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Invariants and relative invariants under compact Lie groups

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This paper presents algebraic methods for the study of polynomial relative invariants, when the group Γ formed by the symmetries and relative symmetries is a compact Lie group and the homomorphism $\sigma : \Gamma \to \mathbf{Z}_m$ has a kernel H of index m, $m \geq 2$, on Γ. For this, we develop the invariant theory of compact Lie groups acting on complex vector spaces. This is the starting point for the study of relative invariants and the computation of their generators. In this paper we first obtain the space of the invariants under the subgroup H of Γ as a direct sum of m submodules over the ring of invariants under the whole group. Based on this decomposition we construct a Hilbert basis of the ring of Γ-invariants from a Hilbert basis of the ring of H-invariants. In both results a knowledge of the relative Reynolds operators defined on H-invariants is shown to be an essential preliminary to an adequate study of invariants under the whole group. The theory is illustrated with some examples.

Key Words: symmetry, invariant theory, relative invariants, finite index subgroups, Hilbert basis
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1. INTRODUCTION

Many natural phenomena possess symmetry properties. In particular, symmetries can occur in a dynamical system, meaning transformations that take solutions to solutions. The equations describing a physical or biological system may have symmetries as a result of its geometry, modeling assumptions and simplifying normal forms. So when this occurrence is taken into account in the mathematical formulation of the physical problem, it can simplify significantly their interpretation. The natural language for describing symmetry properties

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of a dynamical system is that of group theory. The main point is that, in this case, the set Γ formed by these symmetries has structure of group and the differential equations of such a system remain unchanged under the action of Γ. More specifically, consider the system

$$\dot{x} \; = \; G(x) \tag{1}$$

defined on a finite-dimensional vector space V of state variables, where $G : V \to V$ is a smooth vector field. We assume that Γ is a compact Lie group acting orthogonally on V. If G is *purely* $\Gamma-$*equivariant*, i.e.,

$$G(\gamma x) \; = \; \gamma\, G(x) \;, \quad \forall \gamma \in \Gamma, \; x \in V, \tag{2}$$

then $x(t)$ is a solution of (1) if, and only if, $\gamma x(t)$ is a solution of (1), for all $\gamma \in \Gamma$.

It is possible to classify solutions of equivariant dynamical systems based on their symmetry properties. The effect of symmetries in local and global dynamics has been investigated for decades by a great number of authors in many papers (an extensive list of references can be found in [5]) .

There is a parallel class of equivariant dynamical systems, in the sense of (2), which are those in simultaneous presence of symmetries and reversing symmetries. A system is said to have a reversing symmetry if the equations of motion are invariant under that symmetry, in combination with the inversion of time. This contrasts with a symmetry which leaves the equations of motion invariant without time inversion. In this context, the differential equation (1) is invariant under the transformation $(t, x) \mapsto (t, \gamma x)$ if γ is a symmetry and under the transformation $(t, x) \mapsto (-t, \gamma x)$ if γ is a reversing symmetry. In algebraic terms, this corresponds to defining a group homomorphism

$$\sigma : \Gamma \to \mathbf{Z}_2, \tag{3}$$

where $\mathbf{Z}_2$ denotes the multiplicative group $\{1, -1\}$, whose kernel is the subgroup Γ_+ of the symmetries in Γ. The complement $\Gamma_- = \Gamma \setminus \Gamma_+$ is the set of reversing symmetries. Under this formalism, it is obvious that the symmetries of Γ form a normal subgroup of index two of Γ and

$$\Gamma = \Gamma_+ \dot{\cup}\; \delta\Gamma_+,$$

for an arbitrary fixed $\delta \in \Gamma_-$. The mapping $G : V \to V$ is called *reversible-equivariant* if

$$G(\gamma x) \; = \; \sigma(\gamma)\gamma\, G(x) \;, \quad \forall \gamma \in \Gamma, \; x \in V. \tag{4}$$

Then $x(t)$ is a solution of (1) if, and only if, $\gamma x(\sigma(\gamma)t)$ is a solution of (1), for all $\gamma \in \Gamma$.

The interest in reversible-equivariant dynamics is recent. The first impulse to the subject came from the classification of the reversible-equivariant linear systems in terms of group representation theory (see [10]). In [2] we have developed an approach based on singularity theory in combination with group representation theory, in the same lines as Golubitsky et al. [9], for the systematic analysis of reversible-equivariant steady-state bifurcations of the so called "self-dual" case. In another direction, Buono et al. [4] have employed

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equivariant transversality in order to tackle the case of "separable bifurcations". Since then, the subject has become an important area of research in dynamical systems and their applications. More recently, we have used in [3] techniques of group representation theory to describe the decomposition σ-isotypic in the reversible-quivariant context, which is the corresponding isotypic decomposition presented in [9] in the equivariant context.

The starting point for local or global analysis of reversible-equivariant systems is to find the general form of the vector field G in (1) that satisfies (4). The theorems by Schwarz and Poénaru (see Golubitsky et al. [9]) reduce this task to a purely algebraic problem in invariant theory, which, in turn, can be solved by symbolic computation. Therefore, formulae for the number of invariants and reversible-equivariants under the action of Γ are very useful in our study, being used to find the general form of the vector field. In [1] , we have obtained alternative formulae involving the character (trace) of the representation of Γ for calculating the number of reversible-equivariant applications by the action of Γ on V for a given degree. For this, we make use of an important tool in computational invariant theory, called the Hilbert-Poincar series, also known as Molien generating function for the dimensions of the spaces of invariant and equivariant homogeneous polynomials of each degree. In the program GAP [7] there is a function to compute the Hilbert-Poincar series of invariants and equivariants, being necessary to provide, in addition to the generators of the group, the character of the representation. The algorithms for calculating invariants presented by Sturmfels [15] make use of Molien theorem for finite groups, explicit formulae for the Hilbert-Poincar series as an Haar integral under the group. In particular in the equivariant context, algorithms are obtained in [8, 11, 15] and associated Molien functions with a representation of a compact Lie group are investigated in [6, 14].

