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LÊ-GREUEL TYPE FORMULA FOR THE EULER OBSTRUCTION AND APPLICATIONS

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Resumo

A obstrução de Euler de f, definida em [BMPS], pode ser vista como uma generalização do número de Milnor para uma função definida em um espaço singular. Neste trabalho, usando a obstrução de Euler de uma função, damos uma versão da fórmula de Lê-Greuel para germes $f : (V, 0) \to (\mathbb{C}, 0)$ e $g : (V, 0) \to (\mathbb{C}, 0)$ de funções analíticas com singularidade isolada na origem. Usando esta fórmula, também apresentamos neste trabalho uma fórmula integral para a obstrução de Euler de uma função, generalizando as fórmulas de Loeser [Lo] e Kennedy [K].

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ABSTRACT. The Euler obstruction of f, defined in [BMPS], can be looked as a generalization of the Milnor number for functions defined on singular spaces. In this work, using the Euler obstruction of a function, we give a a version of the Lê-Greuel formula for germs $f : (V, 0) \to (\mathbb{C}, 0)$ and $g : (V, 0) \to (\mathbb{C}, 0)$ of analytic functions with isolated singularity at the origin. Using this formula, we also present a integral formula for the Euler obstruction of a function, generalizing the formulae of Loeser [Lo] and Kennedy [K].

1. The Euler obstruction

The Euler obstruction was defined by MacPherson [M] as a tool to prove the conjecture about existence and unicity of the Chern classes in the singular case, since that the Euler obstruction was deeply investigated by many authors as Brasselet, Schwartz, Sebastiani, Lê, Teissier, Sabbah, Dubson, Kato and others. For an over view about the Euler obstruction see [B, BG]. Let us now define some objects in order to define the Euler obstruction.

For all this paper, let us consider the following setting: Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be an equidimensional reduced complex analytic germ of dimension d in an open set $U \subset \mathbb{C}^N$. We consider a complex analytic Whitney stratification $\{V_i\}$ of U adapted to X and we assume that $\{0\}$ is a stratum. We choose a small representative of $(X, 0)$ such that 0 belongs to the closure of all the strata. We will denote it by X and we will write $X = \cup_{i=0}^{q} V_i$ where $V_0 = \{0\}$ and $V_q = X_{reg}$, the set of smooth points of X. We will assume that the strata $V_0, \ldots, V_{q-1}$ are connected and that the algebraic sets $\overline{V_0}, \ldots, \overline{V_{q-1}}$ are reduced. We will set $d_i = \dim V_i$.

Let $G(d, N)$ denote the Grassmanian of complex d-planes in $\mathbb{C}^N$. On the regular part $X_{\text{reg}} = X \setminus \text{Sing}(X)$ of X the Gauss map $\phi : X_{\text{reg}} \to U \times G(d, N)$ is well defined by $\phi(x) = (x, T_x(X_{\text{reg}}))$.

Definition 1.1. *The Nash transformation (or Nash blow up) $\widetilde{X}$ of X is the closure of the image $Im(\phi)$ in $U \times G(d, N)$. It is a (usually singular)*

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complex analytic space endowed with an analytic projection map $\nu : \widetilde{X} \to X$ *which is a biholomorphism away from* $\nu^{-1}(\mathrm{Sing}(X))$.

The fiber of the tautological bundle $\mathcal{T}$ over $G(d,N)$, at the point $P \in G(d,N)$, is the set of the vectors v in the d-plane P. We still denote by $\mathcal{T}$ the corresponding trivial extension bundle over $U \times G(d,N)$. Let $\widetilde{T}$ be the restriction of $\mathcal{T}$ to $\widetilde{X}$, with projection map π. The bundle $\widetilde{T}$ on $\widetilde{X}$ is called *the Nash bundle* of X.

An element of $\widetilde{T}$ is written (x, P, v) where $x \in U$, P is a d-plane in $\mathbb{C}^N$ based at x and v is a vector in P. We have a diagram:

$$\begin{array}{ccc} \widetilde{T} & \hookrightarrow & \mathcal{T} \\ \pi\downarrow & & \downarrow \\ \widetilde{X} & \hookrightarrow & U \times G(d,N) \\ \nu\downarrow & & \downarrow \\ X & \hookrightarrow & U. \end{array}$$

Let us recall the original definition, due to MacPherson [M]:

Let $z = (z_1, \ldots, z_N)$ be local coordinates in $\mathbb{C}^N$ around $\{0\}$, such that $z_i(0) = 0$, we denote by $\mathbb{B}_\varepsilon$ and $\mathbb{S}_\varepsilon$ the ball and the sphere centered at $\{0\}$ and of radius ε in $\mathbb{C}^N$. Let us consider the norm $\|z\| = \sqrt{z_1\overline{z}_1 + \cdots + z_N\overline{z}_N}$. Then the differential form $\omega = d\|z\|^2$ defines a section of the real vector bundle $T(\mathbb{C}^N)^*$, cotangent bundle on $\mathbb{C}^N$. Its pull back and, restricted to $\widetilde{X}$, becomes a section denoted by $\widetilde{\omega}$ of the dual bundle $\widetilde{T}^*$. For small enough ε, the section $\widetilde{\omega}$ is nonzero over $\nu^{-1}(z)$ for $0 < \|z\| \leq \varepsilon$. The obstruction to extend $\widetilde{\omega}$ as a nonzero section of $\widetilde{T}^*$ from $\nu^{-1}(\mathbb{S}_\varepsilon)$ to $\nu^{-1}(\mathbb{B}_\varepsilon)$, denoted by $Obs(\widetilde{T}^*, \widetilde{\omega})$ lies in $H^{2d}(\nu^{-1}(\mathbb{B}_\varepsilon), \nu^{-1}(\mathbb{S}_\varepsilon); \mathbb{Z})$. Let us denote by $\mathcal{O}_{\nu^{-1}(\mathbb{B}_\varepsilon),\nu^{-1}(\mathbb{S}_\varepsilon)}$ the orientation class in $H_{2d}(\nu^{-1}(\mathbb{B}_\varepsilon), \nu^{-1}(\mathbb{S}_\varepsilon); \mathbb{Z})$.

Definition 1.2. *The local Euler obstruction of* X *at* 0 *is the evaluation of* $Obs(\widetilde{T}^*, \widetilde{\omega})$ *on* $\mathcal{O}_{\nu^{-1}(\mathbb{B}_\varepsilon),\nu^{-1}(\mathbb{S}_\varepsilon)}$, *i.e.*

$$Eu_X(0) = \langle Obs(\widetilde{T}^*, \widetilde{\omega}), \mathcal{O}_{\nu^{-1}(\mathbb{B}_\varepsilon),\nu^{-1}(\mathbb{S}_\varepsilon)} \rangle.$$

An equivalent definition for the Euler obstruction was given by Brasselet and Schwartz in the context of vector fields [BS].

