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PARABOLIC PROBLEMS IN H^1 WITH FAST GROWING NONLINEARITIES

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Resumo

Neste artigo, provamos a existência de attratores globais em $H^1(\Omega)$ para problemas parabólicos com não linearidades satisfazendo condição de crescimento subcrítica. Inesperadamente, observamos que para o caso escalar as únicas condições de crescimento provêm da existência local enquanto que para sistemas restrições adicionais no crescimento aparecem devido ao método empregado na prova da propriedade de dissipação. Ao comparar com os nossos resultados anteriores publicados em [C-C-D] sobre existência de atratores em $H^1(\Omega)$ para systemas de equações parabólicas, relaxamos consideravelmente as condições de crescimento no termo não linear para dimensão espacial igual a três.

Palavras Chaves: Equações Parabólicas, crescimento rápido, soluções globais, semigrupos dissipativos, atratores globais.

PARABOLIC PROBLEMS IN H^1 WITH FAST GROWING NONLINEARITIES

ALEXANDRE N. CARVALHO*, TOMASZ DŁOTKO **

ABSTRACT. We prove existence of global attractors in $H^1(\Omega)$ for parabolic problems with nonlinearities satisfying subcritical growth condition. It was an unexpected observation that for the scalar case the only growth restrictions will come from local existence whereas for systems some additional growth restrictions will come from the method employed in proof of dissipativeness. Comparing to our earlier results in [C-C-D], on existence of attractors in $H^1(\Omega)$ for systems of parabolic equations, we relaxed the growth restrictions on the nonlinearity for space dimension $n = 3$.

1. INTRODUCTION

In this paper we study global solvability and asymptotic behavior, in $H_0^1(\Omega)$, for semilinear parabolic problems of the form

$$\begin{aligned} &u_t = Lu + f(u),\ x \in \Omega, \\ &u = 0,\ x \in \partial\Omega, \end{aligned} \tag{1}$$

where, $\Omega \subset \mathbb{R}^n$, L is a uniformly strongly elliptic second order operator in the divergence form and $f : \mathbb{R}^N \to \mathbb{R}^N$ is locally lipschitzian. For simplicity we restrict the presentation (in the introduction) to the case $n = 3$.

Let $X = L^2(\Omega, \mathbb{R}^N)$ and X^α, $\alpha \geq 0$, be the fractional power spaces associated to the operator $A : D(A) \subset X \to X$ defined by

$$D(A) = H^2(\Omega, \mathbb{R}^N) \cap H_0^1(\Omega, \mathbb{R}^N),$$

$$Au = -Lu,\ u \in D(A).$$

We define $X^{-\alpha}$ as the dual of X^α, $\alpha \geq 0$.

For local existence we rewrite the problem (1) in the abstract form

$$u' + Au = f^e(u),\ t > 0, \tag{2}$$

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where $f^e : X^{\frac{1}{2}} \to X^{\beta}$ is defined by $f^e(u)(x) = f(u(x))$. We prove that for a suitable choice of β, $\frac{1}{2} - \beta < 1$, the map f^e is Lipschitz continuous in bounded subsets of $X^{\frac{1}{2}}$ (see [C-O-P-RB]).

Due to Sobolev embedding we need to assume that the function f satisfies subcritical growth condition; that is,

$$|f(u)| \le c(1 + |u|^{5-\epsilon}),$$

for some $\epsilon > 0$. We justify these restrictions later in the paper.

To prove dissipativeness we assume that the function $f = (f_1, \cdots, f_N)$ satisfies the following condition; there are positive constants ξ_i, $1 \le i \le N$, such that

$$f_i(u)u_i < 0, \text{ for, } |u_i| \ge \xi_i, \tag{3}$$

$1 \le i \le N$.
Under the above conditions we prove the following result:

Theorem 1. *For $N = 1$ and under the above conditions the problem (1) has a global attractor in $H_0^1(\Omega)$.*

For $N \ge 2$, to our surprise, we will need to restrict further the growth of the function f to obtain the existence of a global attractor for (1) in $H_0^1(\Omega, \mathbb{R}^N)$. This is related to the *method of invariant regions* used in the proof of dissipativeness. This method requires that the solution of (2) be in $C(\bar{\Omega}, \mathbb{R}^N)$ for $t > 0$. This extra restriction is

$$|f(u)| \le c(1 + |u|^{4-\epsilon}) \tag{4}$$

for some $\epsilon > 0$.
Under this additional restriction we prove the following result:

Theorem 2. *If (4) and (3) are satisfied, the problem (1) has a global attractor in $H_0^1(\Omega, \mathbb{R}^N)$, $N \ge 1$.*

This work extends our previous results in [C-C-D, C-D] on existence of global attractors in the half fractional power space where we have assumed less then cubic growth for systems.
The critical growth case is still untouched. For the case $N = 1$, the global existence can be obtained using the *theory of monotone operators* (see, for example, [BR]). The problem to obtain the existence of a global attractor is the lack of smoothing of the semigroup associated to (1) in the critical case.
It seems that for $N \ge 2$ there is a problem with the method used to obtain dissipativeness which does not allow us to come close to the critical growth.

For completeness of presentation we recall the definition of global attractors employed in this paper. Let $\{T(t), \ge 0\}$, be a semigroup (usually nonlinear) on a Banach space Y. A set $B \subset Y$ is said to *attract* a set $C \subset Y$ under $T(t)$ if $\text{dist}(T(t)C, B) \to 0$ as $t \to \infty$. A set $S \subset Y$ is said to be *invariant* if $T(t)S = S$ for $t \ge 0$. An invariant set $\mathcal{A}$ is said to be a *global attractor* if $\mathcal{A}$ is a maximal compact invariant set which attracts each bounded set $B \subset Y$.

2. Second Order Parabolic Equations in $H^1(\Omega)$

In a bounded smooth domain $\Omega \subset \mathbb{R}^n$ consider a reaction diffusion equation of the form

$$\begin{aligned} &u_t - \sum_{1=1}^n \frac{\partial}{\partial x_i}\left(a_i(x)\frac{\partial u}{\partial x_i}\right) = f(u), \; x \in \Omega, \\ &u = 0, \; x \in \partial\Omega, \end{aligned} \tag{5}$$

with $a_i \in C^1(\bar{\Omega})$, $a_i(x) \geq m_0 > 0$ in $\bar{\Omega}$ and the function $f : \mathbb{R} \to \mathbb{R}$ locally Lipschitz continuous. For space dimension $n \leq 3$ similar problems were studied recently in [C-O-P-RB]. We will thus concentrate on dimensions $n \geq 3$ and fast growing nonlinearity f. We start with a brief discussion of known properties of solutions. Following [HE] we will define the operator $A = -\mathrm{Div}(a \cdot \nabla)$, $a = (a_1, \cdots, a_n)$, and consider it as a sectorial operator on $L^p(\Omega)$ with the domain $D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. It is then well known, that if we choose $p > n$ then, because of the Sobolev embedding $W_0^{1,p}(\Omega) \subset C^0(\bar{\Omega})$ the Lipschitz continuity, in bounded subsets of $W^{1,p}(\Omega)$, of the function $f^e : W_0^{1,p} \to L^p(\Omega)$ defined by $f^e(u)(x) = f(u(x))$, $x \in \Omega$, is given by the local Lipschitz continuity of the function $f : \mathbb{R} \to \mathbb{R}$. Then (c.f. [HE, HA]) local solvability on $W^{1,p}(\Omega)$ follows.

To ensure global existence of solutions of (5) one may use properties of the Liapunov function given by

$$L(\phi) = \frac{1}{2}\int_\Omega \sum_{i=1}^n a_i(x)\left(\frac{\partial\phi}{\partial x_i}\right)^2 dx - \int_\Omega F(\phi)dx, \tag{6}$$

with $F(s) := \int_0^s f(z)dz$. If we further assume the standard dissipativeness condition (c.f. [HA])

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < 0, \tag{7}$$

we obtain that given $\epsilon > 0$ there is $c_\epsilon > 0$ such that

$$F(s) \leq \epsilon s^2 + c_\epsilon, \tag{8}$$

then the property $L(u(t)) \leq L(u_0)$ offers an a priori estimate of the solution u in $H_0^1(\Omega)$;

$$\begin{aligned} \tfrac{1}{2}\|A^{\frac{1}{2}}u(t)\|_{L^2(\Omega)}^2 &\leq L(u_0) + \epsilon_0\|u(t)\|_{L^2(\Omega)}^2 + c_{\epsilon_0}|\Omega| \\ &\leq L(u_0) + c\epsilon_0\|A^{\frac{1}{2}}u(t)\|_{L^2(\Omega)}^2 + c_{\epsilon_0}|\Omega| \end{aligned} \tag{9}$$

with $\epsilon_0 = \frac{1}{4c}$. Note that $\|A^{\frac{1}{2}} \cdot \|_{L^2(\Omega)}$ is an equivalent norm on the space $H_0^1(\Omega)$ as a consequence of Poincaré's inequality which was also used in the final estimate of (9). Then, as a consequence of the results in [C-C-D], there exists a unique global solution $u(t, u_0)$ of (5) for every $u_0 \in D(A^\alpha)$ and for all $\alpha \geq \frac{1}{2}$.

