UNIVERSIDADE DE SÃO PAULO

Instituto de Ciências Matemáticas e de Computação

ISSN 0103-2577

SOLVABILITY NEAR THE CHARACTERISTIC SET FOR A SPECIAL CLASS OF COMPLEX VECTOR FIELDS

PAULO L. DATTORI DA SILVA
EVANDRO R. DA SILVA

Nº 351

NOTAS DO ICMC

SÉRIE MATEMÁTICA

ICMC USP
SÃO CARLOS

São Carlos – SP
Mai./2011

Solvability near the characteristic set for a special class of complex vector fields

Paulo L. Dattori da Silva and Evandro R. da Silva

Abstract

This article deals with the solvability near the characteristic set $\Sigma = \{0\} \times S^1$ of operators of the form $L = \partial/\partial t + (x^n a(x) + ix^m b(x))\partial/\partial x$, $b \not\equiv 0$ and $a(0) \neq 0$, defined on $\Omega_\epsilon = (-\epsilon, \epsilon) \times S^1$, where a and b are real-valued smooth functions in $(-\epsilon, \epsilon)$ and $m \geq 2n$. It is shown that given $f \in C^\infty(\Omega_\epsilon)$, satisfying a finite number of compatibility conditions, there is $u \in C^0$ solution of the equation $Lu = f$ in a neighborhood of Σ; moreover, the C^0 regularity is sharp.

1 Introduction

Let $\Omega_\epsilon = (-\epsilon, \epsilon) \times S^1$, $\epsilon > 0$, and let

$$L = \partial/\partial t + (a(x) + ib(x))\partial/\partial x, \quad b \not\equiv 0, \tag{1.1}$$

be a complex vector field defined on $\Omega_\epsilon = (-\epsilon, \epsilon) \times S^1$, where a and b are real-valued smooth functions in $(-\epsilon, \epsilon)$.

The characteristic set of L is the set of points $(x, t) \in \Omega_\epsilon$ where L fails to be elliptic, that is, the set

$$\Sigma = \{(x, t) \in \Omega_\epsilon;\ L_{(x,t)} \text{ and } \overline{L}_{(x,t)} \text{ are linearly dependent}\}.$$

It is easy to see that $(x, t) \in \Sigma$ if and only if $b(x) = 0$, that is,

$$\Sigma = \{(x, t) \in \Omega_\epsilon;\ b(x) = 0\}.$$

In this article we will assume that $\Sigma = \{0\} \times S^1$; hence, L is elliptic on $\{x \neq 0\}$ and the well-known *Nirenberg-Treves* condition $(\mathcal{P})$ is satisfied.

Let us recall the definition of condition $(\mathcal{P})$: we say that L given by (1.1) satisfies $(\mathcal{P})$ if the function $b(x)$ does not change sign on any integral curve of the real vector field $\partial/\partial t + a(x)\partial/\partial x$ (see [13], theorem 3.7).

We are interested in solving the equation

$$Lu = f$$

near the characteristic set Σ, where $f \in C^\infty(\Omega_\epsilon)$, in sense of Hörmander (see [12]).

We say that L is solvable at Σ if given f belonging to a subspace of finite codimension of $C^\infty(\Omega_\epsilon)$ there exists $u \in \mathcal{D}'(\Omega_\epsilon)$ solving the equation $\mathsf{L}u = f$ in a neighborhood of Σ.

If $a(0) \neq 0$ then it follows from Hörmander's results (see [12]) that L is solvable at Σ; moreover, solutions can be obtained in C^∞ class.

A slight modification of the arguments in [6] shows that if the function $a + ib$ is flat at $x = 0$ then L is not solvable at Σ.

Hence, we can assume that we have $a(x) + ib(x) = x^n a_0(x) + ix^m b_0(x)$, where $(a_0 + ib_0)(0) \neq 0$; moreover, if the function a (resp. b) is not flat at $x = 0$ then n (resp. m) is the order of vanishing of a (resp. b) at $x = 0$; if the function a (resp. b) is flat at $x = 0$ then $n \geq m$ (resp. $m \geq n$). In this article we are interested in the case $m \geq 2$.