Also in [1], we have established algebraic results that reduce the search for the general form of the vector field to the polynomial invariant case. Based on a link existent between the invariant theory for Γ and for its subgroup Γ_+, we have provided a set of generators for the modules of relative invariants and reversible-equivariants, from generators of the ring of invariants under the action of Γ_+ on V, whose solution is well known in several important cases, as in [8, 15, 16]. More specifically, consider σ in (3) the non trivial homomorphism. A polynomial function $f : V \to \mathbb{R}$ is called *anti-invariant* if

$$f(\gamma x) \;=\; \sigma(\gamma)f(x) \;, \tag{5}$$

for all $\gamma \in \Gamma$ and $x \in V$. When σ in (3) is the trivial homomorphism, we recognize the function f satisfying (5) as a Γ-*invariant* function, or simply an *invariant* function.

The space $\mathcal{Q}(\Gamma)$ of all anti-invariant polynomial functions is a finitely generated graded module over the ring $\mathcal{P}(\Gamma)$ of invariant polynomial functions on V – which is also finitely generated according to the Hilbert-Weyl Theorem (see [9, Theorem XII 4.2]). In [1] we have obtained the decomposition

$$\mathcal{P}(\Gamma_+) \;=\; \mathcal{P}(\Gamma) \oplus \mathcal{Q}(\Gamma) \tag{6}$$

as a direct sum of modules over $\mathcal{P}(\Gamma)$. This decomposition is the basis for a second result, namely the algorithm that computes generating sets for anti-invariant polynomials as a module over the ring of the invariant polynomials. The procedure is based on the Reynolds

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operator from $\mathcal{P}(\Gamma_+)$ onto $\mathcal{Q}(\Gamma)$ applied to a Hilbert basis of $\mathcal{P}(\Gamma_+)$, which, in turn, can be readily obtained by the standard methods of computational invariant theory applied to Γ_+.

In this paper we extend the formalism described above when the set of the symmetries H is a normal subgroup of index m greater than 2. The functions now are complex-valued and the representation of Γ is a complex representation. In this new context, we shall also assume that there is a surjective group homomorphism

$$\sigma : \Gamma \to \mathbf{Z}_m, \tag{7}$$

where $\mathbf{Z}_m$ denotes the cyclic group generated by the m-th root of unity, with $H = \ker(\sigma)$.

We remark that when $m = 2$ the existence of the homomorphism (3) is a natural consequence of the fact that Γ_+ to have index 2. The same argument holds for any m prime. If m is not prime, we notice that this may not hold. In fact, when $m = 4$, for $\Gamma = \mathbf{Z}_4 \times \mathbf{Z}_2$ there does not exist a surjective group homomorphism from Γ to $\mathbf{Z}_4$ if $H = \mathbf{Z}_2 \times \mathbf{1}$ (although for this case it does exist for another choice of an index-4 subgroup in Γ). Hence, the existence of the surjective homomorphism (7) is assumed throughout.

If we now fix $\delta \in \Gamma \setminus H$ such that $\delta^m \in H$ (or $\sigma^m(\delta) = 1$), then we have a decomposition of Γ as a disjoint union of left-cosets:

$$\Gamma = H \;\dot{\cup}\; \delta\, H \;\dot{\cup}\; \delta^2\, H \;\dot{\cup}\; \ldots \;\dot{\cup}\; \delta^{m-1}\, H. \tag{8}$$

In section 2, we present our first main result, Theorem 2.1, which is to obtain $\mathcal{P}(H)$ as a direct sum of m submodules over $\mathcal{P}(\Gamma)$. This extends the decomposition (6) when $m = 2$ for the cases $m > 2$. This also extends the analogous result obtained by Larry in [12] in the case that the group is finite for compact Lie groups.

In section 3 we obtain the other two main results, Theorem 3.1 for $m = 2$ and Theorem 3.2 for $m > 2$, which are applications of the decomposition given in section 2. The result is to construct a Hilbert basis of the ring $\mathcal{P}(\Gamma)$ from a Hilbert basis of the ring $\mathcal{P}(H)$. The main idea of this construction is to make use of relative Reynolds operators defined on $\mathcal{P}(H)$ that project onto each of its m components. We point out that the interest of this result is that from the knowledge of the invariants just under the symmetries in Γ we can recover the generators of the invariants under the whole group. Some examples are presented.

2. THE STRUCTURE OF THE σ-RELATIVE INVARIANTS

In this section we give some basic concepts and obtain results about the structure of the set of relative invariant functions.

2.1. The Representation Theory

Let Γ be a compact Lie group acting linearly on a finite-dimensional vector space V over $\mathbf{C}$. This action corresponds to a representation ρ of the group Γ on V through a linear

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homomorphism from Γ to the group $\mathbf{GL}(V)$ of invertible linear transformations on V. We denote by (ρ, V) the vector space V together with the linear action of Γ induced by ρ. We often write γx for the action of the linear transformation $\rho(\gamma)$ of an element $\gamma \in \Gamma$ on a vector $x \in V$, whenever there is no danger of confusion.