We need in this paper some results about the Euler obstruction where this invariant is computed using hyperplane sections. The idea of study the Euler obstruction using hyperplane sections appears in the works of Dubson and Kato, but the approach we follow here is that of [BLS, BMPS].

Theorem 1.3. [BLS] *Let* $(X, 0)$ *and* $\{V_i\}$ *given as before, then for each generic linear form* l, *there is* ε_0 *such that for any* ε, $\varepsilon_0 > \varepsilon > 0$ *and* $t_0 \neq 0$ *sufficiently small, the Euler obstruction of* $(X, 0)$ *is equal to:*

$$\mathrm{Eu}_X(0) = \sum_{i=1}^{q} \chi(V_i \cap \mathbb{B}_\varepsilon \cap l^{-1}(\delta)) \cdot \mathrm{Eu}_X(V_i),$$

where χ denotes the Euler-Poincaré characteristic and $\mathrm{Eu}_X(V_i)$ *is the value of the Euler obstruction of X at any point of V_i, $i = 1, \ldots, q$, and $0 < |\delta| \ll \varepsilon \ll 1$.*

We define now an invariant introduced by Brasselet, Massey, Parameswaran and Seade in [BMPS], which measures in a way how far the equality given in Theorem 1.3 is from being true if we replace the generic linear form l by some other function on X with at most an isolated stratified critical point at 0.

In order to define this invariant the authors defined in [BMPS] a stratified vector field on X, denoted by $\overline{\nabla}_X f$. This vector field is homotopic to $\overline{\nabla} F|_X$ and one has $\overline{\nabla}_X f \neq 0$ unless $x = 0$.

Let $\nu : \widetilde{X} \to X$ be the Nash transform of X, and $\tilde{\zeta}$ be the lifting of $\overline{\nabla}_X f$ as a section of the Nash bundle $\widetilde{T}$ over $\widetilde{X}$ without singularity over $\nu^{-1}(X \cap \mathbb{S}_\varepsilon)$, where $\mathbb{S}_\epsilon = \partial \mathbb{B}_\epsilon$ is the boundary of a small sphere around 0. Let $\mathcal{O}(\tilde{\zeta}) \in H^{2n}(\nu^{-1}(X \cap \mathbb{B}_\varepsilon), \nu^{-1}(X \cap \mathbb{S}_\varepsilon))$ be the obstruction cocycle to the extension of $\tilde{\zeta}$ as a nowhere zero section of $\widetilde{T}$ inside $\nu^{-1}(X \cap \mathbb{B}_\varepsilon)$.

Definition 1.4. *The local Euler obstruction* $\mathrm{Eu}_{f,X}(0)$ *is the evaluation of* $\mathcal{O}(\tilde{\zeta})$ *on the fundamental class of the pair* $(\nu^{-1}(X \cap \mathbb{B}_\varepsilon), \nu^{-1}(X \cap \mathbb{S}_\varepsilon))$.

The following result compares the Euler obstruction of the space X with that of a function on X [BMPS].

Theorem 1.5. *Let $(X, 0)$ and $\{V_i\}$ given as before and $f : (X, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at $0 \in X$. For $0 < |\delta| \ll \varepsilon \ll 1$ we have:*

$$\mathrm{Eu}_{f,X}(0) = \mathrm{Eu}_X(0) - \left(\sum_{i=1}^{q} \chi(V_i \cap \mathbb{B}_\varepsilon \cap f^{-1}(\delta)) \cdot \mathrm{Eu}_X(V_i) \right).$$

In [STV] J. Seade et al show that the Euler obstruction of f is closely related to the number of Morse points of a Morsification of f, as it is stated in the next proposition.

Proposition 1.6 ([STV]). *Let $f : (V, 0) \to (\mathbb{C}, 0)$ be the germ of an analytic function with isolated singularity at the origin. Then*

$$Eu_{f,X}(0) = (-1)^{\dim_{\mathbb{C}}(X,0)} n_{reg},$$

where n_{reg} is the number of Morse points in X_{reg} in a stratified morsification of f.

Let us consider the Nash bundle $\widetilde{T}$ on $\widetilde{X}$, the corresponding dual bundles of complex and real 1-forms are denoted, respectively, by $\widetilde{T}^* \to \widetilde{X}$ and $\widetilde{T}^*_{\mathbb{R}} \to \widetilde{X}$.

Definition 1.7. *Let $(X, 0)$ and $\{V_\alpha\}$ as before. Let ω be a (real or complex) 1-form on X, i.e., a continuous section of either $T^*_{\mathbb{R}}\mathbb{C}^N|_X$ or $T^*\mathbb{C}^N|_X$. A singularity of ω in the stratified sense means a point x where the kernel of ω contains the tangent space of the corresponding stratum.*

This means that the pull back of the form to V_α vanishes at x.

Given a section η of $T^*_{\mathbb{R}}\mathbb{C}^N|_A$, $A \subset V$, there is a canonical way of constructing a section $\tilde{\eta}$ of $\widetilde{T}^*_{\mathbb{R}}|_{\tilde{A}}$, $\tilde{A} = \nu^{-1}A$, thus, if η has an isolated singularity at the point $0 \in X$ (in the stratified sense), then we have a never-zero section $\widetilde{\eta}$ of the dual Nash bundle $\widetilde{T}^*_{\mathbb{R}}$ over $\nu^{-1}(\mathbb{S}_\varepsilon \cap X) \subset \widetilde{X}$. Let $o(\eta) \in H^{2d}(\nu^{-1}(\mathbb{B}_\varepsilon \cap X), \nu^{-1}(\mathbb{S}_\varepsilon \cap X); \mathbb{Z})$ be the cohomology class of the obstruction cycle to extend this to a section of $\widetilde{T}^*_{\mathbb{R}}$ over $\nu^{-1}(\mathbb{B}_\varepsilon \cap X)$. Then define (c.f. [BMPS, EG2]):

Definition 1.8. *The local Euler obstruction of the real differential form η at an isolated singularity is the integer $Eu_X(\eta, 0)$ obtained by evaluating the obstruction cohomology class $o(\eta)$ on the orientation fundamental cycle $[\nu^{-1}(\mathbb{B}_\varepsilon \cap X), \nu^{-1}(\mathbb{S}_\varepsilon \cap X)]$.*

MacPherson's local Euler obstruction $\mathrm{Eu}_X(0)$ corresponds to taking the differential $\omega = d\|z\|^2$ of the square of the function distance to 0.

In the complex case, one can perform the same construction, using the corresponding complex bundles. If ω is a complex differential form, section of $T^*\mathbb{C}^N|_A$ with an isolated singularity, one can define the local Euler obstruction $\mathrm{Eu}_X(\omega, 0)$. Notice that, as explained in [BSS] p.151, it is equal to the local Euler obstruction of its real part up to sign:

$$\mathrm{Eu}_X(\omega, 0) = (-1)^d \mathrm{Eu}_X(\mathrm{Re}\,\omega, 0)\,. \tag{1}$$

This is an immediate consequence of the relation between the Chern classes of a complex vector bundle and those of its dual.