Let $r = \min\{m, n\}$. It is easy to see that if $f \in C^\infty(\Omega_\epsilon)$ is such that there is $u \in C^\ell$, $\ell \geq r$, solution of $Lu = f$ in a neighborhood of Σ then

$$\int_0^{2\pi} \frac{\partial^j f}{\partial x^j}(0, t)dt = 0, \quad j = 0, \ldots, r - 1. \tag{1.2}$$

Now, fixe $k \in \mathbb{Z}_+$. It follows from [3], [10] and [11] that given $f \in C^\infty(\Omega_\epsilon)$, satisfying (1.2), the equation $Lu = f$ has a C^k solution if and only if $2 \leq m \leq 2n - 1$; moreover, if $m > 2n - 1$ then there exists $f \in C^\infty(\Omega_\epsilon)$, flat along Σ, for which there is no $u \in C^1$ solution of the equation $Lu = f$ in any neighborhood of Σ. Thus, the solvability at Σ when $m \geq 2n$ is still an open problem.

Hence, assuming $a_0(0) \neq 0$ and $m \geq 2n$, a natural question appears: given $f \in C^\infty(\Omega_\epsilon)$, satisfying (1.2), is there C^0 solution of the equation $Lu = f$ in some neighborhood of Σ?

Note that when $u \in C^0 \backslash C^1$ the equation $Lu = f$ is to be understood in the distribution sense.

This article concerns with the question above. We will show that for all $f \in C^\infty(\Omega_\epsilon)$, satisfying the compatibility conditions (1.2), there exist $u \in C^0$ solution of $Lu = f$ in a neighborhood of Σ; moreover, we will show that this result is sharp, that is, for each $0 < \alpha < 1$ there exists $f \in C^\infty(\Omega_\epsilon)$, flat along Σ, for which there is no $u \in C^\alpha$ solution of the equation $Lu = f$ in any neighborhood of Σ.

The question addressed here is related to those in [1], [2], [3], [4], [8], [9], [10], [11], and others.

The first author was partially supported by CNPq and FAPESP.

2 Results

Let

$$L = \partial/\partial t + (a(x) + ib(x))\partial/\partial x, \quad b \not\equiv 0, \tag{2.1}$$

be a complex vector field defined on Ω_ϵ, where a and b are C^∞ real-valued functions. Assume that we can write $(a + ib)(x) = x^n a_0(x) + ix^m b_0(x)$, with $n \geq 1$, $m \geq 2n$ and $a_0(x) \neq 0$ for all $x \in (-\epsilon, \epsilon)$. Assume that $\Sigma = \{0\} \times S^1$ is the characteristic set of L.

Theorem 2.1. *Let L be given by (2.1). Given $f \in C^\infty(\Omega_\epsilon)$, satisfying*

$$\int_0^{2\pi} \frac{\partial^j f}{\partial x^j}(0,t)dt = 0, \quad j = 0, \dots, n-1, \tag{2.2}$$

there exists $u \in C^0$ solution of the equation $Lu = f$ in a neighborhood of Σ.

Proof: As done in [3] (see also [6]), given $f \in C^\infty(\Omega_\epsilon)$, satisfying (2.2), there is $v \in C^\infty$ such that $Lv - f$ is flat along Σ. Hence, in order to find a C^0 solution to $Lu = f$ at Σ it is enough to consider f flat.

Let $f \in C^\infty(\Omega_\epsilon)$ be flat along Σ. Now, using partial Fourier series we can write $u(x,t) = \sum_{k\in\mathbb{Z}} \hat{u}_k(x)e^{ikt}$ and $f(x,t) = \sum_{k\in\mathbb{Z}} \hat{f}_k(x)e^{ikt}$. We will use the fact that one has $Lu(x,t) = f(x,t)$ in a neighborhood of Σ if and only if the sequence $(\hat{u}_k)$ satisfies $L_k\hat{u}_k(x) = \hat{f}_k(x)$ in a neighborhood of $x = 0$, $\forall k \in \mathbb{Z}$, where $L_k\hat{u}_k(x) = ik\hat{u}_k(x) + (a+ib)(x)(\hat{u}_k)'(x)$; furthermore, the series $\sum_{k\in\mathbb{Z}} \hat{u}_k(x)e^{ikt}$ converges in the C^0 topology in a neighborhood of Σ.