Now consider the homomorphism $\sigma : \Gamma \to \mathbf{Z}_m$ given in (7). We notice that σ defines a one-dimensional representation of Γ in $\mathbf{C}$, by $x \mapsto \sigma(\gamma)x$ for $x \in \mathbf{C}$. Also, the character afforded by the homomorphism σ is the complex-valued function whose value on $\gamma \in \Gamma$ is $\sigma(\gamma)$; in other words, it is σ itself, but regarded as a complex-valued function, which we shall also denote by σ.

Definition 2.1. For each $j \in \{0, \dots, m-1\}$, a polynomial function $f : V \to \mathbf{C}$ is called σ^j-relative invariant if

$$f(\gamma x) = \sigma^j(\gamma) f(x) \tag{9}$$

for all $\gamma \in \Gamma$ and $x \in V$. $\diamondsuit$

The set of the σ^j-relative invariant polynomial functions is a module over $\mathcal{P}(\Gamma)$ and we shall denoted it by $\mathcal{P}_{\sigma^j}(\Gamma)$. Notice that we can regard a σ^j-relative invariant polynomial function on V as an equivariant mapping from (ρ, V) to $(\sigma^j, \mathbf{C})$. Also, the notion of anti-invariant polynomial function given by the equation (5) is a particular case of the definition of an σ-relative invariant function when $m = 2$. Therefore, the existence of a finite generating set for $\mathcal{P}_{\sigma^j}(\Gamma)$ and $\mathcal{Q}(\Gamma)$ is guaranteed by Poénaru's Theorem (see [9, Theorem XII 6.8]).

In the next lemma we show the relation between the σ-relative invariants and the invariants under the action of the subgroup $H = \ker \sigma$.

Lemma 2.1. *Let σ the homomorphism given in (7), with kernel H. Fix $\delta \in \Gamma \setminus H$ such that $\delta^m \in H$, that is, $\sigma(\delta)$ is the m-th root of unity. Then for each $j \in \{0, \dots, m-1\}$, we have*

$$\mathcal{P}_{\sigma^j}(\Gamma) = \{f \in \mathcal{P}(H) : f(\delta x) = \sigma^j(\delta) f(x)\,,\ \forall\, x \in V\}.$$

Proof. The inclusion " $\subseteq$ " follow directly from the definition of $\mathcal{P}_{\sigma^j}(\Gamma)$. The inclusion " $\supseteq$ " follow directly from (8) and the fact that if $f(\delta x) = \sigma^j(\delta) f(x)$, then $f(\delta^k x) = \sigma^j(\delta^k) f(x)$, for all $x \in V$ and $j, k \in \{1, \dots, m-1\}$. ▮

2.2. The Application of the Invariant Theory

In this subsection we relate the rings of invariants under the action of Γ and H, and the modules of σ^j-relative invariants playing a fundamental role in this construction. We observe that $\mathcal{P}(H)$ also has a module structure over the ring $\mathcal{P}(\Gamma)$.

We start by introducing the σ^j*-relative Reynolds operators* on $\mathcal{P}(H)$, $R_j : \mathcal{P}(H) \to \mathcal{P}(H)$, by

$$R_j(f)(x) \;=\; \frac{1}{m}\sum_{\gamma H} \overline{\sigma^j(\gamma)} f(\gamma x),$$

for each $j \in \{0, \ldots, m-1\}$, where bar means the complex conjugation. So

$$\begin{aligned} R_j(f)(x) &= \frac{1}{m}(f(x) + \overline{\sigma^j(\delta)} f(\delta x) + \ldots + \overline{\sigma^j(\delta^{m-1})} f(\delta^{m-1}x)) \\ &= \frac{1}{m}\sum_{k=0}^{m-1} \overline{\sigma^{jk}(\delta)} f(\delta^k x), \end{aligned} \tag{10}$$

for an arbitrary (and fixed) $\delta \in \Gamma \setminus H$ such that $\delta^m \in H$.

We notice that for $m = 2$, the Reynolds operator $R^{\Gamma}_{\Gamma_+}$ and the Reynolds σ-operator $S^{\Gamma}_{\Gamma_+}$ defined in [1] correspond to R_0 and R_1, respectively. The R_1 operator has been used there to compute a generating set for the σ-relative invariant polynomial functions as a module over the ring of the invariants.

Remark 2. 1. Choose $\delta \in \Gamma \setminus H$ such that $\delta^m \in H$. Then we have

$$\sum_{i=0}^{m-1} \sigma^{ik}(\delta) = 1 + \sigma^k(\delta) + \sigma^{2k}(\delta) + \ldots + \sigma^{(m-1)k}(\delta) = 0, \tag{11}$$

for all $k \in \{1, \ldots, m-1\}$.

Now let us denote by $I_{\mathcal{P}(H)}$ the identity map on $\mathcal{P}(H)$. We then have:

PROPOSITION 2.1. *For each $j \in \{0, \ldots, m-1\}$, the Reynolds operators R_j satisfy the following properties:*

(*i*) *They are homomorphisms of $\mathcal{P}(\Gamma)$-modules and*

$$R_0 + R_1 + \ldots + R_{m-1} \;=\; I_{\mathcal{P}(H)}\, . \tag{12}$$

(*ii*) *They are idempotent projections, with* $\mathrm{Im}(R_j) \;=\; \mathcal{P}_{\sigma^j}(\Gamma)$.
(*iii*) *For all* $1 \leq j \leq m-1$, *we have*

$$\mathcal{P}_{\sigma^j}(\Gamma) \cap (\mathcal{P}(\Gamma) + \ldots + \mathcal{P}_{\sigma^{j-1}}(\Gamma)) = \{0\}.$$

Proof. To prove (i) we just note that if $h, g \in \mathcal{P}(H)$ and $f \in \mathcal{P}(\Gamma)$, then

$$R_j(h+g) \;=\; R_j(h) + R_j(g) \qquad \text{and} \qquad R_j(f\,h) \;=\; f\,R_j(h),$$

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for all $j \in \{0, \ldots, m-1\}$. Moreover, (12) follows directly from (10) and (11).