We note that the idea to consider the (complex) dual Nash bundle was already present in [Sa], where Sabbah introduces a local Euler obstruction $\breve{\mathrm{Eu}}_X(0)$ that satisfies $\breve{\mathrm{Eu}}_X(0) = (-1)^d \mathrm{Eu}_X(0)$. See also [Schu, sec. 5.2].

2. A lemma in subanalytic geometry

Let $X \subset \mathbb{R}^n$ be a closed subanalytic set equipped with a locally finite subanalytic Whitney stratification $\{V_i\}_{\alpha\in\Lambda}$: $X = \cup V_i$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth subanalytic function such that $f^{-1}(a)$ intersects X transversally (in the stratified sense). Then the following partition:

$$X \cap \{f \le a\} = \bigcup V_i \cap \{f < a\} \cup \bigcup V_i \cap \{f = a\},$$

is a Whitney stratification of the closed subanalytic set $X \cap \{f \le a\}$.

Let $g : \mathbb{R}^n \to \mathbb{R}$ be another smooth subanalytic function such that $g_{|X\cap\{f\le a\}}$ admits an isolated critical point p in $X \cap \{f = a\}$ which is not a critical point of $g_{|X}$. If V denotes the stratum of X that contains p, this implies that:

$$\nabla g_{|V}(p) = \lambda(p) \nabla f_{|V}(p),$$

with $\lambda(p) \neq 0$.

Definition 2.1. *We will say that $p \in X \cap \{f = a\}$ is an outward-pointing (resp. inward-pointing) critical point for $g_{|X \cap \{f \leq a\}}$ if $\lambda(p) > 0$ (resp. $\lambda(p) < 0$).*

Now let us suppose that $0 \in X$ and that $\{0\}$ is a single stratum of X. Let $g : U \subset \mathbb{R}^n \to \mathbb{R}$ be a smooth subanalytic function defined on an open neighborhood U of 0. We assume that $g_{|U \setminus \{0\} \cap V}$ has no critical point.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) = x_1^2 + \cdots + x_n^2$. It is known that for $\varepsilon > 0$ small enough, the sphere $S_\varepsilon = f^{-1}(\varepsilon^2)$ intersects X transversally. Let p^ε be a critical point of $g_{|V \cap S_\varepsilon}$. This means that there exists $\lambda(p^\varepsilon)$ such that:

$$\nabla g_{|V}(p^\varepsilon) = \lambda(p^\varepsilon) \nabla f_{|V}(p^\varepsilon),$$

where V is the stratum containing p^ε. Note that $\lambda(p^\varepsilon) \neq 0$ since g has no critical point on $U \setminus \{0\} \cap V$.

Lemma 2.2. *If ε is small enough then $g(p^\varepsilon) \neq 0$. Furthermore p^ε is outward-pointing (resp. inward-pointing) for $g_{|X \cap B_\varepsilon}$ if and only if $\lambda(p^\varepsilon) > 0$ (resp. $\lambda(p^\varepsilon) < 0$).*

Proof. If for ε small enough, there is a critical point p^ε of $g_{|X \cap S^\varepsilon}$ such that $g(p^\varepsilon) = 0$ then by the Curve Selection Lemma, there is a smooth subanalytic curve $p : [0, \nu[\to X$, $p(0) = 0$, such that $p(t)$ is a critical point of $g_{|X \cap S_{\|p(t)\|}}$ and $g(p(t)) = 0$. Since the stratification is locally finite, we can assume that $p(]0, \nu[)$ is included in a stratum V. Hence, we have:

$$0 = (g \circ p)'(t) = \langle \nabla g(p(t)), p'(t) \rangle = \lambda(p(t)) \langle \nabla f_{|V}(p(t)), p'(t) \rangle,$$

because $p'(t)$ lies in $T_{p(t)}V$. Therefore $(f \circ p)'(t) = 0$ and $f \circ p$ is constant. But $f(p(t))$ tends to 0 as t tends to 0 so $f \circ p$ is zero everywhere, which is a contradiction.

Now let us assume that $\lambda(p^\varepsilon) > 0$. By the Curve Selection Lemma, there exists a smooth subanalytic curve $p : [0, \nu[\to X$ passing through p^ε such that $p(0) = 0$, $p(]0, \nu[)$ is included in a stratum V and for $t \neq 0$, $p(t)$ is a critical point of $g_{|V \cap S_{\|p(t)\|}}$ with $\lambda(p(t)) > 0$. Therefore we have:

$$(g \circ p)'(t) = \lambda(p(t)) \langle \nabla f_{|V}(p(t)), p'(t) \rangle = \lambda(p(t))(f \circ p)'(t).$$

But $(f \circ p)' > 0$ for otherwise $(f \circ p)' \leq 0$ and $f \circ p$ would be decreasing. Since $f(p(t))$ tends to 0 as t tends to 0, this would imply that $f \circ p(t) \leq 0$, which is impossible. Hence we can conclude that $(g \circ p)' > 0$ and $g \circ p$ is strictly increasing. Since $g \circ p(t)$ tends to 0 as t tends to 0, we see that $g \circ p(t) > 0$ for $t > 0$. Similarly if $\lambda(p^\varepsilon) < 0$ then $g(p^\varepsilon) < 0$. □

3. Critical points and topology of Milnor fibres

Let $f : X \to \mathbb{C}$ be a holomorphic function which is the restriction of a holomorphic function $F : \mathbb{C}^N \to \mathbb{C}$. A point x in X is a critical point of f if it is a critical point of $F_{|V(x)}$, where $V(x)$ is the stratum containing x. We will assume that f has an isolated singularity (or an isolated critical point) at 0, i.e that f has no critical point in a punctured neighborhood of 0 in X. This implies that $X^f = X \cap f^{-1}(0)$ is a Whitney stratified set of dimension $d-1$, equipped with the stratification $\cup_{i=0}^q V_i \cap f^{-1}(0) = \cup_{i=0}^q V_i^f$.

Let us consider another holomorphic function $g : X \to \mathbb{C}$, restriction of a holomorphic function $G : \mathbb{C}^N \to \mathbb{C}$. We also assume that g has an isolated singularity at 0 so that $X^g = X \cap g^{-1}(0) = \cup_{i=0}^q V_i \cap g^{-1}(0) = \cup_{i=0}^q V_i^g$ is a Whitney stratification of X^g.