For $k = 0$ we have to solve $(a+ib)\dfrac{d}{dx}\hat{u}_0(x) = \hat{f}_0(x)$. Since $\hat{f}_0$ is flat at $x = 0$ we have $\dfrac{\hat{f}_0}{a+ib} \in C^\infty$, hence

$$\hat{u}_0(x) = \int_0^x \frac{\hat{f}_0}{a+ib}(x)\mathrm{d}x$$

is a solution in a neighborhood of $x = 0$.

For $k \neq 0$ we will first solve $L_k\hat{u}_k(x) = \hat{f}_k(x)$ for $x > 0$. There is no loss of generality in assuming that $b \geq 0$ for such values of x. We find for $x > 0$ and for each $k > 0$,

$$\hat{u}_k(x) = \int_0^x e^{-ik(C(x)-C(y))}\frac{\hat{f}_k}{a+ib}(y)\mathrm{d}y, \quad \text{where} \quad C(x) = -\int_x^\eta \frac{1}{a+ib}(y)\mathrm{d}y.$$

Hence,

$$|\hat{u}_k(x)| \leq \int_0^x e^{-k\int_y^x \frac{b}{a^2+b^2}(s)\,ds}\left|\frac{\hat{f}_k}{a+ib}\right|(y)\mathrm{d}y \leq \int_0^x \left|\frac{\hat{f}_k}{a+ib}\right|(y)\mathrm{d}y.$$

Now for $x > 0$ and for each $k < 0$ we find

$$\hat{u}_k(x) = -\int_x^\eta e^{-ik(C(x)-C(y))}\frac{\hat{f}_k}{a+ib}(y)\mathrm{d}y + M,$$

where

$$C(x) = -\int_x^\eta \frac{1}{a+ib}(y)\mathrm{d}y$$

and

$$M = \int_0^\eta e^{-ik(C(0)-C(y))}\frac{\hat{f}_k}{a+ib}(y)\mathrm{d}y.$$

Thus, we have

$$|\hat{u}_k(x)| \leq \int_x^\eta e^{k\int_x^y \frac{b}{a^2+b^2}(s)ds}\left|\frac{\hat{f}_k}{a+ib}\right|(y)\mathrm{d}y + \int_0^\eta e^{k\int_0^y \frac{b}{a^2+b^2}(s)ds}\left|\frac{\hat{f}_k}{a+ib}\right|(y)\mathrm{d}y$$

$$\leq 2\int_0^\eta \left|\frac{\hat{f}_k}{a+ib}\right|(y)dy.$$

Note that in both cases ($k>0$ and $k<0$) $\hat{u}_k(0)=0$.

In an analogous way, for each $k\neq 0$, we can solve the equation $L_k\hat{u}_k=\hat{f}_k$ for $x<0$ and the solution satisfies $\hat{u}_k(0)=0$. Hence, for each $k\in\mathbb{Z}$, we obtain a C^0 solution to $L_k\hat{u}_k=\hat{f}_k$ in a neighborhood of $x=0$.

Since $\dfrac{f}{a+ib}\in C^\infty(\Omega_\epsilon)$, we have that there exists $C>0$ such that

$$\left|\frac{\hat{f}_k}{a+ib}(x)\right|\leq\frac{C}{(1+|k|)^2},\qquad\text{for all } k\in\mathbb{Z} \text{ and for all } x\in(-\epsilon,\epsilon);$$

hence, the series $\sum_{k\in\mathbb{Z}}\hat{u}_k(x)e^{ikt}$ converges uniformely. Therefore, $u(x,t)=\sum_{k\in\mathbb{Z}}\hat{u}_k(x)e^{ikt}$ is a continuous solution of the equation $Lu=f$ in a neighborhood of Σ. □

Theorem 2.2. *Let L be given by (2.1). For each fixed $0<\alpha<1$, there exists $f\in C^\infty(\Omega_\epsilon)$, satisfying (2.2), for which the equation $Lu=f$ does not have a C^α solution in any neighborhood of Σ.*

Proof: Assume that for each function $f\in C^\infty(\Omega_\epsilon)$, satisfying (2.2), there exists $u\in C^\alpha$ solution of the equation $Lu=f$ in a neighborhood of Σ. As done in [2] (see also [5] and [11]), an argument using Baire's theorem and the open mapping theorem implies that there exists $\delta_0>0$ such that given $f\in C^\infty([-\delta_0,\delta_0]\times S^1)$, satisfying (2.2), there exists $u\in C^\alpha([-\delta_0,\delta_0]\times S^1)$ solution of $Lu=f$ in some neighborhood of $[-\delta_0,\delta_0]\times S^1$. In order to keep this work as self-contained as possible let us repeat the arguments here.