To prove (ii) and (iii), we fix $\delta \in \Gamma \setminus H$ such that $\delta^m \in H$. For any $f \in \mathcal{P}(H)$, it is direct from (10) and the Lemma 2.1 that $R_j(f) \in \mathcal{P}_{\sigma^j}(\Gamma)$. Hence, $\mathrm{Im}(R_j) \subseteq \mathcal{P}_{\sigma^j}(\Gamma)$. The other inclusion is immediate, since $R_j|_{\mathcal{P}_{\sigma^j}(\Gamma)} = I_{\mathcal{P}_{\sigma^j}(\Gamma)}$.

We prove (iii) by induction on j. It is easy to see that $\mathcal{P}_\sigma(\Gamma) \cap \mathcal{P}(\Gamma) = \{0\}$. Suppose now that given $1 \leq s \leq m-2$, we have

$$\mathcal{P}_{\sigma^j}(\Gamma) \cap \big(\mathcal{P}(\Gamma) + \ldots + \mathcal{P}_{\sigma^{j-1}}(\Gamma)\big) = \{0\}, \tag{13}$$

for all $1 \leq j \leq s$. We show that (13) is true for $s+1$. In fact, let $f \in \mathcal{P}_{\sigma^{s+1}}(\Gamma) \cap \big(\mathcal{P}(\Gamma) + \ldots + \mathcal{P}_{\sigma^s}(\Gamma)\big)$. Then

$$f(\delta x) = \sigma^{s+1}(\delta) f(x) \qquad \text{and} \qquad f(x) = h_0(x) + \ldots + h_s(x),$$

with $h_j \in \mathcal{P}_{\sigma^j}(\Gamma)$. So

$$(\sigma^{s+1}(\delta) - 1)h_0(x) + \ldots + (\sigma^{s+1}(\delta) - \sigma^s(\delta))h_s(x) = 0,$$

for all $x \in V$. By the induction hypothesis $(\sigma^{s+1}(\delta) - \sigma^j(\delta))h_j(x) = 0$, for all $0 \leq j \leq s$. It follows that $h_j \equiv 0$, for all $0 \leq j \leq s$, as desired. ▌

It follows from the previous proposition that, for each $j \in \{0, \ldots, m-1\}$, the polynomial functions $f \in \mathcal{P}_{\sigma^j}(\Gamma)$ are also characterized by the equality $R_j(f) = f$.

We are now in position to state the following theorem, our first main result, which is now an immediate consequence of the proposition above.

Theorem 2.1. *The following direct sum decomposition of $\mathcal{P}(\Gamma)$−modules holds:*

$$\mathcal{P}(H) \;=\; \mathcal{P}(\Gamma) \oplus \mathcal{P}_\sigma(\Gamma) \;\oplus\; \ldots \;\oplus \mathcal{P}_{\sigma^{m-1}}(\Gamma).$$

3. HILBERT BASIS FOR THE Γ-INVARIANTS

In this section we show how to compute generators for $\mathcal{P}(\Gamma)$, as a ring, from a Hilbert basis of $\mathcal{P}(H)$, where H is a normal subgroup of index m of Γ. The basic idea here is to take advantage of the direct sum decomposition from Theorem 2.1 to transfer generating sets from one ring to the other. The procedure is algorithmic and it holds, as its proof, for any m. However, the case $m = 2$ is special in the sense that a distinguished proof reveals a class of invariants and anti-invariants that does not appear in the proof for $m > 2$. For this reason, we present the two cases separately, as Theorems 3.1 and 3.2. Some examples are given.

We start with index $m = 2$. For this case, if $u : V \to \mathbb{R}$ and $\gamma \in \Gamma$, we denote by $\gamma u : V \to \mathbb{R}$ the polynomial function defined by

$$\gamma u(x) := u(\gamma x), \quad \forall x \in V.$$

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LEMMA 3.1. *Let $u : V \to \mathbb{R}$ be a real-valued function. Then for all $t \in \mathbb{N}$,*

$$u^t + \delta u^t = F_t\big(R_0(u), R_1^2(u)\big), \tag{14}$$

$$\sum_{k=0}^{t} u^k \delta u^{t-k} = G_t\big(R_0(u), R_1^2(u)\big), \tag{15}$$

and

$$u^t - \delta u^t = R_1(u) H_t\big(R_0(u), R_1^2(u)\big), \tag{16}$$

for $F_t, G_t, H_t : \mathbb{R}^2 \to \mathbb{R}$ polynomial functions.

Proof. We check (14) and (15) by induction on t, by using the polynomial identity

$$u^i + \delta u^i \;=\; (u + \delta u)(u^{i-1} + \delta u^{i-1}) - (u\ \delta u)(u^{i-2} + \delta u^{i-2})$$

in the first case and the identity

$$\sum_{k=0}^{i} u^k \delta u^{i-k} = (u^i + \delta u^i) + (u\ \delta u) \sum_{k=0}^{i-2} u^k \delta u^{i-2-k}$$

in the second case.