Lemma 3.1. *The function $g : X^f \to \mathbb{C}$ has an isolated singularity at 0 if and only if the function $f : X^g \to \mathbb{C}$ has an isolated singularity at 0.*

Proof. Let $\Sigma_{g_{|X^f}}$ denote the critical set of $g : X^f \to \mathbb{C}$. By the Curve Selection Lemma, it is easy to prove that $\Sigma_{g_{|X^f}}$ lies in $g^{-1}(0)$. Let x be a point in $\Sigma_{g_{|X^f}}$ different from 0. Then if $V(x)$ denotes the stratum of X that contains x, we have:

$$dG_{|V(x)}(x) = \lambda dF_{|V(x)}(x).$$

Since x is not a critical point of g, λ is different from 0 and:

$$dF_{|V(x)}(x) = \frac{1}{\lambda} dG_{|V(x)}(x).$$

This last equality means that x is a critical point of $f_{|X^g}$. □

For all $i \in \{1, \ldots, q\}$, we denote by $\Gamma^i_{f,g}$ the following relative polar set:

$$\Gamma^i_{f,g} = \left\{ x \in V_i \mid \operatorname{rank}(dF_{|V_i}(x), dG_{|V_i}(x)) < 2 \right\},$$

and we make the assumption that for all $i \in \{1, \ldots, q\}$, $I_0(X^f, \overline{\Gamma^i_{f,g}}) < +\infty$, where $I_0(-,-)$ denotes the intersection multiplicity at the origin. This implies that the only critical point of $g_{|X^f}$ is 0 and, by the previous lemma, that $f_{|X^g}$ has also an isolated singularity at 0. Furthermore, it also implies that the number of critical points of $g_{|X \cap f^{-1}(\delta) \cap \mathring{B}_\varepsilon}$ is finite for $0 < |\delta| \ll \varepsilon \ll 1$. Let us denote by $p_{i1}, \ldots, p_{in_i}$ the critical points of $g_{|V_i \cap f^{-1}(\delta) \cap \mathring{B}_\varepsilon}$ and for each $j \in \{1, \ldots, n_i\}$, let us denote by μ_{ij} the Milnor number of $g_{|V_i}$ at p_{ij}. In the following proposition, we will relate the μ_{ij}'s to the topology of the Milnor fibres $X \cap f^{-1}(\delta) \cap B_\varepsilon$ and $X_g \cap f^{-1}(\delta) \cap B_\varepsilon$.

Theorem 3.2. *For $0 < |\delta| \ll \varepsilon \ll 1$, we have:*

$$\chi\big(X \cap f^{-1}(\delta) \cap B_\varepsilon\big) - \chi\big(X^g \cap f^{-1}(\delta) \cap B_\varepsilon\big) =$$
$$\sum_{i=1}^{q} \sum_{j=1}^{n_i} (-1)^{d_i - 1} \mu_{ij} \left(1 - \chi\big(\operatorname{lk}^{\mathbb{C}}(V_i, X)\big)\right),$$

where $\operatorname{lk}^{\mathbb{C}}(V_i, X)$ is the complex link of the strata V_i in X (see [GMP], p. 161).

Proof. We apply stratified Morse theory to the real and imaginary parts of g. Let us write $g = g_1 + \sqrt{-1} g_2$. Using the Cauchy-Riemann equations and local coordinates, it is not difficult to see that $g_{1|X^f} : X^f \to \mathbb{R}$ and

$g_{2|X^f} : X^f \to \mathbb{R}$ have the same critical points as $g_{|X^f} : X^f \to \mathbb{C}$. Hence $g_{1|X^f}$ and $g_{2|X^f}$ have an isolated singularity at 0. Similarly, $g_{1|X\cap f^{-1}(\delta)\cap \mathring{B}_\varepsilon}$ and $g_{2|X\cap f^{-1}(\delta)\cap \mathring{B}_\varepsilon}$ have the same critical points as $g_{|X\cap f^{-1}(\delta)\cap \mathring{B}_\varepsilon}$.

Let us study the critical points of g_1 on the stratified set $X \cap f^{-1}(\delta) \cap B_\varepsilon$. We can distinguish between two kinds of critical points : those lying in $X \cap f^{-1}(\delta) \cap \mathring{B}_\varepsilon$ and those lying in $X \cap f^{-1}(\delta) \cap S_\varepsilon$. Applying Lemma 2.1 to $g_{1|X_f}$ and taking δ sufficiently close to 0, we can say that the critical points of the second type satisfy the following properties:

(1) they lie outside $\{g_1 = 0\}$,
(2) they are outward-pointing for $g_{1|X\cap f^{-1}(\delta)\cap B_\varepsilon}$ in $\{g_1 > 0\}$,
(3) they are inward-pointing for $g_{1|X\cap f^{-1}(\delta)\cap B_\varepsilon}$ in $\{g_1 < 0\}$.

Let $\tilde{g}_1 : X \cap f^{-1}(\delta) \cap B_\varepsilon \to \mathbb{R}$ be a stratified Morse function close to g_1. Applying Theorem 2.3 of [GMP] to the submersion $\mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}^N$, $(x, v) \mapsto G_1(x) + \sum_{i=1}^N x_i v_i$, where $G = G_1 + \sqrt{-1}G_2$, we can take $\tilde{g}_1$ to be the restriction of a real-analytic function.

For each $i \in \{1, \ldots, q\}$, $j \in \{1, \ldots, n_i\}$, let $\{q_{ij}^k\}$, $k \in \{1, \ldots, m_{ij}\}$, be the set of critical points of $\tilde{g}_{1|V_i}$ lying close to p_{ij}. For each critical point q_{ij}^k, let $M_{\tilde{g}_1}(q_{ij}^k)$ be the local negative Milnor fibre of $\tilde{g}_1$ at q_{ij}^k. It is defined as follows:

$$M_{\tilde{g}_1}(q_{ij}^k) = X \cap f^{-1}(\delta) \cap B_{\tilde{\varepsilon}}(q_{ij}^k) \cap \tilde{g}_1^{-1}(\tilde{g}_1(q_{ij}^k) - \nu),$$

where $0 < \nu \ll \tilde{\varepsilon} \ll 1$.

Let $\tilde{\alpha}$ be a regular value of $\tilde{g}_1$ close to 0. Applying Theorem 3.1 in [Du], we have:

$$\chi(X \cap f^{-1}(\delta) \cap B_\varepsilon \cap \{\tilde{g}_1 \geq \tilde{\alpha}\}) - \chi(X \cap f^{-1}(\delta) \cap B_\varepsilon \cap \{\tilde{g}_1 = \tilde{\alpha}\}) =$$

$$\sum_{i=1}^{q} \sum_{j=1}^{n_i} \sum_{k:\tilde{g}_1(q_{ij}^k) > \tilde{\alpha}} 1 - \chi(M_{\tilde{g}_1}(q_{ij}^k)).$$

Here we remark that the critical points of $\tilde{g}_1$ lying in $\{\tilde{g}_1 > \tilde{\alpha}\} \cap S_\varepsilon$ do not appear in the above equality. This is due to the fact that, since $\tilde{g}_1$ is close to g_1 and α close to 0, these critical points are outward-pointing $\tilde{g}_{1|X\cap f^{-1}(\delta)\cap B_\varepsilon}$. For such critical points, the local negative Milnor fibre has Euler characteristic 1, as explained in [Du], Lemma 2.1.

