify $\Psi(w) = \mathfrak{R}(w)$, we have

$$w_{2k-i+1} = w_{i+1}. \tag{33}$$

In the rest of proof we suppose that w belongs a symmetric cycle.

(a) If $(p, q, m) \in \Theta_1 \cup \Theta_2$, then w verifies Proposition 19(ii), which gives $w_{2k-i+1} = -w_i$. It follows from (33) that $w_{i+1} = -w_i$. This implies that $w_i = (-1)^{i-1} w_1$. Hence w is a 2-periodic sequence. We can take 4 ways of choosing the element w_1. Therefore, $E_m^k = 2$.

Since the sequences such that $\sigma_j = \nu_j$ for all $j \in \mathbb{Z}$ verify $\Psi(w) = \mathfrak{R}(w)$ and belong to 2-cycle of $\mathfrak{R}$, we get $E_m^{r+1} = 1$.

(b) If $(p, q, m) \in \Theta_3 \cup \Theta_4$, then w verifies Proposition 19(iii), which gives $w_{2k-i+1} = w_{i-1}$ for $i = 2, 3, \cdots, k+1$ and $w_{2k} = (-1)^{r-1} w_k$. Since $w \in S^{2k}$, $w_{2k-i+1} = w_{i-1}$ holds for $i \in \mathbb{Z}$ and $w_{2(l+1)k} = (-1)^{r-1} w_{(2l+1)k}$, where $l \in \mathbb{Z}$. It follows from (33) that $w_{i+1} = w_{i-1}$ for $i \in \mathbb{Z}$.

If r is odd, then $w_k = w_{2k}$. It follows from Proposition 15 that k is even. The equation $w_{i+1} = w_{i-1}$ for $i \in \mathbb{Z}$ implies that $w_{2k} = w_{2k-2} = \cdots = w_4 = w_2$ and $w_{2k-1} = w_{2k-3} = \cdots = w_3 = w_1$. Therefore, w is a 2-periodic sequence. We can take 8 ways of choosing the element w_1, w_2 and hence $E_m^k = 4$.

If r is even, then it follows from Proposition 15 that k is odd. The equation $w_{i+1} = w_{i-1}$ implies that $w_{2k} = w_{2k-2} = \cdots = w_4 = w_2$ and $w_{2k-1} = w_{2k-3} = \cdots = w_{k+2} = w_k = w_{k-2} = \cdots = w_3 = w_1$. Since r is even, from Proposition 19(iii) and (33) that $w_k = -w_{2k} = -w_2$, which gives $w_2 = -w_1$. Therefore, w is a 2-periodic sequence. We can take 4 ways of choosing the element w_1 and hence $E_m^k = 2$.

We get E_m^{r+1} by the same arguments as in (a).

□

In the end of this section we prove Theorem 6.

Proof of Theorem 6. Form Corollary 16 it follows that

$$C_{pqm} = \sum_{k \in J_{m,r}} C_{pqm}^k. \tag{34}$$

For $k \neq 0$ we obtain C^k_{pqm} from D^k_m and E^k_m by Proposition 23, Proposition 24, Proposition 27 and Proposition 28. If $(p, q, m) \in \Theta_2 \cup \Theta_3 \cup \Theta_4$, then system (1) has at least one singular point at infinity, which implies that $k \geq 1$. If $(p, q, m) \in \Theta_1 \backslash (\Theta_2 \cup \Theta_3 \cup \Theta_4)$, then $C^0_{pqm} = 2$. The assertions of Theorem 6 follows from (34). □

5. Local phase portrait at the origin and at infinity

We shall prove Theorem 7 and Theorem 8 in this section.

5.1. **At the origin.** Consider the vector field $\bar{X} = (\bar{P}, \bar{Q})$ with

$$\bar{P} = \sum_{i \geq m} P_i(x, y), \quad \bar{Q} = \sum_{i \geq m} Q_i(x, y), \tag{35}$$

where $P_i(x, y)$ and $Q_i(x, y)$ are (p, q)- quasi-homogeneous polynomials of weight degree $p - 1 + i$ and $q - 1 + i$ in the variables x and y respectively.