This theory seems complete, but if one chooses the $L^2(\Omega)$ as base space (instead of $L^p(\Omega)$, $p > n$), then besides local Lipschitz continuity and condition (7) imposed on f, some additional growth restriction on it must be imposed due to the condition of Lipschitz continuity on bounded sets for the function $f^e : H_0^1(\Omega) \to L^2(\Omega)$. Indeed, we need to assume [C-O-P-RB] that

$$|f(u) - f(v)| \leq (1 + |u|^{\gamma-1} + |v|^{\gamma-1})|u - v|, \tag{10}$$

for $\gamma = 3$ and $n = 3$, which is a stronger restriction than the condition

$$|f(u)| \leq c(1 + |u|^\gamma),\ \gamma < \frac{n+2}{n-2}, \tag{11}$$

required for the estimates obtained from (6) to hold. Our task will be to fill up the gap in the theory and allow the same fast growth required in (11) for local existence.

3. Local Existence of Solutions in $H_0^1(\Omega)$

Consider the problem (2) with the sectorial operator $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ defined by

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega),$$
$$Au = -\mathrm{Div}(a(x) \cdot \nabla u),\ \forall u \in D(A).$$

Therefore $Re(\sigma(A)) \geq a > 0$. By X^α we denote, as usual, the domain of the fractional power A^α of A; $X^\alpha = D(A^\alpha)$ endowed with the graph norm. We recall the definition of X^α solution to problem (2).

Definition 1. *By a X^α-solution of (2) we understand a continuous function $u : [0, \tau_{u_0}) \to X^\alpha$ satisfying (2) such that $u_t : (0, \tau_{u_0}) \to X^\alpha$ is continuous and $u(t)$ belongs to $D(A)$ for each $t \in (0, \tau_{u_0})$.*

In order to get local existence of solutions of (5) with locally Lipschitz continuous f satisfying the dissipativeness condition (7) and the growth condition (11) with the full range of the growth allowed in it, we need to enlarge the base space $L^2(\Omega)$ on which the operator A will be considered simultaneously weakening the norm of this space. We will follow the idea of [RB] and extend A to $X^{1-\frac{s}{2}}$ taking values in $H^{-s}(\Omega)$, $s > 0$ where $H^{-s}(\Omega)$ is the dual of $X^{\frac{s}{2}}$. Also note, see [TR] Theorem 5.5.1, that

$$X^\alpha \subset H^{2\alpha}(\Omega),$$

with continuous inclusion and that equality holds for $-\frac{1}{4} \leq \alpha \leq \frac{1}{4}$.

We start by fixing the growth exponent γ in (11) as $\gamma < \frac{n+2}{n-2}$. Next define a real number $s := \max\{0, \frac{\gamma(n-2)-n}{2}\}$ and set as our base space

$$X^{\frac{-s}{2}} = H^{-s}(\Omega). \tag{12}$$

Finally define a number $p > 1$ in such a way that $p\gamma = \frac{2n}{n-2}$. We then have the following:

Lemma 1. *If we assume that the real function f satisfies local Lipschitz condition*

$$|f(y) - f(z)| \leq c(1 + |y|^{\gamma-1} + |z|^{\gamma-1})|y - z|, \tag{13}$$

for all y, z real, then the map $f^e : H^1(\Omega) \to H^{-s}(\Omega)$ is Lipschitz continuous on bounded sets.