For each critical point q_{ij}^k, let λ_{ij}^k be the Morse index of $\tilde{g}_{1|V_i}$ at q_{ij}^k. Since $X \cap f^{-1}(\delta)$ is complex analytic, the normal Morse data of $\tilde{g}_1$ at q_{ij}^k does not depend on $\tilde{g}_1$ nor on q_{ij}^k and has the homotopy type of the pair $(\mathrm{Cone}(\mathrm{lk}^{\mathbb{C}}(V_i, X)), \mathrm{lk}^{\mathbb{C}}(V_i, X))$ (see [GMP, Corollary 1, p. 166]). Moreover, since $\tilde{g}_{1|X\cap f^{-1}(\delta)\cap B_\varepsilon}$ is a Morse function, the local Morse data at a critical point is the product of the tangential Morse data and the normal Morse data (see [GMP, Section 3.7, p. 65]). Hence, we can write :

$$1 - \chi(M_{\tilde{g}_1}(q_{ij}^k)) = (-1)^{\lambda_{ij}^k}(1 - \chi(\mathrm{lk}^{\mathbb{C}}(V_i, X)).$$

Therefore, we have:

$$\chi(X \cap f^{-1}(\delta) \cap B_\varepsilon \cap \{\tilde{g_1} \geq \tilde{\alpha}\}) - \chi(X \cap f^{-1}(\delta) \cap B_\varepsilon \cap \{\tilde{g_1} = \tilde{\alpha}\}) =$$

$$\sum_{i=1}^{q} \sum_{j=1}^{n_i} \sum_{k:\tilde{g_1}(q_{ij}^k) > \tilde{\alpha}} (-1)^{\lambda_{ij}^k} (1 - \chi(\mathrm{lk}^{\mathbb{C}}(V_i, X)).$$

Applying the same method to $-\tilde{g_1}$ and using the fact that the strata have even real dimension, we obtain:

$$\chi(X \cap f^{-1}(\delta) \cap B_\varepsilon \cap \{\tilde{g_1} \leq \tilde{\alpha}\}) - \chi(X \cap f^{-1}(\delta) \cap B_\varepsilon \cap \{\tilde{g_1} = \tilde{\alpha}\}) =$$

$$\sum_{i=1}^{q} \sum_{j=1}^{n_i} \sum_{k:\tilde{g_1}(q_{ij}^k) < \tilde{\alpha}} (-1)^{\lambda_{ij}^k} (1 - \chi(\mathrm{lk}^{\mathbb{C}}(V_i, X)).$$

Summing these two equalities and applying the Mayer-Vietoris sequence, we get:

$$\chi(X \cap f^{-1}(\delta) \cap B_\varepsilon) - \chi(X \cap f^{-1}(\delta) \cap B_\varepsilon \cap \{\tilde{g_1} = \tilde{\alpha}\}) =$$

$$\sum_{i=1}^{q} \sum_{j=1}^{n_i} \sum_{k=1}^{m_{ij}} (-1)^{\lambda_{ij}^k} (1 - \chi(\mathrm{lk}^{\mathbb{C}}(V_i, X)).$$

But now for each $i \in \{1, \ldots, q\}$ and each $j \in \{1, \ldots, n_i\}$, $\sum_{k=1}^{m_{ij}} (-1)^{\lambda_{ij}^k}$ is the Poincaré-Hopf index of the form $dg_{1|V_i}$. Since g_1 is the real part of g, this index is $(-1)^{d_i} \mu_{ij}$ (see for instance [EG, p. 235]). Hence we have proved:

$$\chi(X \cap f^{-1}(\delta) \cap B_\varepsilon) - \chi(X \cap f^{-1}(\delta) \cap B_\varepsilon \cap \{\tilde{g_1} = \tilde{\alpha}\}) =$$

$$\sum_{i=1}^{q} \sum_{j=1}^{n_i} (-1)^{d_i - 1} \mu_{ij} (1 - \chi(\mathrm{lk}^{\mathbb{C}}(V_i, X)),$$

and therefore, since $\tilde{g_1}$ is close to g_1:

$$\chi(X \cap f^{-1}(\delta) \cap B_\varepsilon) - \chi(X \cap f^{-1}(\delta) \cap B_\varepsilon \cap \{g_1 = \alpha\}) =$$

$$\sum_{i=1}^{q} \sum_{j=1}^{n_i} (-1)^{d_i - 1} \mu_{ij} (1 - \chi(\mathrm{lk}^{\mathbb{C}}(V_i, X)),$$

where α is a regular value of g_1 close to 0.

Now we are going to study the critical points of $g_{2|X \cap f^{-1}(\delta) \cap B_\varepsilon \cap \{g_1 = \alpha\}}$. Using the Cauchy-Riemann equations and local coordinates, it is easy to see that $g_{2|X \cap f^{-1}(\delta) \cap B_\varepsilon \cap \{g_1 = \alpha\}}$ has no critical points in $\mathring{B}_\varepsilon$. Similarly $g_{2|X^f \cap \{g_1 = 0\}}$ has an isolated singularity at the origin. Applying Lemma 2.1 to $g_{2|X^f \cap \{g_1 = 0\}}$ and taking δ and α very small, we can control the behaviour of the critical points of $g_{2|X \cap f^{-1}(\delta) \cap B_\varepsilon \cap \{g_1 = \alpha\}}$ lying in S_ε. Namely, we know that:

(1) they lie outside $\{g_2 = 0\}$,
(2) they are outward-pointing for$g_{2|X \cap f^{-1}(\delta) \cap B_\varepsilon \cap \{g_1 = \alpha\}}$ in $\{g_2 > 0\}$,
(3) they are inward-pointing for $g_{2|X \cap f^{-1}(\delta) \cap B_\varepsilon \cap \{g_1 = \alpha\}}$ in $\{g_2 < 0\}$.