Let

$$\mathcal{S}=span<1\otimes\delta^{(j)}>,\quad j=0,\ldots,n-1,$$

and let $\mathcal{S}^\circ$ denote the annihilator of $\mathcal{S}$. For each $\ell\in\mathbb{Z}_+$, define

$$F_\ell=\{(f,u)\in(C^\infty([-\epsilon/2,\epsilon/2]\times S^1)\cap\mathcal{S}^\circ)\times C^\alpha(\overline{\mathcal{U}}_\ell);\ Lu=f \text{ on } \mathcal{U}_\ell\},$$

where $\mathcal{U}_\ell=(-\delta_\ell,\delta_\ell)\times S^1$ and $\delta_\ell\downarrow 0$ with $\delta_1=\epsilon/2$. Note that, for each $\ell\in\mathbb{Z}_+$, F_ℓ is an Fréchet space. Consider

$$\begin{array}{rccl}\pi_\ell: & F_\ell & \rightarrow & C^\infty([-\epsilon/2,\epsilon/2]\times S^1)\cap\mathcal{S}^\circ\\ & (f,u) & \mapsto & f\end{array},$$

the projection on the first factor.

Hence, under our assumptions, we have that

$$\bigcup_\ell \pi_\ell(F_\ell)=C^\infty([-\epsilon/2,\epsilon/2]\times S^1)\cap\mathcal{S}^\circ.$$

Finally, from Baire's theorem and the open mapping theorem, we have that

$$\pi_{\ell_0}(F_{\ell_0})=C^\infty([-\epsilon/2,\epsilon/2]\times S^1)\cap\mathcal{S}^\circ,$$

for some ℓ_0. This means that, for $\delta_0 = \delta_{\ell_0}$, the operator

$$L : C^\alpha([-\delta_0, \delta_0] \times S^1) \to C^\infty([-\delta_0, \delta_0] \times S^1) \cap \mathcal{S}^\circ \tag{2.3}$$

is surjective.

Next, we will contradict (2.3). The argument is similar that in [3], which we will recall.

There is no loss of generality in assuming that $b(x) > 0$ if $-\delta_0 < x < 0$. Define

$$g(x) = \begin{cases} -e^{\frac{1}{x}}, & x < 0 \\ 0, & x \geq 0 \end{cases}.$$

For a fixed $x_0 \in (-\delta_0, 0)$ and for each $k \in \mathbb{Z}$, define $(\hat{f}_k)$ by

$$\hat{f}_k(x) = g(x) \cdot (a + ib)(x) \cdot c_k \cdot \; e^{-ik \int_{x_0}^{x} \frac{a}{a^2+b^2}(y)dy},$$

where (c_k) is a rapidly decreasing sequence of positive real numbers (to be chosen later).

It follows from [3] that the sequence $(\hat{f}_k)$ defines a function $f \in C^\infty([-\delta_0, \delta_0] \times S^1) \cap \mathcal{S}^\circ$, flat along Σ.

We claim that for f defined above there is no $u \in C^\alpha([-\delta_0, \delta_0] \times S^1)$ solving $Lu = f$ in a neighborhood of $[-\delta_0, \delta_0] \times S^1$. Recall that in [3] the authors exhibit a similar function f, for which the equation $Lu = f$ does not have C^∞ solution in any neighborhood of $[-\epsilon_0, \epsilon_0] \times S^1$, for some $0 < \epsilon_0 \leq \epsilon/2$.