Proof: We prove first, that for p chosen above the function $f^e : L^{\frac{2n}{n-2}}(\Omega) \to L^p(\Omega)$ will be Lipschitz continuous on bounded sets. Then we use the duality argument and Sobolev embedding to conclude the proof. Using (13) we have

$$\begin{aligned} \|f^e(u) - f^e(v)\|^p_{L^p(\Omega)} &\leq c_1 \int_\Omega |u - v|^p(1 + |u|^{p(\gamma-1)} + |v|^{p(\gamma-1)})dx \\ &\leq c_2 \left(\int_\Omega |u - v|^{\frac{2n}{n-2}}\right)^{p\frac{2n}{n-2}} \left(\int_\Omega (1 + |u|^{pr'(\gamma-1)} + |v|^{pr'(\gamma-1)})dx\right)^{\frac{1}{r'}}, \end{aligned}$$

where we have used Hölder inequality with $\frac{1}{r'} = 1 - p\frac{n-2}{2n}$. Hence,

$$\|f^e(u) - f^e(v)\|_{L^p(\Omega)} \leq c_3 \|u - v\|_{L^{\frac{2n}{n-2}}(\Omega)} \left(1 + \|u\|_{L^{\frac{2n}{n-2}}(\Omega)} + \|v\|_{L^{\frac{2n}{n-2}}(\Omega)}\right)^{\gamma-1}. \tag{14}$$

For $s > 0$, by the Sobolev embedding, the right side of (14) will be dominated by a similar expression with $L^{\frac{2n}{n-2}}(\Omega)$ norms replaced by $H_0^1(\Omega)$ norms, further, since the Hölder conjugate exponent p' equals

$$p' = \frac{2n}{2n - \gamma(n-2)},$$

then we find, by Sobolev embedding, that

$$X^{\frac{s}{2}} \subset H^s(\Omega) \subset L^{p'}(\Omega),$$

and, through the duality argument, also

$$H^{-s}(\Omega) \supset L^p(\Omega),$$

with both inclusions being continuous. This information leads to

$$\|f^e(u) - f^e(v)\|_{H^{-s}(\Omega)} \leq c_4 \|u - v\|_{H_0^1(\Omega)} \left(1 + \|u\|_{H_0^1(\Omega)} + \|v\|_{H_0^1(\Omega)}\right)^{\gamma-1}.$$

For $s = 0$ this follows trivially from (14) and this concludes the proof of the Lemma.

Remark 1. As a result of the preceding lemma, the nonlinear term $f^e : H^1(\Omega) \to H^{-s}(\Omega)$ is Lipschitz continuous on bounded sets, so with the usual theory (c.f. [HE]) local solvability of (5) in $H^1(\Omega)$ follows. The solutions are in $D(A^{1-\frac{s}{2}})$ for $t > 0$.

4. Global Solvability and dissipativeness

4.1. Liapunov function method.

The choice of $H^{-s}(\Omega)$ as base space, made in the proof of local solvability, is an indubitable advantage. Now the proof of global existence of solutions to (5) and dissipativeness of the semigroup associated to it in $H_0^1(\Omega)$ is extremely fast and easy.

Indeed, it is a familiar consequence of the continuation theorem for local solutions ([HE]) that the local X^α-solution exists globally provided that global a priori estimate of it in X^α is available. But for solutions of the problem (5) such a priori estimate follows from the condition (9)

$$\tfrac{1}{4}\|u(t)\|_{H_0^1(\Omega)} \le L(u_0) + c_{\epsilon_0}|\Omega|. \tag{15}$$

Therefore global solvability in $H_0^1(\Omega)$ is evident. Let $u(t,u_0)$ be the solution of (5) such that $u(0,u_0) = u_0$ and $T(t) : H_0^1(\Omega) \to H_0^1(\Omega)$ be the semigroup defined by $T(t)u_0 = u(t,u_0)$. Next we include some information about the solutions of (5):

Theorem 3. *Assume that the real function f satisfies (7) and the Lipschitz condition (13). Then the local solution u of (5) introduced in Remark 1 will be extended for $t \ge 0$ defining for all $\alpha \in [\frac{1}{2}, 1-\frac{s}{2})$ a strongly continuous semigroup $T(t) : X^\alpha \to X^\alpha$, $t \ge 0$. Moreover, this semigroup is compact for $t > 0$ and bounded in X^α.*