Let β be a small regular value of $g_{2|X\cap f^{-1}(\delta)\cap\{g_1=\alpha\}\cap B_\varepsilon}$. Applying the same method as above, we find that:

$$\chi(X\cap f^{-1}(\delta)\cap B_\varepsilon\cap\{g_1=\alpha\}\cap\{g_2\geq\beta\})- \\ \chi(X\cap f^{-1}(\delta)\cap B_\varepsilon\cap\{g_1=\alpha\}\cap\{g_2=\beta\})=0,$$

$$\chi(X\cap f^{-1}(\delta)\cap B_\varepsilon\cap\{g_1=\alpha\}\cap\{g_2\leq\beta\})- \\ \chi(X\cap f^{-1}(\delta)\cap B_\varepsilon\cap\{g_1=\alpha\}\cap\{g_2=\beta\})=0.$$

Hence, by the Mayer-Vietoris sequence, we get:

$$\chi(X\cap f^{-1}(\delta)\cap B_\varepsilon\cap\{g_1=\alpha\})=\chi(X\cap f^{-1}(\delta)\cap B_\varepsilon\cap\{g_1=\alpha\}\cap\{g_2=\beta\}).$$

Since $f_{|X^g}$ has an isolated singularity at the origin, $f^{-1}(\delta)$ intersects X^g transversally. Hence, if $\alpha+\sqrt{-1}\beta$ is small enough, we have:

$$\chi(X\cap f^{-1}(\delta)\cap B_\varepsilon\cap\{g=\alpha+\sqrt{-1}\beta\})=\chi(X\cap f^{-1}(\delta)\cap B_\varepsilon\cap\{g=0\})= \\ \chi(X^g\cap f^{-1}(\delta)\cap B_\varepsilon).$$

□

4. Lê-Greuel type formula and applications

In order to define the radial index of a 1-form, let us consider first the real case, it means, let $(X,0)\subset(\mathbb{R}^N,0)$ be the germ of a real analytic variety and let ω be a continuous 1-form on a neighborhood of the origin in $\mathbb{R}^N$.

Definition 4.1. *The* 1*-form* ω *is radial on* $(X,0)$ *if, for an arbitrary non-trivial analytic arc* $\varphi:(\mathbb{R},0)\to(X,0)$ *on* $(X,0)$*, the value of the* 1*-form* ω *on the tangent vector* $\varphi'(t)$ *is positive for positive* t *small enough.*

Definition 4.2. *A complex* 1*-form is radial, if its real part is radial.*

Let $\{V_i\}$ be a Whitney stratification of the germ $(X,0)$, where $V_0=\{0\}$. Let ω be an (arbitrary continuous) 1-form on a neighborhood of the origin in $\mathbb{R}^N$ with an isolated singular point on $(X,0)$ at the origin. Let $\varepsilon>0$ be small enough so that in the closed ball B_ε of radius ε centered at the origin in $\mathbb{R}^N$ the 1-form ω has no singular points on $X\setminus\{0\}$. It is easy to see that there exists a 1-form $\tilde\omega$ on $\mathbb{R}^N$ such that: The 1-form $\tilde\omega$ coincides with the 1-form ω on a neighbourhood of the sphere S_ε. The 1-form $\tilde\omega$ is radial on $(X,0)$ at the origin.

Definition 4.3. *The radial index at* 0*,* $ind_{X,0}(\omega)$*, of the* 1*-form* ω *on* X *is defined by*

$$ind_{X,0}(\omega)=1+\sum_{i,j} ind_{PH}(x_j,\tilde\omega,V_i),$$

where $\tilde\omega$ *is taken as above, and the points* x_j*,* $j=1,\cdots,n_i$ *are the singularities of* $\tilde\omega$ *in* V_i*.*

Definition 4.4. *The complex radial index of the complex* 1*-form* ω *on* X *at the origin is* $(-1)^n$ *times the index of the real* 1*-form given by the real part of* ω.

Back to the complex case, let us denote

$$n_i = (-1)^{d-\dim V_i - 1}(\chi(lk^{\mathbb{C}}(V_i, X)) - 1),$$

where $\{V_i\}$ is a Whitney stratification of $(X, 0)$ considered as before. In particular for a open stratum V_i of X, $lk^{\mathbb{C}}(V_i, X)$ is a point and $n_i = 1$. The strata V_i of X are partially ordered: $V_i \prec V_j$, we shall write $i \prec j$, if and only if $V_i \subset \overline{V_j}$ and $V_i \neq V_j$; $i \preceq j$ if and only if $i \prec j$ or $i = j$.

Let us define the Euler obstruction $Eu_{Y,0}(\omega)$ to be equal to 1 for a zero-dimensional variety Y.

Under this conditions Ebeling and Gusein-Zade proved in [EG] the following result:

Theorem 4.5. *Let* $(V, 0) \subset (\mathbb{C}^N, 0)$ *be the germ of a reduced complex analytic space at the origin, with a Whitney stratification* $\{V_i\}$*, for* $i = 0, \ldots, q$ *where* $V_0 = \{0\}$ *and* V_q *the regular part of* X*. Then*

$$ind_{X,0}\omega = \sum_{i=0}^{q} n_i \cdot Eu_{\overline{V_i},0}\omega.$$

Remark 4.6. *Let us consider* $(x_1, x_2, \ldots, x_n)$ *as complex coordinates of* $\mathbb{C}^N$*, where* $x_k = u_k + iv_k$*, that implies that* $(u_1, v_1, \ldots, u_N, v_N)$ *are real coordinates of* $\mathbb{R}^{2N}$*. Let* ω *be a* 1*-form defined by* $\omega = \sum \overline{x_k} dx_k$*, it means that*

$$\omega = \sum (u_k - iv_k)(du_k + idv_k)$$

it implies that

$$\omega = \sum (u_k du_k + v_k dv_k) + i \sum (u_k dv_k - v_k dv_k).$$

In this case we also have that the real 1*-form* $Re\omega = \sum (u_k du_k + v_k dv_k)$ *is a radial* 1*-form, it means that* $ind^{\mathbb{R}}_{X,0} Re\omega = 1$*. As we know* $ind_{X,0}\omega = (-1)^d ind^{\mathbb{R}}_{X,0}$*, therefor we find*

$$ind_{X,0}\omega = (-1)^d ind^{\mathbb{R}}_{X,0}\omega = (-1)^d.$$

As it was remarked before

$$\mathrm{Eu}_X(\omega, 0) \;=\; (-1)^d \mathrm{Eu}_X(\mathrm{Re}\,\omega, 0),$$

using this information and the definition of n_i *we have the next equality*

$$n_i Eu_{\overline{V_i}}(\omega, 0) = (-1)^{d-dimV_i-1}(\chi(\mathrm{lk}^{\mathbb{C}}(V_i, X)) - 1)(-1)^{dimV_i} Eu_{\overline{V_i}}(0).$$

Therefor, by Theorem 4.5 we conclude:

$$(-1)^d = (-1)^d(\sum_{i=0}^{q-1}(1 - \chi(\mathrm{lk}^{\mathbb{C}}(V_i, X)))Eu_{\overline{V_i}}(0) + Eu_X(0)),$$

so we find:

$$(1)\qquad Eu_X(0) = 1 + \sum_{i=0}^{q-1} (\chi(\mathrm{lk}^{\mathbb{C}}(V_i, X)) - 1) Eu_{\overline{V_i}}(0).$$

In one hand, applying the Theorem 4.5 to the form df we have.

$$ind_X df = n_i Eu_{\overline{V_i},0} df = (-1)^{dimV - dimV_i - 1} (\chi(\mathrm{lk}^{\mathbb{C}}(V_i, X)))) Eu_{\overline{V_i},f}(0).$$