Let f be the function defined above. From (2.3) there exists a C^α solution u to $Lu = f$ in a neighborhood of $[-\delta_0, \delta_0] \times S^1$. By using Fourier series, the equation $Lu = f$ can be rewritten as

$$\sum_{k \in \mathbb{Z}} (L_k \hat{u}_k) e^{ikt} = \sum_{k \in \mathbb{Z}} \hat{f}_k e^{ikt},$$

where $L_k \hat{u}_k(x) = ik\hat{u}_k(x) + (a+ib)(x)(\hat{u}_k)'(x)$. Hence, for each $k \in \mathbb{Z}$, we have $L_k \hat{u}_k(x) = \hat{f}_k(x)$ and we can write $\hat{u}_k(x) = v_k(x) + h_k(x)$, where h_k is an arbitrary solution of the homogeneous equation $L_k h_k = 0$ and v_k is a solution of the nonhomogeneous equation.

Define

$$C(x) = \int_{-\delta_0}^{x} \frac{1}{a + ib}(y)\mathrm{d}y.$$

It is easy to see that

$$v_k(x) = \begin{cases} 0, & x \geq 0 \\ -\int_x^0 e^{-ik(C(x)-C(y))} \frac{\hat{f}_k}{a+ib}(y)\mathrm{d}y, & x < 0 \end{cases}$$

is a C^∞ solution to $L_k v_k = \hat{f}_k$ in a neighborhood of $[-\delta_0, \delta_0]$.

A simple calculation shows that for $k > 0$ we have

$$\begin{aligned} v_k(x_0) &= -\int_{x_0}^{0} g(y) c_k e^{k \int_{x_0}^{y} \frac{b}{a^2+b^2}(s)ds} \mathrm{d}y \\ &\geq e^{kM} M' c_k, \end{aligned}$$

where $M = \int_{x_0}^{\frac{x_0}{2}} \frac{b}{a^2+b^2}(s)ds > 0$ and $M' = -\int_{\frac{x_0}{2}}^{0} g(y)dy > 0$.

At this point, for each $k > 0$, we choose $c_k = e^{-Mk}$ to the definition of $\hat{f}_k$. Hence, we have that $v_k(x_0) \geq M'$; consequently, the sequence (v_k) cannot be the coefficients of a C^α function at $x = x_0$.

It is easy to see that, for $x < 0$ and for each $k \neq 0$, the function

$$h_k(x) = h_k(-\delta_0)e^{-ik\int_{-\delta_0}^{x}\frac{1}{a+ib}(y)dy}$$

$$= h_k(-\delta_0)e^{-ik\int_{-\delta_0}^{x}\frac{a}{a^2+b^2}(y)dy}e^{-k\int_{-\delta_0}^{x}\frac{b}{a^2+b^2}(y)dy}.$$

is the general solution to the homogeneous equation $L_k h_k(x) = 0$.

We claim that if $h_k(-\delta_0) \neq 0$ then h_k cannot be extended to a C^α function in any neighborhood of $x = 0$. Indeed, if $h_k \in C^\alpha$ in some neighborhood of $x = 0$ then

$$\lim_{x\to 0}\frac{h_k(x) - h_k(0)}{x^\alpha}$$

should exist. However,

$$\lim_{x\to 0^-}\frac{h_k(x)-h_k(0)}{x^\alpha} = \lim_{x\to 0^-}\frac{h_k'(x)}{\alpha x^{\alpha-1}} = \lim_{x\to 0^-}\frac{-ikh_k(x)}{\alpha(a_0(x)+ix^{m-n}b_0(x))}\cdot\frac{1}{x^{n+\alpha-1}},$$

which, of course, does not exist.

Hence, for each $k > 0$, $\hat{u}_k(x) = v_k(x)$ is the only C^α solution to the equation $L_k\hat{u}_k(x) = \hat{f}_k(x)$ in a neighborhood of $[-\delta_0, \delta_0]$. Therefore, $(\hat{u}_k)$ cannot be the sequence of Fourier coefficients of any C^α periodic function, which contradicts (2.3). The proof is complete.□

References

[1] A. P. Bergamasco and A. Meziani, Solvability near the characteristic set for a class of planar vector fields of infinite type, Ann. Inst. Fourier, Grenoble, **55** 1(2005), 77–112.