We need to check the final claims of the theorem. It is evident, that since the nonlinear term f^e was Lipschitz continuous on bounded sets as a transformation from $X^{\frac{1}{2}}$ to $X^{-\frac{s}{2}}$, then it will be all the more Lipschitz continuous on bounded sets acting from X^α to $X^{-\frac{s}{2}}$. Boundedness of the semigroup on X^α follows from the estimates similar as in ([HE]). Let B be a bounded subset of $X^{\frac{1}{2}}$. For $u_0 \in B$

$$T(t)u_0 = e^{-At}u_0 + \int_0^t e^{-A(t-s)} f^e(T(s)u_0)ds, \tag{16}$$

and, because of the smoothing action of $T(t)$,

$$\begin{aligned} \|T(t)u_0\|_{X^\alpha} &\le Mt^{\frac{1}{2}-\alpha}\|u_0\|_{X^{\frac{1}{2}}} \\ &+ \textstyle\int_0^t Me^{-a(t-s)}(t-s)^{-\alpha}\|f^e(T(s)u_0)\|_X ds \\ &\le Mt^{\frac{1}{2}-\alpha}\|u_0\|_{X^{\frac{1}{2}}} + c\left(\sup_{u_0\in B}\|u_0\|_{X^{\frac{1}{2}}}\right) M\frac{\Gamma(1-\alpha)}{a^{1-\alpha}} \end{aligned} \tag{17}$$

as a consequence of global a priori estimate (15), boundedness of L on bounded subsets of $X^{\frac{1}{2}}$ (see Lemma 2 below) and Lipschitz continuity on bounded sets of $f^e : X^{\frac{1}{2}} \to X^{-\frac{s}{2}}$ (Lemma 1). This proves boundedness of $T(t) : X^\alpha \to X^\alpha$, $\alpha \in [\frac{1}{2}, 1)$. Finally, compactness of the semigroup is a direct consequence of the compactness of the resolvent of A and Theorem 4.2.2 in [HA].

Dissipativeness of the semigroup $T(t)$ is, in general, a consequence of the properties of the Liapunov function defined in (6).

Lemma 2. *The functional $L : H_0^1(\Omega) \to \mathbb{R}$ is well defined, bounded in bounded subsets of $H_0^1(\Omega)$ and defines a Liapunov function in the sense of* [HA] *for the semigroup $T(t)$ on $H_0^1(\Omega)$.*

Since the proof is standard, we only mention that because of the growth restrictions (11) imposed on f its primitive $F(s)$ fulfills:

$$|F(s)| \leq c(1 + |s|^{\gamma+1}),$$

where $\gamma + 1 < \frac{2n}{n-2}$. Then, due to the Sobolev embedding $H_0^1(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega)$, the inclusions are continuous. This justifies the well posedness and boundedness of L.

We also have the following property of the set of stationary solutions to (5).

Lemma 3. *Under the dissipativeness condition (7) the set of stationary solutions of (5) is bounded in $H_0^1(\Omega)$.*

Proof: Stationary solutions of (5) are simultaneously $H_0^1(\Omega)$ solutions of the elliptic problem

$$\begin{aligned} &-\textstyle\sum_{i=1}^m \frac{\partial}{\partial x_i}\left(a_i(x)\frac{\partial v}{\partial x_i}\right) = f(v), \ x \in \Omega, \\ &v = 0, \ x \in \partial\Omega. \end{aligned} \tag{18}$$

Multiplying (18) by v and integrating by parts we obtain

$$\int_\Omega \sum_{i=1}^m a_i(x)\left(\frac{\partial v}{\partial x_i}\right)^2 dx = \int_\Omega f(v)v dx, \tag{19}$$

and as a consequence of (7) for any positive number ϵ there is K_ϵ such that

$$f(s)s \leq \epsilon s^2 + K_\epsilon,$$

for all $s \in \mathbb{R}$. Therefore,

$$\int_\Omega \sum_{i=1}^m a_i(x)\left(\frac{\partial v}{\partial x_i}\right)^2 dx \leq \epsilon \int_\Omega v^2 dx + K_\epsilon|\Omega|. \tag{20}$$

Thus, with ϵ small enough, the assertion follows from the Poincaré inequality.

It is now easy to see that the Liapunov function (6) has all the nice properties required in [HA]. Therefore problem (5) defines a *gradient system* $\{T(t), t \geq 0\}$ on the phase space $H_0^1(\Omega)$ and we are able to claim:

Proposition 1. *The gradient system $\{T(t) : t \geq 0\}$ generated by equation (5) is compact for $t > 0$ and its set of stationary solutions E is bounded in $H_0^1(\Omega)$. It has a global attractor $\mathcal{A}$, moreover this attractor is the unstable manifold of E.*

For a proof of this result see [HA].