To prove (16) just write

$$u^t - \delta u^t = (u - \delta u)\Big(\sum_{k=0}^{t-1} u^k \delta u^{t-1-k}\Big)$$

and use (15). ∎

THEOREM 3.1. *Let Γ be a compact Lie group acting on V and let $\sigma : \Gamma \to \mathbf{Z}_2$ a homomorphism of Lie groups. Let $\{u_1, \ldots, u_s\}$ be a Hilbert basis of the ring $\mathcal{P}(\Gamma_+)$, where Γ_+ is the kernel of the homomorphism σ. Then the set*

$$\{R_0(u_i), R_1(u_i)R_1(u_j),\ 1 \le i, j \le s\}$$

form a Hilbert basis of the ring $\mathcal{P}(\Gamma)$.

Proof. Let us fix $\delta \in \Gamma \setminus \Gamma_+$ and let $\{u_1, \ldots, u_s\}$ be a Hilbert basis of $\mathcal{P}(H)$. We need to show that every polynomial function $f \in \mathcal{P}(\Gamma)$ can be written as

$$f = h\big(R_0(u_i), R_1(u_i)R_1(u_j),\ 1 \le i, j \le s\big),$$

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where $h : \mathbb{R}^{s(s+1)/2} \to \mathbb{R}$ is a polynomial function. The proof is done by induction on the cardinality s of the set $\{u_1, \ldots, u_s\}$.

Let $f \in \mathcal{P}(\Gamma)$. From Proposition 2.1, there exists $\tilde{f} \in \mathcal{P}(\Gamma_+)$ such that $R_0(\tilde{f}) = f$. First we write

$$\tilde{f}(x) \;=\; \sum_{\alpha} a_\alpha \, u^\alpha(x) \;,$$

with $a_\alpha \in \mathbb{R}$, where $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^s$, and $u^\alpha \;=\; u_1^{\alpha_1} \ldots u_s^{\alpha_s}$. Then

$$f \;=\; \sum_{\alpha} a_\alpha \, (u^\alpha + \delta u^\alpha) \;.$$

Assume $s = 1$. Then we may write $f \;=\; \sum_i a_i \, (u^i + \delta u^i)$, where $a_i \in \mathbb{R}$ and $i \in \mathbb{N}$. Now we use (14) to get

$$f \;=\; \sum_i a_i \, h_i\big(R_0(u), R_1^2(u)\big) \;=\; h\big(R_0(u), R_1^2(u)\big) \;,$$

with $h : \mathbb{R}^2 \to \mathbb{R}$ a polynomial function.

Assume that given $l \in \mathbb{N}$, for all sets $\{u_1, \ldots, u_l\}$ with $1 \leq l \leq s-1$ we have

$$\sum_{\alpha} a_\alpha \, (u^\alpha + \delta u^\alpha) \;=\; h_\alpha\big(R_0(u_i), R_1(u_i) R_1(u_j),\; 1 \leq i, j \leq l\big), \tag{17}$$

where $a_i \in \mathbb{R}$, $\alpha \in \mathbb{N}^l$ and $h_\alpha : \mathbb{R}^{l(l+1)/2} \to \mathbb{R}$ is a polynomial function. Consider the set $\{u_1, \ldots, u_l\} \cup \{u_{l+1}\}$ and let $f \in \mathcal{P}(\Gamma)$. Then we may write

$$\begin{aligned}
f &= \sum_{\alpha, \alpha_{\ell+1}} a_{\alpha, \alpha_{\ell+1}} (u^\alpha u_{\ell+1}^{\alpha_{\ell+1}} + \delta u^\alpha \delta u_{\ell+1}^{\alpha_{\ell+1}}) \\
&= \frac{1}{2}\Big[\sum_{\alpha, \alpha_{\ell+1}} a_{\alpha, \alpha_{\ell+1}} (u^\alpha + \delta u^\alpha)(u_{\ell+1}^{\alpha_{\ell+1}} + \delta u_{\ell+1}^{\alpha_{\ell+1}}) + \sum_{\alpha, \alpha_{\ell+1}} a_{\alpha, \alpha_{\ell+1}} (u^\alpha - \delta u^\alpha)(u_{\ell+1}^{\alpha_{\ell+1}} - \delta u_{\ell+1}^{\alpha_{\ell+1}})\Big] \\
&= \frac{1}{2}\;\Big[\sum_{\alpha_{\ell+1}} (u_{\ell+1}^{\alpha_{\ell+1}} + \delta u_{\ell+1}^{\alpha_{\ell+1}}) \sum_{\alpha} a_{\alpha, \alpha_{\ell+1}} (u^\alpha + \delta u^\alpha) + \sum_{\alpha_{\ell+1}} (u_{\ell+1}^{\alpha_{\ell+1}} - \delta u_{\ell+1}^{\alpha_{\ell+1}}) \sum_{\alpha} a_{\alpha, \alpha_{\ell+1}} (u^\alpha - \delta u^\alpha)\Big],
\end{aligned}$$

with u^α, $u_{\ell+1}^{\alpha_{\ell+1}} \in \mathcal{P}(\Gamma_+)$, $\alpha \in \mathbb{N}^\ell$ and $\alpha_{\ell+1} \in \mathbb{N}$.