In other hand, by Theorem 3 of [EG] *we have*

$$ind_X df = (-1)^{dimX} (1 - \chi(f^{-1}(\delta) \cap X \cap B_\varepsilon)).$$

It follows that,

$$1 - \chi(f^{-1}(\delta) \cap X \cap B_\varepsilon) = \sum_{i=0}^{q} (1 - \chi(\mathrm{lk}^{\mathbb{C}}(V_i, X))) Eu_{0\overline{V_i},f}(0)$$

We arrive to the equation:

$$(2)\ Eu_{X,f}(0) = 1 - \chi(f^{-1}(\delta) \cap X \cap B_\varepsilon) + \sum_{i=0}^{q-1} (\chi(\mathrm{lk}^{\mathbb{C}}(V_i, X)) - 1) Eu_{\overline{V_i},f}(0).$$

By the difference $(1) - (2)$ *we arrive to*

$$Eu_X(0) - Eu_{f,X}(0) = \chi(f^{-1}(\delta) \cap X \cap B_\varepsilon) +$$

$$(3)\qquad + \sum_{i=0}^{q-1} (\chi(\mathrm{lk}^{\mathbb{C}}(V_i, X)) - 1))(Eu_{\overline{V_i}}(0) - Eu_{\overline{V_i}}, f(0)).$$

Where we find the equation

$$\chi(f^{-1}(\delta) \cap X \cap B_\varepsilon) = \sum_{i=0}^{q} (1 - \chi(\mathrm{lk}^{\mathbb{C}}(V_i, X)))(Eu_{\overline{V_i}}(0) - Eu_{\overline{V_i,f}}(0)).$$

Let $f, g : (X, 0) \to (\mathbb{C}, 0)$ have an isolated singularity at $0 \in X$ and $f_{|X^g}$ has also an isolated singularity at 0, as it was said before, it implies that the number of critical points of $g_{|X \cap f^{-1}(\delta) \cap \mathring{B}_\varepsilon}$ is finite for $0 < |\delta| \ll \varepsilon \ll 1$. Let us denote by $p_{i1}, \ldots, p_{in_i}$ the critical points of $g_{|V_i \cap f^{-1}(\delta) \cap \mathring{B}_\varepsilon}$ and for each $j \in \{1, \ldots, n_i\}$, let us denote by μ_{ij} the Milnor number of $g_{|V_i}$ at p_{ij}. Let us denote by μ_i the sum $\mu_i = \sum_{j=1}^{n_i} \mu_{ij}$, in other words, μ_i denotes the number of Morse points of a morsification of g on $V_i \cap f^{-1}(\delta)$.

Theorem 4.7. *Let* $f, g : (X, 0) \to (\mathbb{C}, 0)$ *have an isolated singularity at* $0 \in X$ *and* $f_{|X^g}$ *has also an isolated singularity at* 0. *One has:*

$$(-1)^{dimV_q-1}\mu_q = Eu_X(0) - Eu_{f,X}(0) - Eu_{X^g}(0) + Eu_{f,X^g}(0),$$

where $V_q = X_{reg}$.

Proof. Let us prove this result by induction on the depth of the stratification.

The first step to consider is the case when X has isolated singularity at the origin. In this case our stratification will be $\{V_0 = \{0\}, V_1 = V_{reg}\}$.

Applying the Theorem 1.5 we have

$$\begin{aligned} Eu_X(0) - Eu_{f,X}(0) - Eu_{X^g}(0) + Eu_{f,X^g}(0) &= \\ = \chi(V_{reg} \cap B_\epsilon \cap f^{-1}(t_0)) - \chi(V_{reg} \cap B_\epsilon \cap f^{-1}(t_0) \cap \{g = 0\}) \\ = \chi(X \cap B_\epsilon \cap f^{-1}(t_0)) - \chi(X \cap B_\epsilon \cap f^{-1}(t_0) \cap \{g = 0\}). \end{aligned}$$

But, by Theorem 3.2 we have

$$Eu_X(0) - Eu_{f,X}(0) - Eu_{X^g}(0) + Eu_{f,X^g}(0) =$$

$$= \sum_{j=1}^{n_1} (-1)^{d_1}\mu_{1,j}(1 - \chi(lk^{\mathbb{C}}(V_1, X))) = (-1)^{d_1-1}\mu_1,$$

it means that our assumption is true for the case of X with isolated singularity at the origin.

Let us prove the general case. By hypothesis of induction

$$(-1)^{d_i-1}\mu_i = Eu_{\overline{V_i},0} - Eu_{f,\overline{V_i}}(0) - (Eu_{\overline{V_i}\cap\{g=0\}}(0) - Eu_{f,\overline{V_i}\cap\{g=0\}}(0)).$$

Using the Theorem 3.2 we have

$$\sum_{i=1}^{q} (-1)^{d_i-1}\mu_i \left(1 - \chi(lk^{\mathbb{C}}(V_i, X))\right) =$$

$$= \chi(X \cap f^{-1}(\delta) \cap B_\varepsilon) - \chi(X^g \cap f^{-1}(\delta) \cap B_\varepsilon).$$

Using the hypothesis of induction

$$\sum_{i=1}^{q} [Eu_{\overline{V_i},0} - Eu_{f,\overline{V_i}}(0) - (Eu_{\overline{V_i}\cap\{g=0\}}(0) - Eu_{f,\overline{V_i}\cap\{g=0\}}(0))] \left(1 - \chi(lk^{\mathbb{C}}(V_i, X))\right) =$$

$$= \chi(X \cap f^{-1}(\delta) \cap B_\varepsilon) - \chi(X^g \cap f^{-1}(\delta) \cap B_\varepsilon).$$

Equation that we can rewrite as above:

$$(-1)^{d-1}\mu_q = A - B,$$

where

$$A = \chi(f^{-1}(\delta) \cap X \cap B_\varepsilon) - \sum_{i=0}^{q-1} (Eu_{\overline{V_i}}(0) - Eu_{\overline{V_i},f})(1 - \chi(lk^{\mathbb{C}}(V_i, X)))$$

and

$$B = \chi(f^{-1}(\delta) \cap X \cap B_\epsilon \cap \{g = 0\}) -$$
$$\sum_{i=0}^{q-1} (Eu_{\overline{V_i} \cap \{g=0\}}(0) - Eu_{f,\overline{V_i} \cap \{g=0\}}(0))(1 - \chi(lk^{\mathbb{C}}(V_i \cap \{g = 0\}, X \cap \{g = 0\}))).$$

Therefor, from Remark 4.6 (Equation (3)) we have

$$(-1)^{\dim V_q - 1} \mu_q = Eu_X(0) - Eu_{f,V}(0) - [Eu_{X \cap \{g=0\}}(0) - Eu_{f,X \cap \{g=0\}}(0)]$$

□

Remark 4.8. *If we apply the last result in the ICIS case we arrive to the equation*

$$(-1)^{\dim X - 1} \mu_r = \chi(f^{-1}(\delta) \cap X \cap B_\epsilon) - \chi(f^{-1}(\delta) \cap g^{-1}(0) \cap X \cap B_\epsilon),$$

and that is the same of

$$(-1)^{\dim X - 1} \mu_g = \chi(M_f) - \chi(M_{f,g}).$$

In other words, we recover the Lê-Greuel for ICIS.