[2] A. P. Bergamasco, P. D. Cordaro and G. Petronilho, Global solvability for a class of complex vector fields on the two-torus. Comm. in PDE, **29** (2004), 785–819.

[3] A. P. Bergamasco and P. L. Dattori da Silva, Global solvability for a special class of vector fields on the torus, Contemp. Math., **400** (2006), 11–20.

[4] A. P. Bergamasco and P. L. Dattori da Silva, Solvability in the large for a class of vector fields on the torus, J. Math. Pures Appl., **86** (2006), 427–477.

[5] A. P. Bergamasco, P. L. Dattori da Silva and M. R. Ebert, Gevrey solvability near the characteristic set for a class of planar complex vector fields of infinite type, J. Differential Equations **246**(4) (2009), 1673-1702 .

[6] A.P. Bergamasco and G. Petronilho, *Closedness of the range for vector fields on the torus*, J. Differential Equations **154** (1999), 132–139.

[7] S. Berhanu and A. Meziani, Global properties of a class of planar vector fields of infinite type, Comm. PDE **22** (1997), 99–142.

[8] S. Berhanu and A. Meziani, On rotationally invariant vector fields in the plane. Manuscripta Math. **89**(3) (1996), 355–371.

[9] W. A. Cerniauskas and A. Kirilov, C^k-Solvability Near the Characteristic Set for a Class of Vector Fields of Infinite Type, Mat. Contemp. **36** (2009), 91106.

[10] P. L. Dattori da Silva, Nonexistence of Global Solutions for a Class of Complex Vector Fields on two-torus, J. Math. Anal. Appl., **351** (2009), 543–555.

[11] P. L. Dattori da Silva, *C^k-solvability near the characteristic set for a class of planar complex vector fields of infinite type*, Ann. Mat. Pura Appl., **189** (2010), 403–413.

[12] L. Hörmander, *The analysis of linear partial differential operators IV*, Springer-Verlag, 1984.

[13] L. Hörmander, Pseudo-differential operators of principal type, Nato Advanced Study Inst. on Singularities in Boundary Value Problems, Reidel Publ. Co., Dordrecht, 1981, 69–96.

(Paulo L. Dattori da Silva) Universidade de São Paulo, Instituto de Ciências Matemáticas e de Computação, Departamento de Matemática, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail address: dattori@icmc.usp.br

(Evandro R. da Silva) Universidade de São Paulo, Instituto de Ciências Matemáticas e de Computação, Departamento de Matemática, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail address: evandro@icmc.usp.br

SÉRIE MATEMÁTICA

NOTAS DO ICMC

351/11 SILVA, P. L. D. ; SILVA, E. R. - Solvability near the characteristic set for a special class of complex vector fields.

350/11 BONOTTO, E. M. ; DEMUNER, D. P. - Autonomous dissipative semidynamical sustems with impulses.

348/11 ARAGÃO-COSTA, E. R. ; CARABALLO, T. ; CARVALHO, A. N. ; LANGA, J. A. – Stability of gradient semigroups under perturbation.

347/11 DUTERTRE, N. ; GRULHA JR, N. – Lê-Greuel type formula for the Euler obstruction and applications.

346/11 RUAS, M. ; FERNANDES, A. – Regidity of bi-lipschitz equivalence of weighted homogeneous function-germs in the plane.

345/11 MATTOS, D. de; SANTOS, E. L. dos; COELHO, F. R. de C.. – On the existence of G-equivariant maps.

344/11 JORGE-PÉREZ, V. H.; MIRANDA, A. J.; SAIA, M. J. - Counting singularities via fitting ideals.

343/11 AZEVEDO, K. A. G.; BONOTTO, E. M. – On asymptotic stability in impulsive semidynamical systems.

342/11 OLIVEIRA, R. D. S.; ZHAO, YULIN – On the structural stability of planar quasi-homogeneous polynomial vector fields.

341/10 LUCAS, L. A.; MANZOLI NETO, O.; SAEKI, O. – Exteriors of codimension one embeddings of product of three spheres into spheres.

340/10 ARRIETA, J. M.; CARVALHO, A. N.; PEREIRA, M. C.; SILVA, R. P. Nonlinear parabolic problems in thin domains with a highly oscillatory boundary.