By (16) we have

$$u_{l+1}^{\alpha_{l+1}} - \delta u_{l+1}^{\alpha_{l+1}} = R_1(u_{l+1}) H_{\alpha_{l+1}}\big(R_0(u_{l+1}), R_1^2(u_{l+1})\big)$$

and by the induction hypothesis (17) we can write

$$u_{\ell+1}^{\alpha_{\ell+1}} + \delta u_{\ell+1}^{\alpha_{\ell+1}} \;=\; h_{\alpha_{\ell+1}}\big(R_0(u_{l+1}), R_1^2(u_{l+1})\big)$$

and

$$\sum_{\alpha} a_{\alpha, \alpha_{\ell+1}} (u^\alpha + \delta u^\alpha) \;=\; h_{\alpha, \alpha_{\ell+1}}\big(R_0(u_i), R_1(u_i) R_1(u_j),\; 1 \leq i, j \leq l\big),$$

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with $H_{\alpha_{\ell+1}}, h_{\alpha_{\ell+1}}, h_{\alpha,\alpha_{\ell+1}}$ polynomial functions. Then

$$\begin{aligned} f = \frac{1}{2}\, [\sum_{\alpha_{\ell+1}} & h_{\alpha_{\ell+1}}(R_0(u_{l+1}), R_1^2(u_{l+1}))h_{\alpha,\alpha_{\ell+1}}(R_0(u_i), R_1(u_i)R_1(u_j),\ 1 \leq i,j \leq l) \\ & + \sum_{\alpha_{\ell+1}} R_1(u_{l+1})H_{\alpha_{l+1}}(R_0(u_{l+1}), R_1^2(u_{l+1}))\sum_{\alpha} a_{\alpha,\alpha_{\ell+1}}(u^\alpha - \delta u^\alpha)]. \end{aligned} \tag{18}$$

Now we prove by induction on l that for all $1 \leq i \leq l$ and $1 \leq k, m \leq j-1$, $k \neq m$, we have

$$u^\alpha - \delta u^\alpha = \sum_{j=1}^{l} R_1(u_j)H_{\alpha,j}(R_0(u_i), R_1^2(u_i), R_1(u_k)R_1(u_m)), \tag{19}$$

where $H_{\alpha,j} : \mathbb{R}^n \to \mathbb{R}$ are polynomial functions with $n = 2l + \frac{(j-1)(j-2)}{2}$. In fact, for $l = 1$ it is obvious by (16). Assume then that

$$u^{\tilde{\alpha}} - \delta u^{\tilde{\alpha}} = \sum_{j=1}^{l-1} R_1(u_j)H_{\tilde{\alpha},j}(R_0(u_i), R_1^2(u_i), R_1(u_k)R_1(u_m)), \tag{20}$$

where $1 \leq i \leq l-1$, $1 \leq k, m \leq j-1$, $k \neq m$, with $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_{l-1})$ and $u^{\tilde{\alpha}} = u_1^{\alpha_1} \ldots u_{l-1}^{\alpha_{l-1}}$. We write

$$u^\alpha - \delta u^\alpha = \frac{1}{2}[(u^{\tilde{\alpha}} - \delta u^{\tilde{\alpha}})(u_l^{\alpha_l} + \delta u_l^{\alpha_l}) + (u^{\tilde{\alpha}} + \delta u^{\tilde{\alpha}})(u_l^{\alpha_l} - \delta u_l^{\alpha_l})].$$

By the induction hypotheses (17) and (20), we have

$$\begin{aligned} u^\alpha - \delta u^\alpha = \frac{1}{2}[\sum_{j=1}^{l-1} & R_1(u_j)H_{\tilde{\alpha},j}(R_0(u_i), R_1^2(u_i), R_1(u_k)R_1(u_m))h_{\alpha_l}(R_0(u_l), R_1^2(u_l)) \\ & + h_{\tilde{\alpha}_l}(R_0(u_i), R_1(u_i)R_1(u_n))R_1(u_l)H_{\alpha_l}(R_0(u_l), R_1^2(u_l))], \end{aligned}$$

where $1 \leq i, n \leq l-1$ and $1 \leq k, m \leq j-1$, with $k \neq m$.

Thus

$$\begin{aligned} u^\alpha - \delta u^\alpha = [\sum_{j=1}^{l-1} & R_1(u_j)G_{\alpha,j}(R_0(u_i), R_1^2(u_i), R_1(u_k)R_1(u_m)) \\ & + R_1(u_l)G_{\alpha_l}(R_0(u_i), R_1^2(u_i), R_1(u_p)R_1(u_q))], \end{aligned}$$

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where $1 \leq i \leq l$, $1 \leq k, m \leq j-1$, with $k \neq m$, and $1 \leq p, q \leq l-1$, with $p \neq q$, which completes the proof of (19). Therefore

$$\begin{aligned}\sum_{\alpha} a_{\alpha,\alpha_{\ell+1}}(u^{\alpha} - \delta u^{\alpha}) &= \sum_{j=1}^{l} R_1(u_j) \sum_{\alpha} a_{\alpha,\alpha_{\ell+1}} H_{\alpha,j}\big(R_0(u_i), R_1^2(u_i), R_1(u_k)R_1(u_m)\big) \\ &= \sum_{j=1}^{l} R_1(u_j) H_j\big(R_0(u_i), R_1^2(u_i), R_1(u_k)R_1(u_m)\big),\end{aligned}$$

with $1 \leq i \leq l$, $1 \leq k, m \leq j-1$, $k \neq m$. Returning to (18) we obtain

$$\begin{aligned} f = {} & H_{\alpha,\alpha_{\ell+1}}\big(R_0(u_i), R_1^2(u_i), R_1(u_k)R_1(u_m)\big) \\ & + \sum_{j=1}^{l} R_1(u_j)R_1(u_{l+1})H_{j,\alpha_{l+1}}\big(R_0(u_i), R_1^2(u_i), R_1(u_p)R_1(u_q)\big)\big], \end{aligned}$$

with $1 \leq i \leq l+1$, $1 \leq k, m \leq l$, $k \neq m$, and $1 \leq p, q \leq j-1$, $p \neq q$. Therefore

$$f = H\big(R_0(u_i), R_1(u_i)R_1(u_j), 1 \leq i, j \leq l+1\big),$$

as desired. ▮

Example 3.1. $[\Gamma = \mathbf{O}(2)]$ Consider the orthogonal group $\mathbf{O}(2)$ acting on $\mathbf{C} \times \mathbb{R}$, where the rotations $\theta \in \mathbf{SO}(2)$ and the flip κ act as

$$\theta(z, x) = (e^{i\theta} z, x) \qquad \text{and} \qquad \kappa(z, x) = (\bar{z}, -x).$$