We would like to compare the local behavior at origin of the vector field $\bar{X}$ given by (35) and the vector field $X_m = (P_m, Q_m)$. For this we apply the quasi-homogeneous blow-up method as doing in Section 2. In the (p, q)-polar coordinates (r, ϕ) (see (13)) the system associated to $\bar{X}$ is

$$\dot{r} = \sum_{i \geq m} r^i F_i(\phi), \quad \dot{\phi} = \sum_{i \geq m} r^{i-1} G_i(\phi),$$

where

$$F_i(\phi) = \xi_i\left(\mathrm{Cs}\phi, \mathrm{Sn}\phi\right), \quad G_i(\phi) = \eta_i\left(\mathrm{Cs}\phi, \mathrm{Sn}\phi\right), \tag{36}$$

with

$$\begin{aligned} \xi_i(x, y) &= x^{2q-1} P_i(x, y) + y^{2p-1} Q_i(x, y) \\ \eta_i(x, y) &= pxQ_i(x, y) - qyP_i(x, y) \end{aligned}$$

Taking the changes (15), the above system goes over to

$$r' = \sum_{i \geq m} r^{i-m+1} f_i(\theta), \quad \theta' = \sum_{i \geq m} r^{i-m} g_i(\theta). \tag{37}$$

where prime denotes derivative with respect to s and

$$f_i(\theta) = F_i\left(\frac{\mathcal{T}\theta}{2\pi}\right), \quad g_i(\theta) = \frac{2\pi}{\mathcal{T}} G_i\left(\frac{\mathcal{T}\theta}{2\pi}\right). \tag{38}$$

Doing the same transformations in the vector field X_m we obtain that it is equivalent to the following differential system

$$\dot{r} = r f_m(\theta), \quad \dot{\phi} = g_m(\theta), \tag{39}$$

where $f_m(\theta)$ and $g_m(\theta)$ are defined in (38).

Proof of Theorem 7. We split the proof into two cases of Theorem 4.

Case 1. $X_m \in \Omega_{pqm}$ and $\eta_m(1, y)$ has no zeros, $\eta_m(0, 1) \neq 0$. Using Theorem 3 and Theorem 4 we conclude that the origin is a global focus (stable or unstable, depending on the sign of I_{X_m}) of X_m. It follows from (36) and (38) that $g_m(\theta) \neq 0$ in this case.

As the critical point of $\bar{X}$ on $r=0$ are determined by the zeros of $g_m(\theta)$, $r=0$ is a periodic orbit for (37). Furthermore, since the dominant terms of (37) in a neighborhood of $r=0$ are given by rf_m and g_m, the orbit $r=0$ is a limit cycle with the same type of stability for (37) and (39). Therefore the origin is a focus with the same type of stability for $\bar{X}$ and X_m when $\eta_m(1,y)$ has no zeros and $\eta_m(0,1)\neq 0$.

Case 2. $X_m \in \Omega_{pqm}$, *and at least one of the following condition holds: (i)* $\eta_m(1,y)$ *has zeros, (ii)* $\eta_m(0,1)=0$. From Theorem 4 all the zeros of $\eta(1,y)$ are simple if there exists, and $\partial\eta_m(0,1)/\partial x \neq 0$ if $\eta_m(0,1)=0$, which implies that all zeros of $g_m(\theta)$ are simple zeros for $\theta\in[0,2\pi]$ by (36) and (38).