Remark 4.9. *Let us consider $(X, 0)$ a germ of a singular surface with isolated singularity such that admit a 1-parameter smoothing $\pi : \mathcal{X} \to \mathbb{C}$, where $\pi^{-1}(0) = X$ and $X_t = \pi^{-1}(t)$ is smooth. In this setting, if $G : \mathcal{X} \to \mathbb{C}$ is map such that $G|_{X_t} \to \mathbb{C}$ is a Morse map we have*

$$(-1)^{\dim \mathcal{X} - 1} \mu_r = \chi(\pi^{-1}(t_0) \cap \mathcal{X} \cap B_\epsilon) - \chi(\pi^{-1}(t_0) \cap G^{-1}(0) \cap \mathcal{X}_{t_0} \cap B_\epsilon).$$

But in this case we have:

$$\chi(\pi^{-1}(t_0) \cap \mathcal{X} \cap B_\epsilon) = 1 + \mu(X)$$

and

$$\chi(\pi^{-1}(t_0) \cap G^{-1}(0) \cap \mathcal{X}_{t_0} \cap B_\epsilon) = 1 - \mu(X \cap G^{-1}(0)),$$

where $\mu(X)$ is the Milnor number of X defined by the second Betty number of X_{t_0} and $\mu(X \cap G^{-1}(0))$ is the Milnor number of the curve $X \cap G^{-1}(0)$.

Therefor we have

$$(-1)^{\dim \mathcal{X} - 1} \mu_r = \mu(X) + \mu(X \cap G^{-1}(0)).$$

In [LT], the authors show a formula to compute the Euler obstruction of V at p using polar multiplicities, the next result generalize this formula giving a similar formula to the Euler obstruction of a function.

Let us consider $H_i \in \mathbb{CP}^{N-1}$ sufficiently generic hyperplanes through the origin in order to have $H_1 \cap H_2 \cap \cdots \cap H_i$ of codimension i, and $X \cap H_1 \cap H_2 \cap \cdots \cap H_i$ with isolated singularity at the origin.

Theorem 4.10. *Let $(X, 0)$ the germ of a d-dimensional analytic space as before, and $f : (X, 0) \to (\mathbb{C}, 0)$ be the germ of a analytic function with isolated singularity at $0 \in X$. In the same setting as before, we have the formula*

$$\sum (-1)^{dimV_i - 1} \mu_i = Eu_X(0) - Eu_{f,X}(0).$$

Proof. If we denote by $X^i = X \cap H_1 \cap H_2 \cap \cdots \cap H_i$, for $i = 1 \ldots d$, and $X^0 = X$, we have the
equality

$$\sum_1^d (-1)^{dimV_i - 1} \mu_i = \sum_1^d (Eu_{X^{i-1}} - Eu_{f,X^{i-1}} - Eu_{X^i} + Eu_{f,X^i}).$$

But it is a telescopic sum, we just need to know that the last term compute the difference $Eu_{X^d}(0) - Eu_{f,X^d}(0)$,and in this level $X^d = \{0\}$, and hence for $Eu_{\{0\}}(0) = Eu_{f,\{0\}}(0) = 1$. Therefor, $\sum (-1)^{dimV_i - 1} \mu_i = Eu_X(0) - Eu_{f,X}(0)$.

□

In [Lo] the author presents integral formula to the Milnor number, generalizing this formula to de singular case, Kenndy presents in [K] a Gauss-Bonnet type formula. In this spirit, we present here an integral formula to the Euler obstruction of a function and also a Gauss-Bonnet type formula using the Euler obstruction of functions.

In order to do that, we recall two objects used in [Lo].

(1) Let $c^w_{d-1-i}(T_f)$ be the $(d-1-i)^{st}$ Chern Weil form associated to the tangent fiber bundle associated to T_f on $X(t)_{reg}$ with the hermitien structure given by the embedding of V in $\mathbb{C}^N$.
(2) Let us also define $\omega = i/2\pi\partial\bar{\partial}log \parallel z \parallel^2$ the image reciprocated on $\mathbb{C}^N \setminus \{0\}$ of the Kähler on $\mathbb{P}^{N-1}$.

Corollary 4.11. *In the setting described as before we have the formula:*

$$\lim_{\varepsilon \to 0} \lim_{\varepsilon \to 0} \int_{X \cap f^{-1}(\delta) \cap B_\varepsilon} \sum_{k=o}^{d-1} c^w_{d-1-k}(T_f) \wedge \omega^k = Eu_X(0) - Eu_{f,X}(0)$$

$\forall i \in \{1, 2, \ldots, q-1\}$.

Proof.

$$lim_{\varepsilon \to 0} \lim_{\varepsilon \to 0} \int_{X \cap f^{-1}(\delta) \cap B_\varepsilon} c^w_{d-1-k}(T_f) \wedge \omega^k = (-1)^{d-1-k} I_{(X^k,0)(\Gamma^k_f, X^k)};$$

where X^k is section of X by a sufficiently general plane, trough the origin, of codimension k. Using that $I_{(X^k,0)}(\Gamma^k_f, X^k) = \mu_k$, the result follows. □

Applying the last corollary to V_i we have:

$$\lim_{\varepsilon \to 0} \lim_{\varepsilon \to 0} \int_{V_i \cap f^{-1}(\delta) \cap B_\varepsilon} \sum_{k=o}^{d-1} c^w_{dimV_i - 1 - k}(T_f) \wedge \omega^k = Eu_{\overline{V_i}}(0) - Eu_{f,\overline{V_i}}(0).$$

Multiplying by $(1 - \chi(Nlk(V_i, V)))$ and summarizing we have, by the Equation (3) of the Remark 4.6, an Gauus-Bonnet type formula for $X \cap f^{-1}(\delta) \cap B_\varepsilon$, as we can see in the next corollary.

Corollary 4.12. *In the same setting as before we have the formula:*

$$\chi(f^{-1}(\delta) \cap X \cap B_\varepsilon) =$$

$$= \sum_{i=0}^{d} (1 - (\chi(\mathrm{lk}^{\mathbb{C}}(V_i, X)))) \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \int_{V_i \cap f^{-1}(\delta) \cap B_\varepsilon} \sum_{k=o}^{d-1} c^w_{dimV_i - 1 - k}(T_f) \wedge \omega^k.$$

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