Consider $\Gamma_+ = \mathbf{SO}(2)$ and $\delta = \kappa \in \Gamma_-$. We have that $u_1(z, x) = z\bar{z}$ and $u_2(z, x) = x$ form a Hilbert basis of $\mathcal{P}(\mathbf{SO}(2))$. We have

$$R_0(u_1)(z, x) = z\bar{z}, \quad R_1(u_2)(z, x) = x, \quad R_0(u_2)(z, x) = R_1(u_1)(z, x) = 0.$$

By Theorem 3.1, the ring $\mathcal{P}(\mathbf{O}(2))$ is generated by

$$R_0(u_1)(z, x) = z\bar{z} \quad \text{and} \quad R_1^2(u_2)(z, x) = x^2. \qquad \diamondsuit$$

Example 3.2. $[\Gamma = (\mathbf{D}_6 \dot{+} \mathbf{T}^2) \oplus \mathbf{Z}_2]$ Consider the group $\Gamma = (\mathbf{D}_6 \dot{+} \mathbf{T}^2) \oplus \mathbf{Z}_2$ acting on $\mathbf{C}^3$ as follows: for $(z_1, z_2, z_3) \in \mathbf{C}^3$,

(a) the permutations in $\mathbf{D}_6$ permute the coordinates of (z_1, z_2, z_3),
(b) the generator flip in $\mathbf{D}_6$ acts as $(z_1, z_2, z_3) \mapsto (\bar{z}_1, \bar{z}_2, \bar{z}_3)$,

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(c) $\theta = (\theta_1, \theta_2)$ in the torus $\mathbf{T}^2$ acts as

$$\theta \cdot (z_1, z_2, z_3) = (e^{i\theta_1 p} z_1, e^{i\theta_2 p} z_2, e^{-i(\theta_1+\theta_2)p} z_3),$$

(d) the reflection $\kappa \in \mathbf{Z}_2$ acts as minus the identity:

$$\kappa(z_1, z_2, z_3) = (-z_1, -z_2, -z_3).$$

In this example we consider $\delta = \kappa \in \Gamma_-$ and so $\Gamma_+ = \mathbf{D}_6 \dot{+} \mathbf{T}^2$.
Let $v_j = z_j \bar{z}_j$, $j = 1, 2, 3$ and consider the elementary symmetric polynomials in v_j :

$$u_1 = v_1 + v_2 + v_3, \quad u_2 = v_1 v_2 + v_1 v_3 + v_2 v_3, \quad u_3 = v_1 v_2 v_3.$$

Also, let $u_4 = z_1 z_2 z_3 + \bar{z}_1 \bar{z}_2 \bar{z}_3$. It is well known that $\{u_1, \ldots, u_4\}$ is a Hilbert basis for $\mathcal{P}(\mathbf{D}_6 \dot{+} \mathbf{T}^2)$ (see Golubitsky et al. [9, Theorem 3.1(a), p. 156]).

We have $R_0(u_j) = u_j$ and $R_1(u_j) = 0$, for $j = 1, 2, 3$, $R_0(u_4) = 0$ and $R_1(u_4) = u_4$. By Theorem 3.1, the ring $\mathcal{P}(\Gamma)$ is generated by u_1, u_2, u_3 and u_4^2. $\Diamond$

The following theorem is an extension of Theorem 3.1 in the computation of Γ-invariant generators for index m greater than 2. The basic idea here is to take into account that $f \in \mathcal{P}(\Gamma)$ if, and only if, $R_0(f) = f$ and then to deduce the invariant generators under the action of Γ from a Hilbert basis of the ring of invariants under the action of the subgroup H.

THEOREM 3.2. *Let Γ be a compact Lie group acting on V and let H be the kernel of the homomorphism given in (7). Let $\{u_1, \ldots, u_s\}$ be a Hilbert basis of the ring $\mathcal{P}(H)$. For each $j \in \{1, \ldots, m-1\}$, set $\alpha(j) = \sum_{i=1}^{s} \alpha_{ji}$, with $\alpha_{ji} \in \mathbb{N}$. Then the invariants*

$$R_0(u_1), R_0(u_2), \ldots, R_0(u_s),$$

$$R_1(u_1)^{\alpha_{11}} \ldots R_1(u_s)^{\alpha_{1s}} \ldots R_{m-1}(u_1)^{\alpha_{m-1,1}} \ldots R_{m-1}(u_s)^{\alpha_{m-1,s}},$$

such that $\sum_{j=1}^{m-1} j\alpha(j) \equiv 0 \bmod m$, form a Hilbert basis of the ring $\mathcal{P}(\Gamma)$.

Proof. Let us fix $\delta \in \Gamma \setminus H$ such that $\delta^m \in H$ and let $\{u_1, \ldots, u_s\}$ be a Hilbert basis of $\mathcal{P}(H)$. Let $f \in \mathcal{P}(\Gamma)$. From Proposition 2.1, $R_0(f) = f$, which gives

$$f(x) = \frac{1}{m-1} \sum_{k=1}^{m-1} f(\delta^k x), \quad \forall\, x \in V. \tag{21}$$

For each $i \in \{1, \ldots, s\}$ we have $u_i = \sum_{j=0}^{m-1} R_j(u_i)$. So

$$\{R_0(u_i), R_1(u_i), \ldots, R_{m-1}(u_i) : 1 \le i \le s\}$$

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is a new Hilbert basis of $\mathcal{P}(H)$. Using multi-index notation, every polynomial function $f \in \mathcal{P}(\Gamma)$ can be written as

$$f = \sum a_\alpha R_0(u_i)^{\alpha_0} R_1(u_i)^{\alpha_1} \ldots R_{m-1}(u_i)^{\alpha_{m-1}} ,$$

with $a_\alpha \in \mathbf{C}$, where $\alpha = (\alpha_0, \ldots, \alpha_{m-1})$, $\alpha_j = (\alpha_{j1}, \ldots, \alpha_{js}) \in \mathbb{N}^s$ and $R_j(u_i)^{\alpha_j} = R_j(u_1)^{\alpha_{j1}} \ldots R_j(u_s)^{\alpha_{js}}$, for $j \in \{0, \ldots, m-1\}$.

Now, for every $k \in \mathbb{N}$ and $x \in V$, we have

$$R_j(u_i)(\delta^k x) = \sigma^j(\delta^k) R_j(u_i)(x) = \sigma^{jk}(\delta) R_j(u_i)(x),$$

from which we obtain

$$f(\delta^k x) = \sum \sigma(\delta)^{k \sum_{j=1}^{m-1} j\alpha(j)} a_\alpha R_0(u_i)^{\alpha_0} R_1(u_i)^{\alpha_1} \ldots R_{m-1}(u_i)^{\alpha_{m-1}}(x), \tag{22}$$

where $\alpha(j) = \alpha_{j1} + \ldots + \alpha_{js}$, for all $1 \le j \le m-1$.

We now use (22) in (21) to get

$$\sigma(\delta)^N + (\sigma(\delta)^N)^2 + \ldots + (\sigma(\delta)^N)^{m-1} = m - 1,$$

where $N = \sum_{j=1}^{m-1} j\alpha(j) \in \mathbb{N}$. It follows from Remark 2.1 that $\sigma^N(\delta) = 1$, ie, $N \equiv 0 \bmod m$. Therefore the generators of $\mathcal{P}(\Gamma)$, as a ring, are given by

$$R_0(u_1), R_0(u_2), \ldots, R_0(u_s)$$

and by the products

$$R_1(u_1)^{\alpha_{11}} \ldots R_1(u_s)^{\alpha_{1s}} \ldots R_{m-1}(u_1)^{\alpha_{m-1,1}} \ldots R_{m-1}(u_s)^{\alpha_{m-1,s}},$$

for $\sum_{j=1}^{m-1} j\alpha(j) \equiv 0 \bmod m$. ▌

EXAMPLE 3.3. [$\Gamma = \mathbf{Z}_3 \times \mathbf{Z}_3$] Consider the group $\mathbf{Z}_3 \times \mathbf{Z}_3$ acting on $V = \mathbf{C}^2$, where $\xi = (\xi_1, \xi_2) \in \Gamma$ act on $z = (z_1, z_2) \in V$ as

$$\xi(z) = (\xi_1 z_1, \xi_2 z_2).$$

Fix $\sigma : \Gamma \to \mathbf{Z}_3$ such that $H = \ker \sigma = \mathbf{Z}_3 \times 1$ and set $\delta = (1, e^{\frac{2\pi i}{3}}) \in \Gamma \setminus H$. We notice that $\delta^3 \in H$. The functions u_i, $1 \le u_i \le 5$, given by

$$u_1(z_1, \bar{z}_1, z_2, \bar{z}_2) = z_1 \bar{z}_1, \quad u_2(z_1, \bar{z}_1, z_2, \bar{z}_2) = z_1^3, \quad u_3(z_1, \bar{z}_1, z_2, \bar{z}_2) = \bar{z}_1^3,$$

$$u_4(z_1, \bar{z}_1, z_2, \bar{z}_2) = z_2, \quad u_5(z_1, \bar{z}_1, z_2, \bar{z}_2) = \bar{z}_2$$

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form a Hilbert basis of $\mathcal{P}(H)$. In this case, we have $R_0(u_i) = u_i$, for $1 \leq i \leq 3$, $R_1(u_4) = u_4$ and $R_2(u_5) = u_5$. By the Theorem 3.2, the generators of the ring $\mathcal{P}(\Gamma)$ are

$$R_0(u_1),\ R_0(u_2),\ R_0(u_3),\ R_0(u_4),\ R_0(u_5),\ R_1(u_i)R_1(u_j)R_1(u_k),$$

$$R_2(u_i)R_2(u_j)R_2(u_k),\ R_1(u_i)R_2(u_j),$$

for all $1 \leq i, j, k \leq 5$. Therefore the ring $\mathcal{P}(\mathbf{Z}_3 \times \mathbf{Z}_3)$ is generated by the elements

$$z_1\bar{z}_1,\ z_1^3,\ \bar{z}_1^3,\ z_2\bar{z}_2,\ z_2^3,\ \bar{z}_2^3.$$

$\diamond$

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