4.2. Second Order Systems. The method of Invariant Regions.

Following [C-O-P-RB] consider a weakly coupled system of reaction-diffusion equations

$$\begin{aligned} & u_t - \mathrm{Div}(a(x)\cdot\nabla u) = f(u),\ x\in\Omega, \\ & u = 0,\ x\in\partial\Omega, \end{aligned} \tag{21}$$

where $u = (u_1, \cdots, u_N)^\top$, $N \geq 1$, $a(x) = \mathrm{diag}(a_1(x), \cdots, a_N(x))$, $a_i \in C^1(\bar{\Omega})$, $a_i(x) \geq m_0 > 0$, for all $x \in \Omega$, $1 \leq i \leq N$. The coupling is only through the reaction term $f : \mathbb{R}^N \to \mathbb{R}^N$. We restrict ourselves here to space dimensions $n \geq 3$ (the cases $n = 1, 2$ were studied in a more general way in [C-O-P-RB]), and assume the following local Lipschitz condition of the function $f : \mathbb{R}^N \to \mathbb{R}^N$;

$$|f(y) - f(z)| \leq c(1 + |y|^{\gamma-1} + |z|^{\gamma-1})|y - z| \tag{22}$$

with γ fixed in $[1, \frac{n+2}{n-2})$ like in Lemma 1. The system (21) will now be treated in the abstract form (2) if we consider as base space $[H^{-s}(\Omega)]^N$ with $s = \max\{0, \frac{\gamma(n-2)-n}{2}\}$ and define $A = \mathrm{diag}(A_1, \cdots, A_N)$ on the domain $D(A) = [X^{\frac{2-s}{2}}]^N$. The operators A_i, $i = 1, \cdots, N$ are defined by $A_i : X^{\frac{2-s}{2}} \to H^{-s}$ with

$$(A_i\phi) = -\mathrm{Div}(a_i(x)\nabla\phi),\ \phi \in H^2(\Omega)\cap H_0^1(\Omega).$$

There is no difference in the proof of local solvability between the single equation case and the system case. To obtain global existence and dissipativeness we will use *invariant* and *contracting* regions. In order to use such a method the solution should be regular; that is, that $u(t, \cdot) \in [C(\bar{\Omega})]^N$. To that end we need to restrict the range of the power γ to

$$\gamma < \frac{4}{n-2}. \tag{23}$$

Finding then s from the formula $s = \max\{0, \frac{\gamma(n-2)-n}{2}\}$ we get $s < \frac{1}{2}$ and $-s + 2 > \frac{3}{2}$. Since from elliptic regularity it follows that $u(t, \cdot) \in [X^{\frac{2-s}{2}}]^N \subset [H^{2-s}(\Omega)]^N$, $t > 0$, and from Sobolev embedding

$$[H^{2-s}(\Omega)]^N \subset [C(\bar{\Omega})]^N,$$

we have that $u(t, \cdot) \in [C(\bar{\Omega})]^N$, $t > 0$, and the requirements for using *invariant* and *contracting* regions are fulfilled.

Assume that there are positive constants $\xi_i > 0$, $1 \leq i \leq N$, such that the function $f = (f_1, \cdots, f_N)^\top$ satisfies

$$(f_i(u) - \lambda u_i)u_i < 0,\ |u_i| > \xi_i, \tag{24}$$

where λ_i is the first eigenvalue of A_i.

Our aim will be to prove that problem (21) with f as above that in addition satisfies the growth restrictions (22), (23) is locally well posed in X^α for some $\alpha < \frac{1}{2}$. Then, we will prove that the solutions of (21) are globally defined and that the semigroup associated with (21) is point dissipative in X^α. With this information we use the fact that $X^{\frac{1}{2}}$

is compactly embedded in X^{α}, $\alpha < \frac{1}{2}$, and Theorem 2 in [C-C-D] to ascertain that the semigroup associated with (21) has a global attractor in $X^{\frac{1}{2}}$. The following lemma guarantees that the problem (21) is locally well posed.