Comparing (37) and (39) we observe that the critical point on $r=0$ are the same for both systems, and they are determined by the zeros of $g_m(\theta)$. Furthermore the linear part of these systems in a critical point $(0,\theta^*)$ are

$$\begin{pmatrix} f_m(\theta^*) & 0 \\ g_{m+1}(\theta^*) & g_m'(\theta^*) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} f_m(\theta^*) & 0 \\ 0 & g_m'(\theta^*) \end{pmatrix}.$$

By the definition of $f_m(\theta)$ and $g_m(\theta)$ we have that if $g_m(\theta^*)=0$, then $f_m(\theta^*)\neq 0$. So all the critical points of (37) and (39) are hyperbolic and the local phase portrait of (37) and (39) in $(0,\theta^*)$ are locally topologically equivalent, provided that θ^* is a simple zero of $g_m(\theta)$. As the same happens in each singular point and they determine the phase portrait of both systems in a neighborhood of $r=0$, we conclude that $\bar{X}$ and X_m are locally topologically equivalent at the origin. □

5.2. **At infinity.** Now we can get the analogous of the Theorem 7 at infinity.

Consider the vector field $\hat{X}=(\hat{P},\hat{Q})$, with

$$\hat{P}=\sum_{i=0}^{m} P_i(x,y), \quad \hat{Q}=\sum_{i=0}^{m} Q_i(x,y), \tag{40}$$

where $P_i(x,y)$ and $Q_i(x,y)$ are (p,q)-quasi-homogeneous polynomials of weight degree $p-1+i$ and $q-1+i$ respectively.

Applying the (p,q)-polar coordinates (r,ϕ) to the vector field $\hat{X}$ given by (40) the associated system is

$$\dot{r}=\sum_{i=0}^{m} r^i F_i(\phi), \quad \dot{\phi}=\sum_{i=0}^{m} r^{i-1} G_i(\phi),$$

where F_i and G_i are given in (36).

Doing $\rho=\dfrac{1}{r}$ the system (40) is written as

$$\dot{\rho}=-\sum_{i=0}^{m}\frac{1}{\rho^{i-2}}F_i(\phi), \quad \dot{\phi}=\sum_{i=0}^{m}\frac{1}{\rho^{i-1}}G_i(\phi). \tag{41}$$

Reparametrizing the time by $dt/ds=\rho^{m-1}$ the system (41) becomes

$$\begin{aligned} \rho' &= -\rho(F_m(\phi)+\rho F_{m-1}(\phi)+\cdots+\rho^m F_0(\phi)),\\ \phi' &= G_m(\phi)+\rho G_{m-1}(\phi)+\cdots+\rho^m G_0(\phi). \end{aligned} \tag{42}$$

where the derivative is with respect to s.

After these changes of coordinates the proof of Theorem 8 is analogous to the proof of Theorem 7.

Proof of Theorem 8. Applying the same changes of coordinates described before to the vector field $X_m = (P_m, Q_m) \in \Omega_{pqm}$ we obtain

$$\rho' = -\rho F_m(\phi), \quad \phi' = G_m(\phi). \tag{43}$$

The infinity of X_m and $\hat{X}$ corresponds to the invariant circle $\rho = 0$ of the systems (43) and (42). As $X_m \in \Omega_{pqm}$, $\rho = 0$ is a limit cycle or contains a finite number of hyperbolic critical points. Comparing the system (43) and (42) we observe that they have the same dominant terms. Therefore, it follows from the same argument used in the proof of Theorem 7 that both systems are topologically equivalent in a neighborhood of $\rho = 0$. So X_m and $\hat{X}$ are topologically equivalent in a neighborhood of the infinity. $\square$

The converse of Theorems 7 and Theorem 8 are not true. In [10], Proposition 19 is proved that the converse of an analogous theorems to homogeneous vector field are not true. As a homogeneous vector field is a $(1,1)$-quasi homogeneous vector field of weight degree 1 follows the result.

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Regilene D. S. Oliveira, Departamento de Matemática, Instituto de Ciências Matemáticas e de Computaç ão, Universidade de Sao Paulo, Avenida Trabalhador SaoCarlense, 400, Caixa Postal 668, 13560-970, Sao Carlos, SP, Brazil
E-mail address: regilene@icmc.usp.br

Yulin Zhao, Department of Mathematics, Sun Yat-sen University, Guangzhou, 510275, People's Republic of China.
E-mail address: mcszyl@mail.sysu.edu.cn

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