For $1 \leq \gamma < \frac{4}{n-2}$, let $0 < s < \frac{1}{2}$, $0 < r < 1$ be such that

$$\gamma = 1 + \frac{2(s+r)}{n-2r}. \tag{25}$$

Then, we have:

Lemma 4. *The map $f^e : [H^r(\Omega)]^N \to [H^{-s}(\Omega)]^N$, is well defined and Lipschitz continuous on bounded subsets of $[H^r(\Omega)]^N$.*

Proof: Let $m > 0$ and $u, v \in [H^r(\Omega)]^N$ be such that $\|u\|_{[H^r(\Omega)]^N} \leq m$ and $\|v\|_{[H^r(\Omega)]^N} \leq m$. Then,

$$\begin{aligned}\|f^e(u) - f^e(v)\|^p_{H^{-s}(\Omega)} \leq & c\int_\Omega |f(u(x)) - f(v(x))|^p dx \\ \leq & C\int_\Omega \left(1 + |u(x)|^{p(\gamma-1)} + |v(x)|^{p(\gamma-1)}\right)|u(x) - v(x)|^p dx \\ \leq & C(|\Omega|^{\frac{1}{q}} + \|u\|^{p(\gamma-1)}_{[L^{p(\gamma-1)q}(\Omega)]^N} + \|v\|^{p(\gamma-1)}_{[L^{p(\gamma-1)q}(\Omega)]^N})\|u - v\|^p_{[L^{pq'}(\Omega)]^N}\end{aligned}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and $p = \frac{2n}{n+2s}$. Since $[H^r(\Omega)]^N \subset [L^t(\Omega)]^N$ with continuous inclusion for $t = \frac{2n}{n-2r}$, we impose that $pq' = \frac{2n}{n-2r}$; that is, $q' = \frac{n+2s}{n-2r}$. Then, $q = \frac{n+2s}{2(s+r)}$ and we have that $\frac{2n}{2s+n}(\gamma - 1)q = t$. And we have that

$$\begin{aligned}&\|f^e(u) - f^e(v)\|_{[H^{-s}(\Omega)]^N} \\ &\quad \leq cK\left(|\Omega|^{\frac{p(\gamma-1)}{t}} + K^{p(\gamma-1)}(\|u\|^{p(\gamma-1)}_{[H^r(\Omega)]^N} + \|v\|^{p(\gamma-1)}_{[H^r(\Omega)]^N})\right)\|u - v\|_{[H^r(\Omega)]^N} \\ &\quad \leq cK\left(|\Omega|^{\frac{p(\gamma-1)}{t}} + 2\,K^{p(\gamma-1)}m^{p(\gamma-1)}\right)\|u - v\|_{[H^r(\Omega)]^N},\end{aligned}$$

where K is the embedding constant for $[H^r(\Omega)]^N \subset [L^t(\Omega)]^N$ and the result is proved.

Let $u_0 \in X^{\frac{r}{2}}$ with $r < 1$ as in (25). Then, from Lemma 4, there exists a solution $u(t, u_0)$ to the problem (21) satisfying $u(0, u_0) = u_0$. This local solution is such that $u(t, u_0) \in X^{\beta}$, for every $\frac{r}{2} \leq \beta \leq 1$ and $t > 0$ and for as long as it exists. Therefore, $u(t, u_0) \in [C(\bar{\Omega})]^N$ for $t > 0$ and for as long as it exists. If $[0, t_{\max})$ denotes the maximal interval of existence of this solution and $t_0 \in (0, t_{\max})$, we have that $u(t_0, u_0)(x) \in R_\tau$ for some $\tau > 0$. Therefore, from the results in [C-C-D] Section 3.1 we obtain that this solution does not blow up in finite time and as consequence of this, it must exist all the time. Also from the results in [C-C-D] Section 3.1 we obtain that the semigroup associated with (21) is point dissipative in $[C(\bar{\Omega})]^N$. Using the variation of constants formula we obtain that this semigroup is point dissipative in X^{α}, $\alpha \geq \frac{r}{2}$. Theorem 2 in [C-C-D] implies that (21) has a global attractor in X^{β}, $\beta > \frac{r}{2}$, in particular it has a global attractor in $X^{\frac{1}{2}} = [H^1(\Omega)]^N$ and the result is proved.

Remark 2. No gradient structure is assumed for (21). Thus, we have been able to prove the existence of global attractors in $[H^1(\Omega)]^N$ for systems of parabolic problems in the subcritical growth case ($n = 3,\ \gamma < 4$) without assuming that the nonlinearity has a symmetric Jacobian. This is an improvement of our results in [C-C-D] which would only allow for $n = 3,\ \gamma < 3$. We also remark that local existence is proved for $n = 3,\ \gamma < 5$ and because of regularization (see the beginning of this section) we have not been able to prove that there are global solutions for $n = 3,\ 4 \leq \gamma < 5